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PROBABILISTIC SEMANTICS OBJECTIFIED: I. POSTULATES AND LOGICS*

Probabilistic semantics is the study of languages in which the admissible valuations are identified as probability functions. There is now a sizable body of work on this subject, in which binary (conditional) probability functions are defined directly on the syntax.[1] I shall refer to that as 'pre-objective'. In what I call 'objective' probabilistic semantics the syntax is interpreted in an extra-linguistic structure (a model or frame) on which the probabilities are defined. This first paper will be a study of pre-objective probabilistic semantics, with the dual aim of giving reasons to doubt its sufficiency, and of displaying its essential structure as a preliminary step toward 'objectification'. The analogy I have in mind, obviously, is the transition from 'state-description' and 'model set' semantics of modal logic to 'possible world semantics'. But I share the hope that the results will be of such richness as to yield a significant supplement to the familiar truth conditions and worlds semantics. In section 1 I shall make some brief remarks on the philosophical problems that motivate the study. The remainder will propose new probabilistic analyses of some familiar logics, ending with the conclusion that there is no reasonable (pre-objective) treatment of quantification. Part II will discuss arguments with infinite sets of premises, and introduce natural relations and groupings of probability functions that give some idea of what the models will have to be like. And Part III will propose a model theory along those lines.

1. THE IDEA OF PROBABILISTIC SEMANTICS

Various problems in philosophical logic and philosophy of language have led to suggestions that the standard truth condition/possible worlds semantics is inadequate, if not entirely mistaken, as a general framework. Many such suggestions point to pragmatics, but some point (instead, or additionally) to ways in which semantics may be enriched. Among the latter, recourse to probabilities is a recurring theme. In 'Reference and Understanding',

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Hilary Putnam argues that our ability to understand use of the language is not explicable in terms of knowledge of truth conditions; he discusses conditions of warranted or correct assertability, and mentions Carnap's and Reichenbach's related ideas on probabilistically qualified assertion.[2] Hartry Field (1977), arguing that truth and reference are indispensable to semantics, argues also that there is a second semantic dimension, the 'conceptual role' of expressions, which is to be explicated in terms of subjective probability. He mentions especially the vexing problem of "Peter believes that Hesperus is not Phosphorus"; the semantics of propositional attitudes being currently a disaster area in philosophical logic.[3]

Exploring the resources of probability for semantics, there is a great deal of technical work to draw on. For preobjective probabilistic semantics there is a basic result due to Popper, the acticle by Field, and much recent work by Harper, Leblanc, Morgan, and others. For the analysis of classical propositional logic I shall use essentially Popper's postulates, simplified with hindsight. For intuitionistic logic and for quantification theory I shall explore some departures from the extant literature. In the non-triviality and strong completeness proofs I shall exploit the constructions I introduced earlier for a representation theorem.[4]

Before launching the formal development, however, I wish to make some motivating remarks and introduce informally some of the intuitive concepts we draw upon.

What is necessary may not be *a priori* certain; that is, we may not be able to ascertain that it is so without recourse to experience. At least, this is a view that has been cogently argued and widely accepted. Examples offered of necessary truths which are not *a priori* are the statements that Hesperus is identical with Phosphorus (Kripke) and that water is H_2O (Putnam). The history of logic offers other, more recondite and less fashionable, examples of this distinction. If this is correct then one statement may be *a priori* certain and the other not although their truth value is the same in all possible worlds. Hence if this view is accepted we must attempt to develop a semantic theory in which the semantic correlate of a statement is not the set of possible worlds in which that statement is true, and is not determined by this set.

A rough guiding idea that has appeared in other contexts is that intra-person synonymity of two statements is determined by that person's propositional attitudes. This idea is not easily explicable, if only because the objects of the so-called propositional attitudes are not plausibly identified as expressions. A person may believe that A, not realize that sentence B is tautologically equivalent to sentence A, and for that reason not agree that he believes that B. In that case (in the conviction perhaps that he *would* agree, *were* he to realize a certain fact of logic) we would presumably be happy to report that he does believe that B. This does not entirely make nonsense of the attempt to represent a rational person's state of belief by means of a function defined on the sentences of a language; but it does make it at least an idealization.

On the other hand, it is a pertinent idealization. Suppose we think of belief as a theoretical notion related operationally to experiments with bets in the way outlined by such Bayesian writers as De Finetti, Savage, and Jeffrey. Then it is important to point out that a person betting against nature may be in a Dutch Book situation (that is, may have engaged a set of bets such that he will necessarily experience a net loss) because he does not realize that some *a priori* uncertain statement is necessarily true. Coherence in the sense of obedience to the axioms of probability theory cannot rule out such a situation. Hence even with a commonly used idealization, which in effect equates tautologically equivalent statements, we find a genuine and important difference allowed among attitudes toward statements which have the same truth-value in all possible worlds.

