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Savage's Theorem with a Finite Number of States*

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Conditions which guarantee the existence of a (subjective) expected utility representation of preferences, when the state space is finite, are presented. The key assumptions are continuity and an analogue of the independence axiom. *Journal of Economic Literature* Classication Number: D80. © 1992 Academic Press, Inc.

1. INTRODUCTION

Ramsey [17] and Savage [18] have formulated the subjective or personalistic view of probability by imposing consistency or rationality requirements on preferences over bets on events and deducing utilities and probabilities as parameters of these preferences. While the concept of subjective probability is not the only conceivable nor the consensus view of probability, it is accepted to be the only coherent view in some and at least a useful alternative in many discussions of the foundations of probability.

The Savage framework involves a set of states of the world Ω, a set of consequences X, and the set of acts F, which are mappings from Ω to X. The interpretation is that, since the true state of the world $s \in \Omega$ is not known (possibly because it has not yet "occurred"), the individual's preferences over the acts depend on both the consequences of the acts (at each state) and how likely he considers the states to be.[1]

Savage shows that, given a set of "rationality" assumptions on the preferences of the individual, there will exist a unique (finitely additive) probability measure μ on the set of all subsets of Ω and a unique (up to positive affine transformations) utility function on consequences such that

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[1] A detailed analysis and interpretation of the Savage postulates can be found in Savage [18]. Fishburn [8] and Kreps [13] also provide comparisons with the other choice models discussed in this paper.

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the act f will be (weakly) preferred to the act g if and only if the expected utility of f is greater than or equal to the expected utility of g. One of the assumptions imposed by Savage necessitates that Ω be infinite.[2]

It is easy to see why the case of finite Ω is problematic. Assume that Ω and X are both finite and the individual has preferences which are represented by the function $U(f)=\sum_{s\in\Omega} u(f(s))\,\mu(s)$. Furthermore assume that $U(f)\neq U(g)$ whenever $f\neq g$. Since F is also finite, it is clear that changing u and μ slightly will not effect how U ranks the acts in F. But this shows that in general there is little hope for obtaining a unique representation of the preferences. Even more problematic is an example due to Kraft, Pratt, and Seidenberg [11] which shows that if Ω is finite and R is a comparative probability relation satisfying the di Finetti assumptions, on the set of all subsets of Ω, it may be the case that there exists no probability measure which represents R. However, Savage's postulates (P2) and (P4) yield a comparative probability relation and his proof involves constructing a probability measure which represents this relation by utilizing the very assumption (P6) which requires that Ω be infinite. Hence, the example of Kraft, Pratt, and Seidenberg poses a serious problem when Ω is finite.

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2. The Theorem

Following Savage [18], instead of "$f(s)=g(s)$ for all $s\in a$" the statement "$f=g$ on a" will be used.

Assumption 1. $\succsim$ is a preference relation (i.e., complete and transitive).

Assumption 1 is standard and requires no elaboration.

An event a will be called null if $f=g$ on a^c implies $f\sim g$. A state s will be called null if $\{s\}$ is null. It follows from Assumption 1 that a is null if and only if s is null for all $s\in\Omega$.

Assumption 2 below plays a central role in our representation theorem. For the purpose of exposition, assume momentarily that an "objective randomization" device exists. Suppose further that for some $x, y, z\in[m, M]$, x is indifferent to an act which yields y with probability p and z with probability $1-p$. For an expected utility maximizer, this is equivalent to

$$u(x)=pu(y)+(1-p)u(z).$$

So that x lies proportion p of the way from z to y "in utility terms."

[2] Technically, this assumption (P6), by itself, does not require that Ω is infinite. If the individual was indifferent among all acts, a possibility which is precluded by another Savage postulate (P5), (P6) could be satisfied vacuously.

While we do not have an objective randomization device, it is still the case that if $x \sim f$ for some f such that $f = y$ on a and $f = z$ on a^c then

$$u(x) = \text{Prob}(a)\, u(y) + (1 - \text{Prob}(a))\, u(z)$$

So again, x lies in proportion Prob(a) of the way from z to y "in utility terms." Finally note that, by the characteristic linearity property of expected utility theory, if two acts f and g are indifferent, then moving each outcome by any third act h toward corresponding outcomes of f and g by the proportion Prob(a) "in utility terms" preserves the indifference. This is the content of Assumption 2 below.[3]

Assumption 2. $f'(s) \sim af(s) + (1-a)\, h(s)$, $g'(s) \sim ag(s) + (1-a)\, h(s)$ for all $s \in \Omega$ and a is not null implies $f \succ g$ if and only if $f' \succ g'$.

Assumption 2 is analogous to the independence axiom. In words, it states the following: Take arbitrary acts f, h and some non-null (i.e., not null) event a. Consider the act f' constructed from f, h, and a by requiring that the outcome of f' in any state s to be indifferent (as a constant act) to the act which yields $f(s)$ if a occurs and $h(s)$ if a^c occurs (note that we are not requiring that it be possible to construct such an f'—Assumption 2 applies only if such f' can be constructed). Similarly, construct g' as above by using g in place of f. Then f is strictly preferred to g if and only if f' is strictly preferred to g'. As in the independence axiom, we require that preference be preserved when both f and g are "mixed" with the same h in the stylized sense described above. However, the finiteness of Ω makes Assumption 2 much weaker than the independence axiom since, because of this finiteness mixture, space arguments cannot be utilized.

