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Item BarCode: 32101050394863
Item Title: Logic and ontology.
Item Author:
Item Call Number: 6251.594
Item Vol/Part:

Article Title: 'Extension, Intension, and Comprehension'
Article Author: B. van Fraassen
Art Vol/Part: 1973
Beg Page: 101 End Page: 131 Total Pages: 31
Other Info: 101-131
Notes:

TOTAL COUNT: 1

would wish to say simply that the term 'exists' as it appears in connection with the subclass is to be understood *intensionally,* we then face the question as to what *this* means: Is 'exists' to be thought of as definable, or as a primitive, undefined term? Whichever answer is given, it would seem the *same* answer as is given for the use of the term 'exists' in connection with the *subclass,* can be given for the use of the term 'exists' in connection with an *individual,* as in the statement '*a* exists', and so on. We should not, in other words, have gained any deeper understanding of how 'exists' is to be used predicatively by detouring to the use of this term as applied to an entire *subclass.*

Extension, Intension, and Comprehension[1]

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The central arguments in this paper concern adverbs. At first sight, adverbs are not a likely subject for philosophical discussion, nor would they seem to be relevant to existence. But these arguments mean to provide the essential tactical support for my overall strategy: My aim is to show that there are distinctions in traditional logical theory to which the orthodoxies in current logical theory do not do justice. These distinctions do concern possible and actual being, but not solely.

This strategy is pursued in sections 1 through 5. Section 6 is a polemic concerning metaphysics and methodology, and the Appendices provide the technical apparatus for a logic of comprehension and adverbial modification.

I

Distinctions and the Theory Thereof

In this section, I shall both draw and discuss distinctions. The first distinction is that between *being* and existence. I cannot define

[1] I have benefited much from discussions especially with Professor R. H. Thomason, Yale University; Professor T. Parsons, University of Illinois at Chicago Circle; and Miss Hidé Ishiguro, University College (London). This research was supported first by the Canada Council and then by the John Simon Guggenheim Memorial Foundation.

what existence amounts to, though I can give tautological equivalents: To exist is

to have real being,
to belong to the extension of some predicate,
to be identical with some existent.

Being, however, belongs to any subject of discourse, existent or nonexistent, possible or impossible, real or imaginary or unimaginable or inconceivable.

I shall freely say that there are things which do not exist. There are also things that are impossible. I do subscribe to the view that, in moments of high seriousness, a philosopher ought not to use "there is" except when willing to use "there exists." But high seriousness is highly inconvenient in ordinary contexts, and I shall say no more on this methodological point until the final section.

Just as I shall not equate being with existence, I shall not equate being with being possible. But the region of the possibles is as important a subregion of being as the region of existents, for logical theory. Unfortunately, the term "possible" is not univocal. (Perhaps it was to begin in no worse shape than "existent," but we have learned from Quine to insist firmly on the univocity of existence, and I am not inclined to renege on that.) For example, if it is possible that a given thing is possibly a possible entity, does it follow that it is a possible entity? If we say *no,* gradations appear in the region of possible being. But, as is clear from the immediately preceding assertion, we labor under the guidance of a picture in which "possible" has a maximal sense in which it pertains to a region of being including all subregions which qualify as referents of "the region of possible being" in some sense.[2] Hence univocity can be maintained by insisting on that maximal sense.

I turn now to a second set of distinctions, to be drawn between distinctions. Taking the liberty to choose from common but not uniform terminology, I shall describe the medieval distinctions among *distinctions,* extrapolating and reconstructing where I must. Between any two individuals, there is a real distinction; we express this by saying that they are not identical; they could exist separately. In this sense there is no *real* distinction between the evening star and the morning star.

Mobilizing our earlier distinction between being and existence, we might ask whether there is a real distinction between nonexistents.

[2] The grammatical role of *labor* in this sentence is that in "labor under a delusion." Something similar could be said of naïve discourse about sets, though the doubts there would be more acute.

I do not know of any discussion of the matter. Some authors apparently granted only a logical or conceptual status to possibles, and I imagine that there can only be a logical or conceptual distinction between the entities of reason. But the distinction between a real thing and a chimaera is presumably a real distinction.

In any case, distinctions among individuals can be used to distinguish properties in a straightforward way. I shall say that two properties are *existentially* distinct if some existent has the one but not the other, and *conceptually* distinct if some possible has the one but not the other. The question whether the conceptual distinction as well as the existential distinction between properties can be explained in terms of the real distinction is not one that will matter much to us here. What is most interesting is that Medieval philosophers discuss a distinction among properties that is clearly neither of the two above. I refer to the *formal* distinction. Two properties may be formally distinct although they are conceptually identical. Thus Scotus held that the transcendentals are formally distinct.[3] For example, whatever is, is one; whatever is one, is; yet *ens* and *unum* are (formally) distinct properties.

The formal distinction is to be distinguished from a merely verbal distinction. The paradigm of the latter concerns definition: If "human" is completely characterized by saying that it is (introduced as) *short for* "rational animal," then any distinction made between humanity and rational animality must be a purely verbal one (and to think of such a distinction as properly a distinction is to confuse use and mention). There are two defining characteristics of the formal distinction:

(1) The distinction is objective, founded *a parte rei.*
(2) If X and Y are formally distinct, not even God's power could separate them.

I take the second characteristic to be very strong: If X and Y are formally distinct properties, the possible instances of X and those of Y must be the same, in the strongest sense of "possible" that we can have.

In a previous paper I argued that the distinction between Being and Non-Being in Plato's *Sophist* is thus.[4] And a platonist today

[3] While Scotus did not introduce the formal distinction, he made the most extensive use of it. It should also be noted that he amended the theory of transcendentals; what I refer to above he termed the subclass of *passiones convertibiles,* properties convertible with being.
[4] "Logical Structure in Plato's *Sophist,*" *Review of Metaphysics,* Vol. 22 (1969), pp. 482–498.

could easily insist that properties might be distinct, although all their possible instances are the same.[5] But I cannot accept that as an account of the formal distinction; I would require at least non-tautological identity conditions for properties. Distinctions, after all, should not be multiplied beyond necessity. To arrive at an adequate account I shall now detour via some problems concerning adverbs.

II

Predicate Modifiers and Extensionality

Syntactically there are many kinds of predicate modifiers. He speaks glibly, with a forked tongue, and sometimes tongue in cheek; he is a diplomat, he is a two-faced diplomat, he stoops at nothing, he stoops to conquer, by fair means or foul. The examples I have given show predicates modified by adverbs, adverbial phrases, adjectives, infinitives, and prepositional phrases. In some cases the modified predicate is logically equivalent to a complex of unmodified predicates, but not in all. So, in general syntax, we should recognize a special class of functors, the *predicate modifiers* which turn predicates into predicates. In an extended but practical usage we may call them adverbs.

Superficially, at least, predicate modifiers present a danger to extensionality. Suppose for a moment that those who drive are exactly those who walk; it certainly does not follow that those who drive slowly, walk slowly. Hence it seems that we cannot replace predicates with coextensive predicates within adverbial contexts and hope to preserve truth.

But is this appearance perhaps deceptive, disappearing when we speak perspicuously with the learned? I shall examine now, at some length, an analysis of predicate modification which tends to support the affirmative. I base this analysis on Davidson's analysis of action sentences, but some of its more *outré* moments are due to Gilbert Harman and to my devil's advocacy.[6]

Consider the sentence "John walks slowly." Davidson analyzed this into the counterpart: "There is an event which is a walking, and is of (by) John, and is slow." No wonder then that slow drivers need not be slow walkers even if drivers are walkers, for the events or acts of walking are not those of driving. The analysis seems to have two virtues: It saves extensionality, and is not *ad hoc,* in that there is a definite recipe for going from the vulgar speech to its perspicuous counterpart. I shall in no way criticize Davidson's analysis of action sentences. But I shall consider the thesis that this analysis leads straightforwardly to an acceptable theory of predicate modifiers in general, and dispute that.

[5] This was proposed in discussion by Professor F. Fitch, Yale University.

[6] D. Davidson, "The Logical Form of Action Sentences," in N. Rescher (ed.), *The Logic of Decision and Action* (Pittsburgh: University of Pittsburgh Press, 1968), pp. 81–95.

