# Bargaining and Reputation*

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December 1998

**Abstract**

The paper develops a reputation based theory of bargaining. The idea is to investigate and highlight the influence of bargaining "postures" on bargaining outcomes. A complete information bargaining model à la Rubinstein is amended to accommodate "irrational types" who are obstinate, and indeed for tractability assumed to be completely inflexible in their offers and demands. A strong "independence of procedures" result is derived: after initial postures have been adopted, the bargaining outcome is independent of the fine details of the bargaining protocol so long as both players have the opportunity to make offers frequently. The latter analysis yields a unique continuous-time limit with a war of attrition structure. In the continuous-time game, equilibrium is unique, and entails delay, consequently inefficiency. The equilibrium outcome reflects the combined influence of the rates of time preference of the players and the *ex ante* probabilities of different irrational types. As the probability of irrationality goes to zero, delay and inefficiency disappear; furthermore, if there are a rich set of types for both agents, their limit equilibrium payoffs are inversely proportional to their rates of time preference.

*Abreu thanks the Russell Sage Foundation and Gul thanks the Alfred P. Sloan Foundation for their generous support. Both authors gratefully acknowledge financial support from the National Science Foundation.

# 1 Introduction

This paper addresses the following question. Two agents seek to divide some surplus: to what division will they agree? Our approach is to emphasize the role of *reputation* in the determination of this division.

Non-cooperative bargaining theory in its current form has been deeply influenced by the celebrated paper of Rubinstein (1982) which has provided the basic framework for an enormous and still growing literature. His paper provides a natural reference point for our own work. The only parameters in Rubinstein's *complete* information model are the players' costs of waiting (due to impatience, exogenous termination, etc.) for their turn to make an offer. These parameters determine a unique equilibrium.

Our theory replaces the *impatience* between offers of Rubinstein by uncertainty about the *strategic posture* of one's opponent. Following Kreps and Wilson (1982) and Milgrom and Roberts (1982), we have "irrational" types where each type is identified by a fixed offer and acceptance rule. While players are still impatient, the driving force of the theory is the peripheral uncertainty about the inflexible demand (or rule of thumb, bargaining convention, etcetera) one's opponent may be endowed with, or more significantly, may pretend to.

Rubinstein's theory embraces both slight impatience (frequent offers) and substantial discounting (significant delay) between offers. We view the former as a salient case, as does much of the literature. Many of the remarks below must be understood in this light and we will typically drop the qualifier "in the case of frequent offers" when referring to Rubinstein's results. From this perspective, a first objection to the Rubinstein theory is that a "marginal" feature of the problem (*slight* impatience between offers) completely determines the outcome. Of course, a similar objection may be leveled against our own theory (i.e., *slight* irrationality). In a sense, non-cooperative bargaining theory is quintessentially about deriving a determinate division when *a priori* it is difficult to rule out any of a range of possible divisions. In its *purest* form, it is precisely about explaining the division of a *residual* surplus which remains after one has accounted for market forces, outside options, and so on. Thus "perturbations" of one sort or another would appear to be a natural ingredient of bargaining theories.

The issue is then what kinds of perturbations does one want to build a theory upon? We believe that "irrationality" provides a starting point which is perhaps as worthy of attention as the prevailing Rubinstein paradigm based on impatience. A point in favor of irrationality is that in a sequential process slight *ex ante* irrationality can, as a result of observed behavior, become very likely, *ex post*: it is precisely this fact which accounts for the magnitude of reputational effects relative to the underlying uncertainty upon which they are premised. Hence, we find it intuitively plausible that slight irrationality is an important explanatory factor in sequential bargaining. We emphasize the case of *slight* ex ante irrationality, in keeping with the tradition of the reputational literature. One might also take the view that in many contexts the ex ante probability of "type" behavior is not vanishingly small. We note that many of our results are independent of the magnitude of this ex ante probability.

At this point the reader might wonder how the current exercise relates to the large literature on bargaining with asymmetric information.[1] Our work, of course, shares with this literature a departure from the assumption of *complete* information. The cited papers are, however, concerned with uncertainty about "fundamentals," in particular, the discount factor or the reservation values of the bargainers. The motivation for our work is rather different in that we seek to model uncertainty about the strategic intent or strategic posture of the opponent rather than uncertainty about such concrete factors as seller's costs of production or buyer's valuations. Our model has *two* sided incomplete information, *two*-sided offers, and *multiple* types. The corresponding literature, with asymmetric information about valuations is not one which we have been able to build upon. In these models, multiple equilibria and refinement arguments are of the essence. These issues do not arise in our work. The last paragraph of the introduction (to which impatient readers might wish to immediately turn) traces in a connected way the many influences on our work, and records several noteworthy prior contributions. See also Section 6 for further discussion.

We have modeled irrationality in an extremely simple way. An irrational type always demands a particular share, accepts any offer greater or equal to that share and rejects all smaller offers. There are many possible types

[1] See, for instance, Sobel and Takahashi (1983), Cramton (1984), Fudenberg, Levine and Tirole (1985), Rubinstein (1985), Grossman and Perry (1986), Gul, Sonnenschein and Wilson (1986), Chatterjee and Samuelson (1987), Ausubel and Deneckere (1992).

corresponding to different inflexible demands. Agents discount the future at fixed rates and, as usual, the structure of the problem, in particular the discount factors and the probability distributions over irrational types are common knowledge.

We are primarily interested in bargaining games with (irrational types and) *frequent* offers. The logical starting point of the analysis is the consideration of discrete-time bargaining games and limit behavior in the latter as both players are able to make offers more and more frequently. A central issue is whether limit behavior depends upon precisely how one goes to the limit. In fact, we show that a limit outcome exists and is uniquely defined independently of the fine details of the bargaining protocol. The latter analysis is quite involved and we defer it to Section 4.

Sections 2 and 3 deal with the analytically convenient and expositionally less demanding continuous-time formulation. At time zero, each player picks a type to mimic. After this initial choice of types, the defining feature of the continuous-time game is the identification of revealing rationality with conceding to the other player's irrational demand. That is, the players are engaged in a *war of attrition* in which each player seeks to avoid being the one to concede. Note that the war of attrition structure is *derived* in Section 4: as the time between offers goes to zero, equilibrium strategies in effect entail either sticking with the initial "irrational" demand or conceding *completely* to the opponent's demand.

In Section 4, we establish the "independence from procedures" result and the explicit characterization of the limit alluded to above.[2] The key element of the convergence result is a *Coasian* property according to which the first player to reveal rationality does so by either accepting his opponents' irrational demand or by conceding to it "immediately" after revealing rationality. Our analysis here relies heavily upon a brief but incisive discussion of one-sided reputation formation in bargaining, developed in Chapter 8 of Myerson's (1991) game theory text. Our notion of type corresponds exactly to his definition of an *r-insistent strategy*.

Section 5 develops limit results when the *ex ante* probability of irrationality goes to zero. We show that in the limit delay and inefficiency disappear and provide explicit formulae for the players' equilibrium payoffs.

[2]The uniqueness result refers to behavior *after* the initial choice of types to mimic. The overall outcome will depend in general on the *non*-trivial procedural description of the sequence in which initial types are chosen.

Section 6 discusses related literature and Section 7 concludes.

Before proceeding further, we offer an overview of the varied influence on this research and our sense of how it fits into earlier literatures. In terms of the reputational prospective we develop, our work reflects the seminal influence of Kreps and Wilson (1982), Milgrom and Roberts (1982), and most recently Myerson (1991). From the point of view of bargaining theory and its substantive predictions, the natural counterpoint is the fundamental paper by Rubinstein (1982) on *complete* information bargaining, particularly when one adopts the perspective of *slight* irrationality. On the analytical side the reduction to a war of attrition is very closely related to the logic of the Coase conjecture (see Gul, Sonnenschein and Wilson (1986)), and, in particular, Myerson's (1991) reputational perspective on this theme. After the war of attrition structure is in place, much of the subsequent analysis is familiar, say, from the work of Hendricks, Weiss and Wilson (1988), and in a bargaining context, Chatterjee and Samuelson (1988). There are, however, key differences (uniqueness of equilibrium, simultaneous termination of concessions in finite time) which are elaborated upon in Section 2. Indeed the latter features already appear in Kreps and Wilson (1982) in their analysis of two-sided reputation formation with a single irrational type on each side. As compared with Rubinstein (1982) we add general bargaining protocols and incomplete information about types. As compared to Myerson (1991), we introduce general bargaining protocols, two-sided reputation formation, and multiple types. To conclude, we have drawn upon a variety of connected themes to develop a stylized but reasonably full-fledged reputational theory of bargaining which, in particular, allows for *two*-sided reputation formation, and *multiple* reputational types.

# 2 Continuous-Time Bargaining

Here we define the bargaining problem in continuous-time. We then analyze a special case in which each player has only one irrational type. This special case both conveys the flavor of the analysis and is furthermore the basic building block for the multiple type cases studied subsequently. An irrational type of player $i$ is identified by a number $\alpha^i \in (0,1)$; a type $\alpha^i$ always demands $\alpha^i$, accepts any offer greater or equal to $\alpha^i$, and rejects all smaller offers. We denote by $C^i \subset (0,1)$ the finite set of irrational types for player $i$ and by $\pi^i(\alpha^i)$ the conditional probability that $i$ is irrational of type $\alpha^i$ given that he is irrational. Hence, $\pi^i$ is a probability distribution on $C^i$. The initial probability that $i$ is irrational is denoted by $z^i$. Finally, player $i$'s rate of time preference is $r^i$. The continuous-time bargaining problem is denoted $B = \{(C^i, z^i, \pi^i, r^i)_{i=1}^2\}$.

At time 0, player 1 chooses her demand $\alpha^1$. If she is rational, this is a strategic choice; if she is irrational, she merely makes the demand corresponding to her type. After observing $\alpha^1 \in C^1$, player 2 either immediately accepts, strategically if he is rational or because he is irrational and of a type $\alpha^2$ such that $\alpha^1 + \alpha^2 \leq 1$. Or player 2 makes a demand $\alpha^2 \in C^2$ such that $\alpha^1 + \alpha^2 > 1$. Again, this may be because 2 is rational and strategically demands $\alpha^2$ or because 2 is irrational of type $\alpha^2 \in C^2$. After player 2 demands $\alpha^2$, player 1 can concede or a war of attrition ensues. It is shown in Section 4 that this war of attrition structure is uniquely (and robustly) determined as the limit of equilibria of discrete-time games in which both players make offers frequently. We assume that $\max\limits_{\alpha^1 \in C^1} \alpha^1 + \min\limits_{\alpha^2 \in C^2} \alpha^2 > 1$. Hence all types of player 2 are incompatible with the greediest type of player 1. A strategy $\sigma^1$ for player 1 is defined by a probability distribution $\mu^1$ on $C^1$ and a collection of cumulative distributions $F^1_{\alpha^1,\alpha^2}$ on $\Re_+ \cup \{\infty\}$, for all $(\alpha^1, \alpha^2) \in C^1 \times C^2$ such that $\alpha^1 + \alpha^2 > 1$. $F^1_{\alpha^1,\alpha^2}(t)$ is the probability of player 1 conceding to player 2 by time $t$ (inclusive). It follows that,

$$\lim_{t \to \infty} F^1_{\alpha^1,\alpha^2}(t) \leq 1 - \bar{\pi}^1(\alpha^1)$$

where

$$\bar{\pi}^1(\alpha^1) = \frac{z^1 \pi^1(\alpha^1)}{z^1 \pi^1(\alpha^1) + (1 - z^1)\mu^1(\alpha^1)}$$

is the posterior probability that 1 is irrational immediately after it is observed that 1 demands $\alpha^1$ at time zero. Note that $F^1_{\alpha^1,\alpha^2}(0)$ may be strictly positive and represents the probability that 1 may concede immediately to 2's counteroffer $\alpha^2$.

Let $Q$ denote "immediate acceptance." A strategy for player 2 is defined as a collection $\sigma^2 = \left(\mu^2_{\alpha^1}, F^2_{\alpha^1,\alpha^2}\right)$ for $\alpha^1 \in C^1$ and $\alpha^2 > 1 - \alpha^1, \alpha^2 \in C^2$. Here, $\mu^2_{\alpha^1}$ is a probability distribution over $C^2 \cup \{Q\}$ and describes player 2's choice (after observing $\alpha^1$) between $Q$ (immediate acceptance) and $\alpha^2 > 1 - \alpha^1, \alpha^2 \in C^2$. For any $\alpha^2 > 1 - \alpha^1, F^2_{\alpha^1,\alpha^2}$ describes player 2's choice of concession time conditional upon $\alpha^1$ and his choice of $\alpha^2$. Without loss of generality, we require that $F^2_{\alpha^1,\alpha^2}(0) = 0$ for all $\alpha^2 > 1 - \alpha^1$. (Both conceding at $t = 0$ and choosing $Q$ correspond to immediate concession.)

The conditional probability of 2's irrationality given that $(\alpha^1, \alpha^2)$ are demanded initially is

$$\bar{\pi}^2_{\alpha^1}\left(\alpha^2\right) = \frac{z^2\pi^2\left(\alpha^2\right)}{z^2\pi^2\left(\alpha^2\right) + \left(1 - z^2\right)\mu^2_{\alpha^1}\left(\alpha^2\right)}$$

Note two important differences between this bargaining game and the standard war of attrition. In the bargaining game, at time zero players choose a type to mimic; furthermore the probability of eventual concession is less than one. For $\bar{\sigma} = (\sigma^1, \sigma^2)$, $\bar{\alpha} = (\alpha^1, \alpha^2)$ and $t < \infty$ define

$$\begin{aligned} U^1\left(t, \sigma^2 \mid \bar{\alpha}\right) :&= \alpha^1 \textstyle\int_{y<t} e^{-r^1 y} dF^2_{\alpha^1,\alpha^2}(y) \\ &+ \tfrac{1}{2}\left(\alpha^1 - \alpha^2 + 1\right)\left[F^2_{\alpha^1,\alpha^2}(t) - F^2_{\alpha^1,\alpha^2}\left(t^-\right)\right] \cdot e^{-r^1 t} \\ &+ \left(1 - \alpha^2\right)\left[1 - F^2_{\alpha^1,\alpha^2}(t)\right] \cdot e^{-r^1 t} \end{aligned}$$

where $F^2_{\alpha^1,\alpha^2}\left(t^-\right) = \lim_{y \uparrow t} F^2_{\alpha^1,\alpha^2}(y)$. This expression is the expected utility of player 1 who concedes at time $t$, given the strategy profile $\bar{\sigma} = (\sigma^1, \sigma^2)$ and given that $\bar{\alpha} = (\alpha^1, \alpha^2)$ is observed at time zero. For a player who never concedes, i.e., $t = \infty$ the corresponding payoff is

$$U^1\left(\infty, \sigma^2 \mid \bar{\alpha}\right) = \alpha^1 \int_{y \in [0,\infty)} e^{-r^1 y} dF^2_{\alpha^1,\alpha^2}(y).$$

The penultimate expression, and the ones below, assume an equal split of the surplus in the event of simultaneous concession. This tie-breaking assumption may be replaced by any (possibly time-dependent) rule without affecting the results; in equilibrium simultaneous concessions arise with probability zero. The concession behavior of a *rational* player 1 is described by $\frac{1}{(1-\bar{\pi}^1(\alpha^1))} F^1_{\alpha^1,\alpha^2}$; consequently the latter's overall expected utility conditional upon $\bar{\alpha} = (\alpha^1, \alpha^2)$ being observed at time zero is:

$$U^1\left(\bar{\sigma} \mid \bar{\alpha}\right) = \frac{1}{\left(1 - \bar{\pi}^1\left(\alpha^1\right)\right)} \int_{y \in [0,\infty]} U^1\left(y, \sigma^2 \mid \bar{\alpha}\right) dF^1_{\alpha^1,\alpha^2}(y)$$

