Measurable Ambiguity †

Faruk Gul

and

Wolfgang Pesendorfer

Princeton University

August 2008

Abstract

We introduce *subjective expected uncertain utility theory (SEUU)*. In SEUU the decision maker uses a semiprobability to assess the likelihood of events. The semiprobability allows the decision maker to reduce acts to *bilotteries*. A bilottery specifies for each interval of monetary prizes $[x, y]$ the probability that the decision maker will end up with a prize in this interval. A bilottery allows for the possibility that the probability of receiving a prize in the interval $[x, y]$ cannot be reduced to the probability of receiving a prize in the subinterval $[x, w)$ and $[w, y]$. The decision maker evaluates bilotteries by taking the expectation of a utility index u that specifies a utility for each interval $[x, y]$. We provide a Savage style representation theorem for SEUU theory, define uncertainty aversion and characterize the corresponding order on bilotteries.

† This research was supported by grants from the National Science Foundation.

1. Introduction

In subjective expected utility theory (SEU), a decision-maker (DM) evaluates uncertain acts in three stages: first, he assesses the likelihoods of the relevant events. These assessments enable the DM to reduce uncertain acts into gambles in a straightforward manner, and finally, the DM evaluates gambles according to their expected utilities. More precisely, in SEU theory the decision-maker has a probabilistic *event-assessment* criterion that enables him to reduce acts into lotteries and an expected utility *gamble-evaluation* criterion. Both the event-assessment stage and the gamble-evaluation stage reveal the DM's subjective criteria: the event-assessment stage reveals the subjective likelihoods while the gamble-evaluation stage identifies the subjective utility. Moreover, one cardinal index–a probability measure over events–describes the event-assessment criterion and another cardinal index–a von Neumann Morgenstern utility–defines the gamble-evaluation criterion. Hence, SEU theory is simple in two distinct senses: it permits the separation of event-assessment and gamble-evaluation and it provides measurable (i.e., cardinal) theories of event-assessment and gamble-evaluation.

The separation between event-assessment and gamble-evaluation plays an important role in economics. For example, most applied work assumes greater uniformity across individuals' event-assessments than in their gamble-evaluations. Hence, subjective probabilities are often common (or at least common up to differences in information), while utility functions over gambles tend to vary across individuals, at least parametrically. Moreover, many economic models ignore the event-assessment stage entirely and assume that the individual faces choices in reduced form; that is, the individual chooses among gambles. This simplification enables the modeler to avoid specifying the consequences of every possible combination of actions and resolution of uncertainty, and facilitates distributional assumptions over lotteries that make it easy to calibrate and test models. Hence, standard models of finance and macroeconomics start with random variables (i.e., gambles) rather than acts. Many important economic insights stem from the simplicity that results from the separation of event-assessment and gamble-evaluation and the opportunity it provides for constructing models with exogenous probabilities that can be estimated through econometric methods.

Machina and Schmeidler (1992) describe how the separation of event-assessment and gamble-evaluation can be achieved for virtually all models that define gambles as lotteries over prizes. Thus, they provide an interpretation of what probability means in a general framework. The goal of this paper is to bring the same separation to the analysis of the Ellsberg paradox and related questions of ambiguity.

We outline a model of choice under uncertainty–the subjective expected uncertain utility (SEUU) model–that is as simple as the SEU model, achieves separation between event-assessment and gamble-evaluation and is suitable for analyzing ambiguity. The model has a novel event-assessment criterion called *semiprobabilistic sophistication*, reduces acts into gambles called *bilotteries*, and evaluates these gambles according to an expected utility criterion.

A semiprobability μ is a generalization of a probability measure and a special case of a capacity that permits ambiguity and quantifies the ambiguity of each event. Like a probability, a semiprobability is a unique cardinal measure and is a probability whenever all events are unambiguous. A bilottery, p, is a probability distribution over pairs of prizes. The outcome (x, y) should be interpreted as a situation where the prize is between x and y and the the decision maker cannot reduce the uncertain consequences into risk over outcomes.

Like subjective expected utility theory, SEUU theory is defined by two cardinal indices, a utility index and a semiprobability. Unlike SEU, it permits ambiguous events and uncertain consequences.

We consider preferences over acts $\mathcal{F}$ on a state space Ω. Acts return monetary prizes in an interval $[l, m]$. SEUU decision makers are characterized by a pair (μ, u) where μ is a semiprobability and u is a utility index over pairs. The value $u(x, y)$ is the utility of the bilottery that yields the interval $[x, y]$ with probability 1.

For any act f the decision maker can determine the corresponding bilottery as follows. Define

$$\mathbf{b}(x, y) = \mu(s \in \Omega | x \leq f(s) \leq y\}$$

The function $\mathbf{b}(x, y)$ is simply the semiprobability of the collection of states that yield prizes in the interval $[x, y]$. We refer to $\mathbf{b}$ as the *bicumulative*; for every bicumulative there is a unique bilottery and for every bilottery there is a unique bicumulative.[1] The utility index u assigns a utility to each pair and the subjective expected uncertain utility of the act f is given by

$$\int u dp$$

analogous to subjective expected utility theory.

Our main result provides a Savage-style representation theorem for SEUU theory. Our assumptions (axioms) deviate from the standard Savage axioms in two instances. We define unambiguous events ($E \in \mathcal{E}$) to be those events for which Savage's sure thing principle (P2) holds. We drop P2 but assume a rich collection of unambiguous events so that over unambiguous acts (acts that are measurable with respect to the unambiguous events) the decision maker is a subjective expected utility maximizer.

An event is *diffuse* if it and its complement do not contain any unambiguous event. Hence, diffuse events are completely $\mathcal{E}$-non-measurable. Our central hypothesis is that the decision maker considers diffuse events to be interchangeable, that is, a bet on one diffuse event is just as good as a bet on another diffuse event. In our model, the decision maker's ability to quantify the likelihood of events is confined to a collection of unambiguous events. Diffuse events intersect every unambiguous event but contain no unambiguous event and therefore the decision maker cannot differentiate between the likelihoods of different diffuse events. Our axioms are straightforward adaptations of the standard Savage axioms to include this hypothesis.

Consider the bilottery that yields with probability 1/2 each the uncertain interval $[x, y]$ and the uncertain interval $[w, z]$. The decision maker is uncertainty averse if he prefers a δ increase of x (the lower bound of an uncertain interval) to a δ increase of z (the upper bound of some other uncertain interval). Uncertainty aversion implies that the utility index satisfies for all x, y, w, z, δ

$$u(x + \delta, y) - u(x, y) \geq u(w, z + \delta) - u(w, z)$$

[1] The bicumulative is analogous to a standard cumulative distribution in $\mathbb{R}^2$ and must satisfy conditions described in section 2.

The bilottery q is a mean-preserving u-spread of the bilottery p if p and q have the same mean but p puts more weight on smaller intervals. Theorem 1 shows that uncertainty averse decision makers are made worse off by mean-preserving u-spreads. Conversely, if p and q have identical means and q is not a mean-preserving u-spread of p then there is an uncertainty averse decision maker who prefers q to p.

The model is related to the work of Jaffray (1983), Sarin and Wakker (1992), Zhang (2002), Epstein and Zhang (2001). Jaffray (1983) develops the gamble evaluation criterion used in this paper. We discuss the relation to Jaffray's work following Proposition 1 in section 3 below. As in Epstein and Zhang (2001) and Zhang (2002), we define the collection of unambiguous events $\mathcal{E}$ to be those events for which preferences are separable. In particular, preferences satisfy Savage's sure thing principle on unambiguous events.[2]

2. Expected Uncertain Utility

We consider monetary prizes and denote with l the smallest and with m the largest prize. The interval $M = [l, m]$ denotes the set of prizes. A bilottery is a probability measure on the set of pairs $I = \{(x, y) : l \leq x \leq y \leq m\}$. The pair (x, y) is interpreted as getting at least the monetary prize x and at most the monetary prize y. The set of bilotteries is denoted $\mathcal{L}$ with generic element $p \in \mathcal{L}$.[3]

A utility function U on I is *expected uncertain utility* if U is continuous and satisfies $U(\lambda p+(1-\lambda)p') = \lambda U(p)+(1-\lambda)U(p')$. Expected uncertain utility (EUU) is characterized by a bi-utility index $u : I \to I\!R$ as the following proposition demonstrates.

Proposition 1: *The function $U : \mathcal{L} \to I\!R$ is expected uncertain utility if and only if there is a continuous biutility index $u : I \to I\!R$ such that*

$$U(p) = \int u(x, y)dp(x, y) \tag{1}$$

[2] Epstein and Zhang (2001) and Zhang (2002) use weaker notions of separability. Epstein and Zhang's objective is to obtain a decision maker who is probabilistically sophisticated on the collection of unambiguous events whereas our decision maker is an SEU maximizer on the collection of unambiguous events. Sarin and Wakker (1992) consider an exogenous collection of unambiguous events and assume that the decision maker satisfies the Sure Thing Principle for those events.

[3] We endow $\mathcal{L}$ with the topology of weak convergence.

Expected uncertain utility was introduced by Jaffray (1983) who takes simple measures over sets of prizes as a primitive and applies the von Neumann-Morgenstern axioms to derive a linear representation. He argues that only the best and worst element of a set should affect utility and thereby arrives at the representation in equation (1) above.[4]

Next, we define *bicumulatives* which provide a convenient representation of bilotteries. The bicumulative $\mathbf{b} : I\!R^2 \to I\!R$ for the bilottery p is defined as

$$\mathbf{b}(a, b) = p([a, b] \times [a, b])$$

that is, $\mathbf{b}(a, b)$ is the probability that realized pair is in the square $[a, b] \times [a, b]$. The value $\mathbf{b}(a, b)$ corresponds to the measure of the grey triangle in Figure 1 below.

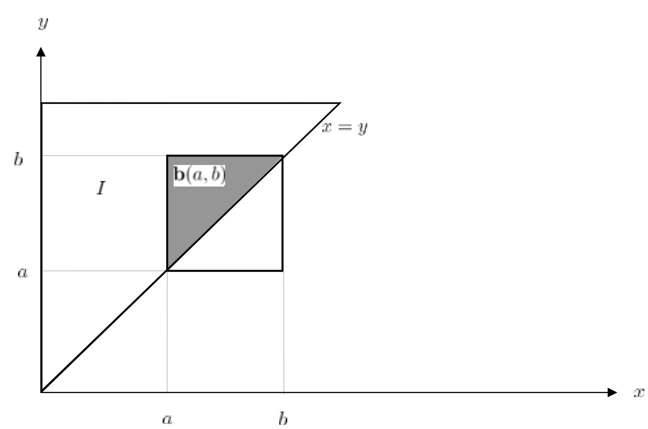

Figure 1

Since p is a (countably additive) probability measure the function $\mathbf{b}$ is continuous from above in the second argument and continuous from below in the first argument. Let A denote a rectangle $[a_1, a_2] \times [b_1, b_2]$ with $a_1 \leq a_2, b_1 \leq b_2$ and define

$$\Delta_A \mathbf{b} = \mathbf{b}(a_1, b_2) - \mathbf{b}(a_1, b_1) - \mathbf{b}(a_2, b_2) + \mathbf{b}(a_2, b_1)$$

[4] More precisely, Jaffray (1983) applies the von Neumann-Morgenstern axioms to preferences over discrete totally monotone capacities to obtain linear preferences over such capacities. He applies the Mobius transform to totally monotone capacities to identify them with probability distributions over sets of prizes. Thus, he interprets linear preferences over capacities with expected utility preferences over lotteries over sets. Finally, he argues that sets that have the same best and worst elements should be indifferent and hence arrives at expected uncertain utility.

As illustrated in Figure 2, it is straightforward to see that $\Delta_A \mathbf{b} = p(A)$ and therefore $\Delta_A \mathbf{b} \geq 0$ for all rectangles A.

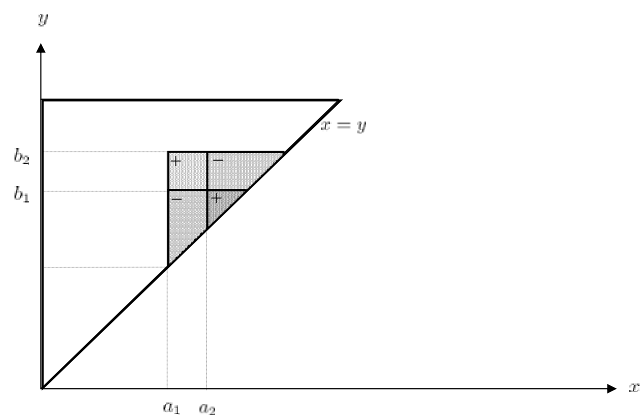

Figure 2

The following proposition shows that to every bicumulative there is a unique measure as long as $\mathbf{b}$ satisfies the conditions above.

Proposition 2: *Suppose that* $\mathbf{b} : R^2 \rightarrow I\!R$ *is continuous from below in the first argument, continuous from above in the second argument, satisfies* $\mathbf{b}(x, y) = 0$ *for* $x > y$ *and* $\Delta_A \mathbf{b} \geq 0$ *for every rectangle* A*. Then there exists a unique measure* p *on* I *such that* $\mathbf{b}(x, y) = p([x, y] \times [x, y])$.

The bicumulative $\mathbf{b}$ is similar to a cumulative distribution function for the measure p. A standard cumulative describes the measure of pairs (a, b) with $a \leq x, b \leq y$ while $\mathbf{b}$ describes the measure of pairs (a, b) with $x \leq a, b \leq y$. In all other respects, $\mathbf{b}$ is exactly analogous to a cumulative for p.

3. Subjective EUU

Let Ω be the state space with the cardinality of the continuum. The decision maker has preferences over acts, that is, functions f from Ω to M. We denote with $\mathcal{F}$ the set of all acts.

A subjective EUU decision maker is characterized by a *semiprobability* μ defined on Ω and a biutility index u on I. A semiprobability is a generalization of a probability measure and a special case of a capacity. The semiprobability captures the decision-makers assessment of events and enables the decision maker to reduce acts to bilotteries.

Let Σ be an algebra on Ω. The pair (μ, Σ) is a *capacity* if $\mu : \Sigma \to [0, 1]$ satisfies (i) $\mu(A) \geq 0$; (ii) $\mu(\Omega) = 1$ and (iii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$. The capacity (μ, Σ) is *supermodular* if

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$$

for all $A, B \in \Sigma$.

Definition: *The supermodular capacity μ is a semiprobability if*

(i) for all A there exists $E \subset A$ such that $\mu(E) = \mu(A)$ and $\mu(E) + \mu(E^c) = 1$.

(ii) $\{A_i\}_{i=1}^{\infty}$ with $A_i \subset A_{i+1}$ implies $\mu\left(\cap_{i-1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$.

The set E should be interpreted as an unambiguous set. Hence, a semiprobability is a supermodular capacity with the property that each set is as likely as some unambiguous subset. Part (ii) of the definition is analogous to the requirement of countable additivity for a standard probability measure.

To map any act into a bilottery, we use the same procedure that would be appropriate for a probabilistically sophisticated decision-maker: each interval of prizes $[x, y]$ gets assigned the probability of all the states that get mapped into prizes in that interval. The resulting function is a bicummulative and, by Proposition 2 above, corresponds to a unique bilottery. Let

$$\begin{aligned}\mathbf{b}_f(x, y) =& \mu(\{\omega \in \Omega \mid f(\omega) \in [x, y]\}) \\ =& \mu(f^{-1}([x, y]))\end{aligned}$$

Proposition 3: *If μ is a semiprobability then $\mathbf{b}_f$ is a bicumulative.*

Proof: Lemma A3 part (i) in Appendix A. $\square$

Propositions 3 implies that each act f corresponds to a unique bicumulative $\mathbf{b}_f$ and therefore, by Proposition 2, to a unique bilottery. We use p_f to denote the bilottery generated by the act f. To simplify the notation we suppress the dependence of p_f on the underlying semiprobability μ.