As a first difficulty consider "He drives in imagination." It may be an act, but it is not an act of driving. The answer will have to be that the recipe that transforms English into canonical English is somewhat more complicated for driving in imagination than for driving in sleet or in anger. We must first transform the sentence into "He imagines that he is driving," or perhaps, "He is imagining in a driving way." Then, perhaps, we can apply the old recipe.

As a second difficulty consider "It is brightly colored." This sentence does not describe or ascribe an action. But an analysis similar to Davidson's can be given if we reify a new category of entities, to which colors belong.[7] "It has a color which is bright." Could the reified entity be a set? Well, not the set of colored things, for this set can also be described by some other predicate say, "has a volume" (or "is macroscopic"; by "is colored" I mean "reflects light" to rule out the more traditional parlor tricks with glass and pink ice cubes). In that case the analysans would logically convert with "It has a volume which is bright" or some such attribution that I assume to be nonsensical. Hence the newly reified category must be that of properties (in some sense in which properties are not sets). No one will doubt that extensionality can be saved at the expense of a bloated ontology, but the price is not right.

Returning to acts, consider "Although swimming fast, John crossed slowly." (This kind of example was already discussed by Davidson.) There is here one act, called slow under one description, fast under another. The solution is to argue that "slow" and "fast" are really relational terms. The analysis is, then, "There is an event, which is a swimiming, and which is a crossing, by John, and which is fast among swimmings but slow among crossings."

The crux of the solution to the preceding difficulty is that the two sets of acts are not identical. What if the world contained only mermen who planned mechanized transport but were unable to construct it? They would, in their frustration, declare even their fastest swimmings to be slow crossings, although all their crossings are by swimming. And what if von Braun, with an eye to the future, remarked that all our present spacecraft are relatively slow? If Armageddon occurred tomorrow, they would be the fastest to have existed, but would that mean that his claim was false?

[7] This solution was proposed in correspondence from Professor G. Harman, Princeton University. A better example is this: Suppose the red things were exactly the hard ones; then "bright" cannot be construed in "It is bright red" as classifying the set of red things, on pain of the consequence "It is bright hard." The structure of the example is as in the walkers-drivers problem.

In these modifications of the example I am clearly bringing in conceptual elements. For it is by comparison to conceived, as opposed to actualized, possibilities that the terms "slow" and "fast" are now being applied. The answer given by the extensionalist must be that I am not to apply the usual recipes here: Another transformation, into discourse that is perhaps partly metalinguistic, is needed.

I have not tried to argue that the thesis that Davidson's analysis of action sentences has a straightforward extension to a general analysis of adverbial modification is faced with unsolved problems. But I shall now object to it on the basis of a pattern that may be discerned in the solutions to the problems I have displayed. In any explication of a specific area of discourse in natural language there are three factors: the phenomena (actual usage), the canonical language, the formal or symbolic language. The first is given, but imperfectly: We do not have a perfect systematic description of the grammar of actual language in use. The third is described, with both syntax and semantics precisely specified, in (some part of) logical theory proper. The second is circumscribed, in a relatively precise way, as for example by Quine in *Word and Object*. The procedures for transition from the canonical to the symbolic are relatively straightforward, for the canonical language is delimited with an eye to the formal language that is to be used. The procedures for transition from the actual language to the canonical language are relatively imprecise and less straightforward, irremediably so as long as the description of the phenomena is not precise and systematic.

This picture is meant to fit all explications of actual language, those I admire with few qualifications as well as those I cannot accept even with many. Now I can state my objection to the thesis, qualified by a series of problems and solutions given above: Every epicycle occurred not in the formal machinery, which is readily accessible to discursive reflection, but in the paraphrase procedure that leads from ordinary discourse to canonical discourse. Every formal explication incorporates such a paraphrase procedure. But that is the most flexible, most malleable, least tractable, and least disputable part of the explication. Hence in fairness to the opposition, that is where epicycles should *not* be added. And not just in fairness, but in fear: in fear of the flibbertigibbet of glibness that can confound a strong spirit.

III

Semantic Correlates of Predicate Modifiers

We have so far been concerned to explore an approach in which, through appropriate paraphrase, the predicate modifiers disappear in the transition from the linguistic phenomena to be saved to what saves them. I shall now outline the most important alternate to this approach. Language can be syntactically analyzed so that each expression is formed from simpler component expressions in a certain systematic way. This syntactic structure has an exact parallel in the semantics: An interpretation gives each expression a value, which is determined in a systematic way by the values it gives to the component expressions. Now, a predicate modifier turns predicates into predicates; hence it is natural to take the value of a predicate modifier to be an operator that turns values of predicates into values of predicates.[8] Designating the value of expression E as $|E|$, the thesis has a simple formulation:

(1) $|\phi(F)| = |\phi|(|F|)$

for any predicate F and predicate modifier ϕ.

But what values do expressions receive? As a first candidate, let us suppose that $|F|$ is the extension of F. Then equation (1) says that the semantic correlate of ϕ is an operator on sets (subsets of the domain of discourse; for convenience I shall restrict myself to monadic predicates for now). However, that candidate fails, for under this supposition the consequence

(2) if $|F| = |G|$ then $|\phi(F)| = |\phi(G)|$

has a corollary

(2a) $(x)(Fx \equiv Gx) \supset (x)(\phi(F)x \equiv \phi(G)x)$

which means that the slow drivers are the slow walkers if the drivers are exactly the walkers.

For this reason we naturally turn to a second candidate: intension. If the predicates are assigned intensions as values, the semantic correlates of adverbs are operators on intensions. But what are intensions? Here there are two distinct answers, one simple-minded and one very powerful. I shall give both answers in pictorial, metaphorical language (but postpone to the end my reasons for calling it pictorial). The first answer is that the intension of a predicate is the collection of possibles of which it is true, while its extension is the set of actuals of which it is true. Writing "$(/x)$" for "for all possible entities x," the intensional corollary to equation (2) is

(2b) $(/x)(Fx \equiv Gx) \supset (/x)(\phi(F)x \equiv \phi(G)x)$

[8] Natural bnt not necessary; the general thesis implies only that $|\phi(F)|$ is some function of $|\phi|$ and $|F|$.

And there are possible, nonexistent walkers who do not drive even if all actual walkers do; hence the reasons against equation (2a) do not count against (2b).

The more powerful approach, taken most recently by Parsons and Thomason, interprets the language with reference to a collection of possible worlds, each of which has inhabitants. A predicate F has an extension in each possible world; its intension $|F|$ is the function that maps each world α into the extension of F in α. (I simplify, but not overly for present purposes.) We can now express the identity of intension as necessary coextension; hence equation (2) has the new corollary.

(2c) $\Box(x)(Fx \equiv Gx) \supset \Box(x)(\phi(F)x \equiv \phi(G)x)$

In what follows, nothing hinges on which explication of intension (whether either of the above, or a mixture of the two which replaces the antecedent of equation (2b) by, say, $\Box(/x)(Fx \equiv Gx)$ or $(/x)\ \Box\ (Fx \equiv Gx)$) is used. I have a preference for the first because of its relative simplicity and economy, virtues not to be slighted if other things are equal.

I shall now bring forward two objections to this candidate. If the objections hold, we shall need to look for a candidate for the value of a predicate other than its extension or intension.

The first objection is a problem raised by Thomason.[9] It does not seem that adverbs can always modify negative predicates. "He reluctantly did not go" makes sense, but "He slowly did not go" does not. Thomason concludes that "reluctantly" ought to be construed as modifying the whole sentence ("Reluctantly, he did not go" in analogy with "Possibly, he did not go") and ends his notes with a problem: "there is a certain asymmetry between the syntax and the semantics. Adverbial phrases are not allowed to modify negative predicates, but there is no semantic way to distinguish 'negative' from 'positive' propositional functions."

The second objection is more nearly analogous to the slow-walkers, slow-drivers problem. Its form is simple: identity of intension of F and G is no guarantee of identity of intension of $\phi(F)$ and $\phi(G)$. The following are pairs of cointensive predicates, it seems to me:

(3) is colored; is extended
has a mass; has a volume
thinks; acts
has a property; is identical with something

[9] R. H. Thomason, "A Semantic Theory of Adverbs," Yale University, May 1970 Xerox.