Finally, a rational player 1's expected utility from the strategy profile $\bar{\sigma}$ is:

$$U^1(\bar{\sigma}) = \sum_{\alpha^1} \mu^1(\alpha^1) \left\{ \alpha^1 \left[ (1-z^2)\, \mu^2_{\alpha^1}(Q) + z^2 \sum_{\alpha^2 \leq 1-\alpha^1} \pi^2(\alpha^2) \right] + \sum_{\alpha^2 > 1-\alpha^1} U^1(\bar{\sigma} \mid \alpha^1, \alpha^2)\left( (1-z^2)\, \mu^2_{\alpha^1}(\alpha^2) + z^2 \pi^2(\alpha^2) \right) \right\}$$

The definitions of $U^2(t, \sigma^1 \mid \bar{\alpha})$ and $U^2(\bar{\sigma} \mid \bar{\alpha})$ are symmetric to $U^1(t, \sigma^2 \mid \bar{\alpha})$ and $U^1(\bar{\sigma} \mid \bar{\alpha})$ with $\bar{\pi}^2_{\alpha^1}, F^1_{\alpha^1,\alpha^2}, \mu^2_{\alpha^1}, \pi^2$and $r^2$ replacing $\bar{\pi}^1, F^2_{\alpha^1,\alpha^2}, \mu^1$, $\pi^1$ and $r^1$ respectively. Finally, define

$$\begin{aligned} U^2(\bar{\sigma}) &= \sum_{\alpha^1} \left( \left(1-z^1\right) \mu^1\left(\alpha^1\right) + z^1 \pi^1\left(\alpha^1\right) \right) \\ &\quad \times \left[ (1-\alpha^1)\mu^2_{\alpha^1}(Q) + \sum_{\alpha^2 > 1-\alpha^1} U^2\left(\bar{\sigma} \mid \alpha^1, \alpha^2\right) \mu^2_{\alpha^1}\left(\alpha^2\right) \right] \end{aligned}$$

Note that both agents get zero utility if no one ever concedes.

We now turn to the analysis of equilibrium in the case in which each player has only one irrational type $\alpha^i$. By our earlier assumption, $\alpha^1 + \alpha^2 > 1$. With a single irrational type we simplify our definition of a strategy; Player $i$'s strategy is just the cumulative $F^i$.[3] Consequently, the continuous-time bargaining game is much like a war of attrition: if player $i$ concedes at time $t$, then his utility is $(1-\alpha^j)\, e^{-r^i t}$ while $j$'s utility is $\alpha^j e^{-r^j t}$.

It is well-known (see for instance, Hendricks, Weiss and Wilson (1988)) that the set of equilibria of this war of attrition are characterized by the following three properties: (i) at most one player concedes with positive probability at time 0, (ii) after time 0 player $i$ concedes at constant hazard rate $\lambda^i = r^j (1-\alpha^i) / (\alpha^j - (1-\alpha^i))$ that makes the opponent indifferent between waiting and conceding, and (iii) unless the game ends with probability 1 at time 0, concession by both players continues forever.

Hence $(F^1, F^2)$ is an equilibrium of the war of attrition if and only if $F^i(t) = 1 - c^i e^{-\lambda^i t}, c^i \in [0,1]$ and $(1-c^1)(1-c^2) = 0$. Observe that $1 - c^i = F^i(0)$ is the probability that $i$ concedes at time 0, hence $(1-c^1)(1-c^2) = 0$ corresponds to $(i)$ above. Also, $(dF^i/dt)(1-F^i) = \lambda^i$ is the constant hazard rate noted in $(ii)$. Finally, note that $F^i$ is strictly increasing whenever $c^i > 0$ as required by $(iii)$.

Our bargaining game differs from the war of attrition since there is a positive prior probability of irrationality for each agent. Nevertheless, the

[3] In order to do away for the need for $\mu^2$, we will allow $F^2(0) > 0$ in the remainder of this section.

familiar arguments from the analysis of the war of attrition suffice to show that equilibrium of the bargaining game share the first two properties.

However, in the bargaining game (iii) is replaced by: (iii′) There exists a finite time $T^0$ at which the posterior probability of irrationality for both agents reaches 1 simultaneously and concessions stop. The last requirement pins down the identity of the player who needs to make a concession at time 0 as well as the probability of such a concession and hence establishes the uniqueness of equilibrium. Thus, equilibrium of the bargaining game is characterized by the following conditions:

$$\begin{aligned} &F^i(t) = 1 - c^i e^{-\lambda^i t} && \text{for all } t \leq T^0 \\ &c^i \in [0,1], (1-c^1)(1-c^2) = 0 \text{ and} && \\ &1 - z^i = F^i(T^0) && \text{for } i = 1, 2\,. \end{aligned}$$

This is similar to the two-sided reputation formation analysis of Kreps and Wilson (1982). As in their work property (iii′) is critical in that it enables us to identify a particular equilibrium of the war of attrition as the only equilibrium of the bargaining game. The intuition for the necessity of having the posterior probability of irrationality reach 1 for both agents at the same time is quite clear. Since concession must be made at a constant rate and only rational players concede, eventually the probability of irrationality for $i$ must reach 1. If player 1's probability of irrationality reached 1 at $\tau^1$ before 2's, then 2, if rational, would surely concede at $\tau^1$. Hence, 1 would be conceded to with strictly positive probability at $\tau^1$. But player 1 would stop conceding at $t < \tau^1$ sufficiently close to $\tau^1$ in anticipation of this "bonus" at time $\tau^1$ contradicting the constant hazard rate requirement. If player $i$ does not concede with positive probability at time 0, then the probability of his irrationality will reach 1 at time $T^i := (-\log z^i)/\lambda^i$. Since only one person can concede at time zero, concessions must continue until time $\tau^1 = \min\{T^1, T^2\}$. The "weaker" player, i.e. the player with the larger $T^i$ must concede with sufficient probability at time zero, so that conditional on not conceding, his probability of irrationality reaches 1 at the same time as his opponent's. Notice that a player's "strength" is increasing in his own probability of irrationality and his opponent's rate of impatience and decreasing in the amount of his demand.

The argument establishing that players' probabilities of irrationality reach 1 at the same time is essentially the same argument that is needed for (i) (i.e. both players cannot concede with positive probability at time 0). Our proof of Proposition 1 entails combining this argument with the well-known war of attrition arguments that establish (ii).

Let $\left(\hat{F}^1, \hat{F}^2\right)$ be the unique strategy profile characterized by (i), (ii), and (iii′). That is, $T^0 = \min\left\{\left(-\log z^1\right)/\lambda^1, \left(-\log z^2\right)/\lambda^2\right\}, c^i = z^i e^{\lambda^i T^0}$

and $\hat{F}^i(t) = 1 - c^i e^{-\lambda^i t}$.

**Proposition 1:** If $C^i = \{\alpha^i\}$ for $i = 1, 2$, then the unique sequential equilibrium of $B$ is $\left(\hat{F}^1, \hat{F}^2\right)$.

**Proof**: Let $\bar{\sigma} = (F^1, F^2)$ define a sequential equilibrium. We will argue that $\bar{\sigma}$ must have the form specified (i.e., uniqueness) and that these strategies do indeed define an equilibrium (existence).

Let $u_s^i$ denote the expected utility of a rational player $i$ who concedes at time $s$. Define $A^i := \left\{t \mid u_t^i = \max_s u_s^i\right\}$. Since $\bar{\sigma}$ is an equilibrium, $A^i \neq \emptyset$ for $i = 1, 2$. Also, let $\tau^i = \inf\left\{t \geq 0 \mid F^i(t) = \lim_{t' \to \infty} F^i(t')\right\}$, where $\inf \emptyset := \infty$. Then:

**a)** $\tau^1 = \tau^2$.

A rational player will not delay conceding once she knows that her opponent will never concede.

**b)** If $F^1$ jumps at $t \in \mathbb{R}$, then $F^2$ does not jump at $t$.

If $F^1$ had a jump at $t$, then player 2 receives a strictly higher utility by conceding an instant after $t$ than by conceding exactly at $t$.

**c)** If $F^i$ is continuous at $t$, then $u_s^j$ is continuous at $s = t$ for $j \neq i$.

This follows immediately from the definition of $u_s^i$ (see equation (1) below.)

**d)** There is no interval $(t', t'')$ such that $0 \leq t' < t'' \leq \tau^1$ where both $F^1$ and $F^2$ are constant on the interval $(t', t'')$.

Assume the contrary and without loss of generality, let $t^* \leq \tau^1$ be the supremum of $t''$ for which $(t', t'')$ satisfies the above properties. Fix $t \in (t', t^*)$ and note that for $\varepsilon$ small there exists $\delta > 0$ such that

$u_t^i - \delta \geq u_s^i$ for all $s \in (t^* - \varepsilon, t^*)$ for $i = 1, 2$. By (b) and (c) there exists $i$ such that $u_s^i$ is continuous at $s = t^*$. Hence, for some $\eta > 0$, $u_s^i < u_t^i$ for all $s \in (t^*, t^* + \eta)$ for this player $i$. Since $F^i$ is optimal, we conclude that $F^i$ is constant on the interval $(t', t^* + \eta)$. The optimality of $F^j$ then implies that $F^j$ is constant on $(t', t^* + \eta)$. Hence, both functions are constant on the latter interval. This contradicts the definition of $t^*$.

As we noted above if $F^i$ is constant on some interval $(t', t'')$ then the optimality of $F^j$ implies that $F^j$ is constant on $(t', t'')$; consequently, (d) implies (e):

**e)** If $t' < t'' < \tau^1$ then $F^i(t'') > F^i(t')$ for $i = 1, 2$.

**f)** $F^i$ is continuous at $t > 0$. To see this recall that if $F^i$ has a jump at $t$ then $F^j$ is constant on the interval $(t - \varepsilon, t)$ for $j \neq i$. This contradicts (e).

From (e) it follows that $A^i$ is dense in $[0, \tau^1]$ for $i = 1, 2$. From (c) and (f) it follows that $u_s^i$ is continuous on $(0, \tau^1]$ and hence $u_s^i =$ constant for all $s \in (0, \tau^1]$. Consequently $A^i = (0, \tau^1]$. Hence $u_t^i$ is differentiable as a function of $t$ and $\frac{du_t^i}{dt} = 0 \; \forall t \in (0, \tau^1)$. Now

$$u_t^i = \int_{x=0}^{t} \alpha^i e^{-r^i x} dF^j(x) + \left(1 - \alpha^j\right) e^{-r^i t} \left(1 - F^j(t)\right) \tag{1}$$

The differentiability of $F^j$ follows from the differentiability of $u_t^i$ on $(0, \tau^1)$. Differentiating (1) and applying Leibnitz's rule, we obtain

$$0 = \alpha^i e^{-r^i t} f^j(t) - (1 - \alpha^j) r^i e^{-r^i t} (1 - F^j(t)) - (1 - \alpha^j) e^{-r^i t} f^j(t)$$

where $f^j(t) = \frac{dF^j(t)}{dt}$. This in turn implies $F^j(t) = 1 - c^j e^{-\lambda^j t}$ where $c^j$ is yet to be determined. At $\tau^1 = \tau^2$ optimality for player $i$ implies that $F^i(\tau^i) = 1 - z^i$. At $t = 0$, if $F^j(0) > 0$ then $F^i(0) = 0$ by (b). Let $T^i$ solve $1 - e^{-\lambda^i t} = 1 - z^i$. Then $\tau^1 = \tau^2 = T^0 := \min\{T^1, T^2\}$ and $c^i, c^j$ are determined by the requirement $1 - c^i e^{-\lambda^i T^0} = 1 - z^i$. So $F^i = \hat{F}^i$ for $i = 1, 2$. If $j$'s strategy is $\hat{F}^j$ then $u_t^i$ is constant on $(0, \tau^1]$ and $u_s^i < u_{T^0}^i \; \forall \; s > \tau^1$. Hence any mixed strategy on this support, and, in particular, $\hat{F}^i$ is optimal for player $i$. Hence $(\hat{F}^1, \hat{F}^2)$ is indeed an equilibrium. *Q.E.D.*

In the unique equilibrium derived above, a rational player $j$'s utility is

$$\hat{F}^i(0)\,\alpha^i + \left(1 - \hat{F}^i(0)\right)\left(1 - \alpha^j\right)$$

since player $j$'s equilibrium strategy entails concession at any time $\varepsilon > 0$ (and before $T^0$). Furthermore, $\hat{F}^i(0) = 1 - c^i$ where $c^i = e^{-\lambda^i\left(T^i - T^0\right)}$, $T^0 = \min\{T^1, T^2\}$. Since $\alpha^j > 1 - \alpha^i$, player $i$ prefers to be conceded to than to concede and it is natural to think of $T^i = (-\log z^i)/\lambda^i$ as a measure of player $i$'s "weakness": $T^i > T^j$ means that player $i$ must concede to player $j$ at time 0. The effect of any change in parameters can be determined by calculating how the change influences the $T^i$'s. For example, as $r^i$ increases player $j \neq i$ must concede *more rapidly* between $t$ and $t + \epsilon$ in order to keep a more impatient player $i$ indifferent between conceding at $t$ and waiting to concede till $t + \epsilon$. Thus player $j$'s probability gets "used up" more quickly (i.e. $T^j$ decreases). Consequently if $c^j < 1$ then $\frac{dc^j}{dr^i} > 0, \frac{dc^i}{dr^i} = 0$ (player $j$ concedes to $i$ at $t = 0$ with smaller probability) and if $c^i < 1$, $\frac{dc^i}{dr^i} < 0$ and $\frac{dc^j}{dr^i} = 0$ (player $i$ concedes to $j$ at $t = 0$ with higher probability). A small increase in $r^i$ makes player $i$ worse off (without affecting player $j$) or player $j$ better off (without affecting player $i$).

Finally, note that the equilibrium exhibits delay (and hence inefficiency). To see this simply and starkly, suppose the model is symmetric: $r^1 = r^2 = r, \alpha^1 = \alpha^2 = \alpha$, and $z^1 = z^2 = z$. Then, in equilibrium, $F^1_{\alpha^1,\alpha^2}(0) = F^2_{\alpha^1,\alpha^2}(0) = 0$. The expected payoff of a rational player 1 is $(1 - \alpha)$ since conceding at zero is in the support of player 1's optimal concession times. The payoff of an irrational player 1 is the payoff of a rational player 1 who concedes at $T^0$ (since conceding at $T^0$ is also optimal, this payoff is $(1 - \alpha)$) *less* the payoff at $T^0$ of conceding if player 2 has not conceded up to $T^0$. Thus, the expected payoff to player 1 is less than $(1 - \alpha)$ and the total utility loss is in excess of $1 - 2(1 - \alpha) = 2\alpha - 1$ which is clearly substantial for $\alpha$ significantly greater than $\frac{1}{2}$. Observe also that the inefficiency is a consequence of *delay* to reaching agreement rather than not reaching agreement at all; the ex ante probability of disagreement is (only) $z \times z = (z)^2$.