Definition: *The binary relation $\succeq$ is a SEUU preference if there is a semiprobability μ and a continuous utility index u such that*

$$f \succeq g \text{ if and only if } \int u dp_f \geq \int u dp_g \tag{1}$$

We say that (μ, u) represents the SEUU preference $\succeq$ if (1) holds for the semiprobability μ and the biutility index u.

We identify $x \in M$ with the constant act that yields x in every state. The decision-maker is described by a binary relation $\succeq$ on $\mathcal{F}$. Our main result (Theorem 1 below) shows that $\succeq$ is a SEUU preference if and only if it satisfies the following 6 axioms.

Axiom 1: *The binary relation $\succeq$ is complete and transitive.*

Axiom 2: *If $f(s) > g(s)$ for all $s \in \Omega$, then $f \succ g$.*

The prizes in our model are interpreted as quantities of money and Axiom 2 expresses the natural implication of that interpretation.

For any $f, g \in \mathcal{F}$ and $A \subset \Omega$, let fAg denote the act h such that $h(s) = f(s)$ for all $s \in A$ and $h(s) = g(s)$ for all $s \in A^c$. Hence, xAy denotes the act that yields x if A occurs and y otherwise. Consider two acts that imply different subacts on the event E but have a common subact on E^c. If the event E is *unambiguous* then the ranking of acts does not depend on the common subact on E^c. Similarly, if two acts differ on E^c but have a common subact on E then the ranking of acts does not depend on the common subact.

Definition: *An event E is it unambiguous if $fEh \succeq gEh$ and $hEf \succeq hEf$ implies $fEh' \succeq gEh'$ and $h'Ef \succeq h'Ef$.*

An event is unambiguous if Savage's sure thing principle holds with respect to E and E^c. Our definition of unambiguous events is related to Zhang (2002), Epstein and Zhang (2001) and Sarin and Wakker (1992). Sarin and Wakker assume an exogenous collection of unambiguous events and require Savage's sure thing principle to hold for those events. Epstein and Zhang define unambiguous events to be those events for which a weakened version of the sure thing principle applies. Our model combines the stronger requirement of Sarin and Wakker with Epstein and Zhang's idea to derive the set of unambiguous events from preferences.

Let $\mathcal{E}$ be the set of all unambiguous events. Henceforth E, E', E_i etc. will denote elements of $\mathcal{E}$. An event A is *null* if $fAh \sim gAh$ for all $f, g, h \in \mathcal{F}$. If A is not null, we call it non-null. Let $\mathcal{E}_+$ denote the set of unambiguous events that are not null.

A *diffuse* event has the property that it and its complement intersects every non-null unambiguous event. Diffuse events are maximally ambiguous in the sense that the decision maker cannot find any (non-null) unambiguous event contained in it or its complement.

Definition: *An event D is diffuse if $E \cap D \neq \emptyset \neq E \cap D^c$ for every $E \in \mathcal{E}_+$.*

Let $\mathcal{D}$ be the set of all diffuse events. Elements of $\mathcal{D}$ are denoted D, D', D_i.

Example: Suppose the state space is the interval $[0, 1]$ and the unambiguous events are given by the Borel sigma algebra on $[0, 1]$. A diffuse event D is any completely non-measurable set, that is, any set with inner Lebesgue measure 0 and outer Lebesgue measure 1. It is well known that such sets exist.

Our maintained hypothesis is that the decision maker cannot discriminate between diffuse events. That is, if given the choice between a bet on the diffuse event D_1 or the diffuse event D_2, the decision maker is indifferent. The decision maker's ability to assess the likelihood of events is confined to unambiguous events. Since each diffuse event intersects every unambiguous event, the decision maker regards them as interchangeable.

Axiom 3 (i) below is identical to Savage's comparative probability axiom (P4) for unambiguous events. Axiom 3(ii) says that diffuse events are interchangeable.

Axiom 3: *If $x > y$ and $x' > y'$, then (i) $xEy \succeq xE'y$ implies $x'Ey' \succeq x'E'y'$ and (ii) $xDy \sim xD'y$.*

Let $\mathcal{F}_o$ denote the set of simple acts, that is, acts such that $f(\Omega)$ is finite. The simple act $f \in \mathcal{F}_o$ is unambiguous if $f^{-1}(x) \in \mathcal{E}$ for all x. Let $\mathcal{F}_e$ denote the set of unambiguous simple acts. A simple act f is diffuse if $f^{-1}(x) \in \mathcal{D} \cup \emptyset$. An act is constant if $f^{-1}(x) \in \Omega \cup \emptyset$. Let $\mathcal{F}_d$ be the collection of diffuse or constant simple acts. Note that constant acts are in $\mathcal{F}_d$ and in $\mathcal{F}_e$.

The standard state independence assumption requires that the ranking of constant acts be the same conditional on any non-null event. Axiom 4 below requires the same for unambiguous events. In that sense, Axiom 4 below weakens the standard state independence assumption. On the other hand, Axiom 4 requires state independence to hold not just for constant acts but for all *diffuse acts* that is, acts that are measurable with respect to the collection of diffuse events. This strengthening of state-independence follows from our hypothesis that diffuse events are interchangeable. To see this, consider the diffuse act xDy. The event $D \cap E$ is a diffuse subset of E as is the event $D^c \cap E$. Therefore, conditional on any unambiguous event E the act xDy yields x on a diffuse subset of E and y on its (diffuse) complement in E. Therefore, xDy is analogous to a constant act; it yields identical diffuse bets conditional on any unambiguous event. If utility is state independent, the ranking of diffuse acts must therefore be preserved when conditioning on a non-null unambiguous event.

Axiom 4: *If E is nonnull, then $f \succ g$ implies $fEh \succ gEh$ for all $f, g \in \mathcal{F}_d$.*

Axiom 5 is Savage's divisibility axiom for unambiguous events. It serves the same role here as in Savage.

Axiom 5: *If $f, g \in \mathcal{F}_e$ and $f \succ g$, then there exists a partition $E_1, \ldots, E_n$ of Ω such that $lE_if \succ mE_ig$ for all i.*

Axiom 6 below is a strengthening of Savage's dominance condition adapted to our setting. We use it to extend the representation from simple acts to all acts, to establish continuity of u and to guarantee countable additivity of the semiprobability μ.

Axiom 6: *(i) If $f_n \in \mathcal{F}_e$ converges pointwise to f, then $g \succeq f_n \succeq h$ for all n implies $g \succeq f \succeq h$. (ii) If $f_n \in \mathcal{F}$ converges uniformly to f, then $g \succeq f_n \succeq h$ for all n implies $g \succeq f \succeq h$.*

Notice that for unambiguous acts $f \in \mathcal{F}_e$ Axiom 6(i) implies Arrow's (1970) monotone continuity axiom, the standard axiom used to establish countable additivity of the probability measure in SEU.

The semiprobability μ is convex-valued if for every A and $r \leq 1$ there is $B \subset A$ such that $\mu(B) = r\mu(A)$.

Theorem 1: *The binary relation $\succeq$ satisfies Axioms $1-6$ if and only if there is a convex-valued semiprobability μ and a biutility index u such that (μ, u) represent the preference $\succeq$. Moreover, the semiprobability is unique and the biutility index is unique up to positive affine transformations.*

Proof: See Appendix.

Next, we provide a brief description of the proof of Theorem 1. When we restrict attention to unambiguous events, Axioms 1-6 yield a standard subjective expected utility theory with a countably additive probability measure ν and a continuous utility index $v : M \rightarrow I\!R$. The semiprobability μ is defined as $\mu(A) = \max_{E \subset A} \nu(E)$. Note the maximum is attained by some unambiguous event since ν is countably additive and since the collection of unambiguous events form a sigma-algebra.

A *partition act* is a simple act f with the following property. There is a partition of Ω into the unambiguous events $(E_1, \ldots, E_k)$ and a collection of diffuse or constant acts $(f_1, \ldots, f_k)$ such that f coincides with f_k on E_k. A key step in the proof of Theorem 1 is to show that for any simple act $\hat{f} \in \mathcal{F}_o$ we can find an equivalent partition act f. Equivalent acts differ only on null events. As part of this argument, we show that Ω can be partitioned into any finite number of diffuse sets. This step uses a Theorem by Birkhoff (1967) which in turn uses the continuum hypothesis.[5]

[5] Birkhoff (1967), Theorem 13 (pg. 266) shows that no nontrival (i.e., not identically equal to 0) countably additive measure such that every singleton has measure 0 can be defined on the algebra of all subsets of the continuum.

A *binary* partition act is a partition act where each f_k is either a constant act or takes the form xDy for some x, y and some diffuse set D. A simple monotonicity argument shows that any partition act is indifferent to a binary partition act. To see this, let D_1, D_2, D_3 be a partition of Ω into three diffuse events and consider the act xD_1yD_2z with $x < y < z$. By monotonicity

$$xD_1yD_2z \succeq xD_1 \cup D_2z$$

and

$$xD_1z \succeq xD_1yD_2z$$

and by Axiom 3,

$$xD_1 \cup D_2z \sim xD_1z$$

and therefore $xD_1 \cup D_2z \sim xD_1yD_2z \sim xD_1z$.

The diffuse act xDy corresponds to the bilottery 1_{xy} that yields the pair (x, y) with probability 1. The biutility index of the pair x, y is defined as the utility of this bilottery, that is,

$$u(x, y) := U(1_{xy})$$

Note that $U(1_{xx}) = v(x) = u(x, x)$. Consider a binary partition act f with partition $(E_1, \ldots, E_k)$ that yields x_iDy_i on E_i. This act corresponds to a simple bilottery

$$p_f = \sum_{i=1}^{k} \mu(E_i) I_i(x_i, y_i)$$

The utility function

$$U(p_f) = \sum_{i=1}^{k} \mu(E_i) u(x_i, y_i)$$

represents the preference for simple acts. The extension to all acts uses Axiom 6 and follows familiar arguments.

3.1 Relation to the Literature

The notion of a semiprobability is related to work by Zhang (2002) and Lehrer (2006). Zhang (2002) considers acts in the Savage setting. He axiomatizes Choquet expected utility with inner probabilities. An inner probability is a special kinds of capacity that is the extension of regular probability measure μ on subalgebra Σ. The extension defines the capacity $\eta(A)$ as the supremum over probabilities $\mu(B)$ as B ranges over the elements in Σ contained in A. It can be shown that every semiprobability is an inner probability and an inner probability is a semiprobability if and only if that supremum is always attained. Lehrer calls a probability measure on a subalgebra a partially-specified probability and defines a new theory of integration with such probabilities. Then, he axiomatizes preferences in the Anscombe-Aumann framework that have a representation as this kind of an integral.

3.2 Two Prizes

In this subsection, we consider the special case with two prizes, $m > l$. Normalize the utility function u so that $u(l,l) = 0, u(m,m) = 1$ and $u(l,m) = t \in (0,1)$. We can identify each act with the set $A = f^{-1}(m)$ and therefore the preference can be identified with a ranking over sets. Applying our representation theorem, we get that $A \succeq B$ if and only if $U(A) \geq U(B)$ where

$$\begin{aligned} U(A) =& \mu(A) + t \cdot (1 - \mu(A) - \mu(A^c)) \\ =& \mu(A) + t\alpha(A) \end{aligned} \tag{$*$}$$

where $\alpha(A) := 1 - \mu(A) - \mu(A^c)$ measures the ambiguity of event A. For unambiguous events $E \in \mathcal{E}$ we have $\mu(E) + \mu(E^x) = 1$ and hence the ranking is independent of the parameter t. Therefore, the semiprobability μ provides a ranking of unambiguous bets. When the sets A or B are not unambiguous then the agent's ranking depends on t. The parameter t measures the weight given to the ambiguous part of the sets A or B. When evaluating the binary act xAy the decision maker is confronted with three distinct prizes; the prize x, the prize y and the prize (x,y). How trade-offs between those three prizes are resolved depends on the decision maker's utility function. In the case of two prizes the utility function corresponds to the parameter t.

The case of two prizes enables us to compare derivation of subjective probabilities in SEU theory to the derivation of a semiprobability in our setting. Let $\succeq$ be a complete, transitive binary relation such that $\Omega \succ \emptyset$ and $A \succeq B$ whenever $B \subset A$. The event $A \subset \Omega$ is *null* if $B \sim A \cup B$ for all $B \subset A$. Let $\mathcal{N}$ denote the collection of all null sets. Let $\mathcal{D}, \mathcal{E}_+$

be two (nonempty) collections of sets in Ω. We say that $(\mathcal{E}_+, \mathcal{D})$ is an *ambiguity structure* if $E \in \mathcal{E}_+$ is not null and

$$D \in \mathcal{D} \text{ if and only if } D \cap E \neq \emptyset \neq D^c \cap E$$

for all $E \in \mathcal{E}_+$. We say that $\succeq$ is adapted to the ambiguity structure $(\mathcal{E}_+, \mathcal{D})$ if the following properties are satisfied:

Property 1: *If $A, B \subset E \in \mathcal{E}_+$ and $C \subset E^c$, then $A \succeq B$ if and only if $A \cup C \succeq B \cup C$.*

Property 1 says that the preference is separable across unambiguous events. It matches our earlier definition of unambiguous events when we restrict to binary acts.

Property 2: *If $D \succ \emptyset$ for some $D \in \mathcal{D}$ then $E_1 \succ E_2$ implies $E_1 \cap D_1 \succ E_2 \cap D_2$ whenever $E_i \in \mathcal{E}_+$, $D_i \in \mathcal{D}$ for $i = 1, 2$.*

Property 2 plays the role of Axioms 2 and 4(ii) in this context. Axiom 2 applied to binary acts requires that

$$D \succ \emptyset \Rightarrow D \cap E \succ \emptyset \tag{†}$$

for all $E \in \mathcal{E}_+$. If we choose $E_1 \subset E_2, E_1 \succ E_2$ then (†) together with Property 1 imply $E_1 \cap D \succ E_2 \cap D$. The decision maker's indifference among diffuse sets (Axiom 4(ii)) implies that the strict preference of $E_1 \cap D$ over $E_2 \cap D$ is preserved even when different diffuse sets are applied to E_1 and E_2.

Property 3: *$A \succ B$ implies there exists a partition $E_1, \ldots, E_n \in \mathcal{E}_+$ of Ω such that $A \succ B \cup E_i$ for all i.*

Property 3 is standard. Properties 4 and 5 are used to establish countable additivity of the semiprobability. Properties 4 and 5 capture the relevant aspects of Axiom 6 for this context.

Property 4: *$\mathcal{E}_+ \cup \mathcal{N}$ is a $\sigma-$algebra.*

Property 5: *If $E, E_i \in \mathcal{E}_+$ and $E_{i+1} \subset E_i$, then $\bigcap E_i = \emptyset$ implies $E \succ E_i$ for some i.*

Theorem 2: *The preference relation $\succeq$ is adapted to the ambiguity structure $(\mathcal{E}_+, \mathcal{D})$ if and only if there exists a convex-valued semiprobability μ and a real number $t \in [0,1]$ such that $A \succeq B$ if and only if $\mu_(A) + t\alpha(A) \geq \mu(B) + t\alpha(B)$.*

The proof of Theorem 2 is in the appendix. We provide a separate proof because it is simpler than the proof of the main Theorem.