But consider the following modifications:

(4) is brightly colored; is brightly extended
has a mass of 1 kg; has a volume of 1 kg
thinks before he acts; acts before he acts
has a property, namely hardness; is identical with something, namely hardness

In each case the second member is either nonsensical, or necessarily false, or false if the first member is true (of whatever subject one cares to add). I imagine the objection may be attacked by throwing doubt on the cointensiveness of my pairs of predicates, but I am prepared to multiply examples (consider having a mother and having a father, or at any rate a navel, if one more will help).

It may be objected here that I am bending the notion of necessity to my own ends. Not by pure logical necessity, but by necessity relative to some accepted or background theory, or theory chosen for the purpose of example, are the predicates coextensive. But in this I am not diverging from usual practice in philosophical argument concerning modal logic. (And what is pure logical necessity anyway? What is tautological in the context of quantification theory is merely true *ex vi terminorum* in the context of sentential logic.)

Given these problems, it is natural to seek a further semantic dimension to predicates, and the tradition offers us at least a label: *comprehension*. We can look for inspiration to the theories of intension, comprehension, connotation, conventional connotation, multiple intension, total contingent intension, and so on, by Mill, Bradley, Keynes, Lewis, Leonard, and so forth. They are not as helpful as we might wish. Alternatively, we can look to theories of implication which reject substitution of tautological equivalents, most notably the relevant logics constructed by Ackermann, Anderson, Belnap, and their followers. I do not think either coterie of writings is negligible as a source of inspiration, but there is in fact a fortunate circumstance to be attended first: Romane Clark's theory of predicate modifiers is not open at least to our second objection.[10]

For present convenience let me recast Clark's theory in the present form. Besides an extension and perhaps an intension, each predicate F has associated with it a mapping $|F|$ of individuals into sets of facts. This mapping is such that

(5) Fa is true if and only if one of the elements of $|F|\ (|a|)$ is the case.

[10] R. Clark, "Concerning the Logic of Predicate Modifiers," *Nous*, Vol. 4 (1970), pp. 311–335.

If I am allowed to interject my own theory of facts at this point, I can identify the set $|F|(|a|)$ with the set $T^*(Fa)$ defined in an earlier paper.[11] I shall not assume acquaintance with this paper, but offer some intuitive comments that show the main idea. If F is "is colored," then facts to be found in $|F|(|a|)$ are, for example, the fact that a is red, the fact that a is green, and so on (most of which are not the case of course). Supposing ϕ to be "brightly," the corresponding facts to be found in $|\phi(F)|(|a|) = |\phi|(|F|)(|a|)$ are, for example, the fact that a is bright red, the fact that a is bright green, and so on. Note that such facts as that a is brightly six or seven cubic feet do not appear. But they will appear in |"is brightly extended"| $(|a|)$, and indeed, I expect that they are all the same, namely the fact that a is a member of the null set.

In other words, the comprehensions of F and G will be distinct if, for some individual a, the fact that a is F is distinct from the fact that a is G. Could this always be taken to be a real individual, and could the facts always be taken to be facts that some real thing belongs to the *extension* of a given predicate? I do not think so if only because the extensions of "is a golden mountain" and "is an existent golden mountain" are the same. But for the pairs of predicates of example (3) it does not seem to matter much whether the facts themselves be construed extensionally or intensionally. For example, if |"has a property"| $(|a|)$ is (roughly) the set

$$\{\text{the fact that } |a| \in X : X \text{ a subset of the domain}\}$$

and |"is identical with something"| $(|a|)$ is

$$\{\text{the fact that } a \in \{x\} : x \text{ an element of the domain}\}$$

then it is easily seen that the comprehension of "is identical with something" is included in the comprehension of "has a property," and not conversely, which is exactly as it should be.

There is an interesting corollary: If Clark's solution can be construed as I have done, the logic of comprehension is Anderson and Belnap's logic of tautological entailment.[12]

[11] "Facts and Tautological Entailments," *Journal of Philosophy*. Vol. 66 (1969), pp. 477–487.

[12] A. R. Anderson and N. D. Belnap Jr., "Tautological Entailments," *Philosophical Studies*, Vol. 13 (1962), pp. 9–24. The technical details supporting this claim are straightforward given the paper cited in footnote 11. For example, to the predicate abstract $[x_1, \ldots, x_n/A]$ give the function which assigns to $\langle \alpha_1, \alpha_2, \ldots \rangle$ the set of facts $T_d{}^*(A)$ where $d(x_i) = \alpha_i$, $i = 1, 2, \ldots$. Define $F \vee G$ to be $[y/Fy \vee Gy]$ where y is alphabetically the first variable not in F or G; similarly for meet and complement. Define $F \leqq G$ to hold exactly if what is given to F is set-theoretically included in what is given to G. This means, for instance, that $[x/Fx] \leqq [y/Gy]$ holds in a model exactly if $Fx \Vdash Gx$ holds for every assignment d of values to the variables. For completeness, consider vacuous abstracts $[x/A]$ with x not occurring in A. (A more straightforward, but less simple, semantic analysis is given in the first Appendix.)

IV

Transcendentals and Definitions

In response to logical problems about adverbs, we now have a new semantic analysis of predication, issuing in a logic of comprehension. If you accept the approach of Section III and regard the proffered problems to be genuine, the outcome that predicates have a semantic dimension beyond their (modal) intension is a necessary one. The exact construal of that dimension is not. In this section I shall examine the formal distinction and Thomason's problem of negative predicates from the point of view of comprehension as here construed.

In the history of logical theory, attention has shifted from properties, to concepts, to predicates. What I called the existential, conceptual, and formal distinction among properties I see as exactly parallel to differences of extension, intension, and comprehension among predicates. Scotus saw a formal distinction between such transcendental properties as being, oneness, necessary-or-contingent, infinite-or-finite. Yet no two are conceptually distinct: All possibles have each property. We can show no difference in intension, but can show a difference in comprehension. For instance, the fact that a is a finite being enters the comprehension of "is finite or infinite"; it does not enter the comprehension of "is one thing" or of "is a necessary or a contingent being." (I would pair "has being" with the abstract $[x/(\exists G)Gx]$ and "is one" with $[x/(/\exists y)(y = x)]$, but details of symbolization could be disputed.)

Syntactically distinct predicates may have the same comprehension. While it is not necessary to do so, we would normally give the simplest kind of comprehension to the simplest kind of predicate, a predicate constant.[13] This means that in such cases, identity of intension entails identity of comprehension. But even distinct complex predicates can have the same comprehension; for example $[x/Fx]$ and $[x/Fx \vee Fx]$ do. The paradigm case of sameness of comprehension, however, occurs in the case of explicit definition. Only the Polish logicians have bothered to attempt a thorough logical analysis of this process in theory construction; in current logical practice this process

[13] In the simple construal of Section III this is done explicitly (though this could be amended). In the Appendix, properties (candidates for comprehensions of predicates) are constructed without reference to the syntax, so there it is easy to give to a predicate constant or parameter a comprehension of any degree of complexity. This is essential for the discussion of, for example, the relation between "is colored" and "is red."

occurs in the language-in-use but is generally not mirrored in the object language.[14] Think of a language being given piecemeal, a bit at a time, and at a certain point a primitive predicate F is characterized as *short for* a predicate G which has been characterized previously. My analysis of this is: F is assigned the comprehension of G. (In this way a simple predicate can receive a very complex comprehension in a straightforward way.)

The process of explicit definition plays an important role in actual theory construction, and misunderstanding of its nature can produce much confusion. I mention it here because reference is made to the process of definition to explain how formal distinctions are to be distinguished from merely verbal distinctions. Only a full-fledged theory of comprehension can answer every question of the form "Is this a formal or merely verbal distinction?" that is, "Is there a difference in comprehension between these intensionally equivalent predicates or not?" But in the case in which one predicate is introduced by the process of explicit definition, as short for the other, the answer must be the same regardless of the exact construal of comprehensions. Hence the aptness of the reference to explicit definitions.

If the subject of definition is not well understood, it might seem that formal distinctions may be made to disappear at will, to be turned into merely verbal distinctions, by the process of *redefinition*. One predicate is *definable* as another if they have the same intension in the sense of having provably (relative to given theory T) the same extension. So then why not go one step further, and *define* the one as the other? Is this not a common practice in the development of the theoretical sciences, and does it not show that differences in comprehension are there eliminated at will?