The idealized model to be used here as intuitive guide for the logical theory is accordingly as follows. A person's epistemic state can be represented by means of a conditional probability function defined on the set of sentences of a language. What is being represented can at present only be described informally as follows. This person envisages a large space K of possibilities, which he first of all divides into two parts, A and B. He is unconditionally sure that the actual situation is identical with one of the possibilities in A. His unconditional degrees of belief are equal to degrees of belief condition on (the actuality lying in) K, hence his degree of belief that (the actual lies in) A, given K equals 1. But he also allocates degrees of belief on the supposition that his unconditional certainty is mistaken, i.e., that the actual lies outside of A. This divides B into two parts, B_1 and B_2. He is sure that actuality lies in B_1 *given that* it lies outside A. Of course he may consider it more probable that the actual situation is in B_1^1 than that it is in B_1^2, given that it lies outside A. It is part of the idealization that such comparisons are numerically representable in a unique way. Further discussion of the plausibility and possible improvement of this idealized model I shall leave to epistemological studies where it is frequently encountered.

2. LOGICAL ANALYSIS OF PROBABILIFIED LANGUAGE

Suppose that a person's epistemic state can be represented by a conditional probability function. Then we can say that for him, *A implies B* exactly if the probability of B is at least as high as that of A, on every specifiable condition. Let SY be a syntax with various sentence connectors, such as &, $\vee$, $\supset$, $\sim$; also two special sentential constants t and f. For convenience, each sentence B can be written variously as $\psi(A)$, $\psi(C)$, etc.; in a single context, $\psi(A)$ and $\psi(C)$ are understood to be the results of substituting respectively A and C for all occurrences of an atomic sentence E, which is foreigh to A and C, in a certain sentence. For example, that certain sentence may be $(A \supset E)$, in which case $\psi(A) = (A \supset A)$ and $\psi(C) = (A \supset C)$.

By the above characterization of implication, A and B have the same place in the web of implicational relations, for a given person whose state is represented by P, exactly if $P(A|C) = P(B|C)$ for all sentences C of SY. I shall use the functional notation $g(\cdot)$ in the obvious way: $g(\cdot) = a$ means that $g(x) = a$ and $g(\cdot) = h(\cdot)$ that $g(x) = h(x)$, for all arguments x in the domain (assumed common) of g (and h). Thus A and B have the same implicational place exactly if $P(A|\cdot) = P(B|\cdot)$. I shall restrict this study entirely to those binary functions defined on SY in which sameness of implicational place is carried over by the sentence connectors:

(2-1) A *real binary valuation* of SY is a binary function m defined on SY, with range in the real numbers and such that if $m(A|\cdot) = m(B|\cdot)$ then $m(\psi(A)|\cdot) = m(\psi(B)|\cdot)$ and $m(\cdot|\psi(A)) = m(\cdot|\psi(B))$.

For such binary valuations we can define the reduction SY/m of SY *modulo* the equivalence relationship $m(A|\cdot) = m(B|\cdot)$. We find that SY/m is an algebra with n-ary operator $[\psi]$ defined by $[\psi]([A_1], \ldots [A_n]) = [\psi(A_1, \ldots, A_n)]$ and partially ordered by the relation $[A] \leqslant [B]$ defined by $m(A|\cdot) \leqslant m(B|\cdot)$, where square brackets indicate the relevant equivalence class in SY. Moreover, m induces a binary function of SY/m, defined by $[m]([A]|[B]) = m(A|B)$, so that m can always be thought of as the composition of an interpretation of the syntax in an algebra, plus a binary valuation on that algebra.

In all the literature I know, the probability functions used have been real binary valuations in this sense, though this is usually guaranteed by the special postulate that if $P(A|B) = P(B|A) = 1$ then $P(\cdot|A) = P(\cdot|B)$.

3. THE POSTULATES

What should a function be like to be called a conditional probability function? To begin, I think that it should be a real binary valuation with bounds *zero* and *one*; secondly it should have some reasonable additivity property if one can be formulated at all (not necessarily as strong as the one I shall use below); and thirdly, to honor the conditionality epithet, it should be 'conditionalizable'. My explication of this third notion is that a class of postulates for a syntax with conjunction should be reasonable in the sense that the following holds:

BASIC LEMMA. If P satisfies the postulates, and P^A is defined as $P(\cdot|-\&A)$, then P^A also satisfies the postulates.

We refer to P^A as *P conditioned* on A. Usually the Basic Lemma is provable by inspection. Popper had a postulate which violates it: $P(A|B) \neq 1$ for some A and some B. This rules out the *abnormality*, the constant binary function with value 1. I take it that there is no good reason for such a postulate. Here is a list of postulates for discussion:

QI. $0 \leqslant P(A|B) \leqslant P(A|A \& B) = P(t|B) = 1$
$P(f|C) = 0$ unless $P(\cdot|C) = 1$

QII. $P(A \& B|C) = P(B \& A|C)$

QIII. $P(A \& B|C) = P(A|C)P(B|A \& C)$

QIV. $P(A \vee B|C) + P(A \& B|C) = P(A|C) + P(B|C)$

QV. $P(A|B \& \cdot) = 1$ iff $P(B \supset A|\cdot) = 1$

QVI. $P(A \supset (B \supset C)|E) = P((A \& B) \supset C|E)$

QVII. $P(A|C) + P(\sim A|C) = 1$, unless $P(\cdot|C) = 1$

Of these, QIV–VI are redundant given the others, $\vee$ and $\supset$ being uniquely definable. Later I shall add postulates for quantifiers. The class of real binary valuations P which satisfy postulates QI–Qn for all sentences A, B, C, E of SY, I shall call the class CQ(I–n). We must first ask which of these classes are reasonable in my sense.