As Theorem 2 demonstrates, the semiprobability alone does not provide enough information to identify the decision maker's preferences over bets. Even in the case of two prizes there are 3 possible outcomes for an SEUU decision maker, l, m and (l, m). The last outcome is interpreted as a situation where the agent receives l or m and cannot reduce the uncertainty further. To evaluate a bet that is not unambiguous we need to know how the agent trades-off (l, m) and the prizes l or m.

4. Uncertainty Aversion and Mean-Preserving u-spreads

In this section, we provide a definition of uncertainty aversion and a corresponding criterion that allows us to rank bilotteries as more or less uncertain. For an SEUU decision maker, the bilottery generated by the act captures all relevant information about an act. Conversely, as we show in the proposition below, for every bilottery with support I there is some act that generates it.

Proposition 4: *Let p be a bilottery with support I and let μ be a semiprobability. Then, there is $f \in \mathcal{F}$ with $p_f = p$*

Proof: Appendix A, Lemma 3 (ii).

Our representation theorem together with Proposition 4 imply that any statement about preferences over acts can be described in terms of bilotteries and, conversely, statements about the ranking of bilotteries imply a corresponding statement about the ranking over acts.

Consider the bilottery

$$^1\!/_2 \cdot 1_{xy} + {}^1\!/_2 \cdot 1_{wz}$$

which yields the pairs (x, y) and (w, z) with equal probability. An uncertainty averse EUU prefers a δ-increase in the lower bound of any uncertain pair to a δ-increase in the upper bound of any pair. More precisely, we define and uncertainty averse EUU as follows.

Definition: *U is uncertainty averse if*

$$U(1/2 \cdot 1_{xy} + 1/2 \cdot 1_{wz}) \geq U(1/2 \cdot 1_{x'y} + 1/2 \cdot 1_{wz'})$$

for $x - x' = z' - z = \delta > 0$.

We can alternatively define uncertainty aversion in terms of the utility index u. The index u is uncertainty averse if for $\delta > 0$

$$u(x + \delta, y) - u(x, y) \geq u(w, z + \delta) - u(w, z)$$

An uncertainty averse decision maker prefers the bet on a fair coin toss $1/2 \cdot 1_{xx} + 1/2 \cdot 1_{yy}$ to the diffuse bet 1_{xy}. To see this, set $\delta = y - x$ to get

$$U(1/2 \cdot 1_{xx} + 1/2 \cdot 1_{yy}) \geq U(1/2 \cdot 1_{xy} + 1/2 \cdot 1_{xy}) = U(1_{xy})$$

Furthermore, if $z = (x + y)/2$ then, for an uncertainty averse U

$$\begin{aligned} U(1_{zz}) &\geq U(1/2 \cdot 1_{xz} + 1/2 \cdot 1_{zy}) \\ &\geq U(1/2 \cdot 1_{xy} + 1/2 \cdot 1_{xy}) \\ &= U(1_{xy})) \end{aligned}$$

Hence, an uncertainty averse decision maker prefers the average of x and y to the diffuse bet 1_{xy}.

Next, we define a criterion that ranks bilotteries according to how uncertain they are. A bilottery q is a mean preserving u-spread of p if two conditions are satisfied. First, q and p must have the same mean. Second, the bilottery p must put more weight on smaller intervals. To formalize this second condition, consider a "southeast" set T, that is, a set that has the property that $(x, y) \in T$ implies $(w, z) \in T$ for $x \leq w \leq z \leq y$. The bilottery

p must put more weight on T than the bilottery q for every southeast set T. Let ι_A denote the indicator function on a set A. Let

$$M(p) := {}^1\!/\!_2 \int (x+y)dp$$

denote the mean of b.

Definition: *The bilottery p is a mean-preserving u-spread of q if $M(p) = M(p)$ and $p(T) \geq q(T)$ for every closed $T \subset I$ with the property that $(x, y) \in T$ implies $(w, z) \in T$ for $x \leq w \leq z \leq y$.*

Theorem 3: *(i) If the bilottery q is a mean-preserving u-spread of p and U is an uncertainty averse EUU then $U(p) \geq U(q)$. (ii) If $M(p) = M(q)$ and $U(p) \geq U(q)$ for every uncertainty averse EUU U then q is a mean-preserving u-spread of p.*

Proof: Appendix D.

Theorem 1 shows that if uncertainty averse decision makers dislike mean-preserving u-spreads and, conversely, that mean-preserving u-spreads are the only mean-preserving transformations that an uncertainty averse decision maker is guaranteed to dislike.

The following example describes a family of bilotteries that is ordered according to mean-preserving u-spreads. Let $M = [0, 1]$ and note that the bicumulative

$$\mathbf{b}_1 = (y - x)$$

for $1 \geq y \geq x \geq 0$ corresponds to a standard uniform distribution on the unit interval. The bicumulative

$$\mathbf{b}_2 = (y - x)^2$$

corresponds to a uniform distribution on I, that is, the measure of any rectangle $A = [a_1, b_1] \times [a_2, b_2]$ is proportional to (twice) its area. It is easy to see that $\mathbf{b}_2$ is a mean preserving u-spread of $\mathbf{b}_1$. More generally, consider any $\lambda > \lambda' \geq 1$. We have the following:

Proposition 5: $\mathbf{b}_\lambda$ *is a mean preserving u-spread of* $\mathbf{b}_{\lambda'}$ *if* $\lambda > \lambda' \geq 1$.

Proof: Appendix D.

5. Implications of SEUU Maximization

5.1 Comparison to Choquet Expected Utility

It is useful to compare SEUU maximization to Choquet expected utility (CEU) theory (Schmeidler (1989)) to illustrate the differences in implications for behavior. A CEU preference is characterized by a continuous nonconstant utility index, $v : M \to I\!R$, and a capacity η. The following proposition characterizes those SEUU preferences that are also Choquet expected utility preferences.

Proposition 6: *The SEUU preference* (μ, u) *is a CEU preference if and only if there is* $t \in [0,1]$ *such that* $u(x,y) = (1-t)u(x,x) + tu(y,y)$.

Consider first the "only if" part of Proposition 5. Let (μ, u) be a SEU preference and consider the diffuse act yDx with $y > x$. Since u is continuous, there is a t such that if $\mu(E) = t$ then $yDx \sim yEx$ and hence $u(x,y) = (1-t)u(x,x) + tu(y,y)$. For a CEU preference it then follows that for all $y' > x'$ we have $y'Dx' \sim y'Ex'$ and therefore $u(x',y') = (1-t)u(x',x') + tu(y',y')$ for all $y' > x'$.

For the converse, let $\nu(E) = \mu(E)$ for all unambiguous sets $E \in \mathcal{E}$. For an arbitrary $A \subset \Omega$, let $\nu(A) = \mu(A) + t(1 - \mu(A) - \mu(A^c))$. Choose $v(x) = u(x,x)$, then the CEU (η, v) represents the SEUU preference (μ, u).

Notice that the parameter t is a parameter of the capacity in the Chocquet representation while it is a parameter of the biutility index in the SEUU representation. Hence, the two theories draw a different line between event assessment and gamble evaluation. When SEUU theory is not a special case of CEUU theory it is because the evaluation of ambiguous bets depends on the prizes (x,y) involved in the bets. Conversely, when CEU theory is not a special case of SEUU theory it is because the evaluation of ambiguous bets depends on the particular diffuse sets involved in the bet.

5.2 Machina Reversals

Recently, Machina (2007) has raised concerns regarding the ability of Choquet expected utility theory (and related models) to accommodate variations of the Ellsberg paradox that appear plausible and even natural. In this subsection, we will show that within SEUU theory the behavior described by Machina is synonymous with the failure of 'constant absolute uncertainty aversion.' Thus, SEUU theory is flexible enough to accommodate Machina's version of the Ellsberg paradox.

We say that a EUU preference u has constant absolute uncertainty aversion if there exist functions v_1 and v_2 such that $u(x,y) = v_1(x) + v_2(y)$. To see why this can be interpreted as constant absolute uncertainty aversion, note that for $x > w$, the utility effect of reducing uncertainty from $[w,y]$ to $[x,y]$ is $u(x,y) - u(w,y)$. Constant absolute uncertainty aversion ensures that this effect is equal to $v_1(x) - v_1(w)$ and hence independent of y.

Next, we describe Machina's version of the Ellsberg paradox: consider four prizes, $\{0, 50, 100, 200\} \subset M$ and a four-element partition A_1, A_2, A_3, A_4 of Ω. To be concrete, suppose a ball will be drawn from an urn that is known to have 20 balls. It is also known that 10 of these balls are marked 1 or 2 and the other 10 balls are marked 3 or 4. The set A_i is the event that a ball marked i will be drawn from the urn. Suppose that some decision-maker, $\succeq$, is indifferent among all bets on A_i:

$$yA_1x \sim yA_2x \sim yA_3x \sim yA_3x \tag{2}$$

for all $x, y \in \{0, 50, 100, 200\}$. Suppose also that

$$yA_1 \cup A_2x \sim yA_3 \cup A_4x \tag{3}$$

and

$$yA_1 \cup A_3x \sim yA_1 \cup A_4x \sim yA_2 \cup A_3x \sim yA_2 \cup A_4x \tag{4}$$

Expression (3) says that the decision-maker is indifferent between the two bets where he knows the number of balls he is betting on. Expression (4) says that the decision-maker is indifferent between any bet on two numbers where he cannot determine the number of balls. But, the decision-maker strictly prefers bets on events with a known number of balls over bets on events with an unknown number of balls:

$$yA_1 \cup A_2x \succ yA_1 \cup A_3x \tag{5}$$

whenever $y > x$.

Next consider acts $f = (x_1, x_2, x_3, x_4)$ where x_i is what f yields if event A_i occurs. Machina (2008) observes that the decision-maker above must be indifferent between $(0, 100, 200, 50)$ and $(0, 200, 100, 50)$ if he is a Choquet expected utility maximizer and that this indifference may not be a desirable restriction for a flexible model of uncertainty aversion. In the language of our model, Machina argues that a decision-maker may not be indifferent between enjoying a fifty-fifty bet over the equally uncertain/risky consequence $(0, 200)$ and $(100, 50)$ versus a fifty-fifty bet over the equally uncertain/risky consequences $(0, 100)$ and $(200, 50)$ because he may prefer "packaging" 200 with 50 rather than 0.

Call it an M-reversal if a EUU preference is not indifferent between (x_1, x_3, x_4, x_2) and (x_1, x_4, x_3, x_2) for some $x_i \in M$, $i = 1, 2, 3, 4$, despite satisfying equations (2)-(5). Then, Machina argues that imposing no M-reversals is a possibly unwarranted restriction on a model of uncertainty. It can be shown that an SEUU decision-maker has no M-reversals if and only if it has constant absolute uncertainty aversion. Thus, within the framework of the SEUU model, non-constant absolute uncertainty aversion allows for M-reversals.

6. Appendix A: Preliminary Results

For any set X, let $\Delta_o(X) = \{\nu \in \Delta(X) \mid \ \nu \text{ has finite support } \}$. We call $\Delta_o(X)$ the set of simple measures. The set $\mathcal{L}_o := \Delta_o(I)$ are the simple bilotteries.

Lemma A1: *(i) If μ is a semiprobability then $\mathcal{E}$ is a σ-algebra. (ii) μ is a countably additive probability measure on $\mathcal{E}$ and $(\Omega, \mathcal{E}, \mu)$ is complete.*

Proof: First, we show that $\mathcal{E}$ is an algebra. First, note that $E \in \mathcal{E}$ implies $E^c \in \mathcal{E}$ by definition. Moreover, it is obvious that $\emptyset \in \mathcal{E}$. Therefore, it suffices to show that $E_1 \cap E_2 \in \mathcal{E}$ for $E_1, E_2 \in \mathcal{E}$. Since μ is supermodular, it suffices to show that $\mu(E_1 \cap E_2) + \mu((E_1 \cap E_2)^c) \geq 1$. Since $\mu(E_i) = 1 - \mu(E_i^c)$ for $i = 1, 2$ and, since μ is supermodular, it follows that

$$\mu((E_1 \cap E_2)^c) = \mu(E_1^c \cup E_2^c) \geq 2 - \mu(E_1) - \mu(E_2) - \mu((E_1 \cup E_2)^c))$$

and therefore it suffices to show that

$$\mu(E_1 \cap E_2) + 1 = \mu(E_1 \cap E_2) + \mu(\Omega) \geq \mu(E_1) + \mu(E_2) + \mu((E_1 \cup E_2)^c))$$

which in turn follows from supermodularity.

Let $E_i \in \mathcal{E}$ with $E_i \supset E_{i+1}$. To show that $\mathcal{E}$ is a σ-algebra we must show that $E = \cap_{i=1}^{\infty} E_i \in \mathcal{E}$. Note that it suffices to show that $\mu(E) \geq 1 - \mu(E^c)$. We have $\mu(E_i) = 1 - \mu(E_i^c)$ for all i and by property (ii) of a semiprobability $\mu(E) \geq 1 - \mu(E_i^c) - \epsilon \geq 1 - \mu(E^c) - \epsilon$ for all $\epsilon > 0$. Therefore, $E \in \mathcal{E}$.

For part (ii) we first show that $\mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2)$ for $E_1 \cap E_2 = \emptyset$. By supermodularity, it suffices to show that $\mu(E_1) + \mu(E_2) \geq \mu(E_1 \cup E_2)$. Note that by supermodularity, $\mu(E_i^c) + \mu(E_1 \cup E_2) \leq 1 + \mu(E_i)$ and $\mu(E_1 \cup E_2) \geq \mu(E_1) + \mu(E_2)$ therefore $\mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2) = 2 - \mu(E_1^c) - \mu(E_2^c) - \mu(E_1 \cup E_2) \geq 0$ as required. To see countable additivity, let $E_i \in \mathcal{E}$ be a collection of disjoint sets. Let $E = \cup_{i=1}^{\infty} E_i$ and define $E^i = \cup_{j=i}^{\infty} E_i$. Note that $\mu(E) = \sum_{i=1}^{j-1} \mu(E_i) + \mu(E^j)$. Note that $\lim \mu(E^j) \to 0$ by property (ii) of a semiprobability and therefore countable additivity follows. That $(\Omega, \mathcal{E}, \mu)$ is complete is obvious. $\square$

Lemma A2: *Assume the continuum hypothesis holds. If μ is a convex valued semiprobability then (i) there exists a diffuse set $D \subset \Omega$. (ii) For any natural number n, there exists a partition $(D_1, \ldots, D_n)$ of Ω with $D_i \in \mathcal{D}$ for $i = 1, \ldots, n$.*

Proof: Birkhoff (1967) page 266, Theorem 13 proves the following: no nontrival (i.e., not identically equal to 0) measure such that every singleton has measure 0 can be defined on the algebra of all subsets of the continuum.

For each $A \subset \Omega$, let $E_1 \subset A$ be such that $\mu(E_1) = \mu(A)$ and let $E_3 \subset A^c$ be such that $\mu(E_3) = \mu(A^c)$. Define $N(A) = (E_1 \cup E_3)^c$. Call $N(A) \cap A$ the completely nonmeasurable part of A. Let $\alpha = \sup\{\mu(N(A)) \mid A \subset \Omega\}$. We note that this α is attained. To see this, let A_i be a sequence such that $\lim \mu(N(A_i)) = \alpha$. Define B_i as follows: $B_1 = A_1 \cap N(A_1)$ and $B_{i+1} = A_i \cap N(A_{i+1}) \cap (\bigcup_{j \leq i} N(A_i)^c)$. Note that $N(B_1 \cup \ldots \cup B_i) = N(A_1) \cup \ldots \cup N(A_i)$ and $\bigcup_{i=1}^{\infty} B_i$ is completely nonmeasurable in $\bigcup_{i=1}^{\infty} N(A_i)$. Since $\lim \mu(N(A_i)) = \alpha$, we have $\mu(\bigcup_{i=1}^{\infty} N(A_i) \geq \alpha$ showing that α is attained.