To see why Assumption 2 is the appropriate analogue of the independence axiom for the case in which Ω is finite, suppose that $\succ$ satisfies the requirements of expected utility theory. Hence, there exist μ, a probability measure on Ω, and $u: X \to \mathbb{R}$, a utility function, such that (μ, u) represents $\succ$ in the sense of Savage [18]. Now consider the acts f, g, and h of Assumption 2. Let P_f, P_g, and P_h be the probability distribution of each of these acts. If there existed act f^* and g^* such that $P_{f^*} = \mu(a)\, P_f + (1-\mu(a)) P_h$ and $P_{g^*} = \mu(a) P_g + (1-\mu(a)) P_h$ then the independence axiom would imply that $f \succ g$ iff $f^* \succ g^*$. Furthermore, it follows from the fact that (μ, u) represents $\succ$ that $f^* \sim f'$ and $g^* \sim g'$.[4] Thus we obtain $f \succ g$ iff $f^* \succ g^*$ if $f' \succ g'$. But since Ω is finite, we cannot count on the existence of acts such as f^* and g^*; hence we use the condition $f \succ g$ iff $f' \succ g'$ (i.e., Assumption 2) instead of $f \succ g$ iff $f^* \succ g^*$ (i.e., the independence axiom).

[3] I am grateful to Mark Machina for suggesting this intuitive explanation of Assumption 2.

[4] This property (i.e., $f^* \sim f'$) is closely related to the isometry condition used in Nakamura [14] to obtain a similar representation theorem.

Assumption 3. $x > y$ implies $x \succsim y$. Furthermore, there exist $a_* \subset \Omega$ such that $a_* x + (1 - a_*) y \sim a_* y + (1 - a_*) x$ for all $x, y \sim X$.

The first part of Assumption 3 imposes monotonicity over constant acts. The second part requires that it be possible to partition the Ω into two "equally likely events." It should not be considered a particularly stringent requirement—provided that there is one coin that the individual considers to be "fair," replacing Ω with $\Omega \times \{\text{Heads, Tails}\}$ would satisfy the requirement. The monotonicity over constant acts is not necessary. The theorem minus the conclusion that u is strictly increasing would still hold if the monotonicity requirement were replaced with the condition that there exists $x, y \in X$ such that $x \succ y$.[5]

Since X is a subset of the reals, it is possible to view F as a subset of $\mathbb{R}^N$ where $N = |\Omega|$. Hence $v \in \mathbb{R}^N$ denotes $f \in F$ if $v_i = f(s_i)$ where $\Omega = \{s_1, s_2, ..., s_N\}$.

Thus, $G \subset F$ is said to be closed if G is a closed subset of $\mathbb{R}^N$.

Assumption 4. For all $f \in F$, the sets $B(f) = \{g \in F \mid g \succsim f\}$ and $W(f) = \{g \in F \mid f \succsim g\}$ are closed.

Assumption 4 is continuity of $\succsim$ in the sense of Debreu [5]. Given the earlier discussion of the difficulty of obtaining existence and uniqueness of numerical probabilities, its importance is clear. It replaces (P6) of Savage which requires that (the subjective) probability measure that is ultimately constructed from preferences be non-atomic. Hence Ω is required to be an infinite set. In a sense, we are substituting one kind of continuity for another.

THEOREM. *If $\succsim$ satisfies Assumptions 1–4, then there exists a probability measure μ on the set of all subsets of Ω and a function u: $X \to \mathbb{R}$ such that*

(i) $f \succsim g$ *iff* $\sum_s u(f(s))\, \mu(s) \geq \sum_s u(g(s))\, \mu(s)$;

(ii) *u is continuous and strictly increasing*;

(iii) *if* (i) *above holds when μ is replaced by the probability measure μ' and u is replaced by u': $X \to \mathbb{R}$, then $\mu' = \mu$ and $u' = cu + b$ for some $c > 0$, $b \in \mathbb{R}$.*

The conclusions (i) and (iii) are the standard conclusions of expected utility theory. Assumption 4 also guarantees that u is continuous. The monotonicity requirement of Assumption 3 guarantees that u is strictly increasing. Thus certainty equivalent are well-defined and f is strictly preferred to g whenever f stochastically dominates g.

[5] The proof, however, would have to be modified. In particular, as I have been informed by Ennio Stacchetti, Lemma 8 of the appendix would require a somewhat lengthier proof.

3. Conclusion

A finite state version of the Savage theorem has been provided. Assumptions 2 and 4 appear to circumvent the problem posed by the Kraft, Pratt, and Seidenberg example in a manner similar to the Anscombe Aumann approach. They use the "roulette lottery" probabilities to determine the "horserace lottery" probabilities; here the richness of the set X is used for the same purpose.