(3-1) If $V \neq n \leqslant$ VII then the Basic Lemma holds for CQ(I–n)

A note on QV: it and QVI were suggested by Birkhoff's axiomatization of intuitionistic logic.[5] They are demonstrably different from the postulates proposed in Morgan and Leblanc (1980) for intuitionistic logic, with negligible overlap in the sets of functions defined. In such areas as quantum logic, and logics of subjunctive conditionals, QVI will not be correct. For the former, QIV is also too strong; in the case of the latter, however, we generally have a material as well as a genuine conditional, and QV–QVI hold for the material one.

Proof of (3-1) is by inspection for all but QV in CQ(I–VI). In that case we argue that $P^E(A \mid B \,\&\, \cdot) = 1$ iff $P(A \mid (B \,\&\, \cdot) \,\&\, E) = 1$ iff $P(A \mid (E \,\&\, B) \,\&\, \cdot) = 1$ (for which, see (3-2) (i) below) iff $P((E \,\&\, B) \supset A \mid \cdot) = 1$ by QV, iff $P(E \supset (B \supset A) \mid \cdot) = 1$ by QVI, iff $P(B \supset A \mid E \,\&\, \cdot) = 1$ by QV again, iff $P(B \supset A \mid \cdot \,\&\, E) = 1$ (see below), iff $P^E(B \supset A \mid \cdot) = 1$, as required. The missing justifications are supplied, *inter alia* by:

(3-2) The algebra SY/P forms a
(i) semi-lattice with [&] as meet, [f] and [t] as *zero* and *one* if P is in CQ(I–III);
(ii) distributive lattice with [$\vee$] as join if P is in CQ(I–IV);
(iii) Heyting algebra with [$\supset$] as relative pseudo-complement if P is in CQ(I–VI)
(iv) Boolean algebra with [$\sim$] as complement if P is in CQ(I–VII).

A *Heyting algebra* (also called *pseudo-Boolean algebra*) is defined as a distributive latice with zero and relative pseudo-complement, i.e. operation $\rightarrow$ such that $x \leqslant y \rightarrow z$ iff $x \wedge y \leqslant z$ for all elements x, y, z. A Boolean algebra is a distributive lattice with complement, i.e. operation* such that $X \wedge X^* = 0$ and $X \vee X^* = 1$.

Result (3-2) is proved by means of elementary calculations which I shall give here for some parts of (i) and (ii) only. Suppose first that $[E] \leqslant [A]$

and $[E] \leqslant [B]$, and we have already established that $[C \& D] \leqslant [C]$ for all C, D. Then we note that $P(E \& A \mid C) = P(E \mid C)P(A \mid E \& C) \geqslant P(E \mid C)$ $P(E \mid E \& C) = P(E \& E \mid C)$. So $[E \& A] \geqslant [E \& E] \geqslant [E]$. But also $P(E \& A \mid C) = P(A \& E \mid C) = P(A \mid C)P(E \mid A \& C) \leqslant P(A \mid C)P(B \mid A \& C) = P(A \& B \mid C)$. Thus $[A \& B] \geqslant [E \& A] \geqslant [E]$. Suppose next that $[A] \leqslant [E]$ and $[B] \leqslant [E]$ and let us prove that $[A] \vee [B] \leqslant [E]$. It follows from (i) and the suppositions that $[A] = [A \& E]$, $[B] = [B \& E]$, and accordingly that $[A \& B \& E] = [A \& B]$. So $P(A \vee B \mid C) = P(A \& E \mid C) + P(B \& E \mid C) - P(A \& B \& E \mid C)$ by these observations and QIV, which equals $P(E \mid C)$ $[P(A \mid E \& C) + P(B \mid E \& C) - P(A \& B \mid E \& C)]$ by QIII, which is $P(E \mid C)$ $P(A \vee B \mid E \& C)$ by QIV, which is less than or equal to $P(E \mid C)$ by QI. So $[A \vee B] \leqslant [E]$ in SY/P.

To prove distributivity, we need to show that $[A \& (B \vee C)] = [(A \& B) \vee (A \& C)]$. But $P(A \& (B \vee C) \mid E) = P(A \mid E)\ [P(B \mid A \& E) + P(C \mid A \& E) - P(B \& C \mid A \& E)]$ by QIII, QIV, and QIII again; which equals $P(A \& B \mid E) + P(A \& C \mid E) - P(A \& (B \& C) \mid E)$ by (i) and those same postulates. By QIV again, that equals $P((A \& B) \vee (A \& C) \mid E)$, as required.

Soundness of classical and intuitionistic propositional calculus for languages with appropriate syntax SY and classes of admissible valuations CQ(I–VII) and CQ(I–VI) follows now at once from familiar results relating those logics to lattices. Note that in intuitionistic logic, negation is defined by $\neg A = (A \supset f)$. For the lattice-theoretic concepts and results I refer to the summary by Fitting, the detailed proofs of Rasiowa and Sikorski, and to Balbes and Dwinger (see bibliography). Note finally that IV–VI are redundant given I–III and VII, this latter set being sufficient to yield a Boolean algebra with join and relative pseudo-complement uniquely definable.