If the structure of a theory is so simple that differences of comprehension between intensionally equivalent, syntactically simple predicates cannot be expressed in it, then it should be assumed, it seems to me, that simple predicates (not introduced by definition) have the simplest kind of comprehension. Hence, for them, identity of intension will imply identity of comprehension, and nothing *is* lost if we say (from some point on) that the one is short for the other. But in an extension of the theory in which differences in comprehension can be expressed, this could generally not be done. In that case, eliminating a primitive term (for example by discarding its old comprehension and giving it the comprehension of another term, *or* by removing it from the syntax altogether) might result in an economy because a notion is *discarded* which proved superfluous for some (though not logically for all) purposes.

[14] Cf. E. W. Beth, *Formal Methods* (Dordrecht: Reidel, 1962), Chapter Six.

To illustrate this critique of loose talk about definition is not so easy, since little attention has been paid to differences in comprehension between the fourteenth century and the twentieth. But an example can be taken at a point between, where the smile yet remained if not the cat: a passage by Leibniz which has been discussed at some length by Hidé Ishiguro.[15] Leibniz introduces a notation for the identity of two concepts A and B, and gives as identity criterion mutual substitutivity *salva veritate* everywhere. He adds that certain apparent exceptions to the criterion must be allowed; for example the concept of the trilateral is identical with the concept of the triangular, although we can not say that a trilateral, as opposed to a triangle, contains 180 degrees by its very nature (*quatenus tale*, in so far as it is of such a kind): *"Est in eo aliquid materiale."*[16]

The point seems to be that triangles contain 180 degrees by definition (the definition of "triangle" being "plane straight-sided figure with three enclosed angles" say), while trilaterals are triangles not by definition although by logical necessity. That is, although nothing about angles appears in the definition of "trilateral," we can infer from its definition that a trilateral has three enclosed angles summing to 180 degrees. Now we could put this as follows: triangularity and trilaterality are distinct features of the figure, although it would not be possible for it to have the one feature and not the other. So the distinction between being a trilateral and having three sides, and so on, is only a verbal distinction, but the distinction between being a trilateral and being a triangle is a formal distinction.

Discussing that passage, Miss Ishiguro points out that Leibniz' closing remark (quoted above) is a reference to the theory of supposition; specifically, to material supposition. Her reconstruction of Leibniz' reaction to such contexts that are opaque to substitutivity of identical concepts is this: Each expression has a meaning which is a function of the meaning of component expressions entering it (or its defining phrase if it is defined). Two expressions may then have distinct meanings, although they stand for identical concepts. It will be clear how I understand that account: I identify what Leibniz-pace-Ishiguro calls meaning with comprehension, concept with intension.

I began by exhibiting the passage as an example of the distinction between what is the case by definition and what follows from the definition. Geometry, as it is normally formulated, does not admit

[15] H. Ishiguro, "Leibniz and the Ideas of Sensible Qualities," in *Reason and Realtiy, Royal Institute of Philosophy Lectures,* V (London: Macmillan Press, 1972), pp. 49–63.
[16] L. Couturat, *Opuscules et Fragments Inédits de Leibniz,* p. 261.

the expression of this distinction. To add the expression "by definition" (or "by its nature" or "*quatenus tale*") would be to extend geometry (and not in a way that would serve a geometric purpose. Within geometry proper predicates with the same intension are substitutable everywhere.) But the passage is first and foremost an example of a recognition of distinctions which go beyond the intensional. A similar example, from arithmetic rather than geometry, is furnished by Frege's notion that "$2 + 2$" and "2^2" have the same reference but a different sense.[17] This sense could not be intension, for Frege thinks of numbers as properties of collections, and surely any possible collection having the property $2 + 2$ (that is, having $2 + 2$ members) has the property 2^2 and conversely. So the sense of "$2 + 2$" is not its intension.

V

Determinables and Negation

In what way are meanings, in the sense of comprehensions, a function of components? I propose that there are two ways; the first is conjunctive and typical of the classical paradigm of definition; the second is disjunctive and typical of the determinable-determinate relationship of modern logic. Let the definition of "human" be "rational animal." Then *humanity* comprehends *rationality* and *animality*. I shall write $\leq$ for this relationship, and we have no doubt the typical conjunction laws

$$\text{(6)}\quad \begin{array}{l} \text{human} \leq \text{rational} \\ \text{human} \leq \text{animal} \\ \text{if A} \leq \text{rational and A} \leq \text{animal, then A} \leq \text{human} \end{array}$$

But now consider the relation between *being colored* and *being red*. To be red is not to be colored and something else (unless that be red). Yet whatever is red is colored by its very nature. To be colored is to be red, or blue, or . . . To have a finite case, I will introduce an artificial example: Let spin be a quality which has only two varieties, spin +1 and spin —1. Then we have the typical disjunction laws:

$$\text{(7)}\quad \begin{array}{l} \text{spin} + 1 \leq \text{spin} \\ \text{spin} -1 \leq \text{spin} \\ \text{if spin} + 1 \leq \text{A and spin} -1 \leq \text{A, then spin} \leq \text{A} \end{array}$$

For color we would of course have an infinite analogue, since there are infinitely many colors. In W. E. Johnson's terminology, color

[17] P. Geach and M. Black (eds.), *Translations from the Philosophical Writings of Gottlob Frege* (Oxford: Blackwell, 1966), p. 154.

and spin (and also volume, mass, temperature, and so on) are determinables, and the specific colors and specific spin values are determinates under them.[18]

The tree of Porphyry and Johnson's determinable-determinate hierarchy graphically represent two distinct patterns in conceptual structure. These two patterns must be combined in some integral way if we are to have an adequate representation of how complex meanings are functions of component meanings. For in the building of concepts, one might proceed according to either pattern at any stage. In the logic of comprehension, construed either as in the preceding section or as in the Appendix, these patterns are indeed systematically combined. The way in which they are combined assumes that comprehension is not affected by the standard logical procedure of reducing to 'normal' form. This is a substantial assumption; its justification can consist only in the comparison of Anderson and Belnap's logic of tautological entailment with alternative attempts along similar lines.[19]

In the form adopted, each property (comprehension of a predicate) can be viewed as a determinable, comprising a set of properties (determinates) under it in a disjunctive way. Then each of those properties is this kind of a 'disjunction' only of itself; but it comprises another set of properties in a conjunctive way. Now *these* properties are simplest of all, and comprise only themselves (in either of those ways). These latter are what I called the simplest kind of comprehension a predicate could have: If two predicates have this simplest kind of comprehension, then they have the same comprehension if they have the same intension.

You may wonder what has happened to negation. Well, when a formula is reduced to normal form in ordinary logic, the negations are driven inside, as far as possible. For example, $\sim(p \vee q)$ is reduced to $(\sim p \,\&\, \sim q)$, by De Morgan's laws. The negation of a disjunction is therefore again a disjunction, which has, however, only one disjunct. In the same way, the very simplest kind of comprehensions have very straightforward complements; complementation of more complex comprehensions is defined in accordance with De Morgan's laws.

Now I can state my solution to Thomason's problem, which I cited in Section III. Most ordinary predicate modifiers (for example, most ordinary adverbs) have an intimate connection with a specific

[18] W. E. Johnson, *Logic*, Part I (Cambridge: Cambridge University Press, 1921), Chapter 11.

[19] This logic is the first-degree fragment of a large family of logics of implication (notably *R-mingle, R,* and *E*), but these do not have to be considered here; we are concerned only with an implication relation among propositions, not with a binary implication operation on propositions.

determinable. They produce nonsense when used to modify something which is strictly outside that determinable. So "is bright not red" ("is brightly uncolored") is nonsense for the same reason that "is brightly extended" is nonsense. And the reason is this: "bright" is intimately connected with the determinable *color,* in this sense, and the comprehensions of "is extended," "is not red," and "is uncolored" are not determinates under this determinable. *Most* adverbs do not sensibly modify negative predicates because complementation *usually* takes one outside the determinable in which the complemented property lay.[20]

Thomason has pointed out that intensions could be grouped in families, and these families called *determinables,* so that exactly this solution to his problem can be had without going beyond intension. This is true; I offer the second Appendix and claim only that the problem has a natural solution in our framework.

VI

Methodology in Philosophical Logic

Contemporary logical theory often looks like a metaphysician's garden of delights: possibles, properties, facts—is there no end to our weird and wonderful reifications?