Although this paper was obviously motivated by Savage [18], the method of proof that is utilized bears a strong resemblance to the approach outlined by Ramsey [17] in that "even chance" events are used to determine the utility function and the resulting utility function is used to calibrate the remaining probabilities. In spite of the fact that the proof relies heavily on Assumption 3, the existence of even chance events are not necessary for the conclusions of the theorem. It is, however, necessary for the "uniqueness" of u that there exist at least two non-null states. Hence, it might be that the theorem holds when the even chance event requirement is replaced which the weaker condition that there are at least two non-null states (in which case all of the conditions would be necessary and sufficient). However, it is clear that a proof without even chance events would require a substantially more complicated argument, which I am unable to furnish.[6]

Alternative axiomatizations of subjective expected utility theory for the case in which Ω is finite can be found in Davidson and Suppes [4] Debreu [7], Hens [10], Suppes [20], Nakamura [14], Stigum [19], and Wakker [22]. Davidson and Suppes [4] deal with the case in which X is also finite. They establish, under somewhat restrictive assumptions, the existence of a subjective expected utility representation (with a non-additive probability measure). Debreu [7] considers the special case in which Ω consists of two equally likely states and Hens [10] and Stigum [19] utilize differentiability conditions to obtain the desired representations. Stigum also imposes quasi-concavity on the preferences.

Nakamura [14] and Wakker [22] provide different axioms which also yield a Savage type representation result for finite Ω. The key assumption of Nakamura is an isometry or bisymmetry condition[7] which apparently plays a role similar to that of Assumption 2. Wakker's related theorem imposes the same topological properties on X and $\succcurlyeq$ as our theorem, but

[6] The lack of necessity of the "even chance" events requirement has recently been established by Chew and Karni [5].

[7] Axioms related to this isometry/bisymmetry condition can be found in Aczel [1], Chew [3], Fishburn [9], Krantz, Luce, Suppes, and Tversky [12], Pfanzagl [15] and Quiggin [16].

his "no contradictory tradeoffs" axiom differs substantially from Assumption 2 and the related isometry condition of Nakamura. Both Nakamura [14] and Wakker [22] provide extensions of their basic framework to accommodate non-additive probability measures.

APPENDIX

Proof of the Theorem. Let $(x, y)_0$ denote $a_* x + (1-a_*) y$.

LEMMA 1. *$x > y$ implies*

(i) $x \succ (x, y)_0 \succ y$

(ii) $(x, z)_0 \succ (y, \bar{z})_0$ *whenever* $z \geqslant \bar{z}$.

Proof. (i) Assume $(x, y)_0 \succcurlyeq x$; then by A4, there axists $\bar{x} \in (y, x)$ such that $(\bar{x}, y)_0 \sim x$. By A3, $(\bar{x}, \bar{x})_0 \succ (y, y)_0$; hence by A2, $(\bar{x}, \bar{x})_0 \succ (x, x)$ which contradicts A3. A symmetric argument establishes that $(x, y)_0 \succ y$.

(ii) By (i) above and A4, there exists $\bar{y}, \bar{x}$ such that $\bar{y} \sim (y, z)_0$ and $\bar{x} \sim (x, z)_0$. But by A3, $x \succ y$, so applying A2 yields $\bar{x} \succ \bar{y}$. Hence, $(x, z)_0 \succ (y, z)_0$. Repeating the argument for $(y, z)_0$ and $(y, \bar{z})_0$ completes the proof. ∎

Note that by Lemma 1 and A4, for all $(x, y)_0$ there exists a unique t such that $t \sim (x, y)_0$.

LEMMA 2. (i) *There exists a continuous function $u: X \to \mathbb{R}$ such that $(x, y)_0 \succcurlyeq (w, z)_0$ iff $u(x) + u(y) \geqslant u(w) + u(z)$, u is continuous and unique up to (positive) affine transformations.*

(ii) *u is strictly increasing and can be taken to be such that $u(X) = [0, 1]$.*

Proof. (i) Theorem 1 of Debreu [4] states that if $\succcurlyeq$ is a preference on $S \times S$ for some connected separable S, and $\succcurlyeq$ satisfies A4 and $(*)$ below, then (i) is satisfied. Noting that X is connected and separable establishes that (i) hinges on showing:

$$(x_2, y_1)_0 \succcurlyeq (x_1, y_2)_0 \text{ and } (x_3, y_2)_0 \succcurlyeq (x_2, y_3)_0 \text{ implies } (y_1, x_3)_0 \succcurlyeq (y_3, x_1)_0. \qquad (*)$$

To prove $(*)$, note that by Lemma 1 and A3, $(M, y_2)_0 \succcurlyeq (x_2, m)_0$ so by the premise of $(*)$, A4, and Lemma 1, there exist $\bar{y}_1 \leqslant y_1$, $\bar{x}_1 \geqslant x_1$ and $t \in X$ such that $(x_2, \bar{y}_1)_0 \sim (\bar{x}_1, y_2)_0 \sim t$. A similar argument yields $\bar{x}_3 \leqslant x_3$, $\bar{y}_3 \geqslant y_3$, and $\bar{t}$ such that $(\bar{x}_3, y_2)_0 \sim (x_2, \bar{y}_3)_0 \sim \bar{t}$.