4. CONSTRUCTION OF CONDITIONAL PROBABILITY FUNCTIONS

Preliminary to a discussion of completeness, and partly to establish non-triviality, I shall describe constructions of functions belonging to the various reasonable classes.

(4-1) A *simple valuation* of a lattice L with zero element is a map v of L into the real numbers such that $v(0) = 0$, $v(a) \leqslant v(b)$ if $a \leqslant b$, and $v(a \vee b) + v(a \wedge b) = v(a) + v(b)$.

In Birkhoff's terminology these are a kind of *isotone valuation.*

The method of construction (see note 4) begins with the observation that if v_1 and v_2 are simple valuations (possibly identical) and $V(a|b)$ is defined to equal $v_1(a \wedge b)/v_1(b)$ if $v_1(b)$ is positive; $v_2(a \wedge b)/v_2(b)$ if $v_2(b)$ but not $v_1(b)$ is positive; and 1 otherwise, then V has the properties expected of conditional probability. There is no reason to restrict the construction to two, or even finitely many, simple valuations.

The intuitive picture we may use is of a person who has a large stock of absolute (i.e., one-place) probability functions, well-ordered by a certain preference relation. He will use the first to give unconditional probability judgments ("I am sure that the mass of the moon in kg. is not a rational number"). But if asked for a judgment conditional on a supposition which contradicts his unconditional certainty, he relies on the first function in his stock which assigns a non-zero probability to a rational number ("If it is a rational number, I am sure it is a rational number above 10"). Renyi spoke here of a dimensional ordering; for example:

The position of the particle is certainly not in cube V_3; but if it is in cube V_3 it is certainly not on its side V_2; if it is on V_2, certainly not on the edge V_1, and if on the edge V_1, certainly not exactly on the vertex V_0.

This seems a reasonable ordering of certainties, since a plane has zero volume, a line zero area, and so forth.

(4-2) Let VA be a non-empty well-ordered class of simple valuations on lattice L with zero and one. Define $V(a|b) = v(a \wedge b)/v(b)$ for the first v in VA such that $v(b) \neq 0$, or $= 1$ if VA has no such element. In that case V has the properties:

V1. $0 \leqslant V(a|b) \leqslant V(a|a \wedge b) = V(1|b) = 1$
$V(0|b) = 0$ unless $V(\cdot|b) = 1$

V2. $V(a \wedge b|c) = V(b \wedge a|c)$

V3. $V(a \wedge b|c) = V(a|c)\, V(b|a \wedge c)$

V4. $V(a \vee b|c) + V(a \wedge b|c) = V(a|c) + V(b|c)$

V5. $V(a|b \wedge \cdot) = 1$ iff $V(b \rightarrow a|\cdot) = 1$ if L is Boolean

V6. $V(a \rightarrow (b \rightarrow c)|e) = V((a \wedge b) \rightarrow c|e)$ if L is a Heyting algebra

V7. $V(a|c) + V(a^*|c) = 1$ unless $V(\cdot|c) = 1$, if L is Boolean.

The obvious lacuna in the construction is that V5 need not hold if L is a Heyting algebra. V6 holds there because in such an algebra, $a \to (b \to c) = (a \wedge b) \to c$. V7 holds if L is Boolean because if v is the first element of VA such that $v(c)$ is positive, then $v(a^* \wedge c) + v(a \wedge c) = v((a^* \vee a) \wedge c) + v(a^* \wedge a \wedge c) = v(c) + v(0)$. In addition, if L is Boolean, $(b \to a) = (b^* \vee a)$ so to demonstrate V5 there it suffices that $v(a \wedge b \wedge x) = v(b \wedge x)$ if and only if $v((b^* \vee a) \wedge x) = v(x)$. But $v((b^* \vee a) \wedge x) = v((b^* \wedge x) \vee (a \wedge x)) = v(b^* \wedge x) + v(b\, x) - v(b^* \wedge a \wedge x) = [v(x) - v(b \wedge x)] + v(a \wedge x) - [v(a \wedge x) - v(b \wedge a \wedge x)$ which does equal $v(x)$ if and only if $v(b \wedge x) = v(b \wedge a \wedge x)$ as required.

There are now two approaches to the problem of constructing binary valuations on a Heyting algebra with properties V1–V6. One produces the trivial range $\{0, 1\}$, and can be used in a completeness proof; I shall leave that to the next section. The functions produced by the other have arbitrarily large ranges inside $[0, 1]$, but are 'classical' in a certain sense. These can be produced by extending simple valuations on the Boolean algebra $R(L)$ of ***regular elements*** of L, that is, elements of L such that $a = a^{**}$, where a^* is defined as $(a \to 0)$. These are exactly the elements $a = b^*$ for some b in L, because $b^{***} = b^*$. Here is a summary of relevant facts and a lemma on classical valuations:

(4-4) If L is a Heyting algebra then

(i) $(a^* \vee a)^{**} = 1; a^{***} = a^*$

(ii) $(a \vee b)^{**} = (a^{**} \vee b^{**})^{**}$

(iii) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(iv) $(a \to b)^* = a^{**} \wedge b^*$

(v) if $a \leqslant b$ then $b^* \leqslant a^*$ and $a^{**} \leqslant b^{**}$

(4-5) If L is a Heyting algebra then the algebra $R(L)$ of its regular elements is a Boolean algebra with the zero, one, meet of L, * as complement and $(a + b) = (a \vee b)^{**}$ as join.