Let me say at once that I do not believe a word of it. I can believe in witches and genies; indeed, I can seriously wonder whether I have met those. But I cannot even imagine wondering seriously whether there are sets or properties. Since I talk freely about sets and properties and much other metaphysical flora and fauna, I suppose I have to explain how I feel that I can do this.

My attitude toward mathematical objects such as sets is at least in practice different from my attitudes to the other categories. Specifically, I *use* or *engage in* mathematical discourse to describe other forms of discourse. So I often find myself asserting or assuming that there are sets. Now I do not really believe that there are any. But I do firmly believe that any adequate philosophy of mathematics must show how, nevertheless, we can play the mathematical language game, to the extent that it is needed for all normal scientific and philosophical purposes, in a perfectly sensible way. I grant that neither nominalists nor intuitionists nor constructivists have managed to show this. That means that, in my opinion, there exists today no adequate philosophical account of mathematics. But it would have been foolish not to use normal discourse about motion before Zeno's paradoxes

[20] Similar but more special hypotheses have been offered by Harman and Lakoff.

were adequately handled; it would be equally foolish not to use set theory today.

But, while using mathematical discourse we can, or ought, to provide rational reconstructions of other forms of discourse: modal, epistemic, deontic, discourse about possibles, about implication, about facts, and much more. In the explication of modality, for instance, I distinguish three moments: a language game with modal qualifiers, an account thereof in pictorial language (about possibles or possible worlds), and a formal reconstruction in mathematical language. The pictorial account is a guide to the formal account, but the sole object of semantic analysis is to provide a precise representation of the structure of the language game. (I hold that in addition, a pragmatic analysis is needed for the purposes of philosophical clarification, but our techniques for such analysis have not yet reached maturity.)

The metaphysical language of the second stage I call *pictorial* for a special reason. In any area of discourse, thought and reasoning are guided (and often bewitched) by a picture. The preceding statement is pictorial also. It is an assertion in the kind of language game that I believe to be adequate, on a relatively superficial but not negligible level, to the description of mental activity. That area of discourse is also guided by a picture; a semantic analysis would provide a precise description of the kind of picture that guides it. What it means to call it *adequate,* however, would not be made clear by a semantic analysis; this is a subject in pragmatics.

Given my ontological views, it may be surprising that I do not work within the Quine-Davidson tradition. Superficially, the break is over Tarski's criterion for an adequate theory of truth.

> A theory of truth entails, for each sentence *s*, a statement of the form "*s* is true if any only if *p*" where in the simplest case "*p*" is replaced by *s*. Since the words "is true if and only if" are invariant, we may interpret them as "means that."[21]

I cannot agree to this. Let some particular fragment of natural language be codified and called $\mathcal{O}$. The idea seems now to be that we can frame a metalanguage M, which is part of Responsible Philosophonese, such that (*a*) M has the structure of the languages studied in ordinary quantificational logic, and (*b*) M contains an exact copy of $\mathcal{O}$. But all sentences of M are bivalent (either true or false, no matter what the facts are like). So how could M contain an exact copy of $\mathcal{O}$ if $\mathcal{O}$ is not a bivalent language?

[21] D. Davidson, "Semantics for Natural Languages," *Linguaggi nella societa e nella tecnica* (Milan: Edizioni di Communita, 1970), p. 184.

To this, the following retort is possible: We are not concerned with what language might be like, but only with what it is like—and this language, the language we have, is bivalent. And perhaps the retort could be trivally justified: Is not every language bivalent if by "false" you mean "not true"? But the trivial justification would not be a justification. For the thesis of those who reject bivalence is not that the word "false" has a certain meaning, but that there are important semantic characteristics of and relations among sentences that cannot be explicated in terms of truth (or satisfaction) alone. So the retort, to be effective, must rest on the nontrivial thesis that all important semantic characteristics and relations for natural language are sufficiently like those studied in orthodox formal semantics.

To substantiate this nontrivial thesis, one usually takes recourse to the surface structure/deep structure distinction. It seems fairly certain that Russell held that language has a skeleton, that the philosopher, like Blake's wild beasts, can cleanly separate the flesh from the bones, and that only the skeleton matters. No philosopher today holds this view, I suppose, but the emphasis on deep structure is reminiscent of it. For it is meant, it is not, that there is a hygienic and domesticated fragment of natural language in which all that could be said at all can be said? And that to display the deep structure of a given sentence, one displays its hygienic, domesticated counterpart (or rather, counterparts, to allow for ambiguity)? But the essential philosophical qualification to the earlier view is the present admission of the relative status of deep structure. Davidson (like Harman) proposes that deep structure be identified with logical form, and then adds:

> to give the logical form of a sentence is to give its logical location in the totality of sentences, to describe it in a way that explicitly determines what sentences it entails and what sentences it is entailed by. The location must be given relative to a specific deductive theory; so logical form itself is relative to a theory. The relativity does not stop here, either, since even given a theory of deduction there may be more than one total scheme for interpreting the sentences we are interested in and that preserves the pattern of entailments. The logical form of a particular sentence is, then, relative both to a theory of deduction and to some prior determinations as to how to render sentences in the language of the theory.[22]

[22] D. Davidson, "Action and Reaction," *Inquiry*, Vol. 13 (1970), pp. 140–148.

So the person assigning logical form or deep structure to a given sentence is in the position of a scientist who displays a model of given phenomena: The physical theory he has gives him a certain stock of models for phenomena and rules for selecting the right model. The enterprise falls therefore under the canons of scientific methodology; its hypothetical character is openly granted, as is its nonuniqueness in principle.

From this point of view the difference between the Davidsonian [or (neo?) Quinean or crypto-Russellian] enterprise and our own is that we proceed with reference to a different logical theory. But that is not all: They always seek for logical form in the sense of a model taken from one logic, orthodox quantificational logic, whereas we produce new logical structures to deal with new problems. So this is an inadequate way to characterize the difference. The correct way, in my opinion, is to say that we choose a different *locus for innovation and complication.* The locus they choose is the procedure for fitting the linguistic phenomena (or rather, their surface structure) to the logical theory (what Davidson called in the passage cited above the "total scheme for interpreting the sentences" and "prior determinations as to how to render sentences in the language of the theory"). The locus we choose is the logical theory.

At the end of Section II, I argued against the first (and for the second) choice of locus. Every scientific theory has a precise part and an imprecise part; the procedures for fitting the phenomena to models must belong to the imprecise part if there is no systematic, relatively theory-independent description of those phenomena; complications and innovations ought to occur in the precise part. In our special case, this means that the relation between surface structure and logical form ought to be as close and as direct as feasible. Of course there are also less-official arguments: A stable, well-established, well-understood theory has greater explanatory power, say the others; there is as great a danger of being tyrannized by logic as by society, say we, but a proper understanding sets you free.

To sum up, then, I have offered three theses concerning philosophical logic. The first is that it can be done in such a way as to impute no ontological commitment through any use of language. The second is that orthodox logical theory is inadequate to the analysis of natural language because there are important semantic properties and relations that cannot be characterized in orthodox semantic terms. (As illustrations, but illustrations only, of these two theses I offer my personal rejection of the existence of abstract entities, possibles, and facts, and of the principle of bivalence.) The third is that, in the special situation of philosophical logic, correct methodology requires innovations and complications to occur on the side of the formal apparatus.

Appendix I

The Logic of Comprehension[23]

In Section III I defined the comprehension of a predicate by combining Clark's theory of predicate modifiers with my representation of facts. The fact *that Fa,* for example, is represented as the unit set {<[F],|a|>} where [F] is the intension of F (or its extension, depending on how deep our analysis needs to go) and |a| is the referent of a. Clearly the facts that make a sentence true are determined then by the intensions (or extensions) of the predicates involved, and by the referents of the singular terms involved. Put it this way, it would seem that the comprehensions of predicates could perhaps be constructed directly from intensions, bypassing the individuals and the facts. This more direct approach I shall pursue in these two appendices.

