The English Auction with Differentiated Commodities

Faruk Gul
Princeton University
and
Ennio Stacchetti
University of Michigan

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1. Introduction

A dynamic auction can be described as a rule for adjusting prices given the observed history of demand (i.e. bids) and a rule for terminating the price adjustment procedure and specifying an allocation (i.e. determining who gets the good(s) and at what prices). The English auction is also identified with the property that prices are non-decreasing. More specifically, the English auction is typically identified with the procedure of increasing the prices as long as there is excess demand.

It is possible to study the English auction from two separate dimensions: as a method of eliciting demand information and also as a strategic game. The popularity of the English or ascending bid auction both in theory and in practice is likely to derive from the attractive characteristics it has along both of these dimensions. Without the requisite strategic properties the desired outcome would not be attained. Conversely, if the English auction was not an "economical" method for getting the desired information, it would be less attractive than the associated direct mechanism since, by the revelation principle, it cannot have better strategic properties than the latter.

The English auction, as a rule for allocating a single object, has been studied extensively. It is known to be an efficient (with respect to revenue or consumer surplus) incentive mechanism under various assumptions. Thus, in the single unit case, the equilibrium behavior in the English auction extracts the necessary preference information and leads to "optimal" allocations (see, for example, Milgrom [9]). When all agents know their valuations (i.e., in the private values case), truthfully revealing one's demand is a dominant strategy. Moreover, the equilibrium, when all agents use their dominant strategies, results in a Walrasian allocation and the revenue collected is the value of the object at the smallest Walrasian price.

Related results have been established in a number of different settings. With multiple goods and unit demand consumers (that is, consumers that get satiated with the consumption of one good), Leonard [8] has shown that the Walrasian allocation with the corresponding minimal prices is incentive compatible. Demange, Gale and Sotomayor [2] have designed an English auction for this setting. They utilize a linear programming algorithm to develop an ascending price auction that stops at the smallest Walrasian price vector. Thus, their auction is a dynamic rule that implements the direct mechanism studied by Leonard [8]. Demange, Gale and Sotomayor [2] do not consider the possibility of strategic behavior in the dynamic mechanism. Nevertheless, Leonard's result for the associated direct mechanism ensures that truthful revelation of demand constitutes a Nash equilibrium of their dynamic game.

More recently, Ausubel [1] has studied the English auction in the setting with multiple units of a *homogeneous* good and consumers with decreasing marginal utilities. In this framework, increasing prices as long as there is excess demand also leads to the smallest Walrasian price vector. More importantly, by keeping track of the level of excess demand, Ausubel [1] is able to recover the Vickrey [11] payments associated with an efficient allocation.

Our work is closely related to and builds upon Kelso and Crawford's [7] study of the core of a matching problem. In their setting, a firm can be matched with multiple workers. However, Kelso and Crawford's model can be reinterpreted so that each firm corresponds to a consumer and the workers correspond to goods. With this interpretation, the core of the matching problem corresponds to the set of Walrasian equilibria of the associated exchange economy, and the algorithm they propose corresponds to a dynamic auction. The key ingredient in Kelso and Crawford's analysis is their gross substitutes (GS) condition. It is easy to verify that Kelso and Crawford's framework includes as a special case the settings of all the papers mentioned above. Nevertheless, there are significant differences in the results offered. First, while Ausubel's and Demange, Gale and Sotomayor's dynamic auctions are easily seen to be generalizations of the English auction for a single object, Kelso and Crawford's mechanism is different: their economy is discretized (i.e., there is a small unit of currency), the consumers make bids on the objects they desire rather than announcing their demand, and there are constraints on the bids agents can make. Second, as noted above, the dynamic price adjustment rules investigated by Demange, Gale and Sotomayor [2] and Ausubel [1] reveal enough information to compute the Vickrey-Clarke-Groves payments. Hence, both of these auctions have the desired strategic properties. In contrast, Kelso and Crawford [7] does not investigate strategic behavior and assumes truthful revelation of demand.

In this paper, we study economies with quasilinear preferences satisfying the (GS) condition. We assume that individuals are endowed with a sufficient quantity of the divisible good, "money", so that they are able to purchase as many of the indivisible goods as they wish. We utilize properties of (GS) preferences that we have proven in a companion paper (Gul and Stacchetti [5], hereafter G&S) to develop an alternative "auction" (or algorithm) to the one studied by Kelso and Crawford [7]. The key element in our analysis is the notion of an excess demand set. With multiple discrete goods, determining which goods are in excess demand is not an easy task. When preferences satisfy the (GS) condition, it is possible to define a criterion for excess demand such that at every price p, there is a feasible allocation (a distribution of the goods that satisfies all demands) if and only if there is no excess demand. Moreover, we show that increasing the prices of all the goods that are in excess demand eventually leads to the smallest Walrasian price vector. Our notion of excess demand is related to the idea of an over-demanded set which has been used to study one-to-one matching problems. Hence, our auction turns out to be the appropriate generalization of the Demange, Gale and Sotomayor [2] auction from settings with unit demand preferences to settings with (GS) preferences .

Finally, we investigate the strategic properties of our auction. Here, our main finding is a negative result. We show that truthful revelation of demand is a perfect Bayesian equilibrium in our auction if the smallest Walrasian price vector corresponds to the Vickrey-Clarke-Groves payments. The latter holds in the case of unit demand consumers but cannot be guaranteed without a joint restriction on the set of preferences in the case of (GS) preferences. More importantly, we show that no dynamic auction can reveal sufficient information to implement the Vickrey mechanism if all (GS) preferences are allowed. Thus, the unit demand case of Demange, Gale and Sotomayor [2] and the multiple homogeneous goods case of Ausubel [1] are the most general environments for which generalizations of the English auction can be used to implement efficient, strategy-proof allocations.

2. An Example

In this paper we provide a generalization of the English auction to the case in which multiple objects are to be sold simultaneously, and each agent may wish to consume a bundle of different objects.

The following example illustrates our dynamic procedure. There are two agents, agents 1 and 2, and two objects, a and b. Agents' utilities for the various subsets of $\{a, b\}$ are given by the following table:

	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
u_1	0	8	9	12
u_2	0	6	8	14

Table I

We wish to mimic the English auction by announcing prices and asking the agents to declare their demand. Specifically, let us announce $p_a = p_b = 0$. At these prices, we find out both agents demand $\{a, b\}$. We conclude that both a and b are in excess demand. We raise the price of both objects. At prices $p_a = p_b = 3$, the demand of agent 1 changes. Now, he wants to purchase either element of the set $D_1(3, 3) = \{\,\{b\}, \{a, b\}\,\}$. Moreover, if the price of a increases any further, he will no longer want to buy $\{a, b\}$. At this stage, we conclude that only b is in excess demand, since it is possible to satisfy agent 1 without a, and there is only one other agent. So we raise the price of b, while keeping constant the price of a at 3. At $p = (3, 4)$, the demand of the first agent becomes $D_1(3, 4) = \{\,\{a\}, \{b\}, \{a, b\}\,\}$. Note that if the prices of a and b are raised simultaneously by a small amount, agent 1's demand becomes $\{\,\{a\}, \{b\}\,\}$, while agent 2's only preferred bundle remains $\{a, b\}$. Thus, if we had an extra unit of a or b, we could satisfy the demand of both agents. We conclude that both a and b are in excess demand, although the level of excess demand is a single unit. Thus, we increase the prices of both objects at the same rate. The next change in demand occurs at price $p = (6, 7)$. Then, the demand of agent 2 becomes $D_2(6, 7) = \{\,\{b\}, \{a, b\}\,\}$. Note that now it is possible to satisfy both agents' demands. Thus the equilibrium outcome of the auction is to allocate a to agent 1 and b to agent 2 at prices 6 and 7 respectively.

A number of features of the "algorithm" above are worth noting. First, observe that at prices $(3, 3)$, there was only one agent interested in consuming object a, and we decided that a was not in excess demand. Only agent 1 insists in consuming a at prices $(3, 4)$ as well. Yet, in that case we concluded that a *was* in excess demand. The key distinction between the two cases is that at prices $(3, 3)$, a second unit of a would not have helped in satisfying both agents' demands, whereas the additional unit would resolve the problem at prices $(3, 4)$. An important step in our specification of the auction is the construction of a criterion for identifying the objects that are in excess demand. A useful criterion for excess demand should have the property that it is possible to satisfy all demands if and only if no object is in excess demand.

The second important feature of this example is that the algorithm stops at prices that are Walrasian. Indeed, $(6,7)$ are the *smallest* Walrasian prices. This is a general feature of our auction.

The procedure developed in this paper "works" if preferences and cost functions satisfy the (GS) condition. That is, with the (GS) condition, the notion of excess demand we offer has the property that demands can be met if and only if no good is in excess demand. Furthermore, increasing of all goods in excess demand, eventually leads to the smallest Walrasian prices. To establish these results, we use theorems from submodular optimization. In particular, the classic result of Edmonds [3] on matroid partitioning plays an important role in our analysis.

3. Utility Functions and Demand Correspondences

In this section we restate definitions and results about the agents' preferences, first introduced in our companion paper G&S which are useful for the current analysis. We then present characterizations of the demand correspondences. As in G&S, we study here economies with a *finite* number of objects. $\Omega = \{\omega_1, \ldots, \omega_m\}$ denotes the set of objects. A *bundle* is any subset B of Ω; the set of all bundles is

$$2^\Omega := \{B \mid B \subset \Omega\}.$$

A price vector $p \in \mathbf{R}^m_+$ contains a price for each *object* in Ω.

Definition: A map $u : 2^\Omega \to \mathbf{R}$ is called a *utility function* on Ω. A utility function assigns a value to each *bundle* of Ω. For each price vector $p \in \mathbf{R}^m_+$ and utility function u we associate the *net utility function* $v : 2^\Omega \times \mathbf{R}^m_+ \to \mathbf{R}$, which is defined by

$$v(A,p) := u(A) - < p, A >, \text{ where } \quad < p, A > := \sum_{a \in A} p_a$$

(and by convention, $< p, \emptyset > := 0$). A utility function u represents *unit demand preferences* if for each $A \subset \Omega$,

$$u(A) = \max_{a \in A} u(\{a\}).$$

Without loss of generality, we normalize every utility function so that $u(\emptyset) = 0$.