(4-6) Let v be a simple valuation on the Boolean algebra $R(L)$ and extend it to L by the equation $v(a) = v(a^{**})$. Then v is also a simple valuation on L.

First $v(0) = 0, v(1) = 1$; second if $a \leqslant b$ then $a^{**} \leqslant b^{**}$ so $v(a) \leqslant v(b)$; thirdly, $v(a \vee b) = v((a \vee b)^{**}) = v((a^{**} \vee b^{**})^{**}) = v(a^{**} + b^{**}) = v(a^{**}) + v(b^{**}) - v(a^{**} \wedge b^{**}) = v(a) + v(b) - v(a \wedge b)$. A valuation produced in this manner will be called *classical.*

(4-7) If VA is a non-empty well-ordered class of classical valuations on Heyting algebra L then V, as defined in (4-2), has properties V1–V6.

In view of (4-2) we need only demonstrate V5. Suppose that $V(a|b \wedge x) \neq 1$ and let v be the first element in VA such that $0 \neq v(b \wedge x)$. Then $v(b \wedge x) \neq v(a \wedge b \wedge x)$. By lemma (4-8) below, $v((b \to a)^* \wedge x) \neq 0$. But $v((b \to a) \wedge (b \to a)^* \wedge x) = 0$, hence $V(b \to a|(b \to a)^* \wedge x) \neq 1$. Suppose secondly that $V(b \to a|x) \neq 1$ and let v be the first element of VA such that $0 \neq v(x)$. Then $v(x) \neq v(x \wedge (b \to a))$. So by lemma (4-8) again, $v(x \wedge b \wedge a^*) \neq 0$, and $V(a|b \wedge x \wedge a^*) \neq 1$. In each case, given that some member v of VA has the property, there will be a first such member, and so we deduce that V5 holds. There remains the lemma.

(4-8) If v is a classical valuation on a Heyting algebra then

(i) $v(x \wedge (b \to a)^*) = 0$ implies $v(x \wedge a \wedge b) = v(x \wedge b)$

(ii) $v(y \wedge b \wedge a^*) = 0$ implies $v(y \wedge (b \to a)) = v(y)$

To prove (i) assume its antecedent. Because v is classical, it will suffice to show that $v(x^{**} \wedge b^{**}) = v((x \wedge a \wedge b)^{**})$. For momentary convenience, let

$$u = x^{**} \wedge b^{**} \wedge (b \to a)^*$$

$$w = x^{**} \wedge b^{**} \wedge (b \to a)^{**}$$

In $R(L)$, $x^{**} \wedge b^{**}$ is the join of u and w. But $v(u) = v((x \wedge b \wedge (b \to a)^*)^{**}) \leqslant v((x \wedge (b \to a)^*)^{**}) = v(0) = 0$. Hence $v(x^{**} \wedge b^{**}) = v(w) = v((x \wedge b \wedge (b \to a))^{**}) = v((x \wedge b \wedge a)^{**})$ as required. For the second part, we know that y^{**} is the + join in $R(L)$ of $[y^{**} \wedge (b \to a)^*]$ and $[y^{**} \wedge (b \to a)^{**}]$; the former is $y^{**} \wedge b^{**} \wedge a^*$, to which v gives the same value as to $y \wedge b \wedge a^*$. So if that is zero, $v(y^{**}) = v(y^{**} \wedge (b \to a)^{**}) = v(y \wedge (b \to a))$.

That there is indeed a significant variety of classical valuations in Heyting algebras is clear from the fact that even in the free Heyting algebra with two generators, $R(L)$ is infinite.[6]

5. STRONG COMPLETENESS OF CLASSICAL AND INTUITIONISTIC PROPOSITIONAL CALCULUS

We can say that $A_1, \ldots, A_n$ *imply* B in P if $(A_1 \& \ldots \& A_n)$ implies B in P. As we shall see in Part Two, generalization to infinitely many premises is not straightforward. However the correct definition must certainly entail the necessary condition:

(5-1) if set X implies B in P, and $P(A \mid C) = 1$ for all A in X, then $P(B \mid C) = 1$.

This will suffice for our proofs of strong completeness.

For classical propositional calculus, this is exceedingly easy. If B cannot be deduced from X in that logic, we know that there is a map of w of SY into $\{0, 1\}$ such that $w(f) = 0$, $w(t) = 1$, $w(E \& C) = w(E)w(C)$, $w(\sim E) = 1 - w(E)$ and assigns 1 to all members of X, 0 to B. The function P defined by $P(E \mid C) = w(E)$ if $w(C) = 1$, or 1 otherwise (reminiscent of construction (4-2)) belongs then to the class CQ(I–VII) – recall that QIV–VI are redundant here – and x does not imply B in P.

For the case of intuitionistic propositional logic I shall adapt the standard strong completeness proof given by R. H. Thomason for an intuitionistic quantification theory.[7] The general argument is this: let sentence B not be deducible from set of sentences X by intuitionistic logic (briefly, X does not *entail* B). Then there is *some* syntax SY to which X and B both belong, and a function P in CQ(I–VI) defined on SY: such that $P(A \mid \cdot) = 1$ for all A in X, but $P(B \mid \cdot) \neq 1$. It will turn out, in fact that if X and A do not jointly entail B, then there is a condition C such that $P(B \mid A \& C) \neq 1$. We could call P a *canonical* probability function for X, since it does this job for all sentences A and B.