If $\alpha < 1$, then we would find A such that $\mu(N(A)) = \alpha$ and use Birkoff result to find $B \subset N(A)^c$ with $\mu(N(B)) > 0$ to get $C = B \cup (A \cap N(A))$ with $\mu(N(C)) > \mu(N(A))$ contradicting the maximality of α. Hence, $\alpha = 1$. Then, choose D such that $\mu(N(D)) = 1$ and note that D is a diffuse set. This proves part (i).

Next, we will show any diffuse set can be partitioned into two diffuse sets. Then, a simple inductive argument yields part (ii). Let D be any diffuse set and define $\Sigma_1 = \{E \cap D \,|\, E \in \mathcal{E}\}$, $\mu_1(E \cap D) = \mu(E)$. Note that since D is diffuse, $\mu(E \cap D) = \mu(E' \cap D)$ implies that E, E' differ by a set of measure 0. Hence, (D, Σ_1, μ_1) is a probability space and $\mu_1(\{s\}) = 0$ for $s \in D$. Since $\inf_{E \supset D} \mu(D) = 1$, D cannot be countable. Then, by the Continuum Hypothesis, the cardinality of D must be the continuum. Repeated the argument in part (i) above yields a diffuse subset of D_1 of D. Then, for any E such that $\mu(E) > 0$, we have $\mu_1(E \cap D) > 0$ and therefore $E \cap D_1 \neq \emptyset$. A symmetric argument yields $E \cap [D \backslash D_1] \neq \emptyset$. Hence, $D_1, D \backslash D_1$ are diffuse in Ω. □

Lemma A3: *(i) If μ is a semiprobability then $\mathbf{b}_f$ is a bicumulative. (ii) For any bicumulative $\mathbf{b}$, there exists $f \in \mathcal{F}$ such that $\mathbf{b}_f = \mathbf{b}$.*

Proof: Part (i): We must show that ψ_f is submodular and satisfies properties (i)-(iii). Property (iii) follows from property (ii) of a semiprobability. Properties (i) and (ii) are obvious. That ψ_f is submodular follows from the supermodularity of μ. □

Part (ii): Let $w = m - l$ and $z_i^n = l + wi2^{-n}$ for all $i = 0, 1, \ldots, 2^n$. For any $x, y \in M$, let $i(n, x) = \max\{i \,|\, z_i^n \leq x\}$ and $j(n, y) = \min\{j \,|\, z_j^n \geq y\}$. The function i is increasing in both arguments while j is decreasing in the first argument and increasing in the second argument.

Define, for all $n \geq 0$, $i = 0, \ldots 2^n - 1$,

$$\begin{aligned}
\alpha_{01}^n &= \mathbf{b}(z_0^n, z_1^n) \\
\alpha_{i(i+1)}^n &= \mathbf{b}(z_i^n, z_{i+1}^n) - \mathbf{b}(z_i^n, z_i^n) \text{ for } i = 1, \ldots, 2^n - 1 \\
\alpha_{ij}^n &= \mathbf{b}(z_i^n, z_j^n) - \mathbf{b}(z_{i+1}^n, z_j^n) - \mathbf{b}(z_i^n, z_{j-1}^n) + \mathbf{b}(z_{i+1}^n, z_{j-1}^n)
\end{aligned}$$

for all j such that $2^n > j > i + 1$. Pick any diffuse set D. Define $f^0 = mDl$ and $E_{01} = \Omega$. Consider any f^n and collection of set E_{ij}^n for $i = 1, \ldots, 2^n$, j satisfying $2^n > j > i + 1$ such that (i) $f^n(\omega) = z_j^n$ if and only if $\omega \in E_{ij}^n \cap D$ for some E_{ij}^n, (ii) $f^n(\omega) = z_i^n$ if and only if $\omega \in E_{ij}^n \cap D^c$ for some E_{ij}^n, (iii) $\mu(E_{ij}^n) = \alpha_{ij}^n$, (iv) $\alpha_{ij}^n = 0$ implies $E_{ij}^n = \emptyset$ and (v) $\{E_{ij}^n\}$ is a partition of Ω. Define f^{n+1} inductively, as follows: for all $t > r$, $\alpha_{rt}^{n+1} > 0$, $r \in \{2i, 2i+1\}$,

$t \in \{2j, 2j-1\}$, choose E_{rt} such that (i) $\alpha_{rt}^n = \mu(E_{rt})$ and (ii) the collection of such E_{rt} is a partition of E_{ij}. Finally, set $E_{rt} = \emptyset$ whenever $\alpha_{rt}^n = 0$. This is possible because

$$\alpha_{(2i)(2j)}^{n+1} + \alpha_{(2i)(2j-1)}^{n+1} + \alpha_{(2i+1)(2j)}^{n+1} + \alpha_{(2i+1)(2j-1)}^{n+1} = \alpha_{ij}^n$$

and the restriction of μ to $\mathcal{E}$ is convex-valued. Then, let $f^{n+1}(\omega) = z_j^{n+1}$ for all $\omega \in E_{ij}^{n+1} \cap D$ and $f^{n+1}(\omega) = z_i^{n+1}$ for all $\omega \in E_{ij}^{n+1} \cap D^c$. Note that $f^n(\omega) = f^{n+1}(\omega)$ whenever $\omega \in E_{(2i)(2j)}^{n+1}$. More precisely, $f^n(\omega) \geq f^{n+1}(\omega)$ for all $\omega \in D$, while $f^n(\omega) \leq f^{n+1}(\omega)$ for all $\omega \in D^c$. Since $f^n(\omega)$ is a monotone sequence, it has a limit $f(\omega) = f^\infty(\omega)$. It is easy to see that for $n = 0, \ldots,$ and $k = n+1, \ldots, \infty$,

$$f^n(\omega) = \begin{cases} z_{j(n,f^k(\omega))}^n & \text{if } \omega \in D \\ z_{i(n,f^k(\omega))}^n & \text{if } \omega \in D^c \end{cases} \tag{$*$}$$

To establish that $\psi(f) = \mathbf{b}$, fix $(x, y) \in I$ such that $y > x$ and define

$$x^n = z_{i(n,x)}^n, \; y^n = z_{j(n,y)}^n$$

Note that $j(n, y) > i(n, x)$ since $y > x$ and for $n = 0, \ldots,$ and $k = n+1, \ldots, \infty$,

$$\begin{aligned} x^n &= z_{i(n,x^k)}^n \\ y^n &= z_{j(n,x^k)}^n \end{aligned} \tag{$**$}$$

First, we show that

$$f^{-1}[x, y] \subset (f^{n+1})^{-1}[x^{n+1}, y^{n+1}] \subset (f^n)^{-1}[x^n, y^n] \tag{1}$$

for all n. To see this, assume $f(\omega) = z \in [x, y]$ and $\omega \in D$. Then, $(*)$ yields

$$x^{n+1} = z_{i(n+1,x)}^{n+1} \leq z_{i(n+1,z)}^{n+1} \leq z_{j(n+1,z)}^{n+1} = f^{n+1}(\omega) \leq z_{j(n+1,y)}^{n+1} = y^{n+1}$$

Hence, $\omega \in (f^{n+1})^{-1}[x^{n+1}, y^{n+1}]$ proving the first inclusion above. A symmetric argument yields $\omega \in (f^{n+1})^{-1}[x^{n+1}, y^{n+1}]$ for $\omega \in D^c$.

Next, suppose $\omega \in (f^{n+1})^{-1}[x^{n+1}, y^{n+1}]$ for $\omega \in D$. $f^{n+1}(\omega) = z \in [x^{n+1}, y^{n+1}]$. Since x^k is increasing and $f^k(\omega)$ is decreasing, we have $x^n \leq x^{n+1} \leq f^{n+1}(\omega) \leq f^n(\omega)$. By $(*)$, we have $f^n(\omega) = z_{j(n,z)}^n$ while $(**)$ and the fact that $z \leq y^{n+1}$ yields $z_{j(n,z)}^n \leq$

$z^n_{j(n,y^{n+1})} = y^n$. Hence, $f^n(\omega) \leq y^n$ and therefore $\omega \in (f^n)^{-1}[x^n, y^n]$. A symmetric argument yields the desired result for $\omega \in D^c$.

Next, we prove

$$f^{-1}[x,y] = \bigcap_n (f^n)^{-1}[x^n, y^n] \tag{2}$$

Since (1) holds, it suffices to prove that $x^n \leq f^n(\omega) \leq y^n$ for all n implies $x \leq f(\omega) \leq y$. Suppose $z = f(\omega) < x$. Assume first that $\omega \in D^c$. Then, there exists x^n for $n > 1$ such that $z < x^n < x$. The fact that $f^k(\omega)$ is increasing for $\omega \in D^c$ implies $f^n(\omega) \leq z < x^n$ and hence $\omega \notin (f^n)^{-1}[x^n, y^n]$. If $\omega \in D$, then there exists z^n_i such that $z < z^n_i < x$ and by $(*)$, $f^n(\omega) = z^n_{j(n,z)} \leq z^n_i$, again proving $\omega \notin (f^n)^{-1}[x^n, y^n]$. The argument for the case $f(\omega) > y$ is symmetric and omitted.

Since μ is a semiprobability, (1) and (2) implies

$$\mu(f^{-1}[x,y]) = \lim_{n\to\infty} \mu((f^n)^{-1}[x^n, y^n]) \tag{3}$$

By construction, $\mu((f^n)^{-1}[x^n, y^n]) = \mathbf{b}(x^n, y^n)$ and since x^n converges to x from below and y^n converges to y from above, we have $\lim_{n\to\infty} \mu((f^n)^{-1}[x^n, y^n]) = \mathbf{b}(x,y)$. Then (3) yields

$$\mu(f^{-1}[x,y]) = \mathbf{b}(x,y)$$

for all x, y such that $l \leq x < y \leq m$. To show that $\mu(f^{-1}(z)) = \mathbf{b}(z,z)$, assume that $z < m$ (if not $z > l$ and a symmetric argument applies). Define $y_n = z + \frac{\epsilon}{n}$ for ϵ sufficiently small so that $[z, y_n] \subset [l,m]$. Note that $f^{-1}[z, y_{n+1}] \subset f^{-1}[z, y_n]$ and $\bigcap_n f^{-1}[z, y_n] = f^{-1}(z)$. Again, since μ is a semiprobability, it follows that $\mu(f^{-1}(z)) = \lim_n \mu(f^{-1}[z, y_n]) = \lim \mathbf{b}(z, y_n) = \mathbf{b}(z,z)$ as desired. Thus, $\mu(f^{-1}[x,y]) = \mathbf{b}(x,y)$ for all x, y proving that $\psi(f) = \mathbf{b}$. $\square$

7. Proof of Propositions 1 and 2

7.1 Proof of Proposition 1

Define $u(x,y) = U(1_{xy})$ and $V(\eta) = \int u(x,y)d\eta$. Obviously, V is affine. Since I is compact, the topology of weak convergence and the Prohorov topology are equivalent. Therefore, V is continuous. Standard arguments familiar from expected utility theory suffice to show that $V(\eta_{\mathbf{b}}) = U(\mathbf{b})$ for all $\mathbf{b} \in \mathcal{L}_o$. Since $\Delta_o(I)$ is dense in $\Delta(I)$ and therefore $\mathcal{L}_o$ is dense in $\mathcal{L}$, it follows that $V(\eta_{\mathbf{b}}) = U(\mathbf{b})$ for all $\mathbf{b} \in \mathcal{L}$ as desired. $\square$

7.2 Proof of Proposition 2

For the bilottery $\mathbf{b}$ define

$$G_{\mathbf{b}}(x,y) = \mathbf{b}(l,y) - \lim_{x^+ \downarrow x} \mathbf{b}(x^+, y)$$

It is straightforward to show that $G_{\mathbf{b}} = G_{\mathbf{b}'}$ if and only if $\mathbf{b} = \mathbf{b}'$. The function $G_{\mathbf{b}}$ satisfies the conditions for a distribution on $I\!R^2$ (see Billingley (1995), pg. 177). Therefore, we can apply Theorem 12.5. in Billingsley to show that there is a unique measure ψ on $I\!R^2$ such that $\mathbf{b}(x,y) = \psi([x,y] \times [x,y])$. $\square$

8. Appendix B: Proof of Theorem 1

In this section we prove Theorem 1. The proof is divided into a series of Lemmas. It is understood that Axioms 1-6 hold throughout.

Lemma B1: *(i) $f(s) \geq g(s)$ for all $s \in \Omega$ implies $f \succeq g$. (ii) $f \succ g$ implies $f \succ z \succ g$ for some $z \in M$. (iii) $f_n, g_n \in \mathcal{F}$, f_n converges uniformly to f, g_n converges uniformly to g, $g \succ f$ implies $g_n \succ f_n$ for some n. (iv) $f_n, g_n \in \mathcal{F}_e$, f_n converges pointwise to f, g_n pointwise to g, $g \succ f$ implies $g_n \succ f_n$ for some n.*

Proof: To prove (i), let $f_n = \frac{1}{n}l + (\frac{n-1}{n})f$ and $g_n = \frac{1}{n}l + (\frac{n-1}{n})g$. Then, f_n converges to f uniformly and g_n converges to g uniformly. By Axiom 2, $f_n \succ g_n$. Then, by Axiom 6, $f \succeq g_n$ and applying Axiom 6 again yields $f \succeq g$ as desired.

To prove (ii), assume $f \succ g$ and let $y = \inf\{z \in M \,|\, z \succeq f\}$ and let $x = \sup\{z \in M \,|\, g \succeq z\}$. By (i) above, x and y are well-defined. Axiom 6 ensures that $y \sim f$ and $z \sim g$ and therefore $y \succ x$. Then, for $z = \frac{x+y}{2}$, we have $f \succ z \succ g$.

To prove (iii), let $g \succ f$ and apply (ii) three times to get z, y, x such that $g \succ z \succ y \succ x \succ f$. Axiom 6 ensures that $g_n \succ y$ and $y \succeq f_n$ for all n large enough. Therefore, $g_n \succ f_n$ for all such n. Analogous argument proves (iv). $\square$

Lemma B2: *The collection $\mathcal{E}$ is a $\sigma-$field.*

Proof: First, we note that $\mathcal{E}$ is a field. That $E \in \mathcal{E}$ implies $E^c \in e$ is obvious as is the fact that $\emptyset \in \mathcal{E}$. Hence, to show that $\mathcal{E}$ is a field, we need to establish that $E, E \in \mathcal{E}$ implies $E \cap E \in \mathcal{E}$.

Suppose $fE \cap E'h \succeq gE \cap E'h$. We must show that $fE \cap E'h' \succeq gE \cap E'h'$. Note that $fE \cap E'h = (fEh)E'h$. Since $E' \in \mathcal{E}$ we have $(fEh)E'h' \succeq (gEh)E'h'$. Next, observe that $(fEh)E'h' = (fE'h')E(hE'h')$. Since $E \in \mathcal{E}$ we have $fE \cap E'h' = (fE'h')E(h'E'h') \succeq (gE'h')E(h'E'h') = gE \cap E'h'$ as required. A symmetric argument yields $h'E \cap E'f \succeq h'E \cap E'g$ if $hE \cap E'f \succeq hE \cap E'g$ and therefore $\mathcal{E}$ is a field.