Definition: Let A, B, and C be any three bundles. Then $\#(A)$ denotes the number of elements in A,

$$A \triangle B := [A \backslash B] \cup [B \backslash A]$$

is the *symmetric difference* between A and B, $\#(A \triangle B)$ is the *Hausdorff distance* between A and B, and

$$[A, B, C] := (A \backslash B) \cup C.$$

If B is a singleton $\{b\}$, we write $[A, b, C]$ instead of $[A, \{b\}, C]$ (and similarly if C is a singleton).

DEFINITION: If $u : 2^\Omega \to \mathbf{R}$ is a utility function, its *demand correspondence* and *minimal demand correspondence* $D, D^* : \mathbf{R}_+^m \to 2^\Omega$ are respectively defined by

$$D(p) := \{A \subset \Omega \,|\, v(A,p) \geq v(B,p) \text{ for all } B \subset \Omega\},$$
$$D^*(p) := \{A \in D(p) \,|\, \#(A) \leq \#(B) \text{ for all } B \in D(p)\}.$$

At any given price p, $D(p)$ is simply the collection of optimal (i.e., net utility maximizing) sets. In the rest of this section, we are concerned with alternative ways of representing this collection.

DEFINITION: A utility function $u : 2^\Omega \to \mathbf{R}$

(i) is *monotone* if for all $A \subset B \subset \Omega$, $u(A) \leq u(B)$.

(ii) satisfies the *gross substitutes* (GS) condition if for any two price vectors p and q such that $q \geq p$, and any $A \in D(p)$, there exists $B \in D(q)$ such that $\{\, a \in A \mid p_a = q_a \,\} \subset B$.

(iii) has *no complementarities* (NC) if for all price vector p, and bundles $A, B \in D(p)$ and $X \subset A \backslash B$, there exists $Y \subset B \backslash A$ such that $[A, X, Y] \in D(p)$.

(iv) has the *single improvement* property (SI) if for all price vector p and bundle $A \notin D(p)$, there exists a bundle B such that $v(B,p) > v(A,p)$, $\#(A \backslash B) \leq 1$, and $\#(B \backslash A) \leq 1$.

In G&S we prove that for monotone utility functions, (GS), (NC) and (SI) are equivalent. For a discussion of these and related properties, the reader should look at G&S. As in G&S, property (SI) is particularly useful to prove many of our main results. In particular, (SI) is closely related to the main property of a matroid (see property (I2) in Appendix 1).

Consider a consumer with utility function u. At a given price vector p and for each bundle A, we want her to determine the minimal number of objects in A that she would need to construct any of her optimal consumption bundles. To minimize her requirement, she will choose an optimal bundle which minimizes the intersection with A.

DEFINITION: For any utility function u, its corresponding *requirement function* $K : 2^\Omega \times \mathbf{R}_+^m \to \mathbf{N}$ is defined by

$$K(B,p) := \min_{A \in D(p)} \#(A \cap B).$$

Conversely, it turns out that the requirement function partially characterizes the demand correspondence. Clearly, if $A \in D(p)$, then $\#(A \cap B) \geq K(B,p)$ for all $B \subset \Omega$. Thus, an arbitrary set $A \subset \Omega$ is a candidate for an optimal set only if it satisfies the previous inequality for all $B \subset \Omega$. Not all sets A that satisfy this condition are optimal. However, any such set must contain an optimal set.

LEMMA 1: *For any price vector p, let*

$$\hat{D}(p) := \{A \subset \Omega \,|\, \#(A \cap B) \geq K(B,p) \textit{ for all } B \subset \Omega\}.$$

Then, for every $A \in \hat{D}(p)$ there exists $C \in D(p)$ such that $C \subset A$.

PROOF: Let $A \subset \Omega$ be such that for each $B \in D(p)$, $B \not\subset A$. That is, for each $B \in D(p)$, $A^c \cap B \neq \emptyset$, and we can choose an element $a_B \in A^c \cap B$. Let $C := \{a_B \mid B \in D(p)\}$. Then, $\#(A \cap C) = 0$ and $K(C, p) > 0$, because for each $B \in D(p)$, $C \cap B$ contains at least a_B, and therefore is nonempty. Hence, $K(C, p) > \#(A \cap C)$, and $A \notin \hat{D}(p)$. ∎

Many proofs use the technique of raising (or lowering) the prices in a given bundle. Therefore, the following notation will be used often.

DEFINITION: The *characteristic vector* e^A of a bundle A is the m-dimensional vector whose coordinates are $e^A_a = 1$ for each $a \in A$ and $e^A_a = 0$ otherwise. If A is a singleton $\{a\}$, we will sometimes write e^a instead of e^A.

LEMMA 2: *Suppose u has the (SI) property, and fix any price vector p. Then, for any $A \in D(p)$, there exists $B \in D^*(p)$ such that $B \subset A$. Thus, for each bundle C,*

$$K(C, p) = \min_{B \in D^*(p)} \#(B \cap C).$$

PROOF: Pick any $A \in D(p) \backslash D^*(p)$ and $B \in D^*(p)$. We show that if $\#(B \backslash A) > 0$, there exists $C \in D^*(p)$ such that $\#(C \backslash A) = \#(B \backslash A) - 1$. Suppose $b \in B \backslash A$. Define the price vector $\hat{p}$ by: $\hat{p}_a = p_a$ for $a \in A \cup B$ and $\hat{p}_a = M > \max\{\, u(A) \mid A \subset \Omega \,\}$ otherwise. Clearly, $A, B \in D(\hat{p}) \subset D(p)$. For each $\epsilon \geq 0$, let $q(\epsilon) := \hat{p} + \epsilon e^b$. Then, $v(A, q(\epsilon)) > v(B, q(\epsilon))$ for all $\epsilon > 0$. Since 2^Ω is finite, (SI) implies that there exists C such that $\#(B \backslash C) \leq 1$, $\#(C \backslash B) \leq 1$, and $v(C, q(\epsilon)) > v(B, q(\epsilon))$ for all $\epsilon > 0$ sufficiently small. Obviously, $C \subset A \cup B$ and $b \notin C$, and by continuity, $C \in D(\hat{p})$. Therefore, $\#(C) \leq \#(B)$, so $C \in D^*(p)$, and $\#(C \backslash A) = \#(B \backslash A) - 1$.

The previous argument shows that

$$\min\{\, \#(B \backslash A) \mid B \in D^*(p) \,\} = 0.$$

Thus, any B that attains the min satisfies $B \subset A$. ∎

DEFINITION: $\rho^\# : 2^\Omega \to \mathbf{N}$ is a *dual rank function*[1] on Ω if for every two bundles A and B, it satisfies the following properties:

(i) $\rho^\#(\emptyset) = 0$ and $\rho^\#(A \cup B) \leq \rho^\#(A) + \#(B)$ (growth bound).

(ii) $\rho^\#(A) \geq \rho^\#(B)$ for all $A \supset B$ (monotonicity).

(iii) $\rho^\#(A \cup B) + \rho^\#(A \cap B) \geq \rho^\#(A) + \rho^\#(B)$ (supermodularity).

Theorem 1 below establishes that for each p, $K(\cdot, p)$ is a dual rank function. Property (iii) (supermodularity) plays a central role in the extension of Hall's Theorem [6] (see Theorem 3 below) and the definition of the algorithm in Section 3. The proof of Theorem 1, as well as those of Theorems 2 – 4 and Lemmas 6 – 7 below, are deferred to Appendix 2.

[1] The notion of a rank function and its dual is standard in matroid theory – see Appendix 1 for a definition.

Theorem 1: *Suppose u is monotone and has the (SI) property. Then, for each $p \in \mathbf{R}_+^m$, $\rho^\# := K(\cdot, p)$ is a dual rank function on Ω.*

The next theorem partially justifies the definition of a utility function with no complementarities. It is also key for the construction of the English auction. Fix a bundle A and starting from a price vector q, raise the prices of some of the objects not in A to obtain the price vector p. Then, a consumer with a utility function that has no complementarities will (weakly) increase her requirement on A and (weakly) decrease her requirement on A^c.

Theorem 2: *Suppose u has the (SI) property. Let p and q be two price vectors, and A be a bundle such that $p \geq q$ and $p_a = q_a$ for each $a \in A$. Then*

(i) $K(A, p) \geq K(A, q)$, *and*

(ii) $K(A^c, p) \leq K(A^c, q)$.

Remark: Equivalently, part (ii) can be restated as follows. Let B be a bundle, and p and q be two price vectors such that $p \geq q$ and $p_a = q_a$ for each $a \in B^c$. Then $K(B, p) \leq K(B, q)$.

4. Walrasian Equilibrium

We confine our analysis to an economy $E = (\Omega; u_1, \dots, u_n)$ with a finite collection of objects $\Omega = \{\omega_1, \dots, \omega_m\}$ and a finite collection of consumers $N := \{1, \dots, n\}$. In addition to the objects in Ω, there is a divisible commodity (money). Each consumer i has a quasilinear preference represented by the function $U_i(A, t) = u_i(A) + t$, $A \subset \Omega$ and $t \in \mathbf{R}$, where u_i is a utility function on Ω and t is an amount of money (consumer i has for the consumption of other goods). We assume that agents are endowed with a sufficient amount of money to guarantee that they can purchase as many of the indivisible goods as they may wish. This will be ensured, for example, if each agent's endowment of money y is greater than his utility of the aggregate endowment, $u(\Omega)$. The economy has free disposal, and we let $N_0 := N \cup \{0\}$. For each i, v_i and K_i denote respectively consumer i's surplus function and requirement function.

Definition: $\mathbf{X} = (X_0, \dots, X_n)$ is a *partition* (or *allocation*) of Ω if (i) $X_i \subset \Omega$ for each i; (ii) $X_i \cap X_j = \emptyset$ for all $i \neq j$; and (iii) $\bigcup_{i \in N_0} X_i = \Omega$. The possibility that $X_i = \emptyset$ for some i is allowed.

A partition $\mathbf{X}$ has the following interpretation: for each $i \in N$, X_i represents the bundle consumed by agent i, and X_0 represents the set of objects that are not consumed by anybody (freely disposed).

Definition: A *Walrasian Equilibrium* of the economy $E = (\Omega; u_1, \dots, u_n)$ is a tuple $(p, \mathbf{X})$, where $p \in \mathbf{R}_+^m$ is a vector of prices, and $\mathbf{X} = (X_0, \dots, X_n)$ is a partition of Ω such that (i) $< p, X_0 > = 0$, and (ii) for each $i \in N$, $v_i(X_i, p) \geq v_i(A, p)$ for all bundle $A \subset \Omega$.