In this first appendix I shall give a representation of *properties.* Each property Z will have a corresponding *attribute* [Z] and in each possible world α, Z will determine a set of existents [Z,α]. When a language is interpreted, the value assigned to a predicate is such a property. If |F| = Z, then we also say that Z is the comprehension of F, [Z] the intension of F, and [Z,α] the extension of F in possible world α. A more-complicated scheme, in which intensions may also differ from world to world, presents only routine difficulties. The value of a predicate modifier will be an operation on the set of properties. While I shall consider only monadic adverbs, which are not themselves given as functions of anything else, the extension to more complex cases again presents only routine difficulties, I think (although it might suggest interesting new logical problems). So, for example, "continually," as in "He continually jumped and ran about," is perhaps best seen as polyadic, since running and jumping cannot be done simultaneously. On the other hand, in "He ran to Bath and to Bournemouth," we should perhaps regard the value of "to" as mapping places into operators on properties.

I shall construe properties in such a way that the values of relational and complex predicates can be properties. The maneuver is simple: as extension of "——$_1$ is father of ——$_2$" I take the set of

[23] An earlier draft of these Appendices was presented in my "Adverbs: Some Logical Problems," University of Toronto, August, 1970. Mimeographed.

all infinite sequences of objects such that the first is father of the second. So extensions are sets of infinite sequences, the members of which tend to become irrelevant after some initial segment. The same holds for intensions: the elements of the sequence are then possible individuals. If, in the language, complex predicates are made up by abstraction or definition, the syntactic operators ought to be regarded as corresponding to operations on properties; for example

$$|[x/\mathrm{F}x \ \&\ \mathrm{G}x]| = |\mathrm{F}| \wedge |\mathrm{G}|$$

whatever we eventually mean by the meet of two properties. Similarly universal quantification will correspond to infinite meet, the indexing being through the possible individuals.

From here on I shall disregard existence, which is a predicate without intension or comprehension, and the sole use of which is to determine extensions. (This limitation does not appear if we regard an intension as a function assigning extensions to possible worlds, but I do not think that we need go to that length to encounter any of the problems *peculiar* to the logic of adverbs.) We assume given a set H (the set of possible individuals, the logical space). H^ω is the set of all denumerable sequences of elements of H. I define

(1) (a) a *point* is an element of H^ω.
  (b) an *attribute* is a subset of H^ω, that is, a set of points.
  (c) a *molecule* is a nonempty set of attributes.
  (d) a *property* is a nonempty set of molecules.

The molecules play a purely expository role. We shall use variables

x over the points
X over the attributes
Y over the molecules
Z over the properties

in each case with or without sub- or superscripts. We say that

(2)(*a*) x has X iff $x \in X$.
  (b) x has Y iff x has each element of Y.
  (c) x has Z iff x has some element of Z.

Clearly if $Z = \{Y\}$, then x has Y if and only if x has Z; for that reason the molecules are really inessential.

Each property determines an attribute as follows

(D3) $[Z] = \{x : x \text{ has } Z\}$

Using the convenient molecules, it is easy to see how this can be defined set-theoretically

(4)(Da) $[Y] = \bigcap\{X : X \in Y\}$
(b) $[Z] = \bigcup\{[Y] : Y \in Z\}$

That is, equations (3) and (4a) imply (4b); (4a) and (4b) imply (3). As the capital letter "D" indicates, I regard (3) rather than (4b) as the correct definition.

The properties are ordered, and this ordering carries over to the corresponding attributes. We read $\leq$ as "coerces."

(D5)(a) $X \leq X'$ iff $X \subseteq X'$
(b) $Y \leq Y'$ iff every X' in Y' is coerced by some X in Y
(c) $Z \leq Z'$ iff every Y in Z coerces some Y' in Z'

(6)(a) If $Z \leq Z'$, then $[Z] \subseteq [Z']$
(b) $Z \leq Z$; and, if $Z \leq Z'$ and $Z' \leq Z''$, then $Z \leq Z''$

We shall now show that we can with much convenience and no loss of generality restrict our attention to a certain kind of properties, *closed properties.*

(D7)(a) $Z^* = \{Y : Y \leq Y' \text{ for some } Y' \text{ in } Z\}$
(b) Z is *closed* iff $Z = Z^*$

(8) $Z \leq Z^*$ and $Z^* \leq Z$

To prove equation (8), note first that $Z \subseteq Z^*$. Hence each molecule in Z is identical with, and hence coerces, some molecule in Z^*. Therefore $Z \leq Z^*$. But conversely, every Y in Z^* coerces some Y' in Z by definition. Therefore $Z^* \leq Z$. Thus Z and Z^* are indistinguishable by our ordering, and there is no difference between calling Z the comprehension of a predicate F, or Z^*. Henceforth the variable

P ranges over closed properties

used with or without sub- or superscripts.

(9) $P \leq P'$ iff $P \subseteq P'$

(If $P \subseteq P'$, then $P \leq P'$, as for any pair of properties. Suppose therefore that $P \leq P'$. Then each Y in P coerces some Y' in P'. But then each Y in P belongs to P'^* by the definition of closure. Since P' is already closed, $P \subseteq P'$.)

The closed properties form a set-theoretic lattice. That is, intersections and unions of closed properties are closed again. Let Y be in $P \cap P'$ and let Y' coerce Y. Then Y is in P and also in P'. Hence Y' is in P and also in P' (they are closed), and therefore Y' is in $P \cap P'$. On the other hand let Y be in $P \cup P'$, and $Y' \leq Y$. Then Y is either in P or in P', and Y' is in the same part. None of this reasoning depends on the finitude of the case, hence we conclude

(10) The closed properties form a complete set-theoretic lattice.

We may note that the null element of this lattice is $\{\{\Lambda\}\}$, which is also the null element among all properties. This gives us automatically the interpretation in comprehension of conjunction, disjunction, and quantification, provided the comprehensions we assign are closed. And there could be no advantage in doing otherwise, as equation (8) shows.

However, for the purely expository purpose of facilitating the discussions of complementation and completeness to come, I shall now give meets and joins for properties in general.

(D11)(a) $X \wedge X' = X \cap X'$; $X \vee X' = X \cup X'$
(b) $Y \wedge Y' = Y \cup Y'$; $Y \vee Y' = \{Y, Y'\}$
(c) $Z \wedge Z' = \{Y \cup Y' : Y \in Z \;\&\; Y' \in Z'\}$
$Z \vee Z' = Z \cup Z'$

(12) $[Z \wedge Z'] = [Z] \cap [Z']$; $[Z \vee Z'] = [Z] \cup [Z']$

Infinite analogues are obvious. For example, if $Z_i = \{Y_{ij}\}, j \in J$ for each $i \in I$, and we denote the set of maps from I into J as J^I. Then

$$\bigwedge_{i \in I} Z_i = \bigwedge_{i \in I} \{Y_{ij}\}, \qquad j \in J$$
$$= \{\bigcup_{i \in I} Y_{if(i)} : f \in J^I\}$$

It can easily be checked that the lattice laws hold among the properties. In addition, $P \wedge P' = P \cap P'$. (That $P \wedge P' \subseteq P \cap P'$ follows from the definition of coercion and the fact that $P \cap P'$ is closed. On the other hand if Y is in both P and P', then $Y \cup Y = Y$ is in $P \wedge P'$.)

How shall we define complementation? There is a natural complement on attributes, and this we can extend to molecules and properties with the De Morgan relations in mind.

$$\text{(D13)(a)}\quad \bar{X} = H^{\omega} - X$$
$$\text{(b)}\quad \bar{Y} = \bigvee\{\bar{X} : X \in Y\} = \{\{\bar{X}\} : X \in Y\}$$
$$\text{(c)}\quad \bar{Z} = \bigwedge\{\bar{Y} : Y \in Z\}$$

$$(14)\quad [\bar{Z}] = H^{\omega} - [Z]$$

I have left the consequences for corresponding attributes mostly unproved so far, but equation (14) is less simple than the others. We have to use the law of *complete distributivity* for sets:

$$(15)\quad \bigcup_{i\in I} \Big(\bigcap_{j\in J} X_{ij}\Big) = \bigcap_{f\in S} \Big(\bigcup_{i\in I} X_{if(i)}\Big)$$

where $S = J^I$, the set of all mappings of I into J.