# 3 The Multiple Type Case

The single type case is a basic building block for our analysis but in itself yields a somewhat limited theory of bargaining. This section is concerned with analyzing the general bargaining game of the previous section. The generalization to multiple irrational types is important for a number of reasons: First, it permits a theory with a richer set of possible equilibrium divisions of the surplus. Second, the generalization enables us to identify some of the more robust conclusions of the analysis of the one-type model. Finally, within the multiple type bargaining model, we identify some implications of our theory that are independent of the exogenously specified distributions over types, $\pi^i$.

Proposition 2 establishes existence and uniqueness of equilibrium. In the single type case of Section 2, a basic equilibrium requirement was that the normal types of both players had to finish conceding at exactly the same instant. Consequently it was (sometimes) necessary for one (and only one) player to concede with positive probability at time zero. An additional requirement now is that the multiple types need to be mimicked with appropriate weights: the resulting *posterior* probabilities of rationality modulate the relative strengths of the types such that all types mimicked with positive probability obtain the same equilibrium payoff. The argument for uniqueness is somewhat involved. Nevertheless, we offer some intuition. The "strength" of a player depends upon the posterior probability of the type she mimics, and the latter probability decreases with the probability with which that type is mimicked. The payoffs to a type being conceded to with positive probability at time zero are strictly increasing in "strength." *Multiple* equilibrium distributions over types being conceded to are in conflict with the requirement that types mimicked with positive probability must have equal payoffs which are not smaller than the payoffs of types that are not mimicked. This conflict is easiest to see when the opposing player has a single type.

These points are elaborated upon below, and the formal proof may be found in the appendix. We focus first on player 2, and consider the intermediate case with one irrational type for player 1 and multiple irrational types for player 2. Since $\mu^1(\alpha^1) = 1$ for $\{\alpha^1\} = C^1$, a strategy for player 1 must specify demanding $\alpha^1$ at time 0. Player 2 will either concede immediately (play $Q$) or demand some $\alpha^2 \in C^2, \alpha^2 > 1 - \alpha^1$. The analysis of the previous section implies that in equilibrium, players 1 and 2 must concede at rates $\lambda^1 = \frac{r^2(1-\alpha^1)}{\alpha^1+\alpha^2-1}$ and $\lambda^2 = \frac{r^1(1-\alpha^2)}{\alpha^1+\alpha^2-1}$ respectively. Let $x \in (0,1)$ denote the probability of irrationality for player 1. Absent concession with positive probability at time zero, the probability of irrationality for player 1 will reach 1 at time $T^1 = (-\log x)/\lambda^1$. If player 2 does not play $Q$ but chooses some

$\alpha^2 > 1 - \alpha^1$, then his probability of irrationality will reach 1 at time

$$T^2 = -\frac{1}{\lambda^2} \log \frac{z^2 \pi^2 (\alpha^2)}{z^2 \pi^2 (\alpha^2) + (1 - z^2) \mu^2_{\alpha^1} (\alpha^2)}.$$

As argued in the previous section the probabilities of irrationality must reach 1 for both agents simultaneously. But if player 2 chooses $\alpha^2$ at time 0, he has no other opportunity to concede at time 0 and since he concedes at rate $\lambda^2$ thereafter, in equilibrium it must be that $T^1 \geq T^2$. If this inequality is strict, then player 1 must concede with positive probability at time 0 so that conditional on not conceding the probabilities of irrationality reaches 1 for both agents simultaneously. That is,

$$\frac{1}{\lambda^1} \log \frac{z^1}{z^1 + (1 - z^1)(1 - q^1)} = \frac{1}{\lambda^2} \log \frac{z^2 \pi^2 (\alpha^2)}{z^2 \pi^2 (\alpha^2) + (1 - z^2) \mu^2_{\alpha^1} (\alpha^2)}$$

where $q^1$ is the probability of conceding at $t = 0$.

Therefore the payoff to player 2 of mimicking $\alpha^2$ is $\alpha^2$ times the probability that 1 concedes to player 2 immediately plus $1 - \alpha^1$ times the probability that 1 does not concede immediately. Note that this payoff is strictly increasing in the probability of immediate concession by player 1, $q_1$, and hence is strictly greater than $1 - \alpha^1$ whenever the latter probability is positive. If $\mu^2$ is to be a part of an equilibrium then all types mimicked with positive probability must yield the same payoff for player 2 which furthermore must be not lower than the payoff associated with mimicking types that are mimicked with zero probability. This elementary property pins-down the equilibrium $\mu^2$ uniquely since the payoff associated with mimicking any $\alpha^2$ is increasing in the probability of immediate concession by player 1 which in turn is decreasing in $T^2$. Finally, from the above formula for $T^2$, the latter is increasing in $\mu^2_{\alpha^1} (\alpha^2)$.

What of player 1 and the general case where cardinality of $C^1$ is greater than 1? We already know that for any $\alpha^1$ and posterior probability of irrationality $x$, the equilibrium $\mu^2$ is uniquely determined. In our proof, we establish that given this unique equilibrium response to $x$, the payoff to player 1 is constant until $x$ reaches some $\underline{x}$ and is strictly increasing thereafter. Furthermore $x$ itself is decreasing in $\mu^1 (\alpha^1)$. This observation enables us to construct an argument for the "uniqueness" of $\mu^1$ similar to the uniqueness for $\mu^2$ outlined above. The added complication is the region below $\underline{x}$ over which strict monotonicity fails; for all $x < \underline{x}$, a rational player 2 mimics $\max C^2$ with probability 1. When a rational player 1 knows that her rational opponent will choose $\max C^2$ with probability 1, irrespective of her choice of $\alpha^1$ the equilibrium $\mu^1$ may be indeterminate. This indeterminacy however, does not translate into a multiplicity of equilibrium outcomes $\tilde{\theta}$, as we establish in the proof of Proposition 2 below.

**Proposition 2:** For any bargaining game $B = \left\{(C^i, z^i, \pi^i, r^i)_{i=1}^2\right\}$ a sequential equilibrium $\left(\left(\mu^1, F^1_{\alpha^1,\alpha^2}\right), \left(\mu^2_{\alpha^1}, F^2_{\alpha^1,\alpha^2}\right)\right)$ exists. Furthermore, all equilibria yield the same distribution over outcomes.

**Proof**: See Appendix.

The proof of proposition 2 contains the main elements of the argument for the following comparative static results. Recall from the discussion preceding Proposition 1 that a player becomes "stronger" as she becomes more patient or when the probability that she is irrational increases. When player $i$ becomes infinitely more patient than player $j$ $(r^i/r^j \to 0)$ player $i$ extracts all bargaining surplus from a rational player $j$. A similar conclusion follows if player $i$ is infinitely more likely to be irrational than player $j$ $(z^j/z^i \to 0)$. These are stated formally below.

**Proposition 3:** Let $B_n$ be a sequence of bargaining games and $v^i_n$ the corresponding sequence of sequential equilibrium payoffs for a rational player $i$,

(a) If all parameters other than $r^1_n$ and $r^2_n$ are constant along the sequence $B_n$, then, for $i \neq j$, $\lim\left(r^i_n/r^j_n\right) = 0$ implies $\liminf v^i_n \geq (1 - z^j) \max C^i$, $\lim\left(r^i_n/r^j_n\right) = \infty$ implies $\limsup v^i_n \leq 1 - (1 - z^j) \max C^j$.

(b) If all parameters other than $z^i_n$ for some $i = 1, 2$ are constant along the sequence
$B_n$, then for $i \neq j$, $\lim z^i_n = 1$ implies $\lim v^i_n \geq (1 - z^j) \max C^i$;
$\lim z^i_n = 0$ implies $\lim v^j_n = \max C^j$.

**Proof**: See Appendix.

It is interesting to compare our model and the proposition above to recent work on reputation in repeated games. See Section 6 for a discussion of differences and similarities.

# 4 The Discrete Model and Convergence

This section considers discrete bargaining games. We analyze the limit of equilibria as both players are able to make offers increasingly frequently. Apart from the preceding requirement, there are essentially no restrictions on the bargaining structure. In contrast to Rubinstein's complete information theory, our model satisfies a strong form of independence from the details of the bargaining protocol. When the time between offers is sufficiently small the equilibrium distribution of outcomes must be close to the *unique* equilibrium of the continuous-time game analyzed in Section 2. Somewhat more precisely, consider any sequence of discrete time games indexed by $n$ where in the game $n$, in any $\epsilon_n-$ time interval both players have at least one opportunity to make an offer. For simplicity, assume that each player has only one irrational type. An equilibrium outcome in game $n$ is a random object $\tilde{\theta}_n$ any realization of which is an agreed to division and a time at which agreement is reached. For any such sequence if $\epsilon_n \to 0$ as $n \to \infty, \tilde{\theta}_n \to \tilde{\theta}$ in distribution, where $\tilde{\theta}$ is the unique equilibrium distribution of the continuous-time game. This convergence result motivates the convenient continuous-time framework of Sections 2 and 3. An important ingredient of this result is that in the discrete-time game revelation of rationality amounts in the limit to conceding to one's opponent's irrational demand. This property is, of course, the defining feature of the continuous time game and is closely related to the Coase conjecture (see for instance, Gul, Sonnenschein and Wilson (1986)).

Recall that the Coase conjecture asserts that when the time between offers is sufficiently small, bargaining between a seller with known valuation $s$ and a buyer who may have one of many reservation values (each higher than $s$) results in almost immediate agreement at the lowest buyer valuation. In his discussion of bargaining, Myerson (1991, pgs. 399-404) offers an important new perspective on this result by recasting it in a reputational setting. The low valuation buyer is replaced by an irrational type who demands some constant amount and accepts no less than this amount. In an alternating offer bargaining game, he shows that as the time between offers goes to zero, agreement is reached without delay at the constant share demanded by the irrational type, just as in the Coase conjecture there is immediate agreement at the lowest buyer valuation. Both results are independent of the *ex ante* probability of the low type and the players' relative discount factors (so long as they are both close to 1, as implied by the assumption that offers are frequent). Thus, Myerson observes that the influence of asymmetric information overwhelms the effect of impatience in determining the division of surplus. We extend Myerson's analysis to the case of two-sided asymmetric information where the Coasian effect noted by Myerson transforms the bargaining game into the game studied in Section 2. In the two-sided case, asymmetric information again overwhelms the effect of timing of offers; *however*, as the preceding section indicates, *the relative impatience of the agents does play a role.*

A key element of Myerson's argument and a central ingredient in the proof of Proposition 4 is the following: When one agent is known to be rational and there is a positive probability that her opponent is irrational, delay is not possible. This means that either the agent known to be rational $i$, gives in to the irrational demand of the other agent $j$, or agent $j$ also reveals himself to be rational. The latter outcome occurs only when revealing himself to be rational yields a payoff no less than $\alpha^j$ to agent $j$. Otherwise, $j$ prefers to pretend to be irrational and be conceded to by $i$ without delay. Thus, in either case $j$ obtains a payoff no less than $\alpha^j$. With this conclusion in place it is easy to see how a war of attrition emerges: at any time $t$, by pretending to be irrational, agent $i$ can allow $j$ to make the offer $1-\alpha^j$ or reveal herself to be rational. In both cases $i$ could obtain a payoff no less than $1-\alpha^j$, since $\alpha^i > 1-\alpha^j$ by assumption and $i$ gets a payoff no less than $\alpha^i$ once $j$ reveals himself to be rational. Thus, agent $i$ has a way to end the game that will yield her $1-\alpha^j$, for $i = 1, 2$ and $j \neq i$. On the other hand, if $j$ chooses to end the game $i$ gets a payoff of at least $\alpha^i$ which means that in equilibrium $i$ will get exactly $\alpha^i$. But this is precisely the set-up of a war of attrition where $i$'s high payoff is $\alpha^i$ and her low payoff is $1-\alpha^j$. Thus, the analysis of Section 2 applies.

Formally, our model of discrete-time bargaining is the following: If no agreement is reached, players receive zero utility. If an agreement is reached at time $t$ and if player 1 receives a share $x \in [0, 1]$ of the pie, players 1 and 2 enjoy utilities $xe^{-r^1 t}$ and $(1-x)e^{-r^2 t}$ where $r^1$ and $r^2$ are their respective rates of time preference. The probability of an irrational type is $z^i$ and $\alpha^i$ is the share demanded by the single irrational type. An extensive form bargaining game is specified in this environment by a function $g : \Re_+ \to \{0, 1, 2, 3\}$ where for $i = 1, 2,\ g(t) = i$ denotes that player $i$ can make an offer in period $t$ (to which player $j \neq i$ may immediately agree or disagree), $g\,(t) = 3$ denotes *simultaneous* offers at $t$ and $g(t) = 0$ means that no one makes an offer in period $t$. Let $I^i = \{t \mid g(t) = i \text{ or } 3\}$, denote the set of times at which player $i = 1, 2$ makes an offer. We assume that $g$ is a discrete bargaining game with an infinite horizon: for all $t, I^i \subseteq [0, t]$ is finite, and $I^i$ is an infinite set. The game is played as follows: At $t \in I^i$ player $i$ makes an offer $x$. If player $j$ agrees, then the game ends with the agreement. If player $j$ rejects, then the game continues and the next offer is made at time $t' := \min \{\hat{t} > t \mid \hat{t} \in I^1 \cup I^2\}$ by $j$ such that $t' \in I^j$. For simultaneous offers the game ends if the offers are compatible; in the event of strict compatibility the surplus is split equally. An irrational player $i$ always demands a share $\alpha^i$ for himself and accepts an offer if and only if the offer yields him a share at least $\alpha^i$. We require $\alpha^1 + \alpha^2 > 1$. Notice that this definition of a bargaining game is very general and in particular accommodates non-alternating, non-stationary, bargaining procedures.

A sequence of discrete bargaining games $(g_n)_{n=1}^{\infty}$ is said to converge to continuous-time if for all $\epsilon > 0$, $\exists \overline{n}$ such that for all $n \geq \overline{n}, t \geq 0$, and $i = 1, 2$, $i \in g_n([t, t + \epsilon])$.

Let $\sigma_n^i$ denote a behavior strategy for player $i$ in the game $g_n$, and $\bar{\sigma}_n$ a behavior strategy profile. Let $\tilde{\theta}_n = (\tilde{x}, \tilde{t})$ denote a random outcome of $g_n$, where a realization of $\tilde{\theta}_n$ is the share $x$ received by player 1 and the time $t$ at which agreement is reached. We identify the outcome in which no agreement is reached with $(1/2, \infty)$.

Recall from Section 2 and the discussion immediately preceding Proposition 1, the unique equilibrium of the continuous-time game when each player has a single irrational type. Let $\tilde{\theta}$ denote the associated random outcome.

**Proposition 4:** Let $(g_n)_{n=1}^{\infty}$ be a sequence of discrete bargaining games converging to continuous-time. Let $\sigma_n$ denote a sequential equilibrium of $g_n$ and $\tilde{\theta}_n$ the random outcome corresponding to $\sigma_n$. Then $\tilde{\theta}_n$ converges in distribution to $\tilde{\theta}$.

**Proof**: See Appendix.

**Remarks** (i) Observe that by establishing the convergence of the equilibria of the discrete-time game to the unique equilibrium of the continuous-time game, Proposition 4 establishes the independence of the bargaining outcome from the bargaining procedure. It does not establish the irrelevance of the relative impatience of the two players, since the outcome of the continuous-time bargaining game itself depends on $r^1$ and $r^2$ as can be seen from the equation defining $T^i$ in Sections 2 and 3. To understand the dependence on procedure of the complete information theory consider a discrete-time bargaining game in which players alternate making offers. Let $\Delta^i$ be the time interval between any rejection by player $i$ and the next possible acceptance or rejection by player $j$. In the complete information theory, player $i$'s equilibrium utility is approximately $\frac{r^j\Delta^j}{r^j\Delta^j+r^i\Delta^i}$ when $\Delta^1, \Delta^2$ are small. Thus, even when agents can make offers very frequently equilibrium shares vary substantially with the ratio $\Delta^1/\Delta^2$.