Definition: A price vector p *supports* a partition $\mathbf{X}$ if $v_i(X_i, p) \geq v_i(B, p)$ for each bundle B and consumer i. The price vector p *supports* a bundle A, if p supports a partition $\mathbf{X}$ such that $X_0 = A^c$.

Note that $(p, \mathbf{X})$ is a Walrasian equilibrium for E iff p supports $\mathbf{X}$ and $< p, X_0 > = 0$.

It follows from Theorem 3 in Kelso and Crawford [7] that if each u_i is monotone and satisfies the (GS) condition, then the economy $E = (\Omega; u_1, \ldots, u_n)$ has a Walrasian equilibrium. In G&S we show that (GS) is equivalent to (SI). Theorem 3 below provides an alternative characterization of Walrasian equilibrium in terms of the requirement functions K_i, $i \in N$. This is a generalization of Hall's Theorem (see Hall [6]) for economies with unit demand consumers. The algorithm described in the next section is stated in terms of requirement functions, and thus this alternative characterization is important to show the convergence of the algorithm.

THEOREM 3: *Suppose each u_i, $i \in N$, is monotone and has the (SI) property. Then, for a given price vector p, $K_N(A,p) := \sum_{i \in N} K_i(A,p) \leq \#(A)$ for all $A \subset \Omega$ iﬁ there exists a partition $(B_0, \ldots, B_n)$ of Ω supported by p such that $K_i(A,p) \leq \#(A \cap B_i)$ for each $i \in N$ and $A \subset \Omega$.*

COROLLARY: *Suppose each u_i, $i \in N$, is monotone and has the (SI) property. Then, $K_N(A,p) \leq \#(A)$ for all bundles A iﬁ there exists a partition $\mathbf{B} = (B_0, \ldots, B_n)$ supported by p.*

PROOF: Suppose p supports the partition $\mathbf{B}$. Then, $B_i \in D_i(p)$ for each $i \in N$, and $K_i(A,p) \leq \#(A \cap B_i)$ for all $i \in N$ and bundle A. Thus,

$$K_N(A,p) \leq \sum_{i \in N} \#(A \cap B_i) = \#(A \cap B_0^c) \leq \#(A).$$

Conversely, assume that $K_N(A,p) \leq \#(A)$ for all bundles A. By Theorem 3, there exists a partition $\mathbf{Y}$ such that $K_i(A,p) \leq \#(A \cap Y_i)$ for each $i \in N$ and bundle A. By Lemma 1, for each $i \in N$ there exists $B_i \in D_i(p)$ such that $B_i \subset Y_i$. Let $B_0 := [\bigcup_{i \in N} B_i]^c$. Then p supports the partition $\mathbf{B} = (B_0, \ldots, B_n)$. ∎

5. THE ENGLISH AUCTION

In G&S we show that the set of Walrasian equilibrium prices for the economy $E = (\Omega; u_1, \ldots, u_n)$ is a complete lattice if E satisfies the (GS) property, and denote by $\underline{p}$ the smallest such price. In this section we propose a tâtonnement process that, starting with all prices equal to 0, converges to $\underline{p}$ in a *finite* number of steps.

ASSUMPTION: In this section we assume throughout that each u_i is monotone and has the (SI) property, $i \in N$.

By the Corollary of Theorem 3, a necessary condition for p to be a Walrasian equilibrium price is that $K_N(A,p) - \#(A) \leq 0$ for all $A \subset \Omega$.

DEFINITION: Let $f : 2^\Omega \times \mathbf{R}^m \to \mathbf{Z}$ be the function

$$f(A,p) := K_N(A,p) - \#(A) \quad \text{for each } A \subset \Omega \text{ and } p \in \mathbf{R}^m,$$

(here $\mathbf{Z}$ denotes the set of integer numbers) and $O : \mathbf{R}^m \to 2^\Omega$ be the correspondence

$$O(p) := \{\, A \subset \Omega \mid f(A,p) \geq f(B,p) \quad \text{for all } B \subset \Omega \,\}.$$

$O(p)$ is the collection of *max-demanded* bundles at prices $p \in \mathbf{R}^m_+$.

The following lemma, due to Ore [10], is well known in the literature.

LEMMA 5: $O(p)$ *is a lattice for each* $p \in \mathbf{R}^m_+$.

PROOF: Fix $p \in \mathbf{R}^m$, and let $A, B \in O(p)$ and $z := f(A,p)$. Since the sum of supermodular functions is supermodular, $K_N(X,p)$ is supermodular in X, and

$$z \geq f(A \cup B, p) \geq f(A,p) + f(B,p) - f(A \cap B, p) \geq 2z - f(A \cap B, p) \geq z.$$

Therefore $z = f(A \cup B, p) = f(A \cap B, p)$. ∎

DEFINITION: For each $p \in \mathbf{R}^m_+$, let $O_*(p)$ and $O^*(p)$ denote respectively the smallest and largest element of $O(p)$. $O_*(p)$ is called the *excess demand set*. Also, let $\delta(p)$ denote the characteristic vector of the excess demand set. That is, $\delta(p) \in \mathbf{R}^m$ has coordinates

$$\delta_a(p) = \begin{cases} 1 & \text{if } a \in O_*(p) \\ 0 & \text{if } a \notin O_*(p). \end{cases}$$

If $f(A,p) > 0$, then at prices p, no matter what bundle each consumer picks from his/her optimal collection, there would be more requests for the objects in A than there are elements in A. Hence, it is not possible to divide among the consumers the objects in the set A in such a way that each consumer can simultaneously start constructing an optimal bundle.

Obviously $f(\emptyset, p) = 0$, and so $f(A,p) \leq 0$ for all $A \subset \Omega$ iff $\emptyset \in O(p)$, and $\emptyset \in O(p)$ iff $O_*(p) = \emptyset$. Starting with $p(0) = 0$, the algorithm specifies a procedure to construct an increasing sequence of prices $\{p(t)\}_{t=0}^T$ such that $O_*(p(t)) \neq \emptyset$ for all $t < T$, but $O_*(p(T)) = \emptyset$. In each iteration t, only the prices of the objects in the excess demand set $O_*(p(t))$ are raised simultaneously.

ALGORITHM: T and the sequence $\{p(t)\}_{t=0}^T$ are defined inductively by the following procedure:

Step 1: $p(0) := 0$, and $t = 0$.

Step 2: If $\delta(p(t)) = 0$, make $T = t$ and stop; otherwise go to Step 3.

Step 3: Define

$$\begin{aligned} \epsilon(t) &:= \sup\{\, s \mid O_*(p(t) + s\delta(p(t)) = O_*(p(t))\,\} \\ p(t+1) &:= p(t) + \epsilon(t)\delta(p(t)). \end{aligned}$$

Increase t by 1, and go to Step 2.

The algorithm stops when $\delta(p(T)) = 0$, or equivalently when $O_*(p(T)) = \emptyset$. We will see that T is finite because $\epsilon(t) > 0$ for all $t < T$ (so $p(t+1) \neq p(t)$ for all $t < T$, and the algorithm does not get stuck in step 3), and $O_*(p(T)) = \emptyset$ for some finite T.

Since each u_i is monotone, $u_i(A) \leq u_i(\Omega) \leq u^*$ for each bundle A, where $u^* := \max_{i \in N} u_i(\Omega)$. Fix $a \in \Omega$. Suppose $p \in \mathbf{R}^m_+$ is any price vector such that $p_a > u^*$ and A is a bundle such that $a \in A$. Then, $v_i(A,p) \leq u_i(\Omega) - p_a < 0 = v_i(\emptyset, p)$, and $A \notin D_i(p)$. Hence, $K_i(A,p) = K_i(A \backslash \{a\}, p)$ for each $i \in N$, and $f(A,p) = f(A\backslash\{a\}, p) - 1$. This implies that $A \notin O(p)$, and thus $a \notin O_*(p)$. Since $p(t) \geq 0$ and $\delta(p(t)) \geq 0$ has at least one coordinate equal to 1 for each $t < T$, it must be that $\epsilon(t) \leq u^*$ for each $t < T$.

LEMMA 6: $O_*(p(t)) \neq \emptyset$ *iff* $\epsilon(t) > 0$.

LEMMA 7: $O_*(p(T)) = \emptyset$ *for some finite* T.

Note that the algorithm stops when $O_*(p(T)) = \emptyset$. That is, it stops when $f(A, p(T)) \leq 0$ for all bundles A. By the Corollary of Theorem 3, there exists a partition $\mathbf{X}$ of Ω supported by $p(T)$. We will see that in fact $p(T)$ is the smallest Walrasian equilibrium price $\underline{p}$. Hence, the partition $\mathbf{X}$ can be chosen to be a Walrasian allocation.

THEOREM 4: $p(T) = \underline{p}$.

We now implement the algorithm as the following *English Auction Game*. To make the implementation easier, let us assume that all utility functions are integer valued. That is, assume that $u_i : 2^\Omega \to \mathbf{N}$, $i \in N$. The rules of the auction are as follows.

Step 1: $p(0) := 0$, and $t = 0$.

Step 2: The players report simultaneously their demands $\tilde{D}_i(p(t))$, $i \in N$, and the seller computes the excess demand set $\tilde{O}_*(p(t))$. If $\tilde{O}_*(p(t))$ is not well defined given the players' reports, make $T = \infty$ and stop. If $\tilde{\delta}(p(t)) = 0$, where $\tilde{\delta}(p(t))$ is the characteristic vector of $\tilde{O}_*(p(t))$, make $T = t$ and stop. Otherwise go to Step 3.

Step 3: Let $p(t+1) := p(t) + \tilde{\delta}(p(t))$. Increase t by 1, and go to Step 2.

If the auction never stops (because the agents keep bidding the prices up forever) or stops with $T = \infty$, each player receives the empty set and pays nothing. Otherwise, the seller finds a partition $\mathbf{X}$ supported by $p(T)$, as insured by Theorem 3 and its Corollary. Then, each player i receives the bundle X_i and pays to the seller $< p(T), X_i >$.

In Step 2 we are considering the possibility that the players do not report their demands honestly (that is, that $\tilde{D}_i(p(t)) \neq D_i(p(t))$), in which case $\tilde{f}(A, p(t))$ may not be supermodular in A, and $\tilde{O}(p(t))$ need not be a lattice. The timing of the auction does not correspond to the timing of the algorithm; the auction moves more slowly because in the notation of the algorithm (assuming the players report honestly), $\epsilon(t)$ may be greater or equal to 2, in which case the Step 3 of the algorithm accomplishes in a single move a change in prices that requires several rounds in the auction.