To prove that the field $\mathcal{E}$ is a $\sigma-$field, it is enough to show that if $E_i \in \mathcal{E}$ and $E_i \subset E_{i+1}$, then $\bigcup E_i \in \mathcal{E}$. Let $E_i \subset E_{i+1}$ for all i. Note that $\hat{f}E_i\hat{g}$ converges pointwise to $\hat{f}\bigcup E_i\hat{g}$ for all $\hat{f}, \hat{g} \in \mathcal{F}$. Hence, if $g\bigcup E_ih' \succ f\bigcup E_ih'$ or $h'\bigcup E_ig \succ h'\bigcup E_if$ for some $f, g, h, h' \in \mathcal{F}_e$, by (iv) above, we have $gE_nh' \succ fE_nh'$ or $h'E_ng \succ h'E_nf$ for some n, proving that $E_i \in \mathcal{E}$ for all n implies $\bigcup_i E_i \in \mathcal{E}$. $\square$

Lemma B3: *There exists a finitely additive, convex-ranged probability measure μ on $\mathcal{E}$ and a function $v : \Omega \to I\!R$ such that the function $V : \mathcal{F}_e \to I\!R$ define by*

$$V(f) = \sum_{x \in M} v(x)\mu(f^{-1}(x))$$

represents the restriction of $\succeq$ to $\mathcal{F}_e$.

Proof: Note that Axiom 1 implies Savages P1, Axiom 2 implies P2. By definition P3 is satisfied for acts in $\mathcal{F}_e$, Axiom 3 yields P4, Axiom 4 yields P5, and finally, Axiom 5 yields P6. Then applying the proof of Savage's Theorem to all acts in $\mathcal{F}_e$ yields the desired conclusion. This is true despite the fact that Savage's theorem assumes that the underlying $\sigma-$field is the set of all subsets of Ω; the arguments work for any $\sigma-$field. $\sigma-$field. Hence, the result follows from Savage's theorem restricted to simple acts (i.e., $\mathcal{F}_o$). $\square$.

Lemma B4: *The probability measure μ on $\mathcal{E}$ is countably additive and complete.*

Proof: To show that μ is countably additive, we need to prove that given any sequence E_i such that $E_{i+1} \subset E_i$ for all i and $E^* := \bigcap_i E_i = \emptyset$, $\lim \mu(E_i) = 0$. Suppose $\lim \mu(E_i) > 0$. Then, by Axiom 5, there exists E such that $\lim \mu(E_i) > \mu(E) > 0$. Hence, $\mu(E_i) > \mu(E)$ for all i; that is $mE_il \succ mEl$ for all i. But $mE_il \in \mathcal{F}_e$ and converges pointwise to mE^*l. Hence, $mE^*l \succeq mEl \succ l$. Therefore, $\mu(E^*) > 0$ as desired.

To see that μ is complete, let $fEg \sim g$ for all f, g. Since $E \in \mathcal{E}$ it follows that for $A \subset E$, $(fAg)Eg \sim g$ for all f, g and therefore $fAg \sim g$ for all f, g. This implies that $A \in \mathcal{E}$ and therefore μ is complete. $\square$

Lemma B5: *The function v is strictly increasing and continuous.*

Proof: That v is strictly increasing follows from $y \succ x$ whenever $y > x$. To prove continuity, let $E_r\mathcal{E}$ be any event such that $\mu(E_r) = r$. Suppose $r' = \lim v(x_n) < v(x)$ for some sequence x_n in X. Then, choose $r \in (r', v(x)$ and note that $x \succ hE_rl \succeq x_n$ for n large. Therefore, $x \succ hE_rl \succeq \lim x_n = x$, a contradiction. Hence, $r' \geq v(x)$. A symmetric argument proves $r' = v(x)$ and yields the continuity of v. $\square$

Let

$$\mu^*(A) = \inf_{\{E_i\}} \sum_i \mu(E_i)$$

where the $\{E_i\}$ ranges over all sequences such that $E_i \in \mathcal{E}$ and $A \subset \bigcup_i E_i$. Since $\mathcal{E}$ is a $\sigma-$field, this definition is equivalent to

$$\mu^*(A) = \min_{A \subset E \in \mathcal{E}} \mu(E)$$

That is, there exists $E \in \mathcal{E}$ such that $A \subset E$ and $\mu(E) = \mu^*(A)$. Call such an E a *sheath* of A. Clearly, the symmetric difference between any two sheaths of a given set A has measure 0.

Lemma B6: *For any set $A \subset \Omega$, there exists a partition $E_1, E_2, E_3 \in \mathcal{E}$ of Ω such that (i) $E_1 \subset A \subset E_1 \cup E_2$ and (ii) $\mu^*(E_2 \cap A) = \mu^*(E_2 \cap A^c) = \mu(E_2)$. (iii) If $\hat{E}_1, \hat{E}_2, \hat{E}_3$ also satisfy (i) and (ii), then $\mu([E_i \cap \hat{E}_i^c] \cup [\hat{E}_i \cap E_i^c]) = 0$ for all $i = 1, 2, 3$*

Proof: Choose sheaths $\hat{E}$ E for A and A^c respectively. Then, let $E_1 = E^c$, $E_2 = E \cap \hat{E}$ and $E_3 = \Omega \backslash (E_1 \cup E_2)$. Clearly, $E_1 \subset A$. Note that $\hat{E}^c \subset A^c \subset E$. Then, $x \notin E_1 \cup E_2$ implies $x \in E_1^c \cap E_2^c = E \cap [E \cap \hat{E}^c]^c = E \cap \hat{E}^c = \hat{E}^c$ and hence $x \notin A$. Thus, $A \subset E_1 \cup E_2$.

Finally, note that $\mu^*(A) = \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \geq \mu^*(E_1) + \mu^*(E_2 \cap A)$. Since μ^* is subadditive, we have $\mu^*(E_1) + \mu^*(E_2 \cap A) \geq \mu^*(E_1 \cup [E_2 \cap A]) = \mu^*(A)$. Thus, $\mu^*(E_2) = \mu(E_2) = \mu^*(E_2 \cap A)$. A symmetric argument yields $\mu(E_2) = \mu^*(E_2 \cap A^c)$ establishing (i) and (ii).

To prove the uniqueness claim, note that the argument above showed that if $\hat{E}$ is a sheath for A and E is a sheath for A^c, then $E_1 = E^c$, $E_2 = E \cap \hat{E}$ and $E_3 = \Omega \backslash (E_1 \cup E_2)$ have the desired properties. It is easy to see that the converse is true as well: if E_1, E_2, E_3 have the desired properties, then $E_1 \cup E_2$ is a sheath for A and $E_2 \cup E_3$ is a sheath for $A^c *$. This establishes the uniqueness assertion. $\square$

Since μ is a countably additive and convex valued probability measure on E we can apply Lemma A2 in Appendix A to conclude that (i) there exist a diffuse set $D \subset \Omega$. (ii) For any natural number n, there exists a partition $D_1, \ldots, D_n \in \mathcal{D}$ of Ω.

Lemma B7: *For any y, x and diffuse act D, there exists a unique $z \in X$ such that $yDx \sim z$.*

Proof: Let $z = \sup\{w \in X \,|\, yDx \succeq w\}$. Since, $yDx \succeq l$ by Axiom 2, z is well-defined. Then, we can construct two sequences $y_n \geq z$ and $z \geq x_n$ such that both sequences converge to z and $y_n \succeq yDx$, $yDx \succeq x_n$. Hence, by Axiom 6, $z \succeq yDx \succeq z$ as desired. $\square$

For any $E \in \mathcal{E}$, let $[E]$ denote the equivalence class of sets in $\mathcal{E}$ that differ from E by a set of measure 0. Then, define $[E] \wedge [E'] = [E^*]$ for $E^* \in [E \cap E']$, $[E] \vee [E'] = [E^*]$ for $E^* \in [E \cup E']$ and $\neg E = [E^*]$ for $E^* \in [E^c]$. Let $S(A) = [E^*]$ for some sheath E^* of $A \subset \Omega$. By Lemma B5, $S(A)$ is well-defined for all $A \subset \Omega$. Let $[\mathcal{E}] = \{[E] \,|\, E \in \mathcal{E}\}$. It is easy to verify that $[\mathcal{E}]$ is a Boolean $\sigma-$algebra partially ordered by the binary relation $\leq$, where $[E] \leq [E']$ if and only if $[E] \wedge [E'] = [E]$. When there is no risk of confusion, we omit the brackets and write $E \vee E', \neg E$ etc.

We say that $\{E_1, \ldots E_n\} \in [\mathcal{E}]$ is partition if (i) $[E_i] \wedge [E_j] \neq [\emptyset]$ if and only if $i = j$ and (ii) $\Omega = E_1 \vee E_2, \ldots, \vee E_n$. A partition act $\bar{f}$ is defined as a one-to-one map from some partition $\mathcal{P}$ of $([\mathcal{E}])$ to the set of nonempty finite subsets of M. We let $\mathcal{P}_{\bar{f}}$ denote the partition that is the domain of $\bar{f}$. The partition act $\bar{f}$ is equivalent to the act f if for all $E \in \mathcal{P}_{\bar{f}}$

$$
\begin{array}{ll}
\text{(i)} & \bar{f}(E) = \bigcap_{\hat{E} \in [E]} f(\hat{E}) \\
\text{(ii)} & \bar{f}(E) \subset f(E') \text{ for all } E, E' \text{ such that } \mu(E) > 0 \text{ and } E' \subset E
\end{array}
\qquad (*)
$$

Lemma B8: *For every simple act $f \in \mathcal{F}_o$, there exists a unique partition act $\bar{f}$ that is equivalent f.*

Proof: Define

$$[\mathcal{E}_f^0] = \{S(f^{-1}(Z)) \in [\mathcal{E}] \,|\, Z \subset M\}$$

and let $[\mathcal{E}_f]$ be the smallest sub $\sigma-$algebra of $[\mathcal{E}]$ that contains $[\mathcal{E}_f^0]$. Note that $[\mathcal{E}_f^0]$ and therefore $[\mathcal{E}_f]$ are both finite. Let

$$\mathcal{P}_f = \{[E] \in [\mathcal{E}_f] \backslash [\emptyset] \,|\, E' \in [\mathcal{E}_f] \text{ implies } E' \wedge E \in \{[E], [\emptyset]\}$$

Hence, $\mathcal{P}_f$ is the set of minimal elements in $[\mathcal{E}_f] \backslash [\emptyset]$. We claim that $\mathcal{P}_f$ is a partition. Clearly, $\mathcal{P}_f$ is nonempty and condition (i) is satisfied by construction. So, we need only show that $\bigvee_{\hat{E} \in \mathcal{P}_f} \hat{E} = \Omega$. Let $\mathcal{T} = \{E \in [\mathcal{E}_f] \,|\, [\emptyset] < E \leq \bigwedge_{\hat{E} \in \mathcal{P}_f} \neg \hat{E}\}$. If $\mathcal{T}$ is nonempty, then it must contain a minimal element E and therefore, $E \in \mathcal{P}_f$. Hence, $[\emptyset] < E \leq \bigwedge_{\hat{E} \in \mathcal{P}_f} \neg \hat{E} \leq \neg E$ and therefore $E = [\emptyset]$, a contradiction. This proves that $\mathcal{T}$ is empty and therefore $\bigwedge_{\hat{E} \in \mathcal{P}_f} \neg \hat{E} = [\emptyset]$ which implies $\bigvee_{\hat{E} \in \mathcal{P}_f} \hat{E} = \Omega$ as desired.

Define $\bar{f}(E) = \bigcap_{\hat{E} \in [E]} f(\hat{E})$ for all $E \in \mathcal{P}_f$. Choose $E' \subset E$ such that $\mu(E') > 0$. Let $Z = f(E')$ and let $E^* = S^{-1}(Z) \in [\mathcal{E}_f^0]$. Note that $f(E') \cap \bar{f}(E) \neq \emptyset$. Since $E^* \in [\mathcal{E}_f^0]$ and $[E] \in \mathcal{P}_f$, we conclude $[E] \leq E^*$ and therefore $\bar{f}(E) \subset f(E^*) = f(E')$ proving $(*)$.

Next, we will show that $\bar{f}$ is a partition act with the domain $\mathcal{P}_{\bar{f}} = \mathcal{P}_f$. If $x \notin \bar{f}(\hat{E})$, then $\mu(f^{-1}(x) \cap \hat{E}) = 0$. Since $\mu(E) > 0$ for $E \in \mathcal{P}_f$ it follows that $\bar{f}(E)$ cannot be empty for $E \in \mathcal{P}_f$. We have established above that $\mathcal{P}_f$ is a partition. So to prove that $\bar{f}$ is a partition act with $\mathcal{P}_{\bar{f}} = \mathcal{P}_f$, we need only show that $\bar{f}$ is one-to-one. Assume that $\bar{f}(E) = \bar{f}(\hat{E}) = Z$, for $E, \hat{E} \in \mathcal{P}_f$. Then, for all $E_0 \in [\mathcal{E}_f^0]$, $E \subset E_0$ if and only if $\hat{E} \subset E_0$. Therefore, the same holds for all $E_0 \in [\mathcal{E}_f]$. Since $E, \hat{E} \in [\mathcal{E}_f]$, we have $E = \hat{E}$ as desired.

Finally, let $\bar{g}$ be some other partition act that is equivalent to f. Choose $E_1 \in \mathcal{P}_{\bar{f}}$ and $E_2 \in \mathcal{P}_{\bar{g}}$ such that $E_1 \wedge E_2 \neq 0$. Then, $(*)$ implies

$$\bar{f}(E_1) = \bigcap_{\hat{E} \subset E_1 \wedge E_2} f(\hat{E}) = \bar{g}(E_2)$$

Then, the one-to-oneness of $\bar{f}$ ensures that $E_1 \wedge E_2 \neq 0 \neq E_1' \wedge E_2$ implies $E_1 = E_1'$ for all $E_1, E_1' \in \mathcal{P}_{\bar{f}}$ and $E_2 \in \mathcal{P}_{\bar{g}}$. Hence, $\mathcal{P}_{\bar{f}} = \mathcal{P}_{\bar{g}}$ and since both $\bar{f}, \bar{g}$ are equivalent to f, we have $\bar{f} = \bar{g}$ as desired. □

Henceforth we write $\bar{f}$ to denote the partition act that is equivalent to f.

Lemma B9: *(i) Let $D_1, \ldots, D_n \in \mathcal{D}$ be a partition of Ω and $y_{i+1} \geq y_i$ for $i = 1, \ldots, n-1$ and define $f : \Omega \to X$ as follows: $f(s) = y_i$ whenever $s \in D_i$. Then, $f \sim y_n D y_1$ for all $D \in \mathcal{D}$. (ii) For any partition act g, there exists some simple act f such that $g = \bar{f}$. (iii) For any partition act $\bar{g}$ and $E \in \mathcal{P}_{\bar{g}}$, there exist $h \in \mathcal{F}_o$ and $f \in \mathcal{F}_d$ such that $hEf = f$ and $\bar{h} = \bar{g}$.*

Proof: By monotonicity, $y_n[D_2 \cup \ldots \cup D_n]y_1 \succeq f y_n D_n y_1$. By Axiom 5, $y_n[D_2 \cup \ldots \cup D_n]y_1 \sim y_n D_n y_1 \sim y_n D y_1$. This proves part (i).

Let $\bar{f}$ be the partition act and n be the maximum of the cardinality of $\bar{f}(E)$ for $E \in \mathcal{P}_{\bar{f}}$. Define an onto function $t_E : \{D_1, \ldots D_n\} \to \bar{f}(E)$ for each $E \in \mathcal{P}_{\bar{f}}$. Let $D_1, \ldots, D_n \mathcal{D}$ be a partition of $\mathcal{D}$. Choose a partition $\mathcal{P} \subset \mathcal{E}$ of Ω such that $\{[E] \, \|, E \in \mathcal{P}\} = \mathcal{P}_{\bar{f}}$. Define the act f as follows: for all $s \in E \cap D_n$ $f(s) = t_E(D_n)$. Then, $\bar{f}$ is equivalent to f. This proves (ii).