Let $f=(\bar{y}_1, \bar{x}_3)_0$, $g=(\bar{y}_3, \bar{x}_1)$, $h=(x_2, y_2)$ and $a=a_*$. Then A2 and A3 yield $f \sim g$ if $(t, \bar{t})_0 \sim (\bar{t}, t)_0$. Thus by A3, $(\bar{y}_1, \bar{x}_3) \sim (\bar{y}_3, \bar{x}_1)$. Then Lemma 1 yields the desired conclusion.

(ii) Follows from (i) and A3 (monotonicity). ∎

LEMMA 3. (i) *For any* $y_0 \in X$ *define* $y_i \sim (y_{i-1}, x)_0$ *for* $i \geqslant 1$. *The sequence* $\{y_i\}$ *converges to* x.

(ii) *Let* $S=(x_1, x_2, x_3, ..., x_n)$. *We say that* y_0 *reaches* x *through* S *iff* $y_i \sim (y_{i-1}, x_i)$ *for* $i=1, 2, ..., n$ *and* $y_n = x$.

For $y_0 \in X$ *and* $x \in (m, M)$ *there exists some (finite)* S *such that* y_0 *reaches* x *through* S.

Proof. (i) Assume wlog that $x > y_0$ (if $x = y_0$ there is nothing to prove; if $x < y_0$, the argument is symmetric). Then by Lemma 1 and A3, y_i is a strictly increasing sequence and $y_i < x$ for all i. Assume $\lim y_i = \bar{y} < x$. Let $\hat{y} \sim (\bar{y}, x)_0$. Again by Lemma 1 and A3, $\bar{y} < \hat{y} < x$. So $(1/2)(\hat{y}+\bar{y}) > \bar{y} > y_{i+1} \sim (y_i, x)_0$. Hence by A3, $(1/2(\hat{y}+\bar{y}) \rangle (y_i, x)_0$. But $\lim (y_i, x)_0 = (\bar{y}, x)_0 \sim \hat{y} \rangle (1/2)(\hat{y}+\bar{y})$ contradicting A4.

(ii) Again wlog assume $x > y_0$. Let $y_i \sim (y_{i-1}, M)_0$. By (i), y_i converges to M. Let $n = \inf\{i \mid y_i > x\} - 1$ (since $\lim y_i = M$, n is well-defined). Hence $y_{n-1} \leqslant x < y_n \sim (y_{n-1}, M)_0$. So by A4, A3, and Lemma 1, there exists z such that $x \sim (y_{n-1}, z)$. Thus setting $x_i = M$ for $i = 1, 2, ..., n-1$ and $x_n = z$ establishes the desired S. ∎

LEMMA 4. *Let* $\bar{S}=(f_1, f_2, ..., f_n)$. *We say that* g_0 *reaches* f *through* $\bar{S}$ *if for all* $s \in \Omega$, $g_i(s) \sim a_* g_{i-1}(s) + (1-a_*) f_i(s)$ *for* $i=1, 2, ..., n$ *and* $g_n = f$.

(i) $g_0 \in F$, $f(s) \in (m, M)$ *for all* $s \in \Omega$ *implies there exists* $\bar{S}$ *such that* g_0 *reaches* f *through* $\bar{S}$.

(ii) *If* g_0 *reaches* f *through* $\bar{S}$ *and* $\hat{g}_0$ *reaches* $\hat{f}$ *through* $\bar{S}$ *then* $g_0 \rangle \hat{g}_0$ *iff* $f \rangle \hat{f}$ *and for all* $s \in \Omega$, $g_0(s) > \hat{g}_0(s)$ *iff* $f(s) > \hat{f}(s)$.

Proof. (i) Follows from a repeated application of Lemma 3.

(ii) The first statement in (ii) follows from a repeated application of A2; the second statement follows from Lemma 1 and A3. ∎

LEMMA 5. $\hat{f}(s) \geqslant \hat{g}(s)$ *for all* $s \in \Omega$ *and* $\hat{f}(s^*) > \hat{g}(s^*)$ *for some* s^* *not null implies* $f \rangle g$ [*This is essentially Savage's Postulate* 3].

Proof. We will establish the result for the case in which $\hat{f} = \hat{g}$ on $\Omega \backslash \{s^*\}$. Then the transitivity of $\rangle$ yields the desired conclusion. By Lemma 4(i), for $x \in X$, there exists $\bar{S}$ such that $\hat{f}$ reaches x through $\bar{S}$. Then by Lemma 4(ii), $\hat{g}$ reaches $\bar{g}$ such that $\bar{g}=x$ on $\Omega\backslash\{s^*\}$ and $\bar{g}(s)=y<x$, through $\bar{S}$. Furthermore, by Lemma 3(ii) we can make y arbitrarily close to x so that $M\rangle g\rangle m$; hence by A4, there exists $\bar{x}\in X$ such that $\bar{x}\sim\bar{g}$. Letting $f=x\rangle y=g$, $h=x$, $a=a_*$ and applying A2 yields $x\rangle\bar{x}\sim\bar{g}$. Hence by Lemma 4(ii), $\hat{f}=\hat{g}$. ∎

For a non-null a define CE(a, f) to be x such that $\bar{f}=x$ on a, $\bar{f}=f$ on a^c implies $f\sim\bar{f}$. Lemma 5 and A4 guarantee that CE(a, f) is well-defined. CE(f) will be used to denote CE(Ω, f).