6. Dynamic Incentive Properties of the English Auction

In the preceding section we did not explore the incentive properties of the English auction. We investigated the consequences of truthful behavior only. If one takes the view that Walrasian equilibrium is an adequate criterion for approximate incentive compatibility or decentralized optimal behavior, then the assumption of honest behavior might be justified. Alternatively, one might wish to seek further restrictions on preferences or specify a different allocation rule to render the auction dynamically incentive compatible.

In this section we will explore the possibility of implementing the Vickrey-Clarke-Groves (VCG) mechanism through a dynamic auction. For any profile of utility functions

$(u_1, \dots, u_n)$ on Ω, subset $I \subset N$, and subset $X \subset \Omega$, let

$$S_I(X) := \max \sum_{i \in I} u_i(X_i)$$
$$\text{s.t.} \quad \bigcup_{i \in I} X_i = X \quad \text{and} \quad X_i \cap X_j = \emptyset \text{ for all } i \neq j.$$

Then, a VCG mechanism Ψ associates with any profile $u = (u_1, \dots, u_n)$ an allocation $\mathbf{X} = (X_1, \dots, X_n)$ and vector of payments $q \in \mathbf{R}^n_+$ such that $\mathbf{X}$ is efficient (i.e., it attains the value $S_N(\Omega)$) and

$$q_i = S_{N \setminus \{i\}}(\Omega) - S_{N \setminus \{i\}}(\Omega \setminus X_i), \quad i \in N.$$

Thus, for each $i \in N$, $\Psi_i(u) = X_i$ is the bundle assigned to player i, and $\Psi_{n+i}(u) = q_i$ is his corresponding payment. It is easy to verify that both the revenue and the agents' utilities are independent of the particular efficient allocation $\mathbf{X}$ chosen. Hence, we speak of *the* VCG mechanism instead of *a* VCG mechanism.

In G&S we show that when every u_i is monotone and satisfies the GS condition, the set of Walrasian equilibrium prices is a lattice. We also prove that a player's payment in the VCG mechanism is never greater than the value of his assigned bundle at the smallest Walrasian prices. That is,

$$\Psi_{n+i}(u) \leq < \underline{p}, X_i >, \quad i \in N.$$

We show by example that these inequalities may be strict. However, if an economy with n agents and m goods is replicated k times, where $k \geq m+1$, then, for the resulting economy, the VCG payments coincide with the values of the allocations at prices $\underline{p}$. For a formal statement of the replicated economy and the result we refer the reader to G&S.

Let $\mathcal{U}$ be any class of monotone and GS preference profiles that satisfies the "no gap" condition:

$$\Psi_{n+i}(u) = < \underline{p}, \Psi_i(u) > \quad \text{for all } i \in N \text{ and } u \in \mathcal{U},$$

where $\underline{p}$ is the smallest Walrasian price vector of the economy $E = (\Omega; u)$. We now show that for this class of preferences, honest behavior is a Bayesian-Nash equilibrium of the sequential English auction, for any probability distribution F over $\mathcal{U}$. Since the VCG mechanism is strategy-proof, it is an optimal strategy for each buyer i to report his type truthfully for any report $u_{-i} \in \mathcal{U}_{-i}$ by his opponents (where $\mathcal{U}_{-i} = \{ u_{-i} \mid (u_i, u_{-i}) \in \mathcal{U} \text{ for some } u_i \}$). Therefore, to report truthfully is also optimal in expectation, when the opponents' report is chosen randomly (perhaps according to the marginal of F). It follows that honesty is incentive compatible in the English auction viewed as a direct mechanism because, for the class of preferences $\mathcal{U}$, it attains the same outcome as the VCG auction. A slightly more delicate observation is that truthful behavior in the English auction is a sequentially rational best response to honest behavior by the opponents. That is, truthful behavior is a perfect Bayesian equilibrium of the English auction with incomplete information.

THEOREM 5: *Suppose that buyers' preferences are drawn randomly from $\mathcal{U}$ according with a probability distribution F. Then, honest report of demands for each price vector is a perfect Bayesian equilibrium of the English auction game (defined in Section 5).*

PROOF: Fix $i \in N$ and the profile $u \in \mathcal{U}$, where u_i is the true utility function of buyer i. Let $\underline{p}$ denote the smallest Walrasian price of the economy $E = (\Omega; u)$. Suppose that player i follows a strategy other than reporting demand truthfully, while his opponents report honestly. First, note that i does not want to report so that the outcome where he receives nothing and pays nothing is achieved, because by reporting honestly he could obtain his allocation in a Walrasian equilibrium, which he (weakly) prefers. Thus, suppose that his strategy leads to an outcome $(\hat{p}, \mathbf{X})$. Consider the utility function $\hat{u}_i$ defined by

$$\hat{u}_i(\{a\}) := \begin{cases} 0 & \text{if } a \notin X_i \\ \hat{p}_a + 1 & \text{if } a \in X_i \end{cases} \qquad \text{and} \qquad \hat{u}_i(A) = \sum_{a \in A} \hat{u}_i(\{a\}), \quad A \subset \Omega.$$

Since $\hat{u}_i$ values only objects in X_i and his preferences are additively separable, $(\hat{p}, \mathbf{X})$ is a Walrasian equilibrium of the economy $\hat{E} = (\Omega; \hat{u}_i, u_{-i})$. Moreover, in every efficient allocation for $\hat{E}$, agent i must receive all the objects in X_i.

Suppose the seller uses the VCG mechanism instead. If i reports $\hat{u}_i$ while his opponents report u_{-i}, he will receive X_i. Since the VCG mechanism is strategy-proof and $u \in \mathcal{U}$,

$$\begin{aligned} &u_i(\Psi_i(u)) - <\underline{p}, \Psi_i(u)> = u_i(\Psi_i(u)) - \Psi_{n+i}(u) \geq \\ &u_i(\Psi_i(\hat{u}_i, u_{-i})) - \Psi_{n+i}(\hat{u}_i, u_{-i}) = u_i(X_i) - \Psi_{n+i}(\hat{u}_i, u_{-i}). \end{aligned}$$

Since VCG payments are never higher than Walrasian equilibrium payments, we also have that $\Psi_{n+i}(\hat{u}_i, u_{-i}) \leq <\hat{p}, X_i>$. Therefore

$$u_i(\Psi_i(u)) - <\underline{p}, \Psi_i(u)> \geq u_i(X_i) - <\hat{p}, X_i>.$$

The last inequality establishes that for player i, in the English auction, honestly reporting his demand is a perfect best response against opponents with preferences u_{-i} that follow the same strategy. Since this is true for any profile u_{-i}, honest behavior is a perfect best response for player i, no matter what his beliefs over $\mathcal{U}^n$ are at each one of his information sets. ∎

The "no gap" condition of Theorem 5 is a joint restriction on preferences. In G&S we show that this no gap condition is satisfied for k-replica economies of E with $k \geq m+1$ (see Theorem 9 of G&S). Leonard [8] establishes that the no gap condition is satisfied in all unit demand economies.

We now argue that when all GS preferences are allowed, no generalization of the English auction can extract enough information to determine an efficient allocation and corresponding Vickrey payments. The idea is simple. To construct Vickrey payments, the seller must be able to compute total surplus when a consumer is missing and when a consumer and the bundle he is allocated are missing. This is not always possible when the price trajectory is restricted to be monotone in each coordinate.

DEFINITION: An *ascending price trajectory* is a function $p : [0,1] \to \mathbf{R}^m_+$ such that for all $s < t$ and $a \in \Omega$, $p_a(s) \leq p_a(t)$. The set of all ascending price trajectories is Π.

We denote by $\mathcal{S}$ the set of all monotone and GS utility functions.

DEFINITION: An *ascending auction* is a pair of functions $\pi : \mathcal{S}^n \to \Pi$ and $\xi : \Pi \to \mathbf{R}^n_+ \times [2^\Omega]^{n+1}$ such that (i) for all ascending path $p \in \Pi$, $\xi_1(p)$ is a vector of payments and $\xi_2(p)$ is a partition of Ω; and (ii) if $u, u' \in \mathcal{S}^n$ are such that $p = \pi(u)$ and $D_i(p(t)) = D'_i(p(t))$ for all $i \in N$ and $t \in [0,1]$, then $\pi(u') = \pi(u)$ (where D_i and D'_i denote respectively i's demand correspondence when his utility function is u_i and u'_i).

The map π of an ascending auction mechanism determines the ascending price trajectory as a function of the preference profile $u = (u_1, \ldots, u_n) \in \mathcal{S}^n$. Given the ascending price trajectory p generated by the auction mechanism, the map ξ determines the players' payments and allocation. That is, $\xi_2(p) = \mathbf{X}$, where $\mathbf{X} = (X_0, \ldots, X_n)$ is a partition of Ω. Condition (ii) requires that the price trajectory be responsive only to the demand correspondences of the players along the price trajectory. Thus, if the profile u generates a price trajectory p, and along p the players' demand correspondences with preference profile u coincide with those of the preference profile u', then the profile u' must generate the same price trajectory as u (and hence the same allocation). This requirement encapsulates our notion of an ascending auction.

THEOREM 6: *If $n \geq 3$ and $m \geq 4$, there is no ascending auction mechanism that yields a Vickrey outcome for each profile $u \in \mathcal{S}^n$.*

PROOF: We construct a parametrized example with three players and four goods where the preference of player 1 is fixed and those of players 2 and 3 depend on two parameters. Let $\Omega = \{a, b, c, d\}$, $R_1 = \{a, b\}$, $R_2 = \{c, d\}$, $C_1 = \{a, c\}$, and $C_2 = \{b, d\}$. The players' preferences for any bundle A are

$$u_1(A) = u_1(A \cap R_1) + u_1(A \cap R_2)$$
$$u_i(A) = u_i(A \cap C_1) + u_i(A \cap C_2), \quad i = 2, 3,$$

and

	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{a,b\}$	$\{c,d\}$	$\{a,c\}$	$\{b,d\}$
u_1	8	8	8	8	9	9	16	16
u_2	6	6	x	0	12	x	6	6
u_3	0	y	6	6	y	12	6	6

TABLE II

where $x, y \in (1, 3)$ are two parameters.