First, let us write henceforth

$$Z = \{Y_i : i \in I\} = \{\{X_{ij} : j \in J\} : i \in I\} = \{\{X_{ij}\}, j \in J\}$$

$i \in I$, and let $S = J^I$. Then we can express the complement by

$$\begin{aligned}(16)\quad \bar{Z} &= \bigwedge_{i\in I} \bar{Y}_i = \bigwedge_{i\in I} \{\{\bar{X}_{ij}\} : j \in J\}\} \\ &= \{\bigcup_{i\in I} \{\bar{X}_{if(i)}\} : f \in S\} \\ &= \{\{\bar{X}_{if(i)} : i \in I\} : f \in S\}.\end{aligned}$$

by equations (13) and (15). (In applying equation (15), let $Z_i = \bar{Y}_i = \{Y_{ij}\}, j \in J$ where $Y_{ij} = \{\bar{X}_{ij}\}$; remember that $\bar{Y}_i$, unlike Y_i, is a property. In the same symbolism,

$$\begin{aligned}(17)\quad [Z] &= \bigcup_{i\in I} [Y_i] = \bigcup_{i\in I} \Big(\bigcap_{j\in J} X_{ij}\Big) \\ &= \bigcap_{f\in S} \Big(\bigcup_{i\in I} X_{if(i)}\Big)\end{aligned}$$

So by De Morgan's laws,

$$(18)\quad H^{\omega} - [Z] = \bigcup_{f\in S} \Big(\bigcap_{i\in I} \bar{X}_{if(i)}\Big)$$

But, from equation (16), we can derive

$$\begin{aligned}(19)\quad [\bar{Z}] &= \bigcup_{f\in S} [\{\bar{X}_{if(i)} : i \in I\}] \\ &= \bigcup_{f\in S} \Big(\bigcap_{i\in I} \bar{X}_{if(i)}\Big)\end{aligned}$$

which agrees exactly with equation (18), thus proving consequence (14). And consequence (14) is, in my opinion, a prime criterion for the correct definition of complement in this context.

Turning now to the formal principles obeyed by complementation, we find

$$(20)\quad \bar{\bar{Z}} = Z$$

To prove equation (20), and writing the bar to the left instead of above when convenient, and "$\bar{X}_{f,i}$" for "$\bar{X}_{if(i)}$," we find

$$\begin{aligned}(21)\quad \bar{\bar{Z}} &= -\{\{\bar{X}_{if(i)} : i \in I\} : f \in S\} \\ &= -\{\{\bar{X}_{f,i} : i \in I\} : f \in S\} \\ &= \{\{\bar{\bar{X}}_{f,g(f)} : f \in S\} : g \in I^S\}\end{aligned}$$

by applying equation (16) *mutatis mutandis* to the case $Z' = \{\{X_f{}'_{,i}\}$ $i \in I\}, f \in S$.

Of course $\bar{\bar{X}} = X$. Hence we see that each molecule $\{\bar{\bar{X}}_{f,g(f)} : f \in S\}$ in $\bar{\bar{Z}}$ is a subset of molecule $Y_{g(f)}$ in Z. To see the converse, let $g(f) = i$, and let X_{ij} be in Y_i. There is certainly an f in S, such that $f(i) = j$. Hence $X_{ij} = X_{if(i)} = X_{f,i} = X_{f,g(f)}$ for some f in S. Therefore the two molecules are identical. But this mapping of Z into z is really *onto* Z, since, for each i in I and f in S, there is a mapping of g such that $g(f) = i$.

This is not enough to qualify the operation formally as a complement; we need also at least

$$(21)\quad \text{If } Z_1 \leqq Z_2, \text{ then } \bar{Z}_2 \leqq \bar{Z}_1.$$

Let $Z_1 = \{Y_i\}$, $i \in I = \{\{X_{ij}\}, j \in J]$, $i \in I$, as above, and $Z_2 = \{Y_k\}$, $k \in K = \{\{X_{km}\}, m \in M\}$, $k \in K$, and let there be a mapping $g : I \to K$ such that Y_i coerces $Y_{g(i)}$. Then there is, in addition, a mapping $h : K \times M$ into J such that $X_{g(i)m}$ is coerced by $X_{ih(g(i)m)}$. That is, the latter is a subset of the former.

Now let us consider an arbitrary member $Y_f = \{\bar{X}_{kf(k)} : k \in K\}$, where f is in M^K, of $\bar{Z}_2$. We must show that, for some g in S, Y_f forces $Y_g = \{\bar{X}_{ig(i)} : i \in I\}$. Hence we must show that for that function g,

$\bar{X}_{ig(i)}$ has a subset $\bar{X}_{kf(k)}$ for an appropriate choice of k. So, for a given i, and $f(k) = m$, we must choose g and k such that $X_{ig(i)} \subseteq X_{km}$. Well, let k be $g(i)$. So we must have $X_{ig(i)} \subseteq X_{g(i)m}$. To have that, let $g(i) = h(g(i), m)$, for we do indeed have $X_{ih(g(i),m)} \subseteq X_{g(i)m}$.

We turn back now to closed properties; clearly our complementation does not preserve closure. However, if $Z_1 \underline{\angle} Z_2$, and conversely, then $\bar{Z}_1 \underline{\angle} \bar{Z}_2$, and conversely, and for each Z there is a unique closed property that coerces and is coerced by Z, namely Z^*. Hence we can define

(D22) $\tilde{Z} = \bar{Z}^*$

and prove that this yields a complement of the same kind on the closed properties. First, $\tilde{\tilde{P}} = \bar{\tilde{P}}^* \leqslant \bar{\tilde{P}} \leqslant \bar{\bar{P}}^* \leqslant \bar{\bar{P}}$ (because, as we said, if $Z_1 \leqslant Z_2$, then $\bar{Z}_1 \leqslant \bar{Z}_2$, by equation 21, and, of course, $\bar{P}^* \leqslant \bar{P}$). Second, if $P_1 \subseteq P_2$, then $\tilde{P}_2 = \bar{P}_2{}^* \leqslant \bar{P}_2 \underline{\angle} \bar{P}_1 \leqslant \bar{P}_1{}^* = \tilde{P}_1$. Therefore the analogues of equations (20) and (21) hold, and

(23) The closed properties form a De Morgan lattice of sets.

This means that the logic of tautological entailment is a sound logic of comprehension (by standard results of the former subject) and we wish now to show that it is also complete.

As appropriate syntax, take the variables $x_1, x_2, \ldots,$ to be nouns, and if A_1, A_2 are nouns, so are $\bar{A}_1$, $A_1 \wedge A_2$, $A_1 \vee A_2$. Call a noun *atomic* if it has either the form x_i or the form $\bar{x}_j$. If A_1, A_2 are nouns, $A_1 \underline{\angle} A_2$ is a statement. A statement is *valid* if it is true under the interpretation of the signs, $\wedge$, $\vee$, $-$, $\underline{\angle}$, regardless of what properties are assigned to the variables and regardless of the choice of set H.

(24) If $A_1, \ldots, A_m, B_1, \ldots, B_n$ are atomic nouns, then $A_1 \wedge \cdots \wedge A_m \underline{\angle} B_1 \vee \cdots \vee B_n$ is valid (if and) only if A_i is the same noun as B_j for some i and j.

This follows because we can clearly choose subsets $X_1, \ldots, X_{m+n}$ of H^ω such that $X_i \nsubseteq X_j$ and $X_i \nsubseteq H^\omega - X_j$ for all $i, j \underline{\angle}\ m + n$. In that case, we note

$$\bigwedge_{i=1}^{m} X_i{}^1 \not\underline{\angle} \bigvee_{j=m+1}^{n} X_j{}^1$$

that is,

$$\{\{X_1{}^1, \ldots, X_m{}^1\}\} \not\underline{\angle} \{\{X_{m+1}{}^1\}, \ldots, \{X_n{}^1\}\}$$

where $X_i{}^1$ is either X_i or $H^\omega - X_i$, for $i = 1, \ldots, m + n$.

Call A a *primitive* conjunction (disjunction) if it has the form $A_1 \wedge \cdots \wedge A_m (A_1 \vee \cdots \vee A_m)$ where $A_1, \ldots, A_m$ are atomic.

(25) If $A_1, \ldots, A_m$ are primitive conjunctions and $B_1, \ldots, B_n$ are primitive disjunctions, then $A_1 \vee \cdots \vee A_m \underline{\angle} B_1 \wedge \cdots \wedge B_n$ is valid (if and) only if $A_i \underline{\angle} B_j$ for each i and j.