(ii) Perry and Reny (1993) is motivated by concerns about the robustness of complete information bargaining to the bargaining procedures employed. Their work does much to clarify the essential logic of bargaining models in the style of Rubinstein (1982). They study an extensive form structure which is "procedureless" in the sense that players can make offers at any time subject to the proviso that a minimal time interval $\Delta_i$ must elapse between consecutive offers of the same player. Since they place no constraints on the negotiation structure other than those imposed by the $\Delta_i$-minimal delay requirement they view their results as being "independent of procedures." Nevertheless, their model is closer in spirit to Rubinstein's than ours since, even when both $\Delta_i$'s are arbitrarily small, the equilibrium distribution of the gains from trade depends on the ratio $\Delta_1/\Delta_2$.

(iii) Proposition 4 considers only the case of a single irrational type for each player. A similar convergence result can be established for the multiple type bargaining model provided that the same player moves first in each of the discrete games in the sequence. With multiple types, the equilibrium outcome of the continuous-time game and hence the limiting equilibrium outcome of the discrete-time game does depend on which player moves first. We do not view the ordering of initial moves as representing an innocuous change in bargaining protocol akin to whether a player needs to pause for breath for 10 seconds as opposed to 5 seconds before making a counter-offer. There ought to be no presumption that being "saddled with" or "having the privilege of" first staking-out a bargaining position should have no impact on the final outcome.

(iv) With different irrational types our result may not hold.[4] Suppose we replaced a $\alpha^1$-type for player 1 by a type which demanded $\alpha^1$ but was in fact willing to accept anything above $.8\alpha^1$. Now suppose in the corresponding Rubinstein complete information game player 1's payoff was $k\alpha^1$, for $k \in (.8, 1)$. Then player 2 would concede/reveal rationality by offering $.8\alpha^1$.

[4] We are grateful to a referee for this example.

An irrational player 1 would accept this offer, while a rational player 1 would instead reject it, reveal his rationality and obtain $k\alpha^1 > .8\alpha^1$ instead. Details of bargaining protocol could indeed (within some range) matter. Nevertheless, it is possible that the independence of protocol result would re-emerge with a "rich" set of types. Such a result would seek to identify a basic subclass of types whose presence yielded the result independently of which other types were present.

# 5 The Limiting Case of Complete Rationality

We turn now to limit results as the (*ex ante*) probability of irrationality of *both* players goes to zero.[5] This is a theme first explored by Kambe (1994) and Compte and Jehiel (1995) in recent papers which build upon the preceding analysis, presented initially in Abreu and Gul (1992). Kambe (1994) proves that in his model (see the discussion following the corollary below) inefficiency disappears as the probability of irrationality goes to zero. In Compte and Jehiel (1995), on the other hand, delay and hence inefficiency, persist in the limit.

The purpose of this section is to conduct a Kambe-type analysis of our model, and furthermore to reconcile the contrasting conclusions of Kambe (1994) and Compte and Jehiel (1995) within our framework.

Let $v^i = (r^i + r^j)$. The proposition below establishes that $\underline{v}^i := \max \{ \alpha \in C^i \cup \{0\} \mid \alpha < v^i \}$ is a lower bound on player $i$'s limit payoff. Suppose that for some small $\varepsilon > 0$, there exists $\alpha^i$ such that $v^i - \varepsilon \leq \alpha^i < v^i, i = 1, 2$. Then, it follows simply from feasibility that these bounds are quite tight, that the efficiency loss from bargaining is small, and that agreement is reached quickly with high probability. This is stated in the corollary below.

Kambe (1994) discovered the key fact which drives this result: Consider a sequence of bargaining games $B_n$ such that $z_n^i \to 0,\ i = 1, 2$, and suppose that the $z_n^i$'s go to zero at the same rate (say, $z_n^i = \frac{1}{n}z$). Let $\mu_n^1$ and $\mu_{\alpha^1,n}^2$ be the corresponding equilibrium distributions and suppose that $\mu_n^1 \to \mu$ and $\mu_{\alpha^1,n}^2 \to \mu_{\alpha^1}^2$. Consider any $(\alpha^1, \alpha^2) \in C^1 \times C^2$ such that $\mu_{\alpha^1}^2(\alpha^2) > 0$, $\alpha^1 + \alpha^2 > 1$ and $r^1(1 - \alpha^2) < r^2(1 - \alpha^1)$. Then, in the limit, conditional

[5] These results are derived in the context of the continuous time game. They also apply to the limit of discrete time games, when the time between offers goes to zero, *before* the probabilities of irrationality go to zero.

on $(\alpha^1, \alpha^2)$ being chosen at the beginning of the game, the rational type of player 2 will concede immediately with probability approaching 1. But if $\alpha^1 = \underline{v}^1$, then $r^1(1-\alpha^2) \geq r^2(1-\alpha^1)$ implies $\alpha^2 < 1 - \underline{v}^1$. Hence player 1 can guarantee $\underline{v}^1$ in the limit. Similarly player 2 can guarantee $\underline{v}^2$ in the limit.

**Proposition 5:** Let $B_n = \left\{(C^i, z_n^i, \pi^i, r^i)_{i=1}^2\right\}$ be a sequence of continuous-time bargaining games.

If (a) $\lim z_n^1 = \lim z_n^2 = 0$, $\lim z_n^1/(z_n^1 + z_n^2) \in (0,1)$ and (b) $v_n^i$ is the sequential equilibrium payoff for player $i$ in the game $B_n$, then $\liminf\ v_n^i \geq \underline{v}^i$ for $i = 1, 2$.

**Proof:** See appendix.

We will say that $B_n = \left\{(C^i, z_n^i, \pi^i, r^i)_{i=1}^2\right\}$ converges to $\left\{(C^i, \pi^i, r^i)_{i=1}^2\right\}$ if $(z_n^1, z_n^2)_{n=0}^{\infty}$ satisfies condition (a) of Proposition 5.

**Corollary**: If $B_n = \left\{(C^i, z_n^i, \pi^i, r^i)_{i=1}^2\right\}$ converges to $\left\{(C^i, \pi^i, r^i)_{i=1}^2\right\}$ and for some $\varepsilon > 0$ and all $x \in [0,1]$ there exists $\alpha^i \in C^i$ for $i = 1, 2$ such that $\mid \alpha^i - x \mid < \varepsilon$, then for $n$ sufficiently large, the equilibrium payoff of agent $i$ is at least $\frac{r^i}{r^1+r^2} - 2\varepsilon$ and hence the inefficiency due to delay is at most $4\varepsilon$.

In Kambe's model (in fact he considers four variants, all of which yield rather similar results) the two bargainers are initially free to make *any* offers. Immediately thereafter, they may, with some known exogenous probability $z^i$, get "irrationally" attached to these offers and are in effect unable to accept or make lower offers. In his model, when $z_n^1$ and $z_n^2$ go to zero at the same rate, limit payoffs are $(v^1, v^2)$ as defined above. (In fact, he also analyzes the case where $z_n^1 = (\beta_1)^n$ and $z_n^2 = (\beta_2)^n$ for arbitrary $\beta_1, \beta_2, \in (0,1)$.)

The corollary above establishes that when the probability of irrationality approaches zero, equilibrium behavior in Kambe's model is similar to equilibrium behavior in our model provided that the set of irrational types is sufficiently rich. In the limit, in both models, rational players choose to be virtually compatible and share surplus in proportion to impatience. Hence, inefficiency disappears.

Compte and Jehiel (1995) analyze a number of alternating offer bargaining games to study the interaction of reputation and outside options. In their model with multiple irrational types, they assume symmetry and show that as the time between offers goes to zero, rational types of both players choose to be as close to compatible as possible. In the limit only the lowest types $\underline{\alpha} > 1/2$ are mimicked with positive probability. While inefficiency is minimized there is nevertheless an efficiency loss : $2\underline{\alpha} - 1$ amount of surplus is dissipated through delay. Hence, unlike Kambe's model and the case with a rich set of types, inefficiency persists in their symmetric model when the probability of rationality goes to one.

Proposition 6 below establishes that even when the set of irrational types is not rich, generically inefficiency disappears. Thus, the symmetric case considered in Compte and Jehiel (1995) and hence their inefficiency result is non-generic in the sense of Proposition 6. Call $(C^1, C^2)$, generic if

$$\frac{r^1}{1-\alpha^1} \neq \frac{r^2}{1-\alpha^2}$$

for all $(\alpha^1, \alpha^2) \in C^1 \times C^2$. Then, for generic $(C^1, C^2)$ delay disappears and equilibrium payoffs converge to the "compromise outcome" $(\alpha_c^1, \alpha_c^2)$ defined in the Appendix. Note that $(\alpha_c^1, \alpha_c^2)$ depends only on $C^1, C^2, r^1$ and $r^2$.

**Proposition 6:** Let $B_n = \left\{(C^i, z_n^i, \pi^i, r^i)_{i=1}^2\right\}$ converge to $\left\{(C^i, \pi^i, r^i)_{i=1}^2\right\}$. For generic $(C^1, C^2)$ there exists a compromise outcome $(\alpha_c^1, \alpha_c^2)$, where $\alpha_c^1 + \alpha_c^2 = 1$, such that the equilibrium payoff of player $i$ converges to $\alpha_c^i$, $i = 1, 2$.

**Proof:** See Appendix.

As in the case of Proposition 5, Proposition 6 shows that the equilibrium payoffs do not depend on which player moves first at the beginning of the game.

# 6 Related Literature

Our work abuts upon numerous literatures including complete and incomplete information bargaining, models involving "wars of attrition," and the literature on reputation formation.

The central component of our model is the notion of an "irrational" type as in Kreps and Wilson (1982) and Milgrom and Roberts (1982) and in a bargaining context, Myerson (1991). These papers focus on one-sided reputation formation with one irrational type. Our model features *two*-sided reputation formation with *multiple* types on each side. As noted earlier Kreps and Wilson (1982) do consider two-sided reputation formation with *one* irrational type on each side.

There are some other recent papers that consider two-sided reputation formation. Schmidt (1993) studies *repeated* games. He identifies conditions that guarantee one of the two players his "commitment" payoff even under two-sided reputation formation. He considers a class of games called games of "conflicting interests" and shows that for a fixed probability distribution over irrational types for the two players, if there is a sufficiently high probability that player 2 is rational and if the discount factor of player 1 is sufficiently close to 1, then in any Nash equilibrium the payoff to player 1 approaches his commitment payoff. This result parallels our analysis from Section 3. Proposition 3 establishes that for a fixed and small probability of irrationality $z^2$ and any probability distribution $\pi^2$, the equilibrium probability that a rational player mimics his greediest type approaches 1 and the probability that a rational player 2 concedes at time 0 approaches 1 as $r^1$ approaches 0. When attention is not restricted to games of conflicting interest equilibrium outcomes are not unique in repeated game models of reputation even with one-sided reputation (see Schmidt (1993)). An important distinction between bargaining and general repeated games is that in a bargaining game, once both agents are revealed to be rational, the equilibrium continuation is unique. Moreover the first player to reveal rationality may choose to do so by accepting her opponent's offer and terminating the game. In particular, there is no opportunity for "punishing" either player after the rationality of both players is revealed. This feature of bargaining plays a key role in both Myerson's (1991) result on one-sided bargaining and our Proposition 4. The literature on one and two-sided reputation formation includes Aoyagi (1996), Celentani, Fudenberg, Levine and Pesendorfer (1996), Celentani and Pesendorfer (1996), Cripps, Schmidt and Thomas (forthcoming).

Two other papers which are very closely related are Kambe (1994) and Compte and Jehiel (1995). These were discussed extensively in the preceding section.

After each player has chosen a type to mimic, maintaining reputations in the continuous-time game is akin to not conceding in a war of attrition. Unlike classical wars of attrition models, ours has a *unique* equilibrium due to the fact that an irrational type never concedes. The most pertinent prior uniqueness result for wars of attrition is Kreps and Wilson (1982).

A few earlier papers on bargaining echo the war of attrition aspects of our work. These include Osborne (1985), Ordover and Rubinstein (1986), Chatterjee and Samuelson (1987), and Ponsati and Sakovics (1995). These papers appear to us to be more in the nature of *pure* concession games as opposed to full-fledged *bargaining* models in that

1. The concession game structure is assumed rather than derived. In our work the revelation of rationality amounts to conceding because of the logic of the Coase conjecture and reputational expressions of the same theme. The convergence result of Section 4 is essential in drawing a connection between bargaining and the concession structure.

2. They confine attention to what in our framework would be a *single* irrational demand.[6] In our view it is essential that a theory of bargaining allow for a *wide* array of possible agreements.

In Chatterjee and Samuelson (1988), the pure concession game structure of Chatterjee and Samuelson (1987) is replaced by a true discrete-time bargaining game and the authors show the existence of an equilibrium that replicates the key properties of the unique equilibrium of their earlier paper. The latter, discrete-time paper, however, does not contain any uniqueness results.

We have presented our work as a counterpoint to the complete information paper by Rubinstein (1982). As in the analysis of the chain-store paradox (Selten (1977), Rosenthal (1981), Kreps and Wilson (1982), and Milgrom and Roberts (1982)), we feel that the limit case of complete information yields unintuitive results. While from a substantive point of view the complete information case is a key benchmark, in terms of the analytics, models of bargaining with incomplete information about valuations or discount factors are potentially more closely related to our work. There are a great variety of such models entailing, among other variations, one and two-sided uncertainty, and one and two- sided offers. With one-sided uncertainty, one set of results is concerned with establishing the Coase conjecture (see for example, Gul, Sonnenschein and Wilson (1986) and the treatment in Myerson (1991)). These results are important building blocks for the convergence result of Section 4 and our derivation of the concession game structure (as noted earlier, the proof of Lemma 1 is adapted from Theorem 8.4 in Myerson's (1991) text). Another strand of the literature emphasizes the *multiplicity* of possible equilibria in the *no-gap* case. These involve the consideration of non-stationary

[6] While Osborne (1985) mentions the desirability of allowing multiple demands, his explicit analysis of this case is limited. We also remark that the focus of his paper is quite different from ours: his primary concern is to show that with risk averse agents the prediction of axiomatic and non-cooperative bargaining models differ.

strategies (Ausubel and Deneckere, 1989). In the latter papers only the uninformed player can make offers. Once an informed player can make offers, serious multiple equilibrium problems arise and subtle refinement arguments are needed to (more or less successfully) narrow down the set of equilibria (see Cramton (1984), Rubinstein (1985), Grossman and Perry (1986), Admati and Perry (1987), Gul and Sonnenschein (1988), Cho (1990), Ausubel and Deneckere (1992)).

In terms of assumptions, our work ought to be closest to models with *two*-sided incomplete information and *two*-sided offers. The literature on this case appears to be limited. We have already mentioned Chatterjee and Samuelson (1987). Another contribution is due to Cramton (1987); he constructs a particular equilibrium which has quite a different flavor from the equilibrium we derive. In Cramton's equilibrium, the player with the most extreme valuation reveals (i.e., concedes) first, but the initial concession does not end the game. Moreover, the two players always divide the *ex post* surplus equally. His model does not generate a unique equilibrium. A recent paper by Watson (1998) analyzes an alternating offer bargaining model with two-sided incomplete information about discount factors. He characterizes the set of possible payoffs under a rationalizability type solution concept and compares this set with the set of perfect Bayesian equilibrium payoffs. Finally, we note an intriguing early paper by Crawford (1982), which explores issues of inefficiency and commitment in a two-period bargaining model.