There are four efficient allocations $\mathbf{X} = (X_1, X_2, X_3)$:

$$\begin{aligned} X_1 &= \{a, c\}, \quad X_2 = \{b\}, \quad X_3 = \{d\}, \quad \text{or} \\ X_1 &= \{a, d\}, \quad X_2 = \{b\}, \quad X_3 = \{c\}, \quad \text{or} \\ X_1 &= \{b, c\}, \quad X_2 = \{a\}, \quad X_3 = \{d\}, \quad \text{or} \\ X_1 &= \{b, d\}, \quad X_2 = \{a\}, \quad X_3 = \{c\}. \end{aligned}$$

For any $A \subset R_j$ with $j = 1$ or 2, u_1 clearly satisfies the (SI) property at A and any price p. But then, since $u_1(A) = u_1(A \cap R_1) + u_1(A \cap R_2)$, u_1 satisfies the (SI) property at every $A \subset \Omega$ and every price p. A symmetric argument establishes that u_2 and u_3 also satify the (SI) property. The unique Walrasian equilibrium price vector of this economy is $p^W = (6, 6, 6, 6)$. It is also easy to see that in all four efficient allocations, the corresponding Vickrey payments are $q_1(X_1) = 12$, $q_2(X_2) = y$ and $q_3(X_3) = x$.

Consider an ascending auction mechanism (π, ξ), and let $p = \pi(u)$. Suppose that there does not exist $t \in [0, 1]$ such that

$$p_c(t) \leq x \quad \text{and} \quad p_a(t) - p_c(t) \geq 6 - x. \tag{$*$}$$

Then, for any $t \in [0, 1]$ and $A \in D_2(p(t))$ we must have that $c \notin A$. This is clear: if $p_c(t) > x$, then player 2 will never want to consume c, and if $p_a(t) - p_c(t) < 6 - x$, then in any situation where he would consider adding c to his demand, he would prefer to add a instead. Let u' be the profile where the preferences of players 1 and 3 are as in u, and those of player 2 correspond to decreasing x to $x - \epsilon$ for some small $\epsilon > 0$. Clearly now, with greater reason, player 2 will never want to consume c. Thus, along the trajectory p, $D_i(p(t)) = D'_i(p(t))$ for all $i = 1, 2, 3$, and $t \in [0, 1]$. Therefore, $\pi(u') = p$ and u and u' induce the same allocation and payments. But then, player 3's payment cannot be x when the preference profile is u and $x - \epsilon$ when the preference profile is u'.

We have then shown that for (π, ξ) to always yield Vickrey outcomes, we must have that $(*)$ be satisfied for some $t \in [0, 1]$. By symmetry, there must also exist $s \in [0, 1]$ such that

$$p_a(s) \leq y \quad \text{and} \quad p_c(s) - p_a(s) \geq 6 - y.$$

Since $p(s) \geq 0$, we have that

$$p_c(s) \geq 6 - y > 3 > x \geq p_c(t),$$

which by monotonicity implies that $s > t$. Symmetrically, since $p(t) \geq 0$,

$$p_c(t) \geq 6 - x > 3 > y \geq p_c(s),$$

which implies that $t > s$. As $s > t$ and $t > s$ are incompatible, (π, ξ) cannot always yield Vickrey outcomes.

When $n > 3$ or $m > 4$, set $u_i \equiv 0$ for $i \geq 4$ and $u_i(A) = u_i(A \cap \{a, b, c, d\})$ to embed the example above into the larger economy. ∎

One can show that any efficient and strategy-proof mechanism must charge a constant translation of VCG prices. More specifically, for any preference profile u, such mechanism must allocate $\Psi_i(u)$ and charge $\theta_i(u_{-i}) + \Psi_{n+i}(u)$ to player i ($\theta_i(u_{-i})$ is a constant that does not depend on i's preferences). Hence, Theorem 6 implies that when all GS preferences are allowed, no ascending price auction can be efficient and strategy-proof.

7. Conclusion

In this paper we have developed a generalization of the English auction for the case in which agents demand bundles of different objects. Our main result is that the auction converges to the smallest Walrasian prices.

We also investigated the strategic properties of our auction. We noted that if the smallest Walrasian prices correspond to VCG payments, then our dynamic auction can be used to strategically implement the VCG mechanism. Finally, we showed that in general, no efficient, dynamic auction can extract enough information to implement any strategy-proof mechanism.

One by-product of our analysis is the resulting connection with matroid theory and submodular optimization. It is likely that this connection will prove useful in the analysis of markets and mechanisms.

8. Appendix 1: Matroid Theory

For an excellent reference on matroid theory the reader is referred to Fujishige [4]. The concept of a matroid was introduced by Whitney (1935) to capture the linear independence structure of the columns of a matrix. Briefly, let M be an $\ell \times m$ matrix and $\Omega = \{1, \ldots, m\}$ be the set of columns of M. $X \subset \Omega$ is an *independent set* if the corresponding columns of M are linearly independent. Let I denote the collection of all independent sets. Then, (Ω, I) is a (matric or linear) matroid. In what follows $\Omega = \{1, \ldots, m\}$, which for our purposes still denotes the collection of objects.

Definition: Let $I \subset 2^{\Omega}$. The pair (Ω, I) is a matroid if

(I0) $\emptyset \in I$.

(I1) $A \subset B \in I$ implies $A \in I$.

(I2) $A, B \in I$ and $\#(A) < \#(B)$ implies that there exists $a \in B \backslash A$ such that $A \cup \{a\} \in I$.

Each $A \in I$ is called an independent set of the matroid (Ω, I), and I is the family of independent sets. An independent set which is maximal in I with respect to set inclusion is called a basis. The family of all bases is denoted by $\mathcal{B}$.

It turns out that $\mathcal{B} \subset 2^{\Omega}$ is a family of bases for the matroid (Ω, I) iff it satisfies

(B0) $\mathcal{B} \neq \emptyset$.

(B1) for all $A, B \in \mathcal{B}$ and each $a \in A \backslash B$, there exists $b \in B \backslash A$ such that $[A, a, b] \in \mathcal{B}$.

Moreover, if $\mathcal{B}$ is a family of bases for (Ω, I) then $I = \{A \mid A \subset B \text{ for some } B \in \mathcal{B}\}$.

Definition: The *rank function* $\rho : 2^{\Omega} \rightarrow \mathbf{Z}$ of the matroid (Ω, I) is defined by

$$\rho(X) := \max\{\#(A) \mid A \subset X \text{ and } A \in I\} \equiv \max\{\#(B \cap X) \mid B \in \mathcal{B}\} \quad X \subset \Omega.$$

In the case of a linear matroid, the rank function assigns to each collection X of columns of the matrix M, the dimension of the subspace spanned by those columns. Then, standard linear algebra results show that the rank function satisfies conditions $(\rho 0)$–$(\rho 2)$ below. Interestingly, these conditions are also sufficient.

A function $\rho : 2^\Omega \to \mathbf{Z}$ is the rank function of a matroid (Ω, I) iff it satisfies

($\rho 0$) $0 \le \rho(X) \le \#(X)$ for all $X \subset \Omega$.

($\rho 1$) $X \subset Y \subset \Omega$ implies that $\rho(X) \le \rho(Y)$.

($\rho 2$) $\rho(X) + \rho(Y) \ge \rho(X \cup Y) + \rho(X \cap Y)$ for all $X, Y \subset \Omega$.

Moreover, $I = \{A \,|\, \rho(A) = \#(A)\}$. It is easy to see that ($\rho 0$) and ($\rho 2$) imply that for all $X \subset \Omega$ and $a \notin X$, $\rho(X \cup \{a\}) \le \rho(X) + 1$.

Thus, a matroid is uniquely defined by its family of independent sets I, or family of bases $\mathcal{B}$, or rank function ρ, and sometimes is denoted by $(\Omega, \mathcal{B})$ or (Ω, ρ).

Definition: The *dual* of a function $f : 2^\Omega \to \mathbf{R}$ is the function $f^\# : 2^\Omega \to \mathbf{R}$ defined by

$$f^\#(X) := f(\Omega) - f(X^c) \qquad X \subset \Omega.$$

It is easy to verify that if f is monotone, then $f^\#$ is monotone, and if f is submodular (when 2^Ω is given the standard set inclusion ordering), then $f^\#$ is supermodular. Furthermore, if $f(\emptyset) = 0$, then $(f^\#)^\# = f$. In particular, if ρ is the rank function of a matroid (Ω, I), then $\rho^\#(\emptyset) = 0$, and $\rho^\#$ is monotone and supermodular.

Let (Ω, I_i), $i = 1, \ldots, n$, be a collection of matroids. The *matroid partitioning problem* is to find n pairwise disjoint sets A_i, $i = 1, \ldots, n$, such that $A_i \in I_i$ for each $i = 1, \ldots, n$, and $\bigcup A_i = \Omega$.

Theorem 7: *(Edmonds [3]) Let (Ω, I_i), $i = 1, \ldots, n$, be a collection of matroids. Then, there exists a base B_i of (Ω, I_i) for each $i = 1, \ldots, n$, such that $\bigcup B_i = \Omega$ iff*

$$\rho_1(X) + \cdots + \rho_n(X) \ge \#(X) \text{ for each } X \subset \Omega.$$

9. Appendix 2: Proofs

The following definitions are used in Lemmas 3 and 4 below. Let u be a utility function on Ω. Pick any $\alpha \in \Omega$, and for any price vector p and bundle A let

$$D^*_-(p, \alpha) := \{B \in D^*(p) \,|\, \alpha \notin B\}, \quad K_-(A, p, \alpha) := \min_{B \in D^*_-(p,\alpha)} \#(A \cap B),$$

$$D^*_+(p, \alpha) := \{B \in D^*(p) \,|\, \alpha \in B\}, \quad K_+(A, p, \alpha) := \min_{B \in D^*_+(p,\alpha)} \#(A \cap B).$$

Lemma 3: *Suppose u is monotone and has the (SI) property, and that $D^*_-(p, \alpha) \ne \emptyset$ and $D^*_+(p, \alpha) \ne \emptyset$. Then*

(i) for each $A \in D^*_+(p, \alpha)$ and $B \in D^*_-(p, \alpha)$, there exists $y \in B \backslash A$ such that $[A, \alpha, y] \in D^*_-(p, \alpha)$.