Roughly following Thomason's terminology, let the syntax with t, f, &, $\vee$, $\supset$, and countable set of atomic sentences C be called the *morphology M generated* by C. Let $M(E)$ be the morphology generated by $E \cup C$ when C generates M. A *prime* theory T in M is a set of sentences of M closed under intuitionistic deduction and such that if $(A \vee B)$ is in T, so either A or B. The following fact from proof theory is needed:

(5-4) Let M be a morphology and E a countable set of atomic sentences foreign to M, X a set in M, B a sentence in M, and Y a set of sentences in the morphology generated by E alone.

> Then if X does not entail B, and Y does not entail f, there exists a prime theory T in $M(E)$ which contains X and Y but not B.

This is really a combination of several facts which are also used or proved in Thomason's completeness proof, and several others: the finitary character of intuitionistic deducibility and the Interpolation Lemma for intuitionistic logic.[8]

Let X now be a specific consistent set in morphology M, E a countable set of atomic sentences foreign to M, and let SY $= M(E)$. Let Z be the class of prime theories in SY which contain X. Where the members of E are ordered as $A_1, A_2, \ldots$ define $Z(n)$ to be the subclass of Z which contains as member the atomic sentence A_n, but no atomic sentences after A_n. Let $Z(0)$ be the remainder of Z; that is, $Z(0)$ consists of those prime theories that contain no members of E (they may of course contain complex sentences made up from members of E).

We now order the countable set of pairs $\langle A, B\rangle$ of sentences A, B in SY such that X and A do not jointly entail B, as the pairs $p_1, p_2, \ldots$ For each pair p_i we choose a number $g(i)$ as follows:

(a) if $\langle A, B\rangle = p_i$ then A and B belong to the morphology $M(\{A_1, \ldots, A_{g(i)-1}\})$

(b) $2 \leqslant g(j) \leqslant g(j+1)$

This can be done inductively, choosing $M(\{A_1, \ldots, A_n\})$ where n is the first number such that A and B belong to that morphology, and then choosing $g(i)$ to be the first number after n and after $g(i-1)$.

Let $T(p_i)$ be the set of prime theories in Z that contain A and $A_{g(i)}$ and $\sim A_n$ for all $n > g(i)$, but not B – where $\langle A, B\rangle = p_i$. That set is a subset of $Z(g(i))$. Also, it is not empty, because of fact (5-4), since X and A do not entail B and $\{A_{g(i)}, \sim A_{g(i)+1}, \sim A_{g(i)+2}, \ldots\}$ is consistent.

We now well-order Z as follows. First we well-order $Z(g(i))$ so as to place a member of $T(p_i)$ first. Since all the classes $Z(n)$ are disjoint for distinct n, this can be done for all of them at once. Next we well-order each class $Z(n)$ with n not in the range of g, in some way or other. Finally, we order Z by saying that T in $Z(m)$ precedes T' in $Z(n)$ exactly if either $m < n$ or $m = n$ and T precedes T' in the previously chosen well-ordering of $Z(m)$. It will now be clear that if $\langle A, B\rangle$ is the couple p_i, then the first theory in Z which contains both A and $A_{g(i)}$, is the first in $Z(g(i))$, and does not contain B.

(5-5) Define P on SY by the condition that for all sentences A', B' of SY, $P(A'|B') = 1$ if A' belongs to the first member of Z which contains B', or if Z has no such member, and zero otherwise.

This is of course similar to the construction in (4-2), and all but QV are easily checked. It T contains A & B it contains A, t, B & A; it does not contain f if it is in Z at all; thus QI and QII hold. For QIII, if the first theory T' that contains C also contains A & B, then it is also the first to contain A & C; so $P(A \& B|C)$, $P(A|C)$, $P(B|A \& C)$ are all 1. If T' does not contain A & B, then it lacks at least one of A and B. If it lacks A then $P(A \& B|C) = P(A|C) = 0$. If it contains A it lacks B, in which case T' is also the first to contain A & C, and we have $P(A \& B|C) = P(B|A \& C) = 0$. Finally suppose there is no member of Z which contains C; then neither can any contain A & C, hence both sides of the equation are 1. For QIV, we note that if there is a first theory in Z that contains C, then (being prime) it contains $A \vee B$ only if it contains at least one of A and B, and contains both $A \vee B$ and A & B exactly if it contains both A and B. For QVI we need only note that $A \supset (B \supset C)$ and $(A \& B) \supset C$ are deducible from each other.

This leaves QV, for which we go to so much trouble. Suppose $P(A|B \& C) \neq 1$. Then the first theory to contain B & C does not contain A, so X, B & C does not imply A. Hence X and C do not imply $B \supset A$. Let $\langle C, B \supset A \rangle$ be the pair p_i; then the first theory in Z which contains C and $A_{g(i)}$ does not contain $B \supset A$. Hence $P(B \supset A|C \& A_{g(i)}) \neq 1$. Conversely, suppose that $P(B \supset A|E) \neq 1$. Then X, E does not imply $B \supset A$, so X, E & B does not imply A. Let $\langle E \& B, A \rangle$ be the pair p_j; then the first theory which contains $E \& B \& A_{g(j)}$ does not contain A, and so $P(A|E \& B \& A_{g(j)}) \neq 1$.