# 7 Conclusion

This paper proposes a reputation-based theory of bargaining. It suggests to us that a reputational perspective provides both a natural and quite powerful framework within which to analyze problems of bargaining. We have worked here with a very minimalist model, and in particular a very simple notion of "irrational types." It would be very interesting to extend this kind of analysis to more specified institutional settings (for example, firm/union bargaining), and to accommodate a somewhat richer set of "irrational" strategies. We remark that models with uncertainty about strategic posture as opposed to uncertainty about valuations are of interest in themselves *and* as a pragmatic response to the great difficulty of obtaining clear-cut results with two-sided uncertainty about valuations.

# 8 Appendix

**Proof of Proposition 2**: For all $\alpha^1 \in C^1$ and $x \in (0,1]$, let $B(\alpha^1, x)$ be the bargaining game obtained from the original game by replacing $C^1$ with $\{\alpha^1\}$ and $z^1$ with $x$; that is, in the bargaining game $B(\alpha^1, x)$ player 1 has a single irrational type and her probability of irrationality is $x$. All other parameters of $B(\alpha^1, x)$ are the same as in the original game $B$.

If the game $B(\alpha^1, x)$ does not end at time 0, then it must be that player 2 has chosen some $\alpha^2 > 1 - \alpha^1$. We know from the analysis of Section 2, that after time 0 each player $i$ must concede at rate $\lambda^i = r^j (1 - \alpha^i) / (\alpha^1 + \alpha^2 - 1)$ for $j \neq i$. Therefore, determining the equilibrium mimicking behavior for player 2 suffices to determine his full equilibrium strategy. Hence we will refer to $\mu^2$, a probability distribution over $C^2 \cup \{Q\}$, as a strategy for player 2.

Since mimicking $\alpha^2 < 1 - \alpha^1$ is never optimal and mimicking $\alpha^2 = 1 - \alpha^1$ is equivalent to quitting, we will assume that $\mu^2(\alpha^2) = 0$ for all $\alpha^2 \leq 1 - \alpha^1$.

First, we will show that there is a unique equilibrium of the game $B(\alpha^1, x)$. If $x = 1$, then in equilibrium $\mu^2(Q) = 1$; a rational player will never delay conceding if he knows that his opponent is irrational. For the remainder of the proof we assume $x < 1$.

Define

$$T^1(\alpha^1, \alpha^2, x) = -\tfrac{\alpha^1+\alpha^2-1}{r^21(1-\alpha1)} \log x \quad \text{and } T^2(\alpha^1, \alpha^2, y) = -\tfrac{\alpha^1+\alpha^2-1}{r^1(1-\alpha^2)} \log y$$

Let $a^2(\alpha^1, \alpha^2, x)$ be the unique value of $a^2$ such that $T^1(\alpha^1, \alpha^2, x) = T^2(\alpha^1, \alpha^2, y(a^2))$ where

$$y(a^2) = \tfrac{z^2\pi^2(\alpha^2)}{z^2\pi^2(\alpha^2)+(1-z^2)a^2}$$

and let $x^*(a^2)$ be the value of $x^*$ that solves $T^1(\alpha^1, \alpha^2, x^*) = T^2(\alpha^1, \alpha^2, y(a^2))$. Let $q^1(\alpha^1, \alpha^2, x, a^2)$ be the value of $q^1$ that solves

$$x^*(a^2) = \tfrac{x}{x+(1-x)(1-q^1)}.$$

Note that we have suppressed the dependence of $x^*$ on $\alpha^1$ and $\alpha^2$ and the dependence of $y$ on $\alpha^2$. It is straightforward to verify that each $T^i(\alpha^1, \alpha^2, \cdot)$ is a continuous, strictly decreasing function on (0,1] with $T^i(\alpha^1, \alpha^2, 1) = 0$ for $i = 1, 2$. Similarly, $a^2(\alpha^1, \alpha^2, \cdot)$ is a continuous, strictly decreasing function on (0,1]. Also, the function $q^1(\alpha^1, \alpha^2, \cdot, \cdot)$ is strictly decreasing and continuous in each of its arguments, for all $a^2 \in [0, 1]$ and $x \in (0, x^*(a^2)]$.

The interpretation of these functions is clear; $T^i(\alpha^1, \alpha^2, \cdot)$ is the time at which player $i$'s probability of irrationality reaches 1, as a function of his probability of irrationality at time 0, given that $\alpha^2$ is mimicked and player 1 does not concede with positive probability at time 0. The number $a^2(\alpha^1, \alpha^2, x)$ is the probability with which player 2 must mimic $\alpha^2$ so that both players' probability of irrationality reaches 1 at the same time, given that player 1 does not concede with positive probability at time 0 after $\alpha^2$ is mimicked. Also, $x^*(a^2)$ determines the initial probability of irrationality for player 1 that leads to both players' probability of irrationality reaching 1 at the same time, given that $\alpha^2$ is chosen with probability $a^2$ and player 1 does not concede with positive probability at time 0. Finally, $q^1(\alpha^1, \alpha^2, x, a^2)$ is the probability with which a rational player 1 must concede at time 0, so that both players' probability of irrationality reaches 1 at the same time, given the initial probability of irrationality $x \le x^*(a^2)$ for player 1.

By the arguments of Section 2, in equilibrium both players' irrationality must reach 1 at the same time, so

**i)** $\mu^2(\alpha^2) \le a^2(\alpha^1, \alpha^2, x)$ for all $\alpha^2 > 1 - \alpha^1$. Therefore, the equilibrium probability that a rational player 1 concedes at time 0 is $q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2))$.

Let $u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2))$ denote the utility of player 2 if he mimics $\alpha^2$ in the game $B(\alpha^1, x)$ given that equilibrium specifies that he mimic $\alpha^2$ with probability $\mu^2(\alpha^2)$. That is,

**ii)**
$$u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) = [x + (1 - x)(1 - q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)))](1 - \alpha^1) + (1 - x) q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \cdot \alpha^2$$

To see why (2) holds note that if player 1 concedes at time 0 then player 2's payoff is $\alpha^2$; if not, then player 2's payoff is $1 - \alpha^1$ since in equilibrium he is indifferent between conceding and not at every $t > 0$.

Observe that the continuity and strict monotonicity of $q^1$ implies the same properties for $u^2(\alpha^1, \alpha^2, x, \cdot)$ on $[0, a^2(\alpha^1, \alpha^2, x)]$. If $\mu^2$ is an equilibrium strategy, then $\mu^2(\alpha^2) > 0$ implies $u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \ge u^2(\alpha^1, \hat{\alpha}^2, x, \mu^2(\hat{\alpha}^2))$

for all $\hat{\alpha}^2 > 1 - \alpha^1$. Let $\Delta$ denote the set of all probability distribution on $C^2 \cup \{Q\}$, and $\Delta(\alpha^1, x) = \{\mu^2 \in \Delta \mid \mu^2(\alpha^2) \leq a^2(\alpha^1, \alpha^2, x)$ for all $\alpha^2 > 1 - \alpha^1$ and $\mu^2(\alpha^2) = 0$ if $\alpha^2 \leq 1 - \alpha^1\}$. For $\mu^2$ such that $\mu^2(Q) < 1$,

$$F(x, \mu^2) = \min_{\alpha^2 \text{ s.t.} \mu^2(\alpha^2)>0} u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)), \qquad \text{and}$$
$$F(x, \mu^2) = 1 - \alpha^1, \qquad \text{otherwise.}$$

Define $G$, a correspondence from $\Delta(\alpha^1, x)$ to $\Delta(\alpha^1, x)$ as follows:

$$G(\mu^2) = \{\hat{\mu}^2 \in \Delta(\alpha^1, x) \mid \hat{\mu}^2(\alpha^2) > 0 \text{ implies } u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \geq u^2(\alpha^1, \hat{\alpha}^2, x, \mu^2(\hat{\alpha}^2)) \text{ for all } \hat{\alpha}^2 \in C^2\}.$$

It may be verified that the monotonicity of $u^2$ in its last argument implies that $\mu^2$ is an equilibrium strategy for player 2 in the game $B(\alpha^1, x)$ if and only if $\mu^2$ solves $\max_{\mu^2 \in \Delta(\alpha^1, x)} F(x, \mu^2)$. Moreover, $\mu^2$ solves this maximization problem if and only if $\mu^2$ is a fixed-point of $G$. It follows form the continuity of $u^2(\alpha^1, \alpha^2, x, \cdot)$ that $G$ is upper hemi-continuous. Obviously, $G$ is convex valued and $\Delta(\alpha^1, x)$ is compact. Therefore, by Kakutani's Fixed Point Theorem an equilibrium $\mu^2$ exists. Since each $u^2(\alpha^1, \alpha^2, x, \cdot)$ is strictly decreasing on $[0, a^2(\alpha^1, \alpha^2, x)]$ the equilibrium $\mu^2$ is unique. But if the equilibrium strategy for player 2 is determined uniquely, then $q^1$, the probability that 1 concedes to $\alpha^2$ at time zero is determined uniquely. After time zero, 1 must concede at rate $\lambda^1$. Thus, the equilibrium strategy of player 1 in the game $B(\alpha^1, x)$ is unique.

Let $u^1(\alpha^1, x)$ be the payoff of player 1 in the unique equilibrium of $B(\alpha^1, x)$. We will show that $u^1(\alpha^1, \cdot)$ is continuous and that there exists $\underline{x}$ such that for $x \leq \underline{x}, u^1(\alpha^1, x) = u^1(\alpha^1, \underline{x})$ and $\mu^2(\max C^2) = 1$ where $\mu^2$ is the equilibrium strategy of player 2 in the game $B(\alpha^1, x)$. We will also show that $u^1(\alpha^1, \cdot)$ is continuous and strictly increasing in $x$ on $[\underline{x}, 1]$.

Note that $F(\cdot, \mu^2)$ is continuous and $F(\cdot, \cdot)$ is upper semi-continuous. Hence a straight-forward extension of the Theorem of the Maximum yields that $\arg\max_{\mu^2} F(x, \mu^2)$ is a continuous function of $x$. This implies $u^1(\alpha^1, \cdot)$ is a continuous function of $x$.

Suppose $\sum_{\alpha^2 > 1 - \alpha^1} a^2(\alpha^1, \alpha^2, x) \leq 1$, then $\mu^2(\alpha^2) < a^2(\alpha^1, \alpha^2, x)$ for some $\alpha^2$ implies $\mu^2(Q) > 0$, so the equilibrium payoff of player 2 in $B(\alpha^1, x)$ must be $1 - \alpha^1$. But $\mu^2(\alpha^2) < a^2(\alpha^1, \alpha^2, x)$ implies $q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) > 0$, hence, by (2), $u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) > 1 - \alpha^1$. Therefore player 2 can get a higher payoff than his equilibrium payoff by mimicking $\alpha^2$, a contradiction. Hence, we have

**iii)** $\mu^2(\alpha^2) = a^2(\alpha^1, \alpha^2, x)$ for all $\alpha^2 > 1 - \alpha^1$ and $\mu^2(Q) = 1 - \sum_{\alpha^2 > 1-\alpha^1} a^2(\alpha^1, \alpha^2, x)$ whenever $\sum_{\alpha^2 > 1-\alpha^1} a^2(\alpha^1, \alpha^2, x) \leq 1$.

**iv)** $u^1(\alpha^1, x) = (\sum_{\alpha^2 \leq 1-\alpha^1} z^2\pi^2(\alpha^2) + (1-z^2)\mu^2(Q))\alpha^1 +$

$$\sum_{\alpha^2 > 1-\alpha^1} (z^2\pi^2(\alpha^2) + (1-z^2)\mu^2(\alpha^2))(1-\alpha^2)$$

Let $\hat{x} = x_*$ solve $\sum_{\alpha^2 > 1-\alpha^1} a^2(\alpha^1, \alpha^2, \hat{x}) = 1$.

It follows from (3) that for $x \in (x_*, 1)$ the $\mu^2(\alpha^2)$ in (4) can be replaced with $a^2(\alpha^1, \alpha^2, x)$. Hence $u^1(\alpha^1, \cdot)$ is strictly increasing on the interval $[x_*, 1)$.

Next we show that for some $\underline{x} \leq x_*$, $u^1(\alpha^1, x) = u^1(\alpha^1, \underline{x})$ and $\mu^2(\max C^2) = 1$ for all $x \leq \underline{x}$. Moreover, $u^2(\alpha^1, \cdot)$ is strictly decreasing on the interval $[\underline{x}, x_*]$.

Let $\bar{\alpha}^2 = \max C^2$ and $\underline{x} = \sup\{x \mid \mu^2(\bar{\alpha}^2) = 1$ where $\mu^2$ is the equilibrium strategy of player 2 in $B(\alpha^1, x)\}$. First we show that $\underline{x}$ is well-defined.

Note that as $x$ approaches 0, $q^1(\alpha^1, \bar{\alpha}^2, x, 1)$ approaches 1 and since $q^1$ is strictly decreasing in its last argument, by (ii), $u^2(\alpha^1, \bar{\alpha}^2, x, \mu^2(\overline{\alpha}^2))$ approaches $\bar{\alpha}^2$. But for any $\alpha^2$, $u^2(\alpha^1, \alpha^2, \cdot, \cdot) \leq \alpha^2$, hence mimicking $\alpha^2 < \bar{\alpha}^2$ cannot be optimal when $x$ is sufficiently small. Since $\mu^2$ is continuous in $x$, $\mu^2(\bar{\alpha}^2) = 1$ for $x = \underline{x}$. It remains to be shown that $u^1(\alpha^1, x) = u^1(\alpha^1, \underline{x})$ for all $x < \underline{x}$ and that $u^1(\alpha^1, \cdot)$ is strictly increasing on the interval $[\underline{x}, x_*]$.

From the optimality of player 2's equilibrium strategy it follows that $\mu^2(\hat{\alpha}^2) > 0$ implies $u^2(\alpha^1, \hat{\alpha}^2, x, \mu^2(\hat{\alpha}^2)) \geq u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2))$ for all $\alpha^2$. Therefore, for $x \leq x_*, \mu^2(\hat{\alpha}^2) > 0$ and (2) imply, for all $\alpha^2 \in C^2$,

**v)** $q^1(\alpha^1, \hat{\alpha}^2, x, \mu^2(\hat{\alpha}^2)) \geq q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \cdot \frac{\alpha^1+\alpha^2-1}{\alpha^1+\hat{\alpha}^2-1}$ for all $\alpha^2$.