LEMMA 6. *If $f=g$ on a^c, $f'=g'$ on a^c, $f=f'$ on a, and $g=g'$ on a, then $f\rangle g$ implies $f'\rangle g'$ [This is Savage's sure-thing principle (Postulate 2)].*

Proof. If $g'(s)\in(m, M)$ for all s then let $\bar{S}$ be any finite sequence such that f reaches $\bar{f}$ through $\bar{S}$ where $\bar{f}=x\in(m, M)$ on a and $\bar{f}=g'$ on a^c. Such an $\bar{S}$ exists by Lemma 4(i). Then by Lemma 4(ii), g reaches some $\bar{g}$ such that $\bar{g}=g'$ on a^c through $\bar{S}$. Now replace each h_i in $\bar{S}$ with h_i' such that $h_i'=h_i$ on a and $h_i=g_i'$ on a^c. Call the resulting sequence $\bar{S}'$. Hence $f\rangle g$ iff $f'\rangle g'$ by Lemma 4(ii). If there exists $s\in a^c$ such that $g(s)\in(m, M)$ then define f_1, g_1, f_1', g_1' by $f_1(s)\sim a_* f(s)+(1-a_*)x$, $g_1(s)\sim a_* g(s)+(1-a_*)x$, $f_1'(s)=a_* f'(s)+(1-a_*)x$ and $g_1'(s)=a_* g'(s)+(1-a_*)x$ for all $s\in\Omega$ and some $x\in(m, m)$. By Lemma 1, $g_1'(s)\in(m, M)$ for all $s\in\Omega$. So apply the above argument to obtain $f_1\rangle g_1$ iff $f_1'\rangle g_1'$. But by A2, $f\rangle g$ iff $f_1\rangle g_1$ and $f'\rangle g'$ iff $f_1'\rangle g_1'$ which establishes the desired result. ∎

Define $(x, y)_a$ to be $ax+(1-a)y$. Let $p(a)=u(\mathrm{CE}(M, m)_a)$ for all $a\subset\Omega$.

LEMMA 7. *$p(a)\,u(x)+(1-p(a))\,u(y)=u(z)$ and $|u(x)-u(y)|=(1/2^n)$ for some $n\in\mathbb{N}$ implies $(x, y)_a\sim(z, z)_a$.*

Proof. If a is null or a^c is null the result is trivial, so assume that both a and a^c are not null. The proof will use induction on n. Let $n=0$; hence $|u(x)-u(y)|=1$ which implies that either $x=M$ and $y=m$, or $x=m$ and $y=M$. For the first case, we have $u(z)=p(a)\,u(x)+(1-p(a))\,u(y)=p(a)$. But by definition $(M, m)_a\sim(z', z')_a$ for some z' such that $u(z')=p(a)$; but u is one-to-one, hence $z'=z$, and therefore $(z, z)_a\sim(x, y)_a$. For the $(m, M)_a$ case, note that by A3, $(M, m)_0\sim(m, M)_0$. Hence by A2, $(\mathrm{CE}(M, m)_a, \mathrm{CE}(m, M)_a)_0\sim(\mathrm{CE}(m, m)_a, \mathrm{CE}(M, M)_a)_0$. Therefore $(\bar{z}, \hat{z})_0\sim(m, M)_0$ for $\bar{z}, \hat{z}$ such that $\hat{z}=\mathrm{CE}(m, M)_a$ and $u(\bar{z})=p(a)$. But by Lemma 2, we have $u(\bar{z})+u(\hat{z})=1$; hence $u(\hat{z})=1-p(a)$. Thus $(m, M)_a\sim(\hat{z}, \hat{z})_a$ for some $\hat{z}$ such that $u(\hat{z})=1-p(a)=p(a)\,u(m)+(1-p(a))\,u(M)=p(a)\,u(x)+(1-p(a))\,u(y)$. But since u is one-to-one, this establishes the desired result for $n=0$.

Assume that the lemma holds for n and let $p(a)\,u(x)+(1-p(a))\,u(y)=u(z)$, $|u(x)-u(y)|=(1/2^{n+1})$. Find $\bar{x}$, $\bar{y}$ such that $|u(\bar{x})-u(\bar{y})|=(1/2^n)$ and either $\bar{x}\geqslant x>y\geqslant \bar{y}$ or $\bar{y}\geqslant y>x\geqslant \bar{x}$. Since u is continuous, such $\bar{x}$, $\bar{y}$ always exist. Without loss of generality, assume $\bar{x}\geqslant x>y\geqslant \bar{y}$. Let u^* be such that $(1/2)[u(\bar{x})+u^*]=u(x)$ and choose w such that $u(w)=u^*$. Note that $u^*=2u(x)-u(\bar{x})\leqslant u(x)\leqslant 1$ and $2u(x)=2[u(y)+(1/2^{n+1})]=2u(y)+(1/2^n)$ and $u(\bar{x})=u(\bar{y})+(1/2^n)$. Hence $u^*=2u(x)-u(\bar{x})=2u(y)-u(\bar{y})\geqslant u(y)\geqslant 0$. Therefore $u^*\in[0,1]$ and w is well-defined. By the induction hypothesis $(\bar{x},\bar{y})_a\sim(\bar{z},\bar{z})_a$ for some $\bar{z}$ such that $u(\bar{z})=p(a)\,u(\bar{x})+(1-p(a))\,u(\bar{y})$ (again, we are using the continuity of u and the fact that $u(X)=[0,1]$).