(ii) for each $B \in D^*_-(p, \alpha)$, there exists $x \in B$ such that $[B, x, \alpha] \in D^*_+(p, \alpha)$.

Proof: For part (i), let $A \in D^*_+(p, \alpha)$ and $B \in D^*_-(p, \alpha)$. Then $A, B \in D^*(p)$ and $A \ne B$. Let $\hat{p}$ be the price vector defined by $\hat{p}_a = p_a$ for all $a \in A \cup B$ and $\hat{p}_a = M > u(\Omega)$ otherwise. Clearly, $A, B \in D(\hat{p}) \subset D(p)$. For each $\epsilon \ge 0$, let $q(\epsilon) = \hat{p} + \epsilon e^\alpha$. Now, for each $\epsilon > 0$, $A \notin D(q(\epsilon)$ and $B \in D(q(\epsilon))$. By (SI), there exists C such that $\#(A \backslash C) \le 1$,

$\#(C\backslash A) \leq 1$, and $v(C, q(\epsilon)) > v(A, q(\epsilon))$ for all $\epsilon > 0$ sufficiently small. The latter implies that $C \neq A$, and obviously $C \subset A \cup B$. By continuity, $v(C, \hat{p}) \geq v(A, \hat{p})$, and thus $C \in D(\hat{p})$. In addition, $\alpha \notin C$, otherwise $v(C, q(\epsilon)) = v(A, q(\epsilon))$ for all $\epsilon > 0$. Thus, $A\backslash C = \{\alpha\}$. Finally, $\#(C\backslash A) = 1$, for otherwise $\#(C\backslash A) = 0$, so $\#(C) = \#(A) - 1$, which contradicts the fact that $A \in D^*(p)$. Therefore, $C\backslash A = \{y\}$ for some $y \in B\backslash A$, and $C = [A, \alpha, y] \in D^*_-(p, \alpha)$.

We now prove part (ii). Let $B \in D^*_-(p, \alpha)$ and pick any $A \in D^*_+(p, \alpha)$. Suppose that $\#(A\backslash B) \geq 2$. Then there exists $z \neq \alpha$ such that $z \in A\backslash B$. That is, $A \in D^*_+(p, z)$ and $B \in D^*_-(p, z)$. By the proof of part (i), there exists $y \in B\backslash A$ such that $A^* := [A, z, y] \in D^*(p)$. Obviously $A^* \in D^*_+(p, \alpha)$ and $\#(A^*\backslash B) = \#(A\backslash B) - 1$. Thus,

$$\min \{ \#(A\backslash B) \,|\, A \in D^*_+(p, \alpha) \} = 1.$$

Let A be any optimal solution of this problem. Since $\#(B) = \#(A)$, $\#(B\backslash A) = 1$, and there exists $x \in B$ such that $[B, x, \alpha] = A \in D^*_+(p, \alpha)$. ∎

Proof of Theorem 1: Fix the price vector p. We first establish that $(\Omega, D^*(p))$ is a matroid with basis $D^*(p)$. By definition $D^*(p) \neq \emptyset$. (In particular, if p is "very" large, then $D^*(p) = \{\emptyset\}$.) We only need to check condition (B1) for a basis. Suppose $A, B \in D^*(p)$ are such that $A \neq B$. Let $x \in A\backslash B$. Then $A \in D^*_+(p, x)$ and $B \in D^*_-(p, x)$, and by part (i) of Lemma 3, there exists $y \in B\backslash A$ such that $[A, x, y] \in D^*(p)$.

It is easy to check that the rank function of the matroid $(\Omega, D^*(p))$ is given by

$$\rho(X) = K(\Omega, p) - K(X^c, p), \quad X \subset \Omega.$$

That is, $\rho = K^\#(\cdot, p)$. Since $K(\emptyset, p) = 0$, $\rho^\# = K(\cdot, p)$. ∎

Lemma 4: *Suppose u is monotone and has the (SI) property, and that $D^*_-(p, \alpha) \neq \emptyset$ and $D^*_+(p, \alpha) \neq \emptyset$. Then, for any bundles A and Z, with $\alpha \notin A$ and $\alpha \in Z$,*
(i) $K_+(A, p, \alpha) \leq K_-(A, p, \alpha)$;
(ii) $K_-(Z, p, \alpha) \leq K_+(Z, p, \alpha)$.

Proof: Throughout the proof, fix α and the price vector p, and for each bundle B, let $\rho^\#_-(B) := K_-(B, p, \alpha)$ and $\rho^\#_+(B) := K_+(B, p, \alpha)$.

(i) Pick any bundle A such that $\alpha \notin A$, and let $B \in D^*_-(p, \alpha)$ be such that $\#(A \cap B) = \rho^\#_-(A)$. By (ii) of Lemma 3, there exists $C \in D^*_+(p, \alpha)$ and $x \in B$ such that $C\backslash\{\alpha\} = B\backslash\{x\}$. Therefore

$$\rho^\#_+(A) \leq \#(A \cap C) = \#(A \cap [B\backslash\{x\}]) \leq \rho^\#_-(A).$$

(ii) Let Z be any bundle such that $\alpha \in Z$, and let $C \in D^*_+(p, \alpha)$ be such that $\#(Z \cap C) = \rho^\#_+(A)$. By (i) of Lemma 3, there exists $B \in D^*_-(p, \alpha)$ and $y \in B$ such that $B\backslash\{y\} = C\backslash\{\alpha\}$. Hence

$$\begin{aligned} \rho^\#_-(Z) &\leq \#(Z \cap B) \leq \#(Z \cap [B\backslash\{y\}]) + 1 \\ &= \#(Z \cap [C\backslash\{\alpha\}]) + 1 = \#(Z \cap C) = \rho^\#_+(Z). \end{aligned}$$ ∎

Proof of Theorem 2: The proof of (i) and (ii) are by induction on the cardinality of the set $C := \{ a \in \Omega \,|\, p_a \neq q_a \} \subset A^c$. Obviously, if $\#(C) = 0$ the result is true. Suppose (i) and (ii) are true for any p and q for which $\#(C) \leq k$. Let p and q be such that $\#(C) = k+1$, and for the rest of the proof, fix $\alpha \in C$. We first show that (i) holds. We consider two cases.

Case 1: Suppose there exists $B \in D^*(q)$ such that $\alpha \notin B$. Define

$$q'_a = \begin{cases} q_a & \text{if } a \neq \alpha \\ p_\alpha & \text{if } a = \alpha, \end{cases}$$

and $C' := \{ a \in \Omega \,|\, p_a \neq q'_a \}$. Then $\#(C') = \#(C) - 1$. Therefore, by inductive hypothesis, $K(A,p) \geq K(A,q')$. If $E, F \in D^*(q)$ are such that $\alpha \notin E$ and $\alpha \in F$, then obviously $v(E,q') < v(F,q')$ and $F \in D^*(q')$. Therefore,

$$D^*(q') = \{ B \in D^*(q) \,|\, \alpha \notin B \} \subset D^*(q),$$

and hence $K(A,q') \geq K(A,q)$. Together with the previous inequality, this implies that $K(A,p) \geq K(A,q)$, as we wanted to show.

Case 2: Now suppose that $\alpha \in B$ for all $B \in D^*(q)$. For $\epsilon \geq 0$, define $r(\epsilon)$ as follows:

$$r_a(\epsilon) = \begin{cases} q_a & \text{if } a \neq \alpha \\ q_\alpha + \epsilon & \text{if } a = \alpha. \end{cases}$$

Since $\alpha \in B$ for each $B \in D^*(q)$, $D^*(r(\epsilon)) = D^*(q)$ for all $\epsilon > 0$ sufficiently small. As ϵ is increased, eventually new minimal bundles will become optimal at prices $r(\epsilon)$. Obviously, any such new bundle B is such that $\alpha \notin B$. Let $\hat{\epsilon}$ be the smallest $\epsilon > 0$ for which $D^*(r(\epsilon)) \neq D^*(q)$. If $\hat{\epsilon} > p_\alpha - q_\alpha$, define q' and C' as in Case 1; that is, make $\epsilon' := p_\alpha - q_\alpha$ and $q' := r(\epsilon')$. Since $\epsilon' < \hat{\epsilon}$, $D^*(q') = D^*(q)$, and by the same argument above, $K(A,p) \geq K(A,q') = K(A,q)$. If instead $\hat{\epsilon} \leq p_\alpha - q_\alpha$, let $\hat{q} := r(\hat{\epsilon})$. Then

$$D^*(\hat{q}) = D^*_-(\hat{q},\alpha) \cup D^*_+(\hat{q},\alpha), \quad D^*_+(\hat{q},\alpha) \equiv D^*(q), \quad \text{and } D^*_-(\hat{q},\alpha) \neq \emptyset.$$

By part (i) of Lemma 4, we have that

$$K(A,\hat{q}) = \min\{K_+(A,\hat{q},\alpha), K_-(A,\hat{q},\alpha)\} = K_+(A,\hat{q},\alpha) = K(A,q).$$

Furthermore, if $\hat{\epsilon} < p_\alpha - q_\alpha$, $\hat{q}$ satisfies the assumptions of Case 1, and if $\hat{\epsilon} = p_\alpha - q_\alpha$, $(p,\hat{q})$ satisfies the inductive hypothesis. Therefore $K(A,p) \geq K(A,\hat{q})$. This concludes the proof of (i).

We now show that (ii) holds. Again, we divide the proof in two cases.

Case 3: Suppose $\alpha \in B$ for some $B \in D^*(p)$. Define

$$p'_a := \begin{cases} p_a & \text{if } a \neq \alpha \\ q_\alpha & \text{if } a = \alpha, \end{cases}$$

and $C' := \{ a \in \Omega \,|\, p'_a \neq q_a \}$. Then $\#(C') = \#(C) - 1$, and by inductive hypothesis, $K(A^c, p') \leq K(A^c, q)$. Also

$$D^*(p') = \{ B \in D^*(p) \,|\, \alpha \in B \} \subset D^*(p).$$

Therefore, $K(A^c, p) \leq K(A^c, p')$, and (ii) follows from the last two inequalities strung together.