Finally we need to check that P is a real binary valuation of SY. Suppose that A is not deducible from X and B. Let $\langle A, B \rangle$ be the pair p_j; then $P(B|A \& A_{g(j)}) \neq 1$. But $P(A|A \& A_{g(j)}) = 1$ by QI; hence $P(A|\cdot) \neq P(B|\cdot)$. Similarly of course if B is not deducible from X and A. Hence if $P(A|\cdot) = P(B|\cdot)$ then A and B are proof-theoretically equivalent relative to X, and hence $\psi(A)$ belongs to a theory in Z if and only if $\psi(B)$ does. *A fortiori* A and B are mutually replaceable everywhere in $P(-|\cdot)$.

It will have been noted that, as in the standard strong completeness proof for quantificational intuitionistic logic, I have used a syntax extension. The above proof does not rule out – especially since I gave only a necessary condition for infinitary implication – that only weak completeness may be provable for the 'right' definition of infinitary implication, for the class of members of CQ(I–VI) defined on a *given* syntax SY. In other words, the condition QV may be what Leblanc calls "essentially substitutional". A similar problem will appear in classical quantification theory; and these problems are bound up with the general problem of implication relationships to be investigated below.

6. CLASSICAL QUANTIFICATION: FIRST ANALYSIS

There is in my opinion no reasonable treatment of quantification within pre-objective probabilistic semantics. I shall describe some treatments that work, in the sense that classical quantificational logic is sound and strongly complete; and I shall opt for one of them. In the next section I shall discuss how it might be made more reasonable in other aspects; this will at once provide some concepts needed for the more abstract problems of Part Two.

Leblanc proposed the postulate (in effect):

(6-1) $P((x)Fx\,|\,B) = \underset{n \to \infty}{\text{limit}}\ P(Ft_1\ \&\ \ldots\ \&\ Ft_n\,|\,B)$ where $t_1, t_2, \ldots$ are all the individual constants of SY.

(Our syntax is now assumed to have individual constants and variables, predicators of various degrees, and the universal quantifier.[9]) But in order not to lose strong completeness he redefined implication so that X implies B in P defined on SY only if X implies B in P' for each P' agreeing with P on SY, but defined on a morphology generated by the primitive vocabulary of SY plus a set of individual constants foreign to SY (a "term extension of SY").

Of course this is equivalent to the sort of procedure I adopted for intuitionistic logic; strong completeness is proved in the sense that if B is not deducible from X, then there is *some* syntax SY′ to which X and B both belong, and a function P defined on SY′ and obeying the postulates, and some sentence C of SY′ such that $P(A\,|\,C) = 1$ for all A in X but $P(B\,|\,C) \neq 1$.

Hartry Field had adopted a slightly different approach. He wishes implication to remain a relationship defined for P in terms of P and no other probability function, and gets the same effect as Leblanc by defining:

(6-2) A *reasonable* probability function defined on SY is the restriction to SY of any function defined on some term extension of SY, which obeys a certain set of postulates (equivalent to QI–VII and (6-1)).

In that case the strong completeness result is that if B is not deducible from X, then there is a reasonable probability function defined on the smallest syntax (with &, v, ⊃, ~, (x)) to which X and B belong, and a sentence C of that syntax such that $P(A \mid C) = 1$ for all A in X but $P(B \mid C) \neq 1$.

I have spent some time outlining these two different, but for all 'practical' purposes equivalent approaches, to draw attention to the fact that strong completeness is a problem here which apparently requires artful dodges – *ad hoc* manoeuvres, epicycles. They are reminiscent of the "substitution interpretation" of the quantifiers, not merely in the explicit reference to the language's vocabulary, but in the mystery they leave us with: Why is strong completeness so great a good that we go to such trouble for it? Field answers that strong completeness is not itself the *desideratum*: it is reasonable for a person to envisage additions to his language, and to take this into account when he constitutes his subjective propbability function. Presumably he may envisage an extension of his language in which he can say: "There are entities not named by any terms in my language, distinct circumstances not differentiable by means of my predicates, . . ." and so on. But is that not simply to admit that probabilistic semantics must be a non-self-sufficient fragment of a larger theory of language?

The postulates I shall propose will allow a strong completeness proof without recourse to either limits of sequences, or term extensions. They will still be unreasonable in another way, on a more basic level.

I use the notation $(t/x)A$ to stand for the result of replacing all free occurrences of x in A by occurrences of constant t.

QVIII. $P((x)A \mid B) = P((t/x)A \mid B)P((x)A \mid B \,\&\, (t/x)A)$

QIX. $P((x)A \mid B) = P(A \mid B)$ if x not free in A

QX. If $P((t'/t)A \mid B \,\&\, \cdot) = 1$ for all constants t' in the domain SY of P, then $P((x)(x/t)A \mid B) = 1$, where x is any variable such that t does not occur in the scope of (x) in A (briefly, such that t is *free* for x in A).