Then by A2, $(\mathrm{CE}(\bar{x},w)_0,\ \mathrm{CE}(\bar{y},w)_0)_a\sim(\mathrm{CE}(\bar{z},w)_0,\ \mathrm{CE}(\bar{x},w)_0)_a$. Note that $u(\bar{y})+\ddot{u}(w)=u(\bar{y})+2u(y)-u(\bar{y})=2u(y)$ and $u(\bar{x})+u(w)=u(\bar{x})+2u(x)-u(\bar{x})=2u(x)$. Therefore by Lemma 2, $(x,y)_a\sim(\mathrm{CE}(\bar{z},w)_0,\mathrm{CE}(\bar{z},w)_0)_a$. Again by Lemma 2, $\mathrm{CE}(\bar{z},w)_0=z'$ such that $u(z')=(1/2)[u(\bar{z})+u(w)]=(1/2)[p(a)\,u(\bar{x})+(1-p(a))\,u(\bar{y})+2u(x)-u(\bar{x})]=(1/2)[2u(x)+(1-p(a))(u(\bar{y})-u(\bar{x}))]=u(x)+(1-p(a))(u(y)-u(x))=p(a)\,u(x)+(1-p(a))\,u(y)$.

Therefore $(x,y)_a\sim(z',z')_a$ for some z' such that $u(z')=p(a)\,u(x)+(1-p(a))\,u(y)$. But this establishes that $z'=z$ and concludes the proof. ∎

Lemma 8. *$p(a)\,u(x)+(1-p(a))\,u(y)=u(z)$, $|u(x)-u(y)|=(h/2^n)$ for some $h,n\in\mathbb{N}$, $h\leqslant 2^n$ implies $(x,y)_a\sim(z,z)_a$.*

Proof. Again by induction, let $L^1(n)$ be the lemma for a fixed n. $L^1(0)$ and $L^1(1)$ follow from Lemma 7. To show that $L^1(n)$ implies $L^1(n+1)$ for $n\geqslant 1$, assume that $|u(x)-u(y)|=(h/2^{n+1})$. Now use induction on h. Let $L^2(l)$ be the proposition when $h=l$ (note that $n+1$ is fixed). $L^2(0)$ is trivial. $L^2(1)$ follows from Lemma 9. Hence what remains to be shown is that $L^2(l)$ implies $L^2(l+1)$ for $l\geqslant 1$. If l is odd, $l+1$ is even, so $(l+1)/(2^{n+1})=(s/2^n)$ for some integer s and hence $L^1(n)$ establishes the desired result. So let l be even. Without loss of generality, assume $u(x)>u(y)$ ($u(y)>u(x)$ is symmetric). Choose x_1, y_1 such that $u(x_1)=u(x)-(1/2^{n+1})$, $u(y_2)=u(y)+(1/2^{n+1})$ and w such that $(x,y)_a\sim(w,w)_a$ (such w exist by continuity). Then by A2, $(\mathrm{CE}(x,x_1)_0,\ \mathrm{CE}(y,y_1)_0)_a\sim(\mathrm{CE}(w,x_1)_0,\ \mathrm{CE}(w,y_1)_0)_a$. Let $t_1=\mathrm{CE}(x,x_1)_0$ and $t_2=\mathrm{CE}(y,y_1)_0$. By Lemma 2, $u(t_1)=u(x)-(1/2^{n+2})$ and $u(t_2)=u(y)+(1/2^{n+1})$. Choose $\bar{t}$ such that $u(\bar{t})=p(a)\,u(t_1)+(1-p(a))\,u(t_2)$. Hence $|u(t_1)-u(t_2)|=u(x)-u(y)-(1/2^{n+1})=(l/2^{n+1})$. But then by $L^2(l)$, $(t_1,t_2)_a\sim(\bar{t},\bar{t})_a$. Let $w_1=\mathrm{CE}(w,x_1)_0$ and $w_2=\mathrm{CE}(w,x_2)_0$. Hence $(\bar{t},\bar{t})_a\sim(w_1,w_2)_a$. By Lemma 2, $u(w_1)=(1/2)[u(w)+u(x_1)]$, $u(w_2)=(1/2)[u(w)+u(y_1)]$ and hence $[u(w_1)-u(w_2)|=(1/2)[u(x_1)-u(y_1)]=(1/2)[u(x)-u(y)]-(1/2^{n+1})=(l/2^{n+1})$. Choose t' such that $u(t')=p(a)\,u(w_1)+(1-p(a))\,u(w_2)$. Then by $L^2(l)$, $(w_1,w_2)_a\sim(t',t')_a\sim(t',t')_a$. Hence by Lemma 1,