Case 4: Now suppose that $\alpha \notin B$ for all $B \in D^*(p)$. For $\epsilon \geq 0$, define $r(\epsilon)$ as follows:

$$r_a(\epsilon) = \begin{cases} p_a & \text{if } a \neq \alpha \\ p_\alpha - \epsilon & \text{if } a = \alpha. \end{cases}$$

We have that $D^*(r(\epsilon)) = D^*(p)$ for all $\epsilon > 0$ sufficiently small, and as ϵ is increased, eventually new minimal bundles will become optimal at prices $r(\epsilon)$. Moreover, any new optimal bundle B will satisfy $\alpha \in B$. Let $\epsilon' := p_\alpha - q_\alpha$ and $\hat{\epsilon}$ be the smallest ϵ for which $D^*(r(\epsilon)) \neq D^*(p)$. If $\epsilon' < \hat{\epsilon}$, let $p' := r(\epsilon')$ and $C' := \{ a \in \Omega \,|\, p'_a \neq q_a \}$; otherwise, let $\hat{p} := r(\hat{\epsilon})$. In the former case, since $\#(C') = \#(C) - 1$, we have as in Case 3 that $K(A^c, p) = K(A^c, p') \leq K(A^c, q)$. In the latter, we have that

$$D^*(\hat{p}) = D^*_-(\hat{p}, \alpha) \cup D^*_+(\hat{p}, \alpha), \quad D^*_-(\hat{p}, \alpha) \equiv D^*(p), \quad \text{and } D^*_+(\hat{p}, \alpha) \neq \emptyset.$$

By part (ii) of Lemma 4, $K(A^c, \hat{p}) = K(A^c, p)$, and if $\epsilon' > \hat{\epsilon}$, $\hat{p}$ satisfies the assumptions of Case 3, otherwise $\epsilon' = \hat{\epsilon}$ and $(\hat{p}, q)$ satisfies the inductive hypothesis. Therefore $K(A^c, \hat{p}) \leq K(A^c, q)$. ∎

Proof of Theorem 3: If such a partition $(B_0, \ldots, B_n)$ exists, then for any bundle A,

$$\sum_{i \in N} K_i(A, p) \leq \sum_{i \in N} \#(A \cap B_i) = \#(A \cap B_0^c) \leq \#(A).$$

Conversely, for each $i \in N$, and $A \subset \Omega$, let $\rho_i(A) := K_i(\Omega, p) - K_i(A^c, p)$. Suppose for the moment that $K_N(\Omega, p) = \#(\Omega) = m$. We will prove the theorem for this case first, and later deal with the case $K_N(\Omega, p) < m$. Then, for each $A \subset \Omega$,

$$\sum_{i \in N} \rho_i(A) = K_N(\Omega, p) - K_N(A^c, p) = \#(\Omega) - K_N(X^c, p).$$

Since by assumption $K_N(A^c, p) \leq \#(A^c)$, we have that $\sum_{i \in N} \rho_i(A) \geq \#(A)$ for every $A \subset \Omega$. Therefore, by Edmonds' theorem, there exists $B_i \in D_i^*(p)$, $i \in N$, such that $\bigcup B_i = \Omega$. By definition of the rank function, $\rho_i(B_i) = \#(B_i)$ for each $i \in N$, so

$$\sum_{i \in N} \rho_i(B_i) = K_N(\Omega, p) - \sum_{i \in N} K_i(B_i^c, p) = K_N(\Omega, p) = \#(\Omega)$$

(since $K_i(B_i^c, p) = \#(B_i^c \cap B_i) = 0$ for each $i \in N$). But, $\sum_{i \in N} \#(B_i) = \#(\Omega) = \#(\bigcup_{i \in N} B_i)$ implies that the sets B_i, $i \in N$, must be disjoint. Thus, if $B_0 := \emptyset$,

$(B_0, \ldots, B_n)$ is a partition of Ω supported by p, and $K_i(A,p) \leq \#(A \cap B_i)$, $i \in N$. This concludes the proof for the case in which $K_N(\Omega, p) = m$.

Now, suppose that $K_N(\Omega, p) = k < m$. Define the "demand correspondence"

$$D_0^*(p) := \{A \subset \Omega \,|\, \#(A) = m - k\}, \quad p \in \mathbf{R}_+^m.$$

It is easy to see that for each p, $D_0^*(p)$ is the basis of a matroid. If we let K_0 be the corresponding requirement function, then

$$K_0(A,p) = \max\{0, \#(A) - k\}.$$

Define $\rho_0(A) := K_0(\Omega, p) - K_0(A^c, p) = m - k - K_0(A^c, p)$ for each $A \subset \Omega$. Then

$$\sum_{i=0}^{n} K_i(\Omega, p) = m \quad \text{and} \quad \sum_{i=0}^{n} K_i(A,p) \leq \#(A) \quad \text{for each } A \subset \Omega.$$

Thus, by the previous case, there exist $B_i \in D_i^*(p)$, $i \in N_0$, such that $(B_0, \ldots, B_n)$ is a partition of Ω. ∎

Proof of Lemma 6: If $\epsilon(t) > 0$, then there exists $s > 0$ such that $O_*(p(t)+s\delta(t)) \neq O_*(p(t))$. This implies that $\delta(t) \neq 0$, and therefore that $O_*(p(t)) \neq \emptyset$.

Conversely, suppose $O_*(p(t)) \neq \emptyset$. For any $s \geq 0$ define

$$q(s) := p(t) + s\delta(p(t)) \quad \text{and} \quad r(s) := p(t) + se^{\Omega}.$$

Obviously, $r(0) \equiv q(0) \equiv p(t)$. We need to show that when $O_*(q(0)) \neq \emptyset$, then $O_*(q(s)) = O_*(q(0))$ for all $s > 0$ sufficiently small. This is a kind of "right continuity" of O_*.

By definition, for each i, $A \in D_i(q(0))$ and $B \notin D_i(q(0))$, we have that $v_i(A, p(t)) > v_i(B, p(t))$. Therefore, there exists $\epsilon_1 > 0$ such that $v_i(A, q(s)) > v_i(B, q(s))$ for all $s \in [0, \epsilon_1]$, $A \in D_i(q(0))$, and $B \notin D_i(q(0))$. Let

$$w := \min_{C \in D_i(p(t))} \#[C \cap O_*(p(t))].$$

Then

$$D_i(q(s)) = \{\, B \in D_i(p(t)) \,|\, \#[B \cap O_*(p(t))] = w \,\},$$

and clearly $D_i(q(s)) \subset D_i(q(0))$. Also, if $\emptyset \notin D_i(r(0))$, then there exists $\epsilon' > 0$ such that $D_i(r(s)) = D_i(r(0))$ for all $s \in [0, \epsilon']$. And if $\emptyset \in D_i(r(0))$, then $D_i(r(s)) = \{\emptyset\}$ for all $s > 0$. Hence, there exists $\epsilon_2 > 0$ such that $D_i^*(r(s)) = D_i^*(r(0))$ for all $s \in [0, \epsilon_2]$. Let $\epsilon_0 := \min\{\, \epsilon_1, \epsilon_2 \,\}$. We now show that $O_*(q(s)) = O_*(q(0))$ for all $s \in [0, \epsilon_0]$.

Pick any $s \in [0, \epsilon_0]$ and any bundle A. Since $D_i^*(r(s)) = D_i^*(r(0))$ for each $i \in N$ and $r(0) = q(0) = p(t)$,

$$f(A, r(s)) = f(A, q(0)). \tag{1}$$

Also, since $D_i(q(s)) \subset D_i(q(0))$ for each $i \in N$,

$$f(A, q(s)) \geq f(A, q(0)). \tag{2}$$

Finally, if $A \subset O_*(q(0))$, part (i) of Theorem 2 implies that

$$f(A, r(s)) \geq f(A, q(s)). \tag{3}$$

When $A = O_*(q(0))$, inequalities (1) – (3) imply that

$$x := f(O_*(q(0)), q(0)) \equiv f(O_*(q(0)), r(s)) \equiv f(O_*(q(0)), q(s)), \tag{4}$$

and when $A = O_*(q(0)) \cap O_*(q(s))$, they imply that

$$\begin{aligned} y :=& f(O_*(q(0)) \cap O_*(q(s)), q(0)) \equiv f(O_*(q(0)) \cap O_*(q(s)), r(s)) \\ \equiv& f(O_*(q(0)) \cap O_*(q(s)), q(s)). \end{aligned} \tag{5}$$

Finally, (1) and (2) also hold when $A = O_*(q(0)) \cup O_*(q(s))$, and from part (ii) of Theorem 2, we also have

$$f(O_*(q(0)) \cup O_*(q(s)), q(0)) \geq f(O_*(q(0)) \cup O_*(q(s)), q(s)). \tag{3$'$}$$

Therefore

$$\begin{aligned} z :=& f(O_*(q(0)) \cup O_*(q(s)), q(0)) \equiv f(O_*(q(0)) \cup O_*(q(s)), r(s)) \\ \equiv& f(O_*(q(0)) \cup O_*(q(s)), q(s)). \end{aligned} \tag{6}$$

Since for all $s' \geq 0$ and bundle A, $f(O_*(q(s')), q(s')) \geq f(A, q(s'))$, we have

$$f(O_*(q(s)), q(s)) \geq y \quad \text{and} \quad f(O_*(q(s)), q(s)) \geq z \tag{7}$$
$$x \geq y \quad \text{and} \quad x \geq z, \tag{8}$$

where the first inequality in (7) follows from (5), the second from (6), and the inequalities in (8) follow from (4) – (6). From the supermodularity of $f(\cdot, q(s))$, the equality $f(O_*(q(0)), q(s)) = x$, and each inequality in (7), it follows that $2x \leq y + z$. Hence $x = y = z$. Again from the supermodularity of $f(\cdot, q(s))$ and (8), it follows that $f(O_*(q(s)), q(s)) + x \leq y + z$, so $f(O_*(q(s)), q(s)) \leq x$, which together with (7) implies that $f(O_*(q(s)), q(s)) = x$. That is, $O_*(q(0)) \in O(q(s))$, and therefore $O_*(q(s)) \subset O_*(q(0))$. Similarly, $O_*(q(s)) \in O(q(0))$, so $O_*(q(0)) \subset O_*(q(s))$. Hence $O_*(q(s)) = O_*(q(0))$ ∎

Proof of Lemma 7: Let $c_t := (m+1) f(O_*(p(t)), p(t)) - \#(O_*(p(t))$. Note that $c_0 \leq (m+1)(n-1)m$ and c_t is a nonnegative integer for each t. Thus, to complete the proof, we show that $c_{t+1} < c_t$ for each $t < T$. To accomplish this, we establish that

(i) $f(O_*(p(t+1)), p(t+1)) \leq f(O_*(p(t)), p(t))$ for all $t < T$.
(ii) if $O_*(p(t))) \neq \emptyset$ and $f(O_*(p(t+1)), p(t+1)) = f(O_*(p(t)), p(t))$, then $\#(O_*(p(t+1))) > \#(O_*(p(t)))$.
(iii) if $c_t > 0$ then $\#(O_*(p(t))) \neq \emptyset$.