In particular (v) holds for $x = \underline{x}, \hat{\alpha}^2 = \bar{\alpha}^2$, and $\mu^2$ such that $\mu^2(\bar{\alpha}^2) = 1$ and $\mu^2(\alpha^2) = 0$ for $\alpha^2 \neq \bar{\alpha}^2$. But then for $x < \underline{x}$, the monotonicity of $q^1$ implies $q^1(\alpha^1, \bar{\alpha}^2, x, \mu^2(\bar{\alpha}^2)) > q^1(\alpha^1, \bar{\alpha}^2, \underline{x}, 1)$. Also, $q^1(\alpha^1, \alpha^2, \underline{x}, 0) = 1 \geq q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2))$ since a rational agent 1 will concede immediately if he knows he is dealing with an irrational opponent. Thus, for $x < \underline{x}$,

$$q^1(\alpha^1, \bar{\alpha}^2, x, \mu^2(\bar{\alpha}^2)) > q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \cdot \frac{\alpha^1+\alpha^2-1}{\alpha^1+\bar{\alpha}^2-1} \text{ for all } \alpha^2 \neq \bar{\alpha}^2.$$

Therefore, for $x < \underline{x}$, $\quad u^2(\alpha^1, \bar{\alpha}^2, x, \mu^2(\bar{\alpha}^2)) > u^2(\alpha^1, \alpha^2, x, \mu^2(\alpha^2))$ for all $\alpha^2 \neq \bar{\alpha}^2$. Hence, the optimality of $\mu^2$ implies $\mu^2(\alpha^2) = 0$ for all $\alpha^2 \neq \bar{\alpha}^2$ and $\mu^2(\bar{\alpha}^2) = 1$ whenever $x \leq \underline{x}$. Then (4) yields the conclusion $u^1(\alpha^1, x) = u^1(\alpha^1, \underline{x})$ for all $x \leq \underline{x}$.

To prove that $u^1(\alpha^1, \cdot)$ is strictly increasing on $[\underline{x}, x_*]$, we first observe that if $\mu^2$ is an equilibrium strategy, then $\mu^2(\alpha^2) > 0$ implies $\mu^2(\hat{\alpha}^2) > 0$ for all $\hat{\alpha}^2 > \alpha^2$. This is easily verified by noting that if $\mu^2(\hat{\alpha}^2) = 0$, then $u^2(\alpha^1, \hat{\alpha}^2, x, \mu^2(\alpha^2)) = (1-x)\hat{\alpha}^2 + x(1-\alpha^1)$, while $u^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \leq (1-x)\alpha^2 + x(1-\alpha^1)$. Thus, (v) implies that there exists $\tilde{\alpha}^2$ such that

**(vi)** $q^1(\alpha^1, \hat{\alpha}^2, x, \mu^2(\hat{\alpha}^2)) = q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \cdot \frac{\alpha^1+\alpha^2-1}{\alpha^1+\hat{\alpha}^2-1}$ for all $\alpha^2, \hat{\alpha}^2 \geq \tilde{\alpha}^2$,
$q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) \geq \frac{\alpha^1+\alpha^2-1}{\alpha^1+\hat{\alpha}^2-1}$ for $\hat{\alpha}^2 \geq \tilde{\alpha}^2 > \alpha^2$,
$\sum_{\hat{\alpha}^2 \geq \tilde{\alpha}^2} \mu^2(\hat{\alpha}^2) = 1$ and $\mu^2(\alpha^2) = 0$ for $\alpha^2 < \tilde{\alpha}^2$.

Some tedious but straightforward calculations reveal that

$$q^1(\alpha^1, \alpha^2, x, \mu^2(\alpha^2)) = \frac{K(\alpha^1,\alpha^2,\mu^2(\alpha^2))-x}{K(\alpha^1,\alpha^2,\mu^2(\alpha^2))(1-x)},$$

where

$$K(\alpha^1, \alpha^2, \mu^2(\alpha^2)) := (y(\mu^2(\alpha^2)))^{\gamma(\alpha^1,\alpha^2)} and\ \gamma(\alpha^1, \alpha^2)$$

$:= \frac{r^2(1-\alpha^1)}{r^1(1-\alpha^2)}$. Substituting this definition of $q^1$ for both $\alpha^2$ and $\hat{\alpha}^2$ and some simplifying yields

**(vii)** $\frac{x(\hat{\alpha}^2-\alpha^2)}{\alpha^1+\hat{\alpha}^2-1} = K(\alpha^1, \hat{\alpha}^2, \mu^2(\hat{\alpha}^2) - K(\alpha^1, \alpha^2, \mu^2(\alpha^2)) \cdot \frac{\alpha^1+\alpha^2-1}{\alpha^1+\hat{\alpha}^2-1}$
for all $\hat{\alpha}^2 > \alpha^2 \geq \tilde{\alpha}^2$

Observe that the left-hand side of (7) increases as $x$ increases to some $\hat{x}$, while the right-hand side does not depend on $x$. Thus, the equilibrium strategy of player 2 must change as $x$ increases. Furthermore, since $K(\alpha^1, \alpha^2, \cdot)$ is decreasing (7) implies that either $\mu^2(\hat{\alpha}^2)$ decreases or $\mu^2(\alpha^2)$ increases for all $\hat{\alpha}^2 > \alpha^2$. Let $\hat{\alpha}^2$ be the largest element of $C^2$ such that $\sum_{\alpha^2 \geq \hat{\alpha}^2} \hat{\mu}^2(\alpha^2) > \sum_{\alpha^2 \geq \hat{\alpha}^2} \mu^2(\alpha^2)$ where $\hat{\mu}^2$ is player 2's equilibrium strategy in $B(\alpha^1, \hat{x})$. If such an $\hat{\alpha}^2$ exists, then for some $\alpha^2 < \hat{\alpha}^2$, we have $\hat{\mu}^2(\hat{\alpha}^2) > \mu^2(\hat{\alpha}^2)$ and $\hat{\mu}^2(\alpha^2) < \mu^2(\alpha^2)$ a contradiction. Therefore, $\sum_{\alpha^2 \geq \hat{\alpha}^2} \hat{\mu}^2(\alpha^2) \leq \sum_{\alpha^2 \geq \hat{\alpha}^2} \mu^2(\alpha^2)$ for all $\hat{\alpha}^2$ and $\hat{\mu}^2 \neq \mu^2$.

That is, $\mu^2$ stochastically dominates $\hat{\mu}^2$ and hence $u^1(\alpha^1, \hat{x}) > u^1(\alpha^1, x)$ by (4). Continuity of $\mu^2$ permits us to extend this conclusion to any $x, \hat{x} \in [\underline{x}, x_*]$ such that $\hat{x} > x$.

Again, we characterize equilibrium distributions $\mu^1$ as the solution to $\max_{\mu^1} \hat{F}(\mu^1)$, where

$\hat{F}(\mu^1) = \min_{\alpha^1 \text{ s.t. } \mu^1(\alpha^1)>0} u^1(\alpha^1, x(\mu^1(\alpha^1)))$ and $x(\mu^1(\alpha^1)) = \frac{z^1\pi^1(\alpha^1)}{z^1\pi^1(\alpha^1)+(1-z^1)\mu^1(\alpha^1)}$.

The continuity of $u^1(\alpha^1, \cdot)$ ensures that an equilibrium $\mu^1$ exists. (See the fixed-point argument establishing existence of an equilibrium strategy $\mu^2$ in $B(\alpha^1, x)$ above).

Let $\bar{u}^1$ be the maximized value above. Hence, $\bar{u}^1$ is the utility that player 1 attains in any equilibrium. Clearly, $\bar{u}^1 \geq u^1(\alpha^1, \underline{x})$ for all $\alpha^1$. Let $\mu^1, \hat{\mu}^1$ be two equilibrium strategies for player 1. If $\bar{u}^1 > u^1(\alpha^1, \underline{x})$ then $\mu^1(\alpha^1) = \hat{\mu}^1(\alpha^1)$. To see this note that either $u^1(\alpha^1, 1) > \bar{u}^1$ in which case there is a unique $a^1$ such that $u^1(\alpha^1, x(a^1)) = \bar{u}^1$ and hence $\mu^1(\alpha^1) = \hat{\mu}^1(\alpha^1) = a^1$ or $u^1(\alpha^1, 1) \leq \bar{u}^1$, in which case $\mu^1(\alpha^1) = \hat{\mu}^1(\alpha^1) = 0$ by the monotonicity of $u^1(\alpha^1, \cdot)$. Let $D^1 = \{\alpha^1 \in C^1 \mid u^1(\alpha^1, \underline{x}) = \bar{u}^1\}$. Recall that $\underline{x}$ depends on $\alpha^1$. We have already noted that $\mu^1(\alpha^1) = \hat{\mu}^1(\alpha^1)$ for all $\alpha^1 \in C^1 \backslash D^1$ and hence $\sum_{\alpha^1 \in D^1} \mu^1(\alpha^1) = \sum_{\alpha^1 \in D^1} \hat{\mu}^1(\alpha^1)$.

We will conclude the proof that $\mu^1$ and $\hat{\mu}^1$ lead to the same random outcome $\tilde{\theta}$ by first verifying that the probability that player 1 chooses some $\alpha^1 \in D^1$ and agreement is reached at time 0 is the same with either $\mu^1$ and $\hat{\mu}^1$. This will imply that the random outcome, conditional on agreement at time 0 is the same with either $\mu^1$ or $\hat{\mu}^1$. Finally, we will show that for each $\alpha^1 \in D^1$, the probability that a rational player 1 will mimic $\alpha^1$ and not concede is the same with either $\mu^1$ and $\hat{\mu}^1$.

Let $A(\mu^1)$ be the probability that player 1 mimics some $\alpha^1 \in D^1$ and agreement is reached at time 0 given the equilibrium strategy $\mu^1$. Since $\alpha^1 \in D^1$ implies $\mu^2_{\alpha^1}(\bar{\alpha}^2) = 1$, it follows that $\alpha^1 \geq 1 - \bar{\alpha}^2$; otherwise player 1 would achieve a higher utility by mimicking $\max C^1 > 1 - \bar{\alpha}^2$. So,

$$\begin{aligned} A(\mu^1) &= \sum_{\alpha^1 \in D^1} q^1\left(\alpha^1, \bar{\alpha}^2, x\left(\mu^1\left(\alpha^1\right)\right), 1\right) \cdot \left(1 - x\left(\mu^1\left(\alpha^1\right)\right)\right) \cdot \\ &\quad \times \left(z^1\pi^1\left(\alpha^1\right) + \left(1 - z^1\right) \cdot \mu^1\left(\alpha^1\right)\right) \\ A(\mu^1) &= \sum_{\alpha^1 \in D^1} \frac{K\left(\alpha^1, \bar{\alpha}^2, 1\right) - x(\mu^1(\alpha^1))}{K(\alpha^1, \bar{\alpha}^2, 1)} \left(z^1\pi^1\left(\alpha^1\right) + \left(1 - z^1\right)\mu^1\left(\alpha^1\right)\right) \\ &= \sum_{\alpha^1 \in D^1} \left(z^1\pi^1\left(\alpha^1\right) + \left(1 - z^1\right)\mu^1\left(\alpha^1\right)\right) - \sum_{\alpha^1 \in D^1} \frac{z^1\pi^1\left(\alpha^1\right)}{K(\alpha^1, \bar{\alpha}^2, 1)} \end{aligned}$$

But since $\sum_{\alpha^1 \in D^1} \mu^1(\alpha^1) = \sum_{\alpha^1 \in D^1} \hat{\mu}^1(\alpha^1)$ we have $A(\mu^1) = A(\hat{\mu}^1)$.

For any $\alpha^1 \in D^1$, the probability that a rational player 1 will mimic $\alpha^1$ and not concede at time 0 is $\mu^1(\alpha^1)(1 - q^1(\alpha^1, \bar{\alpha}^2, x(\mu^1(\alpha^1)), 1))$
$= \mu^1(\alpha^1) \frac{x(\mu^1(\alpha^1))(1-K(\alpha^1,\bar{\alpha}^2,1))}{(1-x(\mu^1(\alpha^1)))K(\alpha^1,\bar{\alpha}^2,1)} = \frac{\pi^1(\alpha^1)(z^1)(1-K(\alpha^1,\bar{\alpha}^2,1))}{(1-z^1)K(\alpha^1,\bar{\alpha}^2,1)}$
Hence, this term is independent of $\mu^1$ and therefore the same for both $\mu^1$ and $\hat{\mu}^1$. *Q.E.D.*

**Proof of Proposition 3:** (a) Pick a subsequence of $B_n$ such that $\mu^1_{n_k}, F^1_{\alpha^1,\alpha^2,n_k}(0), v^1_{n_k}, \mu^2_{\alpha^1,n_k}, v^2_{n_k}$ converge to their respective limits $\mu^1, F^1_{\alpha^1,\alpha^2}(0), v^1, \mu^2_{\alpha^1}$ and $v^2$.

Without loss of generality assume that the subsequence is the sequence itself. Suppose $\lim \frac{r^1_n}{r^2_n} = 0$. It follows from the definition of $a^2(\alpha^1, \alpha^2, x)$ (see the preceding proof) that $\lim a^2(\alpha^1, \alpha^2, x_n) = 0$ and hence $\mu^2_{\alpha^1}(Q) = 1$, where $x_n = \frac{z^1\pi^1(\alpha^1)}{z^1\pi^1(\alpha^1)+(1-z^1)\mu^1_n(\alpha^1)}$ whenever $\lim \frac{r^1_n}{r^2_n} = 0$. Let $x = \lim x_n$. Consequently, the payoff associated with mimicking each $\alpha^1$ is at least $(1 - z^2)\alpha^1$ for player 1, hence his equilibrium payoff is at least $(1 - z^2) \max C^1$ in the limit. Now suppose $\lim \frac{r^1_n}{r^2_n} = \infty$. Let $a^1\left(\alpha^1, \alpha^2, \mu^2_{\alpha^1,n}(\alpha^2)\right)$ be the value of $a^1$ such that $T^1(\alpha^1, \alpha^2, x(a^1)) = T^2\left(\alpha^1, \alpha^2, y\left(\mu^2_{\alpha^1,n}(\alpha^2)\right)\right)$ where $x(a^1) = \frac{z^1\pi^1(\alpha^1)}{z^1\pi^1(\alpha^1)+(1-z^1)a^1}$ and $y(\cdot)$ is defined in the previous proof. Then, for $\alpha^1$ such that $\mu^1_n(\alpha^1) > 0$ and any $\alpha^2 > 1 - \alpha^1, \lim b^1(\alpha^1, \alpha^2, x_n, \mu^2_n(\alpha^2)) = 0$. So $q^1(\alpha^1, \alpha^2, x, \mu^2_{\alpha^1}(\alpha^2)) = 1 - x$. Hence, the conditional probability of 1 conceding after she demands $\alpha^1$ initially and 2 counters with $\alpha^2$ is equal to the conditional probability that 1 is rational. Hence, no matter what 2 demands, 1 concedes if she is rational. But then it must be that after any $\alpha^1$, 2 demands $\max C^2$ for sure. Thus, $\liminf v^2_n \geq (1 - z^1) \max C^2$ and hence $\limsup v^1_n \leq 1 - (1 - z^2) \max C^2$ as desired. This proves (a). The proof of part (b) is similar and omitted. *Q.E.D.*

**Proof of Proposition 4:** The proof is organized as follows: first, we characterize the equilibrium payoffs after a history in which only one player reveals himself to be rational (Lemma 1); then we argue that the game prior to anyone revealing her rationality is analogous to a war of attrition in which if $i$ wins she gets $\alpha^i e^{-r^i t}$ and $j$ gets $(1-\alpha^i)e^{-r^j t}$, where $t$ is the time at which $j$ gives in.

**Lemma 1:** For any $\epsilon > 0 \; \exists \; \bar{n}$ such that in any sequential equilibrium of $g_n$ for $n \geq \bar{n}$ after any history $h_t$ such that $i$ is known to be rational and $j$ is not, the payoff to $i$ is at most $1 - \alpha^j + \epsilon$ and the payoff to $j$ is at least $\alpha^j - \epsilon$ (evaluated at time $t$).