Let $\bar{g}(E) = \{y_1, \ldots, y_n\}$ and $D_1, \ldots, D_n$ be a diffuse partition, the existence of which is guaranteed in Lemma B6(ii). By part (ii), we can choose $h' \in \mathcal{F}_o$ so that $\bar{h}' = \bar{g}$. Define

$f(s) = y_i$ if and only if $s \in D_i$. Let $h = fEh'$. Hence, $\bar{h} = \bar{g}$ as well. Note that h, f have the desired properties. $\square$

For any partition act $\bar{f}$ and $E \in \mathcal{P}_{\bar{f}}$, we define

$$x(E, \bar{f}) := \min \bar{f}(E)$$
$$y(E, \bar{f}) := \max \bar{f}(E)$$

Let $\mathcal{E}_{xy}(\bar{f}) = \{E \in \mathcal{P}_{\bar{f}} \,|\, x = x(E, \bar{f}), y = y(E, \bar{f})\}$ and define

$$E_{xy}(\bar{f}) = \bigvee_{E \in \mathcal{E}_{xy}(\bar{f})} E$$

We define

$$\mathcal{P}(\bar{f}) = \{E_{xy}(\bar{f}) \ x, y \in M\}$$

Note that $\mathcal{P}(\bar{f})$ is a partition that is coarser than $\mathcal{P}_{\bar{f}}$; that is, for $E \in \mathcal{P}_{\bar{f}}$ there exists a unique $\hat{E} \in \mathcal{P}(\bar{f})$ such that $E = E \wedge \hat{E}$. Finally, we define the partition act f^* on $\mathcal{P}(\bar{f})$ as

$$f^*(E_{xy}) = \{x\} \cup \{y\}$$

We call f^* the binary partition act of f.

Lemma B10: *(i) $\bar{f} = \bar{g}$, then $f \sim g$. (ii) If $f^* = \bar{g}$, then $f \sim g$.*

Proof: For all f, g such that $\bar{f} = \bar{g}$ and $E \in \mathcal{P}_{\bar{f}}$, let $T(E) = 0$ if there exists $E' \in [E]$ such that $f(s) = g(s)$ for all $s \in E'$ and $T(E) = 1$ otherwise. We will prove the result by induction on the cardinality of the set of $E \in \mathcal{P}_{\bar{f}}$ such that $T(E) = 1$. If this set is empty, then f, g differ on a set $E \in \mathcal{E}$ such that $\mu(E) = 0$. Hence, $gE^c m \succeq f \succeq gE^c l$ by Lemma B1(i). Similarly, we have $fE^c m \succeq f \succeq fE^c l$. Since E is null, we have $fEm = gEm \sim gEl = fEl$ and therefore $f \sim g$. Next, assume the assertion holds whenever the cardinality of $E \in \mathcal{P}_{\bar{f}}$ such that $T(E) = 1$ is k and consider $E \in \mathcal{P}_{\bar{f}}$ for some f, g for which this cardinality is $k + 1$. Choose $E' \in [E]$ such that $T(E) = 1$ and let n be the cardinality of $\bar{f}(E)$. Since $T(E) = 1$, $n > 1$. Hence, $\bar{f}(E) = \{y_1, \ldots, y_n\}$ for $y_i < y_{i+1}$. Choose a partition $D_1, \ldots D_n \in \mathcal{D}$ and let h be the act yields y_i on D_i for all

i. Let $D_i^* = [D_i \cap E^c] \cup [f^{-1}(y_i) \cap E]$ for all i. It is easy to verify that D_i^* is diffuse for all i. Consider the act h' that yields y_i on each D_i^*. By Lemma B9(i) and Axiom 4(ii), $h' \sim h$. That is, $fEh \sim h$. A similar argument yields $fEh \sim h$, therefore $fEh \sim gEh$ and finally, $fEg \sim g$. But notice that for f and fEg, the cardinality of the set of $[E'] \in \mathcal{P}_{\bar{f}}$ such that $T(E') = 1$ is k and hence by the inductive hypothesis $f \sim fEg$ and therefore $f \sim g$ proving part (i).

The proof is by induction on the cardinality of the set

$$\{E \in \mathcal{P}_{\bar{f}} \,|\, \bar{f}(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$$

When that cardinality is 0, the one-to-oneness of partition functions functions ensures that $\bar{f} = \bar{g} = f^*$ and then part (i) yields $f \sim g$. Suppose the cardinality of that set is $k+1$ and pick any element E of that set. Let $\hat{E}$ be the element of $\mathcal{P}_{\bar{g}}$ such that $E \cap \hat{E} = E$. Choose $f' \in \mathcal{F}_d$ and $h \in \mathcal{F}_o$ such $hEf' = f'$ and $\bar{h} = \bar{f}$. Similarly, choose $g' \in \mathcal{F}_d$ and $h_* \in \mathcal{F}_o$ such $h_*\hat{E}g' = g'$ and $\bar{h}_* = \bar{g}$. Since $\bar{f}(E) \neq \bar{g}(\hat{E})$, we know that both the cardinality of $\bar{f}(E)$ and that of $\bar{g}(\hat{E})$ must be greater than 1. Hence, by Lemma B9(i), $h \sim h_*$. Set $f' = h^*Ef$. It follows from Axiom 3 that $f' \sim hEf = f$. Note that $\mathcal{P}_{f'} = \mathcal{P}_f$ and the cardinality of the set $\{E \in \mathcal{P}_{\bar{f}'} \,|\, \bar{f}'(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$ is one smaller than that of $\{E \in \mathcal{P}_{\bar{f}} \,|\, \bar{f}(E) \neq \bar{g}(\hat{E}) \text{ for } \hat{E} \text{ such that } E \leq \hat{E}\}$. Hence, by the inductive hypothesis, $f' \sim g$ yielding $f \sim g$. $\square$

Let

$$p_f(\{(x,y)\}) = \mu(E_{xy}(\bar{f}))$$

Since $\mu(E') = \mu(E)$ for all $E' \in [E]$, the meaning of $\mu([E])$ is clear and $\mu([E]) = \mu(E)$. Furthermore, p_f is clearly a discrete bilottery. Let $\psi_o(f)$ denote the function that assigns p_f to each f. Hence, ψ maps $\mathcal{F}_o$ into $\mathcal{L}_o$. The next Lemma shows that this map is onto.

Lemma B11: *The function ψ_o is onto and $\psi_o(f)$ is a bilottery for all $f \in \mathcal{F}_o$.*

Proof: Let $p = \sum_{i=1}^n \alpha_i 1_{x_i y_i}$, where $x_i \leq y_i$, $\alpha_i > 0$ for each i and $\sum_i \alpha_i = 1$. Choose any partition $E_1, \ldots, E_n \in \mathcal{E}$ of Ω such that $\mu(E_i) = \alpha_i$. Define $g(E_i) = \{x_i\} \cup \{y_i\}$. Then, g is a partition act. Let f be an act such that $\bar{f} = g$. Clearly, $\psi_o(f) = p$. $\square$

Definition: *Let $u : I \to I\!R$ be defined as $u(x, y) = v(z)$ for z such that $yDx \sim z$.*

Lemma B12: *The function u is increasing and continuous.*

Proof: Suppose $yDx \sim z$ and $\hat{y}D\hat{x} \sim \hat{z}$. If $\hat{y} > y$ and $\hat{x} > x$, then Axiom 2 implies $\hat{z} \succ z$ and applying Axiom 2 again yields $\hat{z} > z$ as desired. If $\hat{y} \geq y$ and $\hat{x} \geq x$, then by Lemma B1(i), $\hat{z} \succeq z$. Then, applying Axiom 2 again yields $\hat{z} \geq z$.

To prove continuity, assume $y_iDx_i \sim z_i$ for $i = 1, \ldots$ and $\lim(x_i, y_i) = (x, y)$. Since z_is are in a compact set in proving continuity, we can assume this sequence converges to some z. Suppose $yDx \succ z$ and note that since y_iDx_i converges uniformly to y_iDx_i and the act z_i converges uniformly to z, we have by Lemma B1(iii), $y_iDx_i \sim z_i$ for some i, a contradiction. A symmetric argument yield $y_iDx_i \sim z_i$ and establishes continuity. $\square$

For $f \in f_o$ define

$$U(f) = \int u d\psi_o(f)$$

Lemma B13: *(i) For all $f \in \mathcal{F}_o$,*

$$U(f) = \sum_{E \in \mathcal{P}_{f^*}} \mu(E)u(x(E, f^*), y(E, f^*))$$

(ii) If $u(x(E, f^*), y(E, f^*)) = u(z, z)$ for $E \in \mathcal{P}_{f^*}$, then $U(zEf) = U(f)$.

Proof: Part (i) is obvious since for any $p = \sum_i \alpha_i 1_{x_i, y_i}$,

$$\int u dp = \sum_i \alpha_i u(x_i, y_i) \qquad (*)$$

If $\mathcal{P}_{f^*} = \mathcal{P}_{(zEf)^*}$, part (ii) follows immediately from part (i). If not, then there exists $E' \in \mathcal{P}_{f^*}$ such that $f^*(E') = \{z\}$. Then, part (i) together with the fact that $\mu(E \cup E') = \mu(E) + \mu(E')$ yield the desired conclusion. $\square$

Let $d(f)$ be the cardinality of the set $\{E \in \mathcal{P}_{f^*} \mid f^*(E) \text{ is not a singleton}\}$. Hence, if $d(f) = 0$, then $f \in \mathcal{F}_e$.

Lemma B14: *The function U represents the restriction of $\succeq$ to $\mathcal{F}_o$.*

Proof: Let $\mathcal{F}^n = \{f \in \mathcal{F}_o \,|\, d(f) \leq n\}$. The proof is by induction on $\mathcal{F}^n$. Note that for $f \in \mathcal{F}^0$

$$\sum_{x \in M} v(x)\mu(f^{-1}(x)) = \int ud\psi_0(f) = U(f)$$

Hence, by Lemma B3, the restriction of U to $\mathcal{F}^0$ represents $\succeq$. Suppose U represents the restriction of $\succeq$ to $\mathcal{F}^n$ and choose $f, g \in \mathcal{F}^{n+1}$. Define h_f as follows: if $f \in \mathcal{F}^n$, then $h_f = f$. Otherwise, choose $E \in \mathcal{P}_{f^*}$ such that the cardinality of $f^*(E)$ is not 1. Hence, $y(E, f^*) > x(E, f^*)$. Lemma B12 ensures that there exists a unique z such that $u(z, z) = u(x(E, f^*), y(E, f^*))$. By construction, $y(E, f^*)Dx(E, f^*) \sim z$. Hence, by Axiom 3, $zEf \sim f$. By Lemma B13(ii), $U(zEf) = U(f)$. Hence, set $h_f = zEf$. Construct an h_g in the same fashion. Then, $f \succeq g$ if and only if $h_f \succeq h_g$. By the inductive hypothesis, $h_f \succeq h_g$ if and only if $U(h_f) \geq U(h_g)$. Since $U(h_f) = U(f)$ and $U(h_g) = U(g)$, the desired result follows. □

Recall that p_f is defined as the bilottery corresponding to the bicumulative $\mathbf{b}_f(x, y) = \mu(f^{-1}([x, y])$. By Proposition 2, p_f is a well defined bilottery. Lemma B13 below shows that for a simple act $f \in \mathcal{F}_o$ the bilottery p_f coincides with the bilottery $\psi_o(f)$ defined before Lemma B11 above.

Lemma B13: *For $f \in \mathcal{F}_o$, $p_f = \psi_o(f)$.*

Proof: We must show that for any $\mathcal{F} \in \mathcal{F}_o$, $\psi_o(f)([x, y] \times [x, y]) = \mu(f^{-1}[x, y])$. Let $\bar{f}$ be the partition act associated with f and let $E_1^f, \ldots, E_n^f$ be the corresponding partition. Define

$$J_{xy} = \{i = 1, \ldots, n \,|\, x \leq \min \bar{f}(E_i), \max \bar{f}(E_i) \leq y\}$$

Then let $J_2 = \{i = 1, \ldots, n, \,|\, i \notin J_{xy}, \bar{f}(E_i) \cap [x, y] \neq \emptyset\}$ and $J_3 = \{i = 1, \ldots, n, \,|\, i \notin J_{xy} \cup J_2\}$. Set $E_1 = \bigcup_{i \in J_{xy}} E_i$, $E_2 = \bigcup_{i \in J_2} E_i$, and $E_3 = \bigcup_{i \in J_3} E_i$ and note that E_1, E_2, and E_3 is a partition for $f^{-1}[x, y])$ of the type guaranteed in Lemma B6. It follows that $\mu(f^{-1}[x, y]) = \mu(E_1) = \psi_o(f)([x, y] \times [x, y])$. □

For $f \in \mathcal{F}$ define

$$U(f) = \int udp_f$$

By Lemma B13 this definition if $U(f)$ coincides with the earlier definition on simple acts $f \in f_o$. Lemma B14 shows that U as defined above represents the preference for all acts. Lemma B14 completes the proof of Theorem 2.

Lemma B14: *The function U represents $\succeq$.*

Proof: Note that for all f, there exists x_f such that $U(x_f) = u(x_f, x_f) = U(f)$. This follows from that fact that u is increasing in both arguments and continuous which implies $u(m, m) \geq U(f) \geq u(l, l)$ and by the intermediate value theorem $u(x_f, x_f) = U(f)$ for some $x_f \in [l, m]$. The monotonicity of u ensures that this x_f is unique. Next, we show that $f \sim x_f$.

Without loss of generality, assume $l = 0$ (if not let $l^* = 0$ and $m^* = m - l$ and identify each f with $f^* = f - l$ and apply all previous results to acts $\mathcal{F}^* = \{f - l \,|\, f \in \mathcal{F}\}$.) Define for any $x \geq 0$ and $\epsilon > 0$, $z^*(x, \epsilon) = \min\{n\epsilon \,|\, n = 0, 1, \ldots \text{ such that } n\epsilon \geq x\}$. Similarly, let $z_*(x, \epsilon) = \max\{n\epsilon \,|\, n = 0, 1, \ldots \text{ such that } n\epsilon \leq x\}$. Clearly,

$$0 \leq z^*(x, \epsilon) - x \leq z^*(x, \epsilon) - z_*(x, \epsilon) < \epsilon \tag{4}$$

and the first two inequalities above are equalities if and only if x is a multiple of ϵ.

Set $f^n(\omega) = z^*(f(\omega), m2^{-n})$ and $f_n(\omega) = z_*(f(\omega), m2^{-n})$ for all $n = 0, 1, \ldots$. Equation (4) above ensures that $f^n \geq f \geq f_n$ and f^n, f_n converge uniformly to f. Note also that $f^n, f_n \in \mathcal{F}_o$.

Let (i) $\mathbf{b} = \mathbf{b}_f$, $G = G_{\mathbf{b}}$, and $p = p_f$ (ii) $\mathbf{b}^n = \mathbf{b}_{f^n}$, $G^n = G_{\mathbf{b}^n}$, and $p^n = p_{\mathbf{b}^n}$, (iii) $p_n = p_{f_n}$, $G_n = G_{\mathbf{b}_n}$, and $p_n = p_{\mathbf{b}_n}$. Recall that

$$G_{\mathbf{c}}(x, y) = \mathbf{c}(l, y) - \lim_{x^+ \downarrow x} \mathbf{c}(x^+, y)$$

is the standard cumulative corresponding to the bicumulative $\mathbf{c}$.