Let t be such that $O_*(p(t))) \neq \emptyset$ (that is, $t < T$). As in the previous lemma, let $q(s) := p(t) + s\delta(p(t))$, and let $\epsilon(t)$ be as defined in step 3 of the algorithm. Then

$p(t+1) = q(\epsilon(t))$. Let $\underline{\epsilon} < \epsilon(t)$ be large enough so that $D_i(q(s))$ remains constant for each i and $s \in (\underline{\epsilon}, \epsilon(t))$. Then, for any $s \in (\underline{\epsilon}, \epsilon(t))$, we have

$$\begin{aligned} & f(O_*(p(t+1)), p(t+1)) \leq f(O_*(p(t+1)), q(s)) \\ & \leq f(O_*(p(t)), q(s)) \leq f(O_*(p(t)), p(t)). \end{aligned} \tag{9}$$

The first inequality follows from the observation made after the specification of the algorithm: $D_i(q(s)) \subset D_i(p(t+1))$ for all $s \in (\underline{\epsilon}, \epsilon(t))$. By definition of $\epsilon(t)$, $O_*(q(s)) = O_*(p(t)) \in O(q(s))$ for all $s \in (\underline{\epsilon}, \epsilon(t))$; this implies the second inequality. The last inequality follows from part (ii) of Theorem 2. This establishes (i).

Suppose equality attains in (i). From (9), it must be that $f(O_*(p(t+1)), q(s)) = f(O_*(p(t)), q(s))$. Hence $O_*(p(t+1)) \in O(q(s))$ for all $s \in (\underline{\epsilon}, \epsilon(t))$, and since $O_*(p(t))$ is the minimal element of $O(q(s))$ for all $s \in (\underline{\epsilon}, \epsilon(t))$, we must have that $O_*(p(t)) \subset O_*(p(t+1))$. So, if $O_*(p(t+1)) \neq O_*(p(t))$, we have that $\#(O_*(p(t+1))) > \#(O_*(p(t)))$. This establishes (ii). Finally, (iii) is obvious. ∎

LEMMA 8: *Let $\{\omega_1, \ldots, \omega_k\}$ be a collection of distinct objects in Ω. Consider the enlarged economy $\hat{E} = (\Omega; u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k})$, where consumer $n+i$ has the unit demand utility function*

$$u_{n+i}(A) = \begin{cases} 0 & \text{if } \omega_i \notin A \\ \underline{p}_{\omega_i} & \text{if } \omega_i \in A. \end{cases}$$

Then the set $\hat{P}^E$ of Walrasian equilibrium prices for $\hat{E}$ coincides with the set P^E of Walrasian equilibrium prices for E.

PROOF: Let $\mathbf{X}$ be any efficient allocation in E such that $X_0 = \emptyset$, and define the allocation $\hat{\mathbf{X}}$ of $\hat{E}$ as follows: $\hat{X}_i = X_i$ for each $i = 1, \ldots, n$, and $\hat{X}_{n+i} = \emptyset$ for each $i = 1, \ldots, k$. Pick any $p \in P^E$. By definition, $p_{\omega_i} \geq \underline{p}_{\omega_i}$ for each $i = 1, \ldots, k$. If $p_{\omega_i} = \underline{p}_{\omega_i}$, consumer $n+i$'s optimal choices are to consume nothing or to consume $\{\omega_i\}$, while if $p_{\omega_i} > \underline{p}_{\omega_i}$, consumer $n+i$ definitely prefers to consume nothing. In either case, at prices p, $\hat{X}_{n+i}$ is an optimal choice for consumer $n+i$, and $(p, \hat{\mathbf{X}})$ is a Walrasian equilibrium of $\hat{E}$.

Conversely, note that by the first theorem of welfare economics, $\hat{\mathbf{X}}$ is an efficient allocation of $\hat{E}$. Therefore, if p is a Walrasian equilibrium price in $\hat{E}$, $< p, X_0 > = 0$ and p supports $\hat{\mathbf{X}}$ in $\hat{E}$. But this implies that p supports $\mathbf{X}$ in E, and thus $(p, \mathbf{X})$ is a Walrasian equilibrium of E. ∎

LEMMA 9: *If p supports $\mathbf{X}$, then $< p, X_0 > \ \geq \ < \underline{p}, X_0 >$.*

PROOF: Let $\mathbf{Y}$ be any efficient allocation with $Y_0 = \emptyset$. Then for each $i \in N$,

$$u_i(X_i) - < p, X_i > \ \geq \ u_i(Y_i) - < p, Y_i >,$$

which implies that

$$\sum_{i \in N} u_i(X_i) - \sum_{i \in N} u_i(Y_i) \geq -< p, X_0 >. \tag{10}$$

Also, for each $i \in N$,

$$u_i(Y_i) - < \underline{p}, Y_i > \;\geq\; u_i(X_i) - < \underline{p}, X_i >,$$

and therefore

$$\sum_{i \in N} u_i(Y_i) - \sum_{i \in N} u_i(X_i) \geq \; < \underline{p}, X_0 >. \tag{11}$$

Adding inequalities (10) and (11) we obtain $0 \geq \; < \underline{p}, X_0 > - < p, X_0 >$. ∎

Proof of Theorem 4: We first show that $p(T) \leq \underline{p}$. Suppose not. Then there exists t and $s \in [0, \epsilon(t))$ such that $q(s) := p(t) + s\delta(p(t)) \leq \underline{p}$ and $O_*(q(s)) \cap W(q(s)) \neq \emptyset$, where for any price p,

$$W(p) = \{\, a \in \Omega \,|\, p_a = \underline{p}_a \,\}.$$

Note that since $s < \epsilon(t)$, $O_*(q(s)) = O_*(p(t)) \neq \emptyset$. Let $W_1 := W(q(s)) \cap O_*(q(s))$ and $W_2 := O_*(q(s)) \backslash W(q(s))$.

For $\gamma > 0$, define the price vector p as follows: $p_a = q(s)$ for $a \notin W_2$ and $p_a = q(s) + \gamma$ for $a \in W_2$. Choose $\gamma > 0$ small so that $p_a < \underline{p}_a$ for $a \in W_2$, and

$$D_i(p) = \{\, X \in D_i(q(s)) \,|\, \#(X \cap W_2) = K_i(W_2, q(s)) \,\}.$$

For some $X \in D_i(p)$, $K_i(W_1, p) = \#(X \cap W_1)$ and $K_i(W_2, q(s)) = \#(X \cap W_2)$. Therefore

$$\begin{aligned} K_i(O_*(q(s)), q(s)) \leq & K_i(W_1 \cup W_2, q(s)) \leq \#((W_1 \cup W_2) \cap X) \\ = & K_i(W_1, p) + K_i(W_2, q(s)), \end{aligned}$$

and thus $K_i(W_1, p) \geq K_i(O_*(q(s)), q(s)) - K_i(W_2, q(s))$. Summing over i and subtracting $\#(W_1) = \#(O_*(q(s))) - \#(W_2)$, we obtain

$$f(W_1, p) \geq f(O_*(q(s)), q(s)) - f(W_2, q(s)).$$

Since W_2 is a strict subset of $O_*(q(s))$, $f(W_2, q(s)) < f(O_*(q(s)), q(s))$. Therefore $f(W_1, p) > 0$. By Theorem 2, then, $f(W_1, \underline{p}) > 0$, which contradicts the fact that $\underline{p}$ is a Walrasian equilibrium.

Now we show that $p(T) \geq \underline{p}$. Let X be a partition which is supported by $p(T)$. The previous lemma then implies that $< p(T), X_0 > \;\geq\; < \underline{p}, X_0 >$. Therefore, $p_a(T) = \underline{p}_a$ for all $a \in X_0$.

Suppose $X_0 = \{\omega_1, \ldots, \omega_k\}$. Let $\hat{E} = (\Omega; u_1, \ldots, u_n, u_{n+1}, \ldots u_{n+k})$ be the enlarged economy where each u_{n+i}, $i = 1, \ldots, k$ is as defined in Lemma 8 above. At prices $p(T)$, consumer $n+i$'s optimal bundles are $\emptyset$ and $\{\omega_i\}$. Therefore, $(p(T), \hat{X})$, where $\hat{X}_0 = \emptyset$, $\hat{X}_i = X_i$ for $i \in N$, and $\hat{X}_{n+i} = \{\omega_i\}$ for $i = 1, \ldots, k$, is a Walrasian equilibrium of $\hat{E}$. Hence, by Lemma 8, $p(T)$ is a Walrasian equilibrium price for E, and thus $p(T) \geq \underline{p}$. ∎

References

Ausubel, L. (1995): "An Efficient Ascending-bid Auction for Multiple Objects," mimeo, University of Maryland.

Demange, G., D. Gale, and M. Sotomayor (1986): "Multi-Item Auctions," *Journal of Political Economy*, **94**, 863–872.

Edmonds, J. (1970): "Submodular Functions, Matroids, and Certain Polyhedra," in *Proceedings of the Calgary International Conference in Combinatorial Structures and Their Applications*, ed. R. Guy, H. Hanani, N. Sauer and J. Schönheim. New York: Gordon and Breach, 69–87.

Fujishige, S. (1991): *Submodular Functions and Optimization.* The Netherlands: North-Holland.

Gul, F. and E. Stacchetti (1998): "Walrasian Equilibrium with Gross Substitutes," mimeo, Michigan.

Hall, P. (1935): "On Representatives of Subsets," *J. London Math. Soc.*, **10**, 26–30.

Kelso, A. S. and V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, **50**, 1483–1504.

Leonard, H. (1983): "Elicitation of Honest Preferences for the Assignment of Individuals to Positions," *Journal of Political Economy*, **91**, 1–36.

Milgrom, P. (1985): "Auction Theory," in *Advances in Economic Theory, Fifth World Congress of the Econometric Society*, ed. T. Bewley. Cambridge: Cambridge University Press, 1–32.

Ore, O. (1956): "Studies on Directed Graphs, I.," *Annals of Mathematics*, **63**, 383–405.

Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *J. of Finance*, **16**, 8–37.