**Proof**: This proof is adapted from the proof of Theorem 8.4 in Myerson (1991). We will show that the payoff to $j$ if $j$ continues to act irrationally converges to $\alpha^j$ as $n \to \infty$. This will imply the desired conclusion. For the remainder of the proof, we assume that $j$ continues to act irrationally while $i$ conforms with his equilibrium strategy.

Note that $z_t^j$, the probability that $j$ is irrational after history $h_t$, is either zero or no less than $z^j$. This is an immediate consequence of Bayes' Law. But since $z_t^j > 0$, by assumption we have $z_t^j \geq z^j$.

Next we will argue that the game ends with probability 1 in finite time, given history $h_t$ if $j$ continues to behave irrationally. To see this note that $i$ can always get utility at least $(1-\alpha^j)z_t^j$ by seeking (almost) immediate agreement with the irrational type of player $j$. On the other hand, if $i$ uses with positive probability any strategy that could extend the game until period $\hat{t}+t$ against an irrational opponent, then $i$'s payoff is at most $1 - z_t^j + z_t^j e^{-r^i \hat{t}}(1-\alpha^j)$. Thus, such a strategy can be optimal only if

$$(1-\alpha^j)z_t^j \leq 1 - z_t^j + z_t^j e^{-r^i \hat{t}}(1-\alpha^j).$$

That is

$$z_t^j \leq \frac{1}{1 + (1-\alpha^j)(1-e^{-r^i \hat{t}})} := \delta.$$

Conditional on the game not ending until time $t + \hat{t}$, we can repeat the above argument to conclude that the probability $z_t^j$ must be less than $\delta^2$ for player $i$ to optimally follow a strategy at time $t$, which will not concede to the irrational player $j$ until time $t + 2\hat{t}$. Similarly, for the game to last until time $t + k\hat{t}$, it must be that $z_t^j \leq \delta^k$. But since $\delta^k$ goes to zero as $k$ goes to infinity and $z_t^j \geq z^j$, there will come a $k^*$ for which this inequality cannot be satisfied. Thus, player $i$ must end the game against an irrationally behaving opponent by time $t + k^*\hat{t}$.

This argument applies to any game $g_n$ and shows that the game ends in finite time $t + \bar{t}(n)$ if $j$ behaves irrationally.

We will now argue that $\bar{t}(n)$ converges to zero as $n$ goes to infinity. If this assertion is false, then we can find a subsequence of $g_n$'s (wlog assume this is the sequence $g_n$ itself), an $\hat{\varepsilon} > 0$, a collection of histories $h_{t_n}$, and $\bar{t}(n)$'s such that the game $g_n$ conditional on the history $h_{t_n}$ ends at time $t_n + \bar{t}(n)$ where $\bar{t}(n) > \hat{\varepsilon}$. To simplify the subsequent notation, we will rescale the units of time so that $r^j = 1$ and $r^i = r$. Consider the last $\epsilon-$time units of the game if $j$ continues to behave irrationally. It must be that $i$ is using some strategy (with positive probability) that does not end the game for at least $\epsilon$ longer. Let $x$ be $i$'s expected payoff if $j$ agrees to an offer worse than $\alpha^j$ by time $\beta\,\epsilon$ for $\beta \in (0,1)$. Let $y$ be $i$'s payoff if $j$ does not agree to such an offer by time $\beta\,\epsilon$ and let $\zeta$ be the probability that $i$ assigns to the event that $j$ will not agree to such an offer by time $\beta\,\epsilon$. Now $i$'s rejection of $\alpha^j$ implies that

$$1 - \alpha^j \leq (1-\zeta)x + \zeta y. \tag{2}$$

This implies

$$\zeta \leq \frac{x - 1 + \alpha^j}{x - y} \text{ whenever } x - y > 0. \tag{3}$$

Note that for $j$ to agree to a payoff less than $\alpha^j$, he must be rational. But then $j$ knows that if he holds out for $\epsilon$ longer, he will get $\alpha^j$. Therefore $x \leq 1 - e^{-\epsilon}\alpha^j$. Similarly, if $j$ does not agree to an offer by $\beta\epsilon$ then the best that $i$ can do after that time is $1 - e^{-(1-\beta)\epsilon}\alpha^j$. So $y \leq e^{-\beta r \epsilon}(1 - e^{-(1-\beta)\epsilon}\alpha^j)$. Notice that this last inequality implies for $\epsilon$ small enough that $y < 1 - \alpha^j$ whenever $\beta > \frac{\alpha^j}{\alpha^j + r(1-\alpha^j)}$. To see this note that $e^{-\beta r \epsilon}(1 - e^{-(1-\beta)\epsilon}\alpha^j) < 1 - \alpha^j$ if and only if $\alpha^j < \frac{1-e^{-\beta r \epsilon}}{1-e^{-(\beta r + (1-\beta))\epsilon}}$. By l'Hospital's rule, this inequality holds for all $\epsilon \in (0, \bar{\epsilon})$ whenever $\beta > \frac{\alpha^j}{\alpha^j + r(1-\alpha^j)}$.

Hence, for $\beta > \frac{\alpha^j}{\alpha^j + r(1-\alpha^j)}$ if (2) is to hold, we must have $x \geq 1 - \alpha^j > y$. Therefore, (3) holds and from (2) and the above bounds on $x$ and $y$ we have

$$\zeta \leq \frac{\alpha^j\,(1 - e^{-\epsilon})}{1 - \alpha^j e^{-\epsilon} - e^{-\beta r \epsilon} + \alpha^j e^{-(\beta r + (1-\beta))\epsilon}} \tag{4}$$

for all $\epsilon \in (0, \bar{\epsilon})$. Again using l'Hospital's rule, we find that the rhs of (4) converges to $\frac{\alpha^j}{\beta r + \alpha^j (1-r)\beta}$. Therefore, since $\alpha^j + r(1-\alpha^j) > \alpha^j$ for all $r > 0$, for $\beta \in (\frac{\alpha^j}{\alpha^j + r(1-\alpha^j)}, 1)$ and $\epsilon$ close to zero, the rhs of 3 is less than 1. In particular, we can fix $\beta$ and choose $\epsilon'$ small enough so that the rhs of 3 is less than some $\delta < 1$ for all $\epsilon \in (0, \epsilon')$.

Thus, in the final $\epsilon$ amount of time, the probability that player $j$ will continue to behave like an irrational player during the first $\beta$ percentage of the time must be less than $\delta$. But after $\beta\epsilon$ time elapses, the same argument may be repeated to show that at time $t$ the probability that player $j$ will continue to behave irrationally until the final $(1-\beta)^2\epsilon$ amount of time must be less than $\delta^2$. Similarly, the probability of player $j$ resisting until the final $(1-\beta)^k\epsilon$ amount of time must be less than $\delta^k$. Choosing $k$ such that $\delta^k < z^j$ establishes a contradiction since, as argued earlier, $z_t^j \geq z^j$ This argument relies on player $i$ being able to make offers sufficiently close to time $t_n + [1-(1-\beta)^m]\epsilon$ for $m = 1, 2, ...k$. Hence, we need the requirement that $g_n$ converges to a continuous-time game. *Q.E.D.*

Lemma 1 implies that after a player reveals that he is rational, agreement must be reached almost immediately, at terms arbitrarily close to the irrational demand of his still possibly irrational opponent. Hence, as in the continuous-time game we may identify revealing rationality with conceding to one's opponent's irrational demand. This convergence of post-revelation equilibrium payoffs underlines the convergence of the overall equilibrium.

Let $\{g_n\}$ be a sequence of discrete bargaining games and $\{\bar{\sigma}_n\}$ a corresponding sequence of sequential equilibria.

For each $\bar{\sigma}_n$ define $F_n^i : I\!R \to [0,1]$ where $F_n^i(t)$ is the cumulative probability that player $i$ takes an action not consistent with being an irrational type at or before time $t$, conditional on player $j \neq i$ having acted like an irrational player until time $t$. To prove the Proposition we will show that:

**(a)** Every subsequence of $(F_n^1, F_n^2)$ has a convergent (sub)-subsequence.

**(b)** The limit points of $(F_n^1, F_n^2)$ do not have common points of discontinuity.

**(c)** If $(F_n^1, F_n^2)$ converges to $(F^1, F^2)$, and $F^1$ and $F^2$ do not have common points of discontinuity, then $(F^1, F^2)$ is an equilibrium of the continuous-time game.

Thus, (a), (b), and (c) imply that $(F_n^1, F_n^2)$ converges to the equilibrium outcome of the continuous-time game. Then, we invoke Proposition 2 to conclude that the limit of $(F_n^1, F_n^2)$ is equal to $\left(\hat{F}^1, \hat{F}^2\right)$, the unique equilibrium of the continuous-time game. Also, by Lemma 1 we conclude that $\tilde{\theta}_n$ converges in distribution to the equilibrium outcome of the continuous-time game.

Step 1 There exists $\bar{n}$ and $T$ such that $F_n^i(t) = 1 - z^i$ for all $t \geq T$ and $n \geq \bar{n}$. This argument is identical to the first part of the proof of Lemma 1 (take $T = k^*\hat{t}$) which establishes that a rational player $i$ must concede in finite time $k^*\hat{t}$ to an opponent $j$ who persists in irrational behavior (i.e., demands $\alpha^j$ and accepts no less).

Define the functions $G_n^i$ where $G_n^i(t) = \frac{F_n^i(t)}{1-z^i}$. Then the $G_n^i$'s are distribution functions. (without loss of generality truncate the first $\bar{n}$ terms and renumber the sequence).

Step 2 There exists a subsequence $\{F_{n_k}^i\}$ and a nondecreasing, right-continuous function $F^i$ such that $\lim_k F_{n_k}^i(t) = F^i(t)$ at continuity points $t$ of $F$. Furthermore $F^i(t) = 1 - z^i$ for all $t \geq T$.

Proof of Step 2: Helly's Theorem (see Billingsley (1986) Theorem 25.9) applied to the sequence $G_n^1$ yields a subsequence $G_{n_j}^1$ and a right–continuous non-decreasing function $G^1$ such that $\lim_{n_j} G_{n_j}^1(t) = G^1(t)$ at every continuity point of $G^1$. Noting that $G_n^1(T) = 1$ for all $n$ establishes that $G^1$ is a distribution function. Now, let $F_{n_j}^1 = (1 - z^1) G_{n_j}^1$ and $F^1 = (1 - z^1) G^1$.

Apply the same argument to the sequence $G_{n_j}^2$ to get the subsubsequence $\left(F_{n_{j_k}}^1, F_{n_{j_k}}^2\right)$ and $(F^1, F^2)$ with the desired properties. *Q.E.D.*

We will again renumber so that $\left(F_{n_{j_k}}^1, F_{n_{j_k}}^2\right)$ will be $(F_n^1, F_n^2)$.

Steps 1 and 2 establish that every subsequence of $(F_n^1, F_n^2)$ has a convergent (sub)subsequence. Steps 3-6 will prove that this limit is an equilibrium of the continuous-time game. In Section 3, we had shown that the continuous-time game has a unique equilibrium. Hence, steps 1-6 show that $(F_n^1, F_n^2)$ converges to the unique equilibrium of the continuous-time game.

Step 3 $F^1, F^2$ have no common discontinuity points.

Proof of Step 3: Assume to the contrary, that at point $t$ both $F^1$ and $F^2$ are discontinuous. Let $J^i > 0$ be the size of the discontinuity of $F^i$ at $t$.

Let $E_\Delta = [t - \Delta, t + \Delta]$. Let $p_n^i$ be the probability that the first action inconsistent with rationality will occur in $E$ and that player $i$ undertakes this action.

By choosing $\Delta$ such that $t - \Delta$ and $t + \Delta$ are continuity points of $F^1$ and $F^2$, we can ensure that $\lim_{n\to\infty} (p_n^1 + p_n^2) = [1 - (1 - F^1(t + \Delta))(1 - F^2(t + \Delta))] - [1 - (1 - F^1(t - \Delta))(1 - F^2(t - \Delta))]$ where the first term is probability that the game ends no later than $t + \Delta$ and the second term is the probability that the game ends no later than $t - \Delta$. Hence

$$\lim_{n\to\infty} \left(p_n^1 + p_n^2\right) = F^1(t + \Delta) - F^1(t - \Delta) + F^2(t + \Delta) - F^2(t - \Delta)$$

$$-F^1(t + \Delta) F^2(t + \Delta) + F^1(t - \Delta) F^2(t - \Delta) =$$

$$J_\Delta^1 + J_\Delta^2 - J_\Delta^1 F^2(t + \Delta) - J_\Delta^2 F^1(t + \Delta) + J_\Delta^1 \cdot J_\Delta^2$$

where $J_\Delta^i = F^i(t + \Delta) - F^i(t - \Delta)$.

Hence for any $\epsilon > 0$, by choosing $\Delta$ sufficiently close to zero, we can ensure that

$$\lim \left(p_n^1 + p_n^2\right) \leq J^1 + J^2 - J^1 F^2(t) - J^2 F^1(t) + J^1 J^2 + \epsilon$$

Pick a subsequence of $F_n^1, F_n^2$ along which both $(p_n^1, p_n^2)$ converge to $p^1$ and $p^2$ respectively. Hence for any $\epsilon > 0$

$$p^1 + p^2 \leq J^1 + J^2 - J^1 F^2(t) - J^2 F^1(t) + J^1 J^2 + \epsilon \tag{5}$$

Note that by using a strategy that puts all of the mass $J_\Delta^1$ on time $t+\Delta$ player 1 can guarantee that the corresponding probability that 2 is the first to reveal himself to be rational in $[t-\Delta, t+\Delta]$ is $J_\Delta^2(1 - F^1(t - \Delta))$ $= J_\Delta^2(1 - F^1(t + \Delta) + J_\Delta^1)$ which for small $\Delta$ is close to $J^2(1 - F^1(t) + J^1)$. If the actual probability $p_n^2$ with which 2 is the first person to reveal his rationality in $E_\Delta$ is less than $J^2(1 - F^1(t) + J^1)$ the strategy in which 1 waits until $t + \Delta$ will do better for him. Thus it must be that

$$p^2 \geq J^2 \left(1 - F^1(t) + J^1\right) - \epsilon$$

A symmetric argument yields

$$p^1 \geq J^1 \left(1 - F^2(t) + J^2\right) - \epsilon$$

hence,

$$p^1 + p^2 \geq J^1 + J^2 - J^1 F^2(t) - J^2 F^1(t) + 2J^1 J^2 - 2\epsilon$$

Thus equation (5) above yields

$$J^1 J^2 - 3\epsilon \leq 0$$

Since this argument can be made for all $\epsilon$, we have $J^1 J^2 \leq 0$, contradicting $J^1 > 0$ and $J^2 > 0$.

Step 4: Let $h : I\!\!R^2 \to I\!\!R^1$ be a Lebesgue measurable function and let $D_h$ denote the set of discontinuity points of $h$. Let $\mu_n$ be a sequence of probability measures converging in distribution to some probability measure $\mu$ where $\mu$ has bounded support. Then $\mu_n h^{-1}$ converges in distribution to $\mu h^{-1}$ if $\mu(D_h) = 0$. Hence, $\int h d\mu_n$ converges to $\int h d\mu$.

Proof: That $\mu_n h^{-1}$ converges to $\mu h^{-1}$ is established in Theorem 29.2 of Billingsley (1986).

Then by Theorem 29.1 of Billingsley (1986):

$$\lim \int_{I\!\!R} x d\mu_n h^{-1} = \int_{I\!\!R} x d\mu h^{-1}$$

(since $\mu$ has bounded support, the unboundedness of $f(x) = x$ is of no consequence).