Next, we will show that p^n and p_n converge weakly to p. Fix (x, y) be any continuity point of G and let $T = \{(\hat{x}, \hat{y}) \in I \,|\, \hat{x} \leq x, \hat{y} \leq y\}$. Then, $p(T) = G(x, y)$. Define, $x^n = z^*(x, m2^{-n}), y^n = z^*(y, m2^{-n}), x_n = z_*(x, m2^{-n}), y_n = z_*(y, m2^{-n})$ and also set

$T^n = \{(\hat{x}, \hat{y}) \in I \,|\, \hat{x} \leq x^n, \hat{y} \leq y^n\}$ and $T_n = \{(\hat{x}, \hat{y}) \in I \,|\, \hat{x} \leq x_n, \hat{y} \leq y_n\}$. By equation (4) and the observation made immediately after its statement, we have

$$G(x^n, y^n) = \eta(T^n) = \eta^n(T^n) = \eta^n(T) \geq \eta_n(T) = \eta_n(T_n) = \eta(T_n) = G(x_n, y_n) \quad (5)$$

Since (x, y) is a continuity point of G and (x^n, y^n), (x_n, y_n) both converge to (x, y), we conclude that $p^n(T) = G^n(x, y)$ and $p_n(T) = G_n(x, y)$ converge to $G(x, y)$, proving the weak convergence of p^n and p_n to p.

Since p^n weakly converges to p, $U(f^n)$ converges to $U(f)$. Also, since $f^n \geq f$, we have $U(f^n) \geq U(f) = U(x_f)$ for all n. Since U represents the restriction of $\succeq$ to $\mathcal{F}_0$, we conclude that $f^n \succeq x$ for all n. Then, Axiom 6 implies $f \succeq x$. A symmetric argument with f_n replacing f^n yields $x_f \succeq f$ and therefore $x_f \sim f$ as desired.

To conclude the proof of the Lemma, suppose $f \succeq g$, then $U(x_f) = U(f)$ and $U(x_g) = U(g)$ and $x_f \sim f \succeq g \sim x_g$. Since U represents the restriction of $\succeq$ to $\mathcal{F}_o$, we conclude that $U(x_f) \geq U(x_g)$ and hence $U(f) \geq U(g)$. Similarly, if $U(f) \geq U(g)$ we conclude $f \sim x_f \succeq x_g \sim g$ and therefore $f \succeq g$. $\square$

The proof of uniqueness follows from the standard uniqueness argument and is therefore omitted. $\square$

9. Appendix C: Proof of Theorem 2

In this section, we provide the proof of Theorem 2. The proof is divided into a series of steps.

Proof: Let $\mathcal{E} := \mathcal{E}_+ \cup \mathcal{N}$ and consider the restriction of $\succeq$ to $\mathcal{E}$. Then, $\succeq$ is a qualitative probability and hence property 3 applied to $\mathcal{E}$ together with Savage's qualitative probability representation theorem yields the existence of a finitely additive, convex valued probability measure ν such that $E \succeq E'$ if and only if $\nu(E) \geq \nu(E')$. Clearly, $\nu(E) = 0$ if and only if $E \in \mathcal{N}$. For $A \subset \Omega$ define

$$\mu(A) = \inf_{E \subset A, E \in \mathcal{E}} \nu(E)$$

Consider any disjoint collection of sets E'_i and let $E' = \bigcup_{i=1}^{\infty} E'_i$. Let $E_i = \bigcup_{j=i}^{\infty} E_j$. Since μ is convex valued, by Property 5, for all $\epsilon > 0$, there exists i such that $\mu(E_i) < \epsilon$. Hence, $\lim \mu(E_i) = 0$. Therefore,

$$\mu(E') = \mu\left(E_i \cup \bigcup_{j=1}^{i-1} E'_j\right) = \mu(E_i) + \sum_{j=1}^{i-1} \mu(E'_j)$$

Taking a limit as i goes to infinity yields $\mu(E') = \sum_{j=1}^{\infty} \mu(E'_j)$ proving the countable additivity of μ on $\mathcal{E}$.

Step 1: *The set function μ is a semiprobability.*

Proof: It is straightforward to show that μ is supermodular. Therefore, it remains to show that (i) for all A there exists $E \in \mathcal{E}$ such that $E \subset A$ and $\mu(E) = \mu(A)$ and (ii) $\mu\left(\cap_{i-1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ for any $\{A_i\}_{i=1}^{\infty}$ with $A_i \subset A_{i+1}$. To see (i) let $E_i \in \mathcal{E}$ be any sequence of sets such that $E_i \subset A$ and $\lim \mu(A_i) = \mu(A)$. Then, let $E = \bigcup_i E_i$ and note that E is the desired set. Since μ is countably additive on $\mathcal{E}$ (ii) follows from (i). $\square$

Step 2: *For any A, there exists $E \in \mathcal{E}$ such that $A \sim E$.*

Proof: Let $\alpha = \sup\{\mu(E) \,|\, E \in \mathcal{E}, A \succeq E\}$. Clearly $\alpha \in [0,1]$ is well-defined. Since μ is convex valued, there exists E such that $\mu(E) = \alpha$. We claim that $A \sim E$. To see this note that if $A \succ E$, then we must have $\alpha < 1$ and by Property 3, there exists a partition $E_1, \ldots, E_n \in \mathcal{E}_+$ such that $A \succ E \cup E_i$ for all i. So, $\mu(E \cup E_i) > \mu(E) = \alpha$ for some i, contradicting the definition of α. Conversely, if $E \succ A$, then there exists E' such that $E \succ A \cup E'$. Hence, $\mu(E) > \mu(E')$ so that we can without loss of generality, assume $E' \subset E$. Then, by Property 1, $E \backslash E' \succ A \backslash E'$. Then, by Property 3, there exists a nonnull $E^* \subset E'$ such that $\mu(E^*) > 0$ and $E \backslash E' \succ A \backslash E' \cup E^*$. Then, taking the union of both sides with $E' \backslash E^*$ and applying property 1 yields $E \backslash E^* \succ A \cup E' \succ A$. But, $\mu(E \backslash E^*) < \alpha$, again contradicting the definition of α. So, $E \sim A$ as desired. $\square$

Step 3: *If $\mu(E_1) = \mu(E_2)$, and $D_1, D_2 \in \mathcal{D}$, then $E_1 \cap D_1 \sim E_2 \cap D_2$.*

Proof: If $D \sim \emptyset$ for some D then, by Property 3, $D \sim \emptyset$ for all $D \in \mathcal{D}$ and the result follows. Hence, assume that $D \succ \emptyset$ for all $D \in \mathcal{D}$. First, we prove the result for $E_1, E_2 = \Omega$.

If $D_1 \succ D_2$, then by Property 3, there exists $E \in \mathcal{E}_+$ such that $D_1 \succ D_2 \cup E$, which by Property 1 (and monotonicity) implies $D_1 \backslash E \succ D_2 \backslash E$. Hence, by Property 3, there exists $E_* \subset E$ such that such that $\mu(E_*) > 0$ and $D_1 \backslash E \succ [D_2 \backslash E] \cup E_*$. Note that by Property 2, $D_2 \cap [E^c \cup E_*] \succ D_1 \backslash E$. But $D_2 \cap [E^c \cup E_*] \subset [D_2 \backslash E] \cup E_*$, hence monotonicity yields a contradiction. A symmetric argument yields $D_1 \sim D_2$. Note that $\mu(E) = 1$ implies $E \cap E_i \neq \emptyset$ for all $E_i \in \mathcal{E}_+$ it follows that $E \cap D \in \mathcal{D}$ for $D \in \mathcal{D}$. Therefore, the above argument proves the result for all E_1 with $\mu(E_1) = 1$.

If $\mu(E_1) < 1$ and $E_1 \cap D \succ E_2 \cap D$ then, by Property 3, $E_1 \cap D \succ E_2 \cap D \cup E_i$ for some $E_i \in \mathcal{E}_+$, $E_i \in E_2^c$. Since $E_2 \cup E_i \succ E_1$ Property 2 implies that $E_2 \cap D \cup E_i \succeq [E_2 \cup E_i] \cap D \succ E_1 \cap D$, a contradiction. $\square$

Define t as follows: $t = \mu(E)$ for $E \sim D$, $D \in \mathcal{D}$. By Step 3, t does not depend on the choice of $D \in \mathcal{D}$. We say that a $\alpha \in (0, 1]$ is diadic if $\alpha = \frac{k}{2^n}$ for natural numbers k, n; such a number i of order n if k is odd. Hence, every diadic number can be stated uniquely as $\frac{k}{2^n}$ where n is its order.

Step 4: *For any diadic number α, and $E_1, E_2 \in \mathcal{E}_+$, if $\mu(E_1) = t\alpha$ and $E_2 = \alpha$, then $E_1 \sim E_2 \cap D$.*

Proof: The proof is by induction on the order of α. If α is of order 0, then $E_2 \cap D \in \mathcal{D}$ and hence the result follows from the definition of t. Suppose the result is true when α is of order $n-1$. Assume $\alpha = k2^{-n}$ and E_1, E_2 satisfy the hypotheses of the assertion. Choose disjoint sets E_2^j, $j = 1, \ldots k+1$ with $\mu(E_2^j) = 2^{-n}$ and choose $E_1^j \subset E_2^j$ with $\mu(E_1^j) = t2^{-n}$. By Step 3, $E_2^j \cap D \sim E_2^i \cap D$ for all i, j and if $E_1^j \succ E_2^j \cap D$ for some j we have $E_1^j \succ E_2^j \cap D$ for all j. By repeated application of Property 1, $\cup_{j=1}^{k+1} E_1^j \succ \cup_{j=1}^{k+1} [E_2^j \cap D] = [\cup_{j=1}^{k+1} E_2^j] \cap D$, contradicting the inductive hypothesis. A symmetric argument concludes the proof. $\square$

Step 5: *$E_1 \succeq E_2 \cap D$ if and only if $\mu(E_1) \geq t\mu(E_2)$.*

Proof: By Property 3, if $E_1 \succ E_2 \cap D$, we can find $\hat{E} \succ E_2$ such that $E_1 \succ \hat{E} \cap D$. Choose any diadic number α in the interval $(\mu(E_2), \mu(\hat{E}))$ and E' such that $\mu(E') = \alpha$. By Property 2, $E_1 \succ E' \cap D$. Choose E such that $\mu(E) = t\alpha$ and, by Step 4, we have $E_1 \succ E' \cap D \sim E$. Hence, $\mu(E_1) > \alpha t > t\mu(E_2)$. A symmetric argument yields

$\mu(E_1) < t\mu(E_2)$ whenever $E_2 \cap D \succ E_1$. If $\mu(E_1) > t\mu(E_2)$, then there exists a diadic α such that $\mu(E_1) > t\alpha > t\mu(E_2)$. Pick E such that $\mu(E) = \alpha$ and E' such that $\mu(E') = t\alpha$. Note that Step 4 yields $E' \cap D \sim E$. Since $\mu(E_1) > \mu(E')$ and $\mu(E) \geq \mu(E_1)$, by Property 2, we have $E_1 \succ E' \sim E \cap D \succ E_2 \cap D$, yielding $E_1 \succ E_2 \cap D$. A symmetric argument yields $E_2 \cap D \succ E_1$ whenever $t\mu(E_2) > \mu(E_1)$. $\square$

Call (E_1, E_2, D) a *decomposition* of A if $E_1 \cap E_2 = \emptyset$ and $A = E_1 \cup [E_2 \cap D]$. A decomposition of A is null if $\mu(E_1) = \mu(E_2) = 0$.

Step 6: *(i) Every A has a decomposition. (ii) If (E_1, E_2, D) and $(\hat{E}_1, \hat{E}_2, \hat{D})$ are two decompositions of A, then such that $\mu(E_1) = \mu(\hat{E}_1)$ and $\mu(E_2) = \mu(\hat{E}_2)$. (iii) If A is null then it has a null decomposition.*

Proof: To prove (i) note that since μ is a semiprobability we can find $E_1 \subset A, E_3 \subset A^c$ such that $\mu(E_1) = \mu(A)$ and $\mu(E_3) = \mu(A^c)$. Let $E_2 = [E_1 \cup E_2]^c$. Pick and $D \in \mathcal{D}$ and let $\hat{D} = [D \cap E_1] \cup [A \cap E_2] \cup [D \cap E_3]$. We claim that $\hat{D} \in \mathcal{D}$. To see this note that if $E \in \mathcal{E}_+$, then $\mu(E \cap E_i) > 0$ for some i. If this is true for $i = 1$ or $i = 3$ we have, $\hat{D} \cap E = \emptyset$ and $\hat{D}^c \cap E = D^c \cap E \neq \emptyset$. Next, suppose $\mu(E \cap E_2) > 0$. If $\hat{D} \cap E \cap E_2 = A \cap E \cap E_2 = \emptyset$ then $E \cap E_2 \subset A^c$ and hence $[E \cap E_2] \cup E_3 \subset A^c$. But we have $\mu([E \cap E_2] \cup E_3) = \mu(E \cap E_2) + \mu(E_3) > \mu(A^c)$ a contradiction. A symmetric argument yields $\hat{D}^c \cap E \cap E_2 \neq \emptyset$. Hence, $\hat{D} \in \mathcal{D}$. Then, note that $E_1 \cup [E_2 \cap \hat{D}] = A$.

To prove (ii), let (E_1, E_2, D) and $\hat{E}_1, \hat{E}_2, \hat{D}$ be two decompositions of A. Note that this implies (E_3, E_2, D^c) is a decomposition of A^c. In particular, $E_2 \cap D^c \subset A^c$. If $\mu(E_1 \backslash \hat{E}_1) > 0$, we have $E_1 \backslash \hat{E}_1 \subset \hat{E}_2 \cap A = \hat{E}_2 \cap \hat{D}$. But we also have $\hat{D}^c \cap (E_1 \backslash \hat{E}_1) \neq \emptyset$ and therefore $D^c \cap D \neq \emptyset$, a contradiction. A symmetric argument ensures $\mu(E_1) = \mu(\hat{E}_1)$. If $\mu(E_2 \backslash E_2) > 0$, then $E_2 \backslash E_2 \subset \hat{E}_1$. But then, $\emptyset \neq D^c \cap E_2 \backslash \hat{E}_2 \subset E_1 \subset A$ and $D^c \cap E_2 \subset A^c$ and hence $A^c \cap A \neq \emptyset$, a contradiction.

(iii) That A null if it has a null partition is obvious. $\square$

Step 7: *If $A_1, A_2 \subset E \in \mathcal{E}$ and $A_3, A_4 \in E^c$, and $A_3 \sim A_4$, then $A_1 \succeq A_2$ if and only if $A_1 \cup A_3 \succeq A_2 \cup A_4$.*

Proof: To see this, note that the result is obvious if E is null. So, assume it is not. Then, from repeated applications of property 2, we have (i) $A_2 \cup A_3 \sim A_2 \cup A_4$ and (ii) $A_1 \succeq A_2$ if and only if $A_1 \cup A_3 \succeq A_2 \cup A_3$. Then, (i) and (ii) yield the desired conclusion. $\square$

Let $T(A) = \mu(E_1) + t\mu(E_2)$ where (E_1, E_2, D) is some decomposition of A. We claim that $A \succeq B$ if and only if $T(A) \geq T(B)$. Let (E_1^A, E_2^A, D^A) be a decomposition of A and (E_1^B, E_2^B, D^B) be a decomposition of B. By Property 1, we can, without loss of generality, assume $E_1^A \cap E_1^B = E_2^A \cap E_2^B = \emptyset$. Let $E = E_1^A \cap E_2^B$ and $E' = E_1^B \cap E_2^A$. Pick $E_t \subset E$ and $E_t' \subset E$ such that $\mu(E_t) = t\mu(E)$ and $\mu(E_t') = t\mu(E')$. By Steps 5 and 7, we have $E_1^A \cup (E_2^A \cap D^A) \succeq E_1^B \cup (E_2^B \cap D^B)$ if and only if

$$(E_1^A \backslash E_t) \cup ((E_2^A \backslash E') \cap D^A) \succeq (E_1^B \backslash E_t') \cup ((E_2^B \backslash E) \cap D^B)$$

Note that $(E_1^A \backslash E_t, E_2^A \backslash E', D)$ is a decomposition of $A \backslash (E_t \cup E')$ and $(E_1^B \backslash E_t', E_2^B \backslash E, D^B)$ is a decomposition of $B \backslash (E_t' \cup E)$. Note that $T(A) \geq T(B)$ if and only if $T(A \backslash (E_t \cup E')) \geq T(B \backslash (E_t' \cup E))$.