Let $|A_i| = \{\{X_1{}^i, \ldots, X_{m_i}{}^i\}\}$ and $|B_j| = \{\{V_1{}^j\}, \ldots, \{V_{m_j}{}^j\}\}$, where $|A|$ is the value of noun A. For

$$\bigvee_{i=1}^{m} |A_i|$$

to coerce

$$\bigwedge_{j=1}^{n} |B_j|$$

each $|A_i|$ must do so. But then $|A_i|$ must coerce $\{\{V_{f(1)}{}^1, \ldots, V_{f(n)}{}^n\}\}$ for a certain mapping f such that $f(j)$ is in $\{1, \ldots, m_j\}$. So each $V_{f(j)}{}^j$ must have a subset $X_{g(f(j))}{}^i$ for a certain mapping g such that $g(f(j)) \in \{1, \ldots, m_i\}$. In that case, however $\{\{X_1{}^i, \ldots, X_{m_i}{}^i\}\}$ coerces $\{\{V_{f(j)}{}^j\}\}$, and hence $|B_j|$.

Because of the normal form theorems for tautological entailment, these two results imply the completeness of this logic as a calculus of comprehension.

Appendix II

The Class of Aspect Modifiers

The usual pattern of predicate modification at the center of the discussions by Reichenbach, Davidson, and Thomason, is this: A subject is qualified by a number of expressions, each of which qualifies this predicate in some particular respect. These qualifiers answer such questions as:

Where?
How?
With what?
When?
To whom?
While what else happened?
How much?

I shall call this *aspect modification*. As an example, we have, say,

> He buttered the toast *at midnight*$_1$ *in the shower*$_2$ *with a knife*$_3$.

If this is done perspicuously, the aspect in question can be determined by looking at the modifier. The modifiers do not interfere (hence their order is, at most, of stylistic importance), and all the modifiers clearly pertain to the original predicate. The following three principles hold:

$$\begin{array}{rl} \text{I.} & \phi(F)x \vdash Fx \\ \text{II.} & \phi\phi'(F)x \vdash \phi'\phi(F)x \\ \text{III.} & \phi(F)x \;\&\; \phi'(F)x \vdash \phi\phi'(F)x \end{array}$$

where ϕ, ϕ' range over aspect modifiers.

I maintain that the aspect modifiers constitute the paradigm and most pervasive class of predicate modifiers in our language. Of course, no theory holding only for them is an adequate theory of predicate modification, but any adequate theory must give a special account of them. Now I shall offer as theses some further principles beyond the three above and a construction that is in accordance with them; I hope that this will qualify as an acceptable rational reconstruction of this class of modifiers, and will serve as a guide to the reconstruction of other wider such classes.

$$\begin{array}{rl} \text{IV.} & \phi\phi = \phi \\ \text{V.} & \phi(Z \vee Z') = \phi(Z) \vee \phi(Z') \\ \text{VI.} & \phi(Z \wedge Z') = \phi(Z) \wedge \phi(Z') \end{array}$$

I realize that, in offering theses IV through VI, I run the danger of seriously limiting the class to a subclass of that discussed by Reichenbach and Davidson.[24] For example, with respect to thesis VI, does Davidson recognize the existence of acts described by conjunctions which cannot be analyzed into sets of acts done conjointly? I don't know.

In addition to the six theses I have listed, I offer the informal thesis that each aspect modifier is intimately connected with a certain determinable, in the sense that it makes nonsense of what does not fall under that determinable. For example, "bright(ly)" in "brightly colored" or "bright red" is an aspect modifier (indicating, roughly, the

[24] Cf. Lakoff, "Is There a Distinction between Adverbs and Modal Operators?" University of Michigan, August 1970, pp. 5–7. Mimeographed.

degree of a certain aspect of the property). It belongs intimately, in this sense, to the determinable *color*. (Of course, words in natural language tend to play many roles; we also speak of bright minds. But this complication I shall ignore here.) Suppose that the intension of "is red" is X and its comprehension is of the simplest kind, namely $\{\{X\}\}$. Then the intension of "is bright red" is, say X_1, a proper subclass of X, and its comprehension should be $\{\{X_1\}\}$. Similarly, if the comprehension of "is hard" is $\{\{X_2\}\}$, then the comprehension of "is bright hard" should be $\{\{\Lambda\}\}$. In this sense "bright(ly)" makes nonsense of "is hard." Note that the comprehension of this *nonsensical* predicate is very different from that of the *self-contradictory* predicate "is hard and is not hard," which is $\{\{X_2, H^\omega - X_2\}\}$.

What exactly is the effect of "bright(ly)" on "is red"? The example above is not accurate because we let the comprehension be a property that is not closed. *Really* we should say that

$$\text{"is red"} = \{\{X\}\}^*$$

and this has as members $\{X_1\}$ as well as $\{X'\}$ for every other subset X' of X. So the effect of the modifier is to reduce $\{\{X\}\}^*$ to its proper subclass $\{\{X_1\}\}^*$. We can do this by saying that ϕ, the corresponding operator, acts as follows (for a certain family $\mathbf{F}$):

$$\begin{array}{ll} \text{(a)} & \phi(X') = \begin{cases} X' & \text{if } X' \subseteq X_1 \in \mathbf{F} \\ \Lambda & \text{otherwise} \end{cases} \\ \text{(b)} & \phi(Y) = \{\phi(X) : X \in Y\} \\ \text{(c)} & \phi(Z) = \{\phi(Y) : Y \in Z\} \end{array}$$

When ϕ fulfills equations (a) through (c) for some family $\mathbf{F}$ of subsets of H^ω I shall call ϕ a *projection*. Now I offer as a basic thesis, from which equations I through VI follow:

VII. An aspect modifier is an operator ϕ^* defined by

$$\phi^*(Z) = \phi(Z)^*$$

for some projection, ϕ, on all (closed) properties Z

The need to use ϕ^* instead of ϕ is clear if we wish to deal with the family of closed properties alone; it must be remembered however that this is mainly a matter of mathematical regimentation.

I shall briefly analyze two examples. Consider the sentence

> It is bright (red and hard).

Perhaps no one would say that; perhaps there are no nonartificial examples of this sort. Let us assume that "is red" and "is hard" both have simple comprehensions, $\{\{X\}\}^*$ and $\{\{X'\}\}^*$. Then "is red and hard" has $\{\{X, X'\}\}^*$. Now this is exactly $Z = \{Y : Y$ contains some subset of X and some subset of $X'\}$. The operator ϕ corresponding to "bright(ly)" takes subsets of X_1 into themselves and subsets of X' into Λ; where $X_1 \subseteq X$. Hence

$$\begin{aligned}\phi(Z) &= \{Y : Y \text{ contains } \Lambda \text{ and some subset of } X_1\}^* \\ &= \{\{X_1, \Lambda\}\}^*\end{aligned}$$

So the above sample sentence amounts exactly to

It is bright red and bright hard.

This is not pure nonsense, it is mixed sense and nonsense; and cannot be true, because the mixture is conjunctive.

As second example, consider then

It is bright (red or hard).

with the same assumptions about "is red" and so on. Then

$$\text{"is red or hard"} = \{\{X\}, \{X'\}\}^*$$

which is

$$\begin{aligned}Z' &= \{\{X\}\}^* \cup \{\{X'\}\}^* \\ &= \{Y : Y = \{X_2\} \text{ for some subset } X_2 \text{ of } X \text{ or of } X'\}\end{aligned}$$

Hence $\phi(Z') = \{Y : Y = \{\Lambda\}$ or $Y = \{X_2\}$ for some $X_2 \subseteq X_1\}^*$ but since $\Lambda \subseteq X_1$ also, that means

$$\phi(Z') = \{\{X_2\} : X_2 \subseteq X_1\}^* = \{\{X_1\}\}^*$$

So the sentence amounts to exactly

It is bright red.

which is the same as

It is bright red or it is bright hard.

just because we have construed pure nonsense to imply everything (as contradictions would in the modal approach).

This last consequence is probably less welcome than the others; the way to eliminate it is to leave the values of some predicates [for example, $\phi(Z)$ for some Z] undefined. The technical details could be adopted from Thomason's handling of similar problems within an intensional (modal) context.[25] I am prepared to be faced with counterexamples and undesirable consequences; but I hope I have given evidence of the possibility of systematic reconstruction of classes of predicate modifiers in the present framework.

[25] R. H. Thomason, "A Semantic Theory of Sortal Incorrectness," *Journal of Philosophical Logic* 1(1972), pp. 209–258.