$\bar{t}=t'$ and therefore $u(t')=u(\bar{t})=p(a)\,u(t_1)+(1-p(a))\,u(t_2)$. Hence $p(a)\,u(t_1)+(1-p(a))\,u(t_2)=p(a)\,u(w_1)+(1-p(a)\,u(w_2)$ which implies $p(a)(u(x)-(1/2^{n+2}))+(1-p(a))(u(y+(1/2^{n+2}))=p(a)(1/2)(u(w)+u(x)-(1/2^{n+1}))+(1-p(a))(1/2)(u(w)+u(y)+(1/2^{n+1}))$.

Therefore $u(w)=p(a)\,u(x)+(1-p(a))\,u(y)=u(z)$. Hence $w=z$ and $(z,z)_a\sim(w,w)_a\sim(x,y)_a$. ∎

LEMMA 9. *$(x,y)_a \succsim (w\,z)_a$ iff $p(a)\,u(x)+(1-p(a))\,u(y)\geqslant p(a)\,u(w)+(1-p(a))\,u(z)$.*

Proof. The result follows if it can be shown that $(x,y)\sim t$ iff $p(a)\,u(x)+(1-p(a))\,u(y)=u(t)$. It follows from Lemma 8 that $p(a)=1-p(a^c)$. Furthermore the result is trivial if a is null, a^c is null or $x,y\notin(m,M)$. Hence wlog assume that a and a^c are not null, and $x,y\in(m,M)$. To prove the only if part of the statement, let $t=\mathrm{CE}(x,y)_a$ and $\{x_i\}$ be a sequence which converges to x from above and satisfies $|u(x_i)-u(y)|=(k_i/2^{n_i})$ for some integers k_i, n_i. Since the set $\{(k/2^n)\mid k,n\in\mathbb{N}$, and $k\leqslant 2^n\}$ is dense in $[0,1]$ and since u is strictly increasing and with $u(x)=[0,1]$, such a sequence exists. Let t_n be such that $u(t_n)=p(a)\,u(x_n)+(1-p(a))\,u(y)$. Then by Lemma 7, $t_n\sim(x_n,y)_a$ and by Lemma 5, $t_n\sim(x_n,y)_a\succ(x,y)\sim t$. Hence $u(t_n)>u(t)$ for all n. X is compact, so t_n has a convergent subsequence; wlog assume it converges to $\bar{t}$. Then by the continuity of u, $\lim[p(a)\,u(x_n)+(1-p(a))\,u(y)]=p(a)\,u(x)+(1-p(a))\,u(y)\geqslant u(\bar{t})\geqslant u(t)$. The reverse inequality is established by a symmetric argument. To prove the if part of the statement, note that $(x_n,y)\succ t$ implies (by A4), $(x,y)\succsim t$. Again a symmetric argument completes the proof. ∎

LEMMA 10. *Let $a,b\subset\Omega$, $a\cap b=\varnothing$, $f=x$ on a, $f=y$ on b, $g=z$ on $a\cup b$, $g=f$ on $(a\cup b)^c$, then $p(a)\,u(x)+p(b)\,u(y)=p(a\cup b)\,u(z)$ implies $f\sim g$.*

Proof. First we will show that $p(a\cup b)=p(a)+p(b)$.

Note that if a is null, then $\mathrm{CE}(M,m)_a=m$; hence $p(a)=u(\mathrm{CE}(M,m)_a)=0$.

Therefore, if a and b are both null, then obviously $a\cup b$ is null so $p(a\cup b)=0=p(a)+p(b)$. If only one of a and b is null (say a) then $\mathrm{CE}(M,m)_b=\mathrm{CE}(M,m)_{a\cup b}$. So $p(a\cup b)=p(b)=p(b)+p(a)$. If neither a nor b is null, then let $\bar{z}=\mathrm{CE}(a\cup b,(M,m)_a)$.

Hence $(M,m)_a\sim\mathrm{CE}(M,m)_a\sim(\bar{z},m)_{a\cup b}$. Therefore by Lemma 9,

$$p(a)=p(a\cup b)\,u(\bar{z}). \tag{1}$$

By Lemma 6, $(m, M)_b \sim (\bar{z}, M)_{a \cup b}$. Let $t = \mathrm{CE}(m, M)_b$; then by Lemma 9,

$$p(b^c) = 1 - p(b) = u(t) = p(a \cup b)\, u(\bar{z}) + 1 - p(a \cup b). \qquad (2)$$

Equations (1) and (2) yield $p(a \cup b) = p(a) + p(b)$.