Also,

$$\int_{I\!\!R} x d\mu_n h^{-1} = \int_{I\!\!R^2} h d\mu_n \quad \text{and similarly}$$

$$\int_{I\!\!R} x d\mu h^{-1} = \int_{I\!\!R^2} h d\mu$$

which establishes that $\lim \int h d\mu_n = \int h d\mu$ as desired.

Step 5: If $G_n^1$ converges in distribution to $G^1$ and $G_n^2$ converges in distribution to $G^2$, then $\mu_n$ the product measure (on $I\!R^2$) associated with $G_n^1, G_n^2$ converges to $\mu$, the product measure associated with $G^1$ and $G^2$.

Proof: By Skorohod's Theorem (Billingsley (1986), Theorem 29.6), there exists probability spaces $(\Omega^i, \mathcal{F}^i, P^i)$ and random variables $X_n^i, X^i$ for $i = 1, 2$ such that $X_n^i(\omega)$ converges to $X^i(\omega)$ at every $\omega$, $X_n^i$ has distribution $G_n^i$ and $X^i$ has distribution $G^i$.

Let $\Omega = \Omega^1 \times \Omega^2, \mathcal{F} = \sigma(\mathcal{F}^1 \times \mathcal{F}^2)$ and $P = P^1 \times P^2$ the product measure of $P^1$ and $P^2$. Evidently, $(X_n^1, X_n^2)$ converges pointwise to $(X^1, X^2)$ and hence $(X_n^1, X_n^2)$ converges in distribution to $(X^1, X^2)$. *Q.E.D.*

Step 6: If $\hat{G}_n^i$ converges to $\hat{G}^i$ for $i = 1, 2$ and

$$\hat{G}^1(t) = \hat{G}^2(t) = \hat{G}_n^1(t) = \hat{G}_n^2(t) = 0 \qquad \forall t < 0$$

$$\hat{G}_n^i(t) = \hat{G}^i(t) = 1 \qquad \forall t \geq T$$

and $\hat{G}^1$ and $\hat{G}^2$ have no common points of discontinuity, then

$$\lim U^i\left(\left(1 - z^1\right)\hat{G}_n^1, \left(1 - z^2\right)\hat{G}_n^2\right) = U^i\left(\left(1 - z^1\right)\hat{G}^1, \left(1 - z^2\right)\hat{G}^2\right)$$

where the utility functions have been defined in Section 3. In particular, if $t_n \to t$, and $t$ which is a continuity point of $\hat{G}^2$ then $\lim U^1\left(t_n, (1 - z^2)\hat{G}_n^2\right) = U^1\left(t, (1 - z^2)\hat{G}^2\right)$ and if $t$ is a continuity point of $\hat{G}^1$, then $\lim U^2\left((1 - z^1)\hat{G}_n^1, t_n\right) = U^2\left((1 - z^1)\hat{G}^1, t\right)$. (Recall that the arguments $\tau = t, t_n$ etc. are shorthand for the degenerate strategy in which the rational type concedes with probability 1 at $\tau$.)

Proof:

$$\text{Let } h^1(t_1, t_2) \begin{array}{ll} = 1 - \alpha^2 & \text{if } t_1 < t_2 \\ = \alpha^1 & \text{if } t_1 > t_2 \\ = \dfrac{1 - \alpha^2 + \alpha^1}{2} & \text{if } t_1 = t_2 \end{array}$$

then simple manipulation of the definition of $U^1$ in Section 3 yields

$$U^1\left(\left(1-z^1\right)\hat{G}^1,\left(1-z^2\right)\hat{G}^2\right)=\left(1/\left(1-z^1\right)\right)\int h^1 d\left(\hat{G}^1\times\hat{G}^2\right)$$

Since $\hat{G}^i_n$ converges in probability to $\hat{G}^i$, Step 5 implies that $\hat{G}^1_n\times\hat{G}^2_n$ the product measure converges to $\hat{G}^1\times\hat{G}^2$. Since $\hat{G}^1$ and $\hat{G}^2$ have no common points of discontinuity, the set $D_h:=\{(t_1,t_2)\,|t_1=t_2\}$ has zero $G^1\times G^2$ measure. Since $D_h$ is the set of discontinuity points of $h$, Step 4 yields the desired result. *Q.E.D.*

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The following completes the proof of the proposition: For any $t>0$ and $\epsilon>0$, let $\tilde{\sigma}^1_n$ be a strategy in $g_n$ in which player 1 behaves according to $\sigma^1_n$ until time $t_n$ where $t_n$ is the last time player 2 makes an offer prior to $t+\bar{\epsilon}$ (for some $\bar{\epsilon}>0$) and at time $t_n$ player 1 accepts $1-\alpha^2$. Let $U^1_n$ denote the utility function of player 1 in the game $g_n$.Then, there exists integers $N^1,N^2,N^3$ and $\bar{\epsilon}>0$ sufficiently close to 0, such that $t+\bar{\epsilon}$ is a continuity point of $F^2$ and

(5) $U^1\left(t,F^2\right)-U^1\left(t+\bar{\epsilon},F^2\right)<\epsilon$

(6) $U^1\left(t+\bar{\epsilon},F^2\right)-U^1\left(t_n,F^2_n\right)<\epsilon \qquad \forall n\geq N^1$

7) $U^1\left(t_n,F^2_n\right)-U^1_n\left(\tilde{\sigma}^1_n,\sigma^2_n\right)<\epsilon \qquad \forall n\geq N^2$

(8) $U^1_n\left(\tilde{\sigma}^1_n,\sigma^2_n\right)-U^1_n\left(\sigma^1_n,\sigma^2_n\right)\leq 0 \qquad \forall n$

(9) $U^1_n\left(\sigma^1_n,\sigma^2_n\right)-U^1\left(F^1_n,F^2_n\right)<\epsilon \qquad \forall n\geq N^2$

(10) $U^1\left(F^1_n,F^2_n\right)-U^1\left(F^1,F^2\right)<\epsilon \qquad \forall n\geq N^3$

Equation (5) follows immediately from the definition of $U^1$. That is, $U^1(\,\cdot\,,F^2)$ is continuous at continuity points of $F^2$ and if $t$ is not a continuity point of $F^2$ then for $\bar{\epsilon}$ small the left-hand side of 1 is strictly negative. Since $t+\bar{\epsilon}$ is a continuity point of $F^2$, (6) follows from Step 6. Equation (7) follows from the definition of $\tilde{\sigma}^1_n$. Equation (8) is the consequence of the fact that $(\sigma^1_n,\sigma^2_n)$ is an equilibrium. Equation (9) is an application of Lemma 1; player $i$ can never get more than $1-\alpha^j$ after revealing herself. Moreover, since her opponent makes offers frequently, she can reveal herself to be rational in a manner that guarantees $1-\alpha^j$. Equation (10) follows from Steps (7) and (10). Choosing $n \geq \max\{N^1, N^2, N^3\}$ and adding equations (5) - (10) yields

$$U^1\left(t, F^2\right) - U^1\left(F^1, F^2\right) < 5\epsilon$$

Since this is true for any $\epsilon > 0$, it must be that

$$U^1\left(t, F^2\right) - U^1\left(F^1, F^2\right) \leq 0.$$

Hence $F^1$ is a best response to $F^2$ and by a symmetric argument $F^1, F^2$ is the Nash equilibrium of the continuous-time game. To conclude the proof, note that if player $i$ is the first to reveal himself to be rational, she can guarantee $1-\alpha^j$ by accepting $j's$ offer. This would yield $j$ utility $\alpha^j$. If $i$ reveals herself to be rational in some other way then, by Lemma 1, $j$ is still, in the limit, guaranteed $\alpha^j$. Thus, the first player $i$ to reveal herself to be rational receives $1-\alpha^j$ and her opponent receives $\alpha^j$. This can only happen if agreement is reached immediately at these terms. Hence, convergence in expected payoffs implies convergence in distribution to $\tilde{\theta}$. *Q.E.D.*

**Proof of Proposition 5:** Without loss of generality, we will assume that $\left(\mu^1_n, \mu^2_{\alpha^2,n}\right)$ converges to some $(\mu^1, \mu^2_{\alpha^2})$.

Assume $\mu^1(\alpha^1) > 0$ and $\alpha^1 > \frac{r^2}{r^1+r^2}$ and $\alpha^2 < \frac{r^1}{r^1+r^2}$. The key observation is the following: If $\alpha^1, \alpha^2$ are demanded at time zero, 1 must concede to 2 with unconditional probability $\mu^1(\alpha^1)$. To see this, recall that $b^1$, the conditional probability that 1 does not concede to 2 must solve

$$\frac{\log \bar{\pi}^1(\alpha^1)}{r^2(1-\alpha^1)} = \frac{\log \bar{\pi}^2_{\alpha^1}(\alpha^2)}{r^1(1-\alpha^2)}$$

where $\bar{\pi}^1(\alpha^1) = \frac{z^1\pi^1\left(\alpha^1\right)}{z^1\pi^1(\alpha^1)+(1-z^1)b^1}$ and $\bar{\pi}^2_{\alpha^1}(\alpha^2) = \frac{z^2\pi^2\left(\alpha^2\right)}{z^2\pi^2(\alpha^2)+(1-z^2)\mu^2_{\alpha^1}(\alpha^2)}$

that is $\frac{\gamma^1}{\gamma^2} = \frac{\log\left(1+\frac{\left(1-z^1\right)b^1}{z^1\pi^1\left(\alpha^1\right)}\right)}{\log\left(1+\frac{(1-z^2)\mu^2_{\alpha^1}\left(\alpha^2\right)}{z^2\pi^2\left(\alpha^2\right)}\right)}$ where $\gamma^i = \frac{r^i}{1-\alpha^i}$. Since $\alpha^1 > \frac{r^2}{r^1+r^2}$ and $\alpha^2 < \frac{r^1}{r^1+r^2}$, we have $\frac{\gamma^1}{\gamma^2} > 1$.

But since $\mu^1(\alpha^1) > 0$ and $z^1$ and $z^2$ are converging to 0 at the same rate, $b^1$ must converge to 0 as well. If the conditional probability of 1 not conceding after $(\alpha^1, \alpha^2)$ is realized, is going to 0, the unconditional probability of conceding must go to $\mu^1(\alpha^1)$.

Thus, by choosing any $\alpha^2 < \frac{r^1}{r^1+r^2}$ player 2 can guarantee that his opponent concedes immediately if he is rational and has initially demanded $\alpha^1 > \frac{r^2}{r^1+r^2}$. If 1 has demanded $\alpha^1 < \frac{r^2}{r^1+r^2}$ then 2 can guarantee at least $\frac{r^1}{r^1+r^2}$ by accepting this demand. Hence 2 can guarantee a payoff of $\underline{v}^2 = \max\left\{\alpha^2 \in C^2 \mid \alpha^2 < \frac{r^1}{r^1+r^2}\right\}$. A similar argument establishes the player 1 can guarantee $\underline{v}^1$. *Q.E.D.*

**Proof of Proposition 6**

Consider the following artificial constant-sum game: $i$ chooses $\alpha^i \in C^i \subset (0,1)$; $i/\alpha^i$ wins iff

$$\frac{r^i}{1-\alpha^i} < \frac{r^j}{1-\alpha^j}.$$

Note that by the genericity assumption there are no ties. We will consistently assume $j \neq i$. The payoff to $i$, if he wins, is $\alpha^i$, and $j$'s payoff, if he loses, is $1-\alpha^i$.

Let

$$\tilde{\alpha}^j(\alpha^i) = \min\{\{\alpha^j \in C^j \mid \alpha^j > 1-\alpha^i\} \cup \{\max C^j\}\}$$
$$\hat{\alpha}^i = \max\left\{\{\alpha^i \in C^i \mid \alpha^i + \tilde{\alpha}^j(\alpha^i) > 1 \text{ and } \alpha^i \text{ beats } \tilde{\alpha}^j(\alpha^i)\} \cup \{0\}\right\}$$
$$\hat{\alpha}^i_+ = \min\left\{\{\alpha^i \in C^i \mid \alpha^i > \hat{\alpha}^i\} \cup \{\max C^i\}\right\}$$

We will demonstrate that our artificial constant-sum game has a *pure* strategy equilibrium $(\bar{\alpha}^1, \bar{\alpha}^2)$. If $\bar{\alpha}^i$ is the winner in this equilibrium, we set $\alpha^i_c = \bar{\alpha}^i$ and $\alpha^j_c = (1-\bar{\alpha}^i)$. Furthermore, in this equilibrium $\bar{\alpha}^1 + \bar{\alpha}^2 > 1$, a fact which simplifies the final step of the proof.

The argument is as follows: By assumption there exists $(\alpha^1, \alpha^2) \in C^1 \times C^2$ such that $\alpha^1 + \alpha^2 > 1$. Furthermore, we may assume that this pair satisfies $\alpha^j = \tilde{\alpha}^j(\alpha^i)$, $i = 1, 2$. Suppose $\alpha^1$ beats $\alpha^2$. Then clearly $\hat{\alpha}^1 > 0$. Thus $\hat{\alpha}^i > 0$ for some $i = 1, 2$.

Suppose $\hat{\alpha}^1 > 0$ and $\left(\hat{\alpha}^1, \tilde{\alpha}^2\left(\hat{\alpha}^1\right)\right)$ is *not* an equilibrium. Then it must be the case that $\hat{\alpha}^1_+ > \hat{\alpha}^1$ and $\hat{\alpha}^1_+$ beats $\tilde{\alpha}^2(\hat{\alpha}^1)$. By the definition of $\hat{\alpha}^1$, $\tilde{\alpha}^2(\hat{\alpha}^1_+)$ beats $\hat{\alpha}^1_+$. Let $\alpha^{2*} = \max\left\{\alpha^2 \in C^2 \mid \alpha^2 \text{ beats } \hat{\alpha}^1_+\right\}$. Clearly $(\hat{\alpha}^1_+, \alpha^{2*})$ is an equilibrium. Thus, we have demonstrated the existence of an equilibrium with the required properties.

To complete the proof, suppose that $\bar{\alpha}^1$ beats $\bar{\alpha}^2$. By the argument above $\alpha^1 > \bar{\alpha}^1$ implies

$\frac{r^1}{1-\alpha^1} > \frac{r^2}{1-\bar{\alpha}^2}$

and hence from the proof of Proposition 5, it follows that if $\mu^1(\alpha^1) > 0$ and $\alpha^1 > \bar{\alpha}^1$, conditional on $\alpha^1$ being demanded in period 0, 2 can guarantee himself at least $\bar{\alpha}^2 > 1-\bar{\alpha}^1$. This means that 1's payoff is strictly less than $\bar{\alpha}^1$. Again, by the argument of Proposition 5, by demanding $\bar{\alpha}^1$, 1 can guarantee $\bar{\alpha}^1$ and also by demanding less than $\bar{\alpha}^1$, 1 guarantees that her payoff will be less than $\bar{\alpha}^1$. So $\mu^1(\bar{\alpha}^1) = 1$. Since 1 guaranteed herself $\bar{\alpha}^1$, 2 cannot get more than $1-\bar{\alpha}^1$ which he can get by accepting 1's initial offer. Hence, the equilibrium payoffs converge to $(\bar{\alpha}^1, 1-\bar{\alpha}^1)$. The argument for the case where $\bar{\alpha}^2$ is the winner, is very similar, and is omitted. *Q.E.D.*

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