Hence, it suffices to prove the result for disjoint A and B. But by Step 7 again, we can assume (i) $E_1^A = \emptyset$ or $E_1^B = \emptyset$ and (ii) $E_2^A = \emptyset$ or $E_2^B = \emptyset$. Recall that A has a null decomposition if it is null. Hence, the result is obvious if either A or B are null. So, assume neither is null. This leaves two symmetric cases, without loss of generality, assume $A = E$ and $B = E' \cap D$. Then, (5) yields the desired conclusion.

Let (E_1^A, E_2^A, D^A) be a decomposition of A and $E_3^A = (E_2^A \cup E_2^A)^c$. To conclude the proof, note that $T(A) \geq T(B)$ if and only if $\mu(E_1^A) + t\mu(E_2^A) \geq \mu(E_1^B) + t\mu(E_2^B)$ if and only if $\mu(E_1^A) + t[1 - \mu(E_1^A) - \mu(E_3^A)] \geq \mu(E_1^B) + t[1 - \mu(E_1^B) - \mu(E_3^B)]$ if and only if $\mu(A) + t[1 - \mu(A) - \mu(B)] \geq \mu(B) + t[1 - \mu(B) - \mu(A)]$ $\square$

10. Appendix D

10.1 Proof of Theorem 3

To prove part (ii) consider p and q with $q(T) > p(T) + \epsilon$ for some closed southeast set T. Let

$$d_h(A, B) = \max\{\max_{(x,y)\in A} \min_{(w,z)\in B} |x - w| + |y - z|, \max_{(w,z)\in B} \min_{(x,y)\in A} |x - w| + |y - z|\}$$

and define $u_\alpha : I \to I\!R$ as follows:

$$u_\alpha(x, y) = 2(x + y) + \max\{\alpha - d_h(T, \{(x, y)\}), 0\}$$

Note that u_α is monotone and continuous. To prove that u_α is uncertainty averse, it suffices to show that

$$d_h(T, \{(x, y)\}) - d_h(T, \{(x + \delta, y)\}) \geq d_h(T, \{(w, z)\}) - d_h(T, \{(w, z + \delta)\})$$

for all (x, y) and (w, z). Since T is a southeast set, we have

$$\begin{aligned} d_h(T, \{(x, y)\}) \geq& d_h(T, \{(x + \delta, y)\}) \\ d_h(T, \{(w, z + \delta)\} \geq& d_h(T, \{(w, z)\}) \end{aligned}$$

and hence u_α is uncertainty averse.

Let $T^\delta := \{(x, y) : d_h(T, \{(x, y)\}) < \delta\}$ and note that since T is closed there is $\bar{\delta} > 0$ such that $p(T^{\bar{\delta}} \backslash T) < \epsilon/2$. Set $\alpha = \bar{\delta}$ and let $\Delta := T^\alpha \backslash T$. Then,

$$\begin{aligned} \int u_\alpha dq - \int u_\alpha dp &= \int \max\{\alpha - d_h(T, \{x, y\}), 0\} dq - \int \max\{\alpha - d_h(T, \{x, y\}), 0\} dp \\ &\geq q(T) - p(T) - p(\Delta) \\ &\geq \epsilon/2 \end{aligned}$$

which proves part (ii) of the theorem.

To prove part (i) we consider first simple bilotteries p, q with $p(\{(x, y)\}) \in \{0, 1/n\}$ and $q(\{(x, y)\}) \in \{0, 1/n\}$ for all (x, y). Assume $q(T) \leq p(T)$ for all southeast sets T. Let S_p be the support of p and let S_q be the support of q and note that both have n elements. We claim that there is a bijection $g : S_p \to S_q$ such that $(x', y') = g(x, y)$ implies $x' \leq x \leq y \leq y'$. We will prove this using Hall's theorem. Let A be a subset of S_q and let $T(A)$ denote the subset of M such that for each $(x, y) \in T(A)$ there is $(x', y') \in A$ with $x' \leq x \leq y \leq y'$. Clearly, $A \subset T(A)$ and $T(A)$ is a southeast set. Since q is a mean-preserving u-spread of p it follows that the cardinality of $S_p \cap T(A)$ is at least as large as the cardinality of $S_q \cap A \subset S_q \cap T(A)$. Hall's theorem therefore guarantees that every $(x, y) \in S_p$ can be matched to a distinct $(x', y') \in S_p$ such that $x' \leq x \leq y \leq y'$. Since S_q and S_p have the same number of elements it follows that there is a bijection g with the desired properties. Let $S_p = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ and let $S_q = \{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$ with $x'_i \leq x_i \leq y_i \leq y'_i$. We must show that

$$\sum_{i=1}^{n} u(x_i, y_i) \geq \sum_{i=1}^{n} u(x'_i, y'_i)$$

For a given $((x_1^k, y_1^k), \ldots, (x_n^k, y_n^k))$ let $\delta_1^k = 0$ and $i^k = n+1$ if $x_i^k \geq x_i$ for all i. Otherwise let $\delta_1^k := \min_{\{i : x_i > x_i^k\}} x_i - x_i$ and $i^k := \min\{i : x_i - x_i^k \geq \delta_1^k\}$. Similarly, let $\delta_2^k = 0$ and $j^k = n+1$ if $y_i \geq y_k^i$ for all i. Otherwise, let $\delta_2^k := \min_{\{i : y_i < y_i^k\}} y_i^k - y_i$ and $j^k := \min\{i : y_i^k - y_i \geq \delta_2^k\}$. Define $\delta^k = \min\{\delta_1^k, \delta_2^k\}$. For $i = 1, \ldots, n$, let

$$x_i^{k+1} = \begin{cases} x_i^k & \text{if } i \neq i^k \\ x_i^k + \delta^k & \text{if } i = i^k \end{cases}$$

$$y_i^{k+1} = \begin{cases} y_i^k & \text{if } i \neq j^k \\ y_i^k - \delta^k & \text{if } i = j^k \end{cases}$$

Let $((x_i^0, y_i^0)) = ((x_i', y_i'))$. If $M(p) \geq M(q)$ then it is easy to see that $(x_i^n, y_i^n) \leq (x_i, y_i)$ for all i (equality holds of $M(p) = M(q)$. By uncertainty aversion

$$\sum_{i=1}^n u(x_i^{k+1}, y_i^{k+1}) \geq \sum_{i=1}^n u(x_i^k, y_i^k)$$

for all k and by monotonicity

$$\begin{aligned} \sum_{i=1}^n u(x_i, y_i) &\geq \sum_{i=1}^n u(x_i^n, y_i^n) \\ &\geq \sum_{i=1}^n u(x_i^0, y_i^0) = \sum_{i=1}^n u(x_i', y_i') \end{aligned}$$

and therefore part (i) of the theorem follows for this case.

The argument above shows that $\int udp \geq \int udq$ for all uncertainty averse u if p, q are simple bilotteries such that (i) $p(\{(x,y)\}), q(\{(x,y)\})$ are rational numbers; (ii) $M(p) \geq M(q)$ and (iii) $p(T) \geq q(T)$ for every southeast set T.

Next, consider arbitrary simple bilotteries p, q such that (i) $M(p) \geq M(q) + \epsilon$ and (ii) $p(T) \geq q(T)$ for every southeast set T. Let p_n, q_n be sequences of simple bilotteries such that $p_n(x,y), q_n(x,y)$ is rational and $p_n(x,y) \to p(x,y)$ and $q_n(x,y) \to q(x,y)$. Let $q_n^k = {}^{k-1}\!/_k q_n + {}^1\!/_k 1_{lm}$. Choose $k > \frac{l+m}{\epsilon}$. Then, for n sufficiently large $M(q_n^k) < M(p_n)$ and $\int \iota_T dp_n \geq \int \iota_T dq_n^k$ for every southeast set T. Hence, $\int udp_n \geq \int udq_n^k$ for all uncertainty averse u. Since $p_n \to p$ and $q_n^k \to q$ as $n \to \infty, k \to \infty$ it follows that $\int udp \geq \int udq$.

Next, consider arbitrary bilotteries p, q such that (i) $M(p) \geq M(q) + \epsilon$ and (ii) $p(T) \geq q(T)$ for every southeast set T. Let $x_i = l + (m-l)\frac{i}{n}$ and let $S^n = \{(x_i, x_j) : i =$

$0, \ldots, n, j \geq i\}$. The bilottery p^n has support S^n and is defined as

$$p^n(x_i, x_j) = p(x_{i-1}, x_{j+1}) - p(x_i, x_{j+1}) - p(x_{i-1}, x_j) + p(x_i, x_j)$$

The bilottery q_n has support S^n and is defined as

$$q_n(x_i, x_j) = q(x_i, x_j) - q(x_{i-1}, x_j) - q(x_i, x_{j-1}) + q(x_{i-1}, x_{j-1})$$

Note that by construction

$$q_n(T) \leq q(T) \leq p(T) \leq p^n(T)$$

for every southeast set T. Note also that $p^n \to p$ and $q_n \to q$ as $n \to \infty$. Therefore, $M(p^n) > M(q_n)$ for n sufficiently large and $\int u dp^n \geq \int u dq_n$ for all uncertainty averse u. This in turn implies that $\int u dp \geq \int u dq$. Since ϵ was arbitrary the Theorem follows. □

10.2 Proof of Proposition 5

Note that

$$\int \iota_T \mathbf{b}_\lambda = \int_0^1 \int_x^1 \iota_T \frac{\partial^2}{\partial x \partial y}(y-x)^\lambda dy dx$$

Observe that $\frac{\partial^2}{\partial x \partial y}(y-x)^\lambda$ depends on $t = y - x$ only. Let ν denote Lebesgue measure and for $t \in [0, 1)$ define

$$\alpha_T(t) = \frac{\nu(\{x : (x, x+t) \in T\})}{\nu(\{x : (x, x+t) \in I\})} = \frac{\nu(\{x : (x, x+t) \in T\})}{1-t}$$

Note that $\alpha_T(t)$ is non-increasing if T is a southeast set. Let $F_\lambda = t^\lambda$ denote the cumulative distribution of t and observe $F_{\lambda'}$ first order stochastically dominates F_λ if $\lambda' > \lambda$. Hence,

$$\int \iota_T \mathbf{b}_{\lambda'} = \int_0^1 \alpha_T(t) dF_{\lambda'} \geq \int_0^1 \alpha_T(t) dF_\lambda = \int \iota_T \mathbf{b}_\lambda$$

for $\lambda' > \lambda$ as desired. □

10.3 Proof of Proposition 6

Without loss of generality assume that $l = 0$ and therefore $M = [0, m]$. Let (μ, u) be a SEU preference with $u(x, y) = (1-t)u(x, x) + tu(y, y)$ and let $v(x) := u(x, x)$. Define

$\eta(A) = \mu(A) + t \cdot \alpha(A)$. Clearly, η is a capacity. We will show that the preference is represented by the function $W(f)$ where

$$W(f) = \int^c v d\eta = \int_0^\infty \eta(\{s | f(s) \geq x\}) dx$$

is Chocquet expected utility.

Consider a simple act $f \in \mathcal{F}_0$ and let $\bar{f}$ denote the corresponding partition act as defined before Lemma B8. Let $\mathcal{P}_{\bar{f}}$ denote the corresponding partition. Then,

$$\begin{aligned} W(f) &= \sum_{E_i \in \mathcal{P}_{\bar{f}}} \int_{E_i}^c f(s) d\eta \\ &= \sum_{E_i \in \mathcal{P}_{\bar{f}}} \mu(E_i)((1-t)x(E, \bar{f}) + ty(E, \bar{f})) \\ &= \int u d\psi_f \end{aligned}$$

as desired.

For an arbitrary act f, define f_n, f^n as in the proof of Lemma B14 and note that f_n, f^n are simple acts with $f_n \leq f \leq f^n$ converging uniformly to f. Note that $\int^c v(f) d\eta$ is well defined since $\eta(\{s | f(s) \geq x\})$ is monotone. Note further that $\int^c v(f_n) d\eta \leq \int^c v(f) d\eta \leq \int^c v(f^n) d\eta$ and therefore continuity of v implies that $\int^c f_n d\eta \to \int^c f d\eta$. Since $\int u d\psi_{f_n} = \int v(f_n) d\eta$ for all n it follows that $\int u d\psi_f = \int^c v(f) d\eta$ as desired.

Too see necessity consider the act yDx for $x < y$ and D a diffuse set. Continuity of u implies that there is $t \in [0,1]$ such that $yDx \sim xEy$ with $\mu(E) = t$. If $\succeq$ is Chocquet expected utility then it follows that $x'Dy' \sim x'Ey'$ for all x', y' with $x' < y'$. This, in turn, implies that $u(x', y') = (1-t)u(x', x') + tu(y', y')$ for all x', y' with $y' > x'$. $\square$

References

Ellsberg, D. (1961): "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics*, 75, 643–669.

Epstein, L. G., (1999): "A Definition of Ambiguity Aversion," *Review of Economic Studies*, 66, 579–608.

Epstein, L. G., and J. Zhang (2001): "Subjective Probabilities on Subjectively Unambiguous Events," *Econometrica*, 69, 265–306.

Ghirardato, P. and M. Marinacci (2001): "Ambiguity Made Precise: A Comparative Foundation," forthcoming *Journal of Economic Theory*.

Gilboa, I, and D. Schmeidler (1994): "Additive Representations of Non-additive Measures and the Choquet Integral," *Annals of Operations Research,* 52, 43–65.

Jaffray, J.Y. (1989): "Linear Utility Theory for Belief Functions," *Operations Research Letters,* 8, 107–112.

Jaffray, J.-Y., and Wakker, P. P. (1993): "Decision Making with Belief Functions: Compatibility and Incompatibility with the Sure-Thing Principle," *Journal of Risk and Uncertainty,* 7, issue 3, 255–71.

Kopylov, I. (2007): "Subjective Probabilities on Small Domains," *Journal of Economic Theory,* 133, Issue 1, March 2007, 236–265.

Lehrer, E. (2007): "Partially Specified Probabilities: Decisions and Games," mimeo.

Maccheroni, F., M. Marinacci and A. Rustichini (2006): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Economometrica,* 74, 1447–1498.

Machina, M. J. (2007): "Risk, Ambiguity, and the Rank-Dependence Axioms," mimeo.

Machina, M. J. and D. Schmeidler (1992): "A More Robust Definition of Subjective Probability,"*Econometrica*, 60, 745–780.

Savage, L. J. (1954): The Foundations of Statistics, Wiley, New York.

Siniscalchi, M., (2007): "Vector Expected Utility and Attitudes towards Variation," mimeo.

Zhang, J., (2002): "Subjective Ambiguity, Expected Utility and Choquet Expected Utility," *Economic Theory*, 20, 159–181.