If either a or b is null, then obviously $f \sim g$. If both a and b are not null, then let $z' = \mathrm{CE}(a \cup b, f)$. By Lemma 6, $(x, y)_a \sim (z', y)_{a \cup b}$. Let $t' = \mathrm{CE}(x, y)_a$. By Lemma 9,

$$p(a)\, u(x) + (1 - p(a))\, u(y) = u(t') = p(a \cup b)\, u(z') + (1 - p(a \cup b))\, u(y).$$

Noting that $p(a \cup b) = p(a) + p(b)$ yields $p(a \cup b)\, u(z') = p(a)\, u(x) + (1 - p(a))\, u(y) = p(a \cup b)\, u(z)$; hence $z = z'$ which is the desired conclusion. ▮

Let $U(f) = \sum_S u(f(s))\, p(s)$. We establish the existence of the desired representation by showing that $f \succsim g$ iff $U(f) \geqslant U(g)$ (note that the additivity of p has been proven in Lemma 10).

Let $\Omega_0 = \{s_1, s_2, ..., s_{\bar{N}}\}$ denote the set of non-null states. For $f \in F$ define $f_1, f_2, ..., f_{\bar{N}}$ as follows: $z_1 = f(s_1)$, $f_1 = f$. For $n \geqslant 2$, $f_n = z_n$ on a_n and $f_n = f_{n-1}$ on a_n^c where $a_n = \bigcup_{i=1}^n \{s_i\}$ and z_n is such that $p(a_n)\, u(z_n) = p(s_n)\, u(f(s_n)) + p(a_{n-1})\, u(z_{n-1})$. By construction $U(f_n) = U(f_{n+1})$ and by Lemma 9, $f_n \sim f_{n+1}$ for all $n \geqslant 1$. Furthermore $f_{\bar{N}} = z_{\bar{N}}$ on Ω^0. Then $f \sim z$ and $U(f) = U(f_{\bar{N}-1}) = u(z)$ (the last equality follows from the fact that $p(\Omega) = 1$, p is additive and $p(s) = 0$ for $s \in \Omega \backslash \Omega^0$, hence $p(\Omega^0) = 1$). Repeating the same argument for g yields z' such that $U(g) = u(z')$ and $z' \sim g$. If $f \succsim g$, then by Lemma 1, $z \geqslant z'$. Hence $U(f) = u(z) \geqslant u(z') = U(g)$ (since u is increasing). Similarly if $U(f) \geqslant U(g)$, then $u(z) \geqslant u(z')$. Hence $f \sim z \succsim z' \sim g$.

The uniqueness (up to affine transformations) of u follows from Lemma 2. But the uniqueness of p follows from the uniqueness of u. Since $(M, m)_a \sim x$ implies $p(a)\, u(M) + (1 - p(a))\, u(m) = u(x)$ so that $p(a) = (u(x) - u(m))/(u(M) - u(m))$ which is invariant across affine transformations of u. ▮

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Insert for "Savage's Theorem with a Finite Number of States"
by Faruk Gul

Finally even if it is assumed that the comparative probability relation implied by (P4) can be represented by some probability measure, an expected utility representation does not follow from the remaining Savage postulates when (P6) is abandoned.

The purpose of this paper is to provide an analog to Savage's theorem when Ω is finite. This will be done by replacing Savage's "continuity" assumption (P6) on Ω with a continuity assumption on X and his sure thing principle (P2), preference among consequences postulate (P3), and qualitative probability postulate (P4) with an assumption which formally resembles the independence axiom of von Neumann and Morgenstern [21].

Given the importance of Savage's theorem it would appear that it is worthwhile to inquire what is at stake in obtaining a Savage representation when Ω is finite. Furthermore unlike Savage's continuity postulate the type of continuity that is assumed in this paper is familiar from the theory of the consumer. Thus the finite state version of the theory enables a more unified treatment of individual choice (with or without uncertainty). In the same vein, the similarity between the assumption which replaces the (P2), (P3) and (P4) postulates and the independence axiom enables a more unified approach to Savage's theorem and the work that assumes extraneous probabilities (such as von Neumann and Morgenstern [21] and Anscombe and Aumann [2]). Finally, since the objective is to obtain a normative theory, an alternative axiomatization of subjective expected utility, to the extent that the axioms are considered reasonable, should serve as a useful compliment to Savage's theorem.

The statement of the assumptions and the theorem are provided in the following section. A proof of the theorem is in the appendix. The paper concludes with a brief discussion of the result and the related literature.

Let Ω be a finite set of states, $X = [m, M] \subset R$ where $m < M$ and $F = \{f \mid f : \Omega \to X\}$. The individual's preferences on F will be described by a binary relation $\succsim \subset F \times F$. The binary relation $\succsim$ is said to be a preference relation if it is transitive and complete. The symbols $\succ$ and $\sim$ are used to denote the strict preference and indifference relations associated with $\succsim$.

An act f is said to be a constant act if there exists $x \in X$ such that $f(s) = x$ for all $s \in \Omega$. A constant act is often identified with its unique consequence x. Hence we write $x \succsim g$ in place of $f \succsim g$ when f is a constant act. For any event $a \subset \Omega$ and $x, y \in X$, $ax + (1 - a)y$ denotes the act f such that $f(s) = x$ for all $s \in a$ and $f(s) = y$ for all $s \in \Omega \backslash a$. The event $\Omega \backslash a$ is denoted by a^c.