It will be apparent that the Basic Lemma holds for CQ(I–X). To prove the soundness of quantificational logic for a language with syntax SY and class of admissible valuations CQ(I–X) we should of course first settle on some formulation of that logic. The following is convenient; it is the one Leblanc used. Note that the sentences never contain free variables; but when A occurs as a part of a formula it may have free variables in it. The axioms are the sentences in set QAX defined inductively by:

1. if C is a theorem of classical propositional logic, then C is in QAX
2. if C has the form
   (a) $(x)B \supset (t/x)B$
   (b) $B \supset (x)B$, where x not free in B
   (c) $(x)(A \supset B) \supset \cdot\, (x)A \supset (x)B$
   then C is in QAX
3. if C is in QAX and t is free for x in C, then $(x)(x/t)C$ is in QAX

The proviso, that t be free for x, in QX and clause 3 here has of course the same reason: we do not wish to assert $(x)(x/t)((y)(\exists x)(\sim Rxy) \supset (\exists x)(\sim Rxt))$.

The only rule of this logical system is *modus ponens*; B is a consequence of X exactly if there is a derivation of B from members of $X \cup QAX$ by repeated applications of that rule.

It will suffice for soundness to prove that if A belongs to QAX and P to CQ(I–X), then $P(A\,|\cdot) = 1$. If A belongs to QAX and the member of quantifiers in A is zero, then A is a theorem of classical propositional calculus, and the considerations of section 3 suffice. Suppose then that the thesis holds for all members of QAX having at most n quantifiers in them, and that A has $n + 1$ quantifiers in it. If A has the form $(x)B \supset (t/x)B$ then $P(A\,|\cdot) = 1$ by QI, QV, and QVIII; it it has the form $B \supset (x)B$, and x is not free in B, then $P(A\,|\cdot) = 1$ by QV and QIX. (Recall here that if P is in CQ(I–VI) then A implies B in P iff $P(A \supset B\,|\cdot) = 1$.)

For 2(c) suppose that C has form $(x)(B \supset C) \supset [(x)B \supset (x)C]$. For brevity, call a sentence D a *P-tautology* or *a priori* in P exactly if $P(D\,|\cdot) = 1$. We note that

$$(x)A \supset (t/x)A \text{ and } (x)(A \supset B) \supset (t/x)(A \supset B), \text{ i.e.}$$

$$(x)(A \supset B) \supset [(t/x)A \supset (t/x)B]$$

are P-tautologies by preceding reasoning; relying on the soundness in the present context of propositional logic we infer that

$$[(x)(A \supset B) \,\&\, (x)A] \supset (t/x)B$$

is a P-tautology too. By QV then,

$$P((t/x)B \mid (x)(A \supset B) \,\&\, (x)A \,\&\, \cdot) = 1$$

and since this has been established for an arbitrary constant t, we can generalize to all constants t'; and since $(t'/t)\,(t/x)$ is (t'/x), we apply QX to derive

$$P((x)B \mid (x)(A \supset B) \,\&\, (x)A \,\&\, \cdot) = 1$$

whence by QV again, $P([(x)(A \supset B) \,\&\, (x)A] \supset (x)B \mid \cdot) = 1$, and our conclusion follows by QVI. (We note that t must be chosen free for x in $(t/x)B$).

Finally, to validate clause 3, suppose that A is $(x)(x/t)B$ and B, with t free for x, is in QAX. In that case every sentence $(t'/t)B$ is also in QAX, an easily attested fact of our proof theory. But $(t'/t)B$ has at most n quantifiers in it, so by our hypothesis of induction, $P((t'/t)B \mid \cdot) = 1$ for all constants t'. Hence by QX, $P((x)(x/t)B \mid \cdot) = 1$.

Turning now to completeness, we can again draw on a standard strong completeness result. Suppose that B is not deducible from X in quantificational logic; let SY be the minimal syntax with all the given connectors and a universal quantifier, to which both X and B belong. Then we must prove that there is a function P and a sentence E of SY, such that P is defined on SY, P is in CQ(I–X), $P(A \mid E) = 1$ for all A in X and $P(B \mid E) \neq 1$.

A *model* for SY consists of a non-empty domain D and an *interpretation function I* which assigns an n-ary relation $I(P)$ on D to each n-ary predicate P. A countable sequence s of members of D *satisfies* a sentence A in this model exactly if certain familiar conditions obtain, of which I shall list two:

(i) s satisfies $Pb_1 \ldots b_n$ exactly if $\langle s(b_1), \ldots, s(b_n)\rangle$ is in $I(P)$, where $s(b)$ is the n^{th} member of s when b is then the n^{th} individual constant of SY (in the given, "alphabetical" ordering)

(ii) s satisfies $(x)(x/b)B$ exactly if all sequences s' which are like s except perhaps in the n^{th} place satisfy B (where b is the n^{th} individual constant)

The standard result needed is that if B is not deducible from X in quantificational logic then there exists such a model M_o, and a sequence s_o in that model which satisfies all members of X, but not B. Using this model and sequence, we can construct a function P in CQ(I–X) of the sort required for the probabilistic completeness proof.

Let J be the set of all sequences in M_o, well-ordered in such a way that s_o is placed first. Define, for all sentences E, C in SY:

> $P(E\,|\,C) = 1$ if the first sequence s which satisfies E satisfies C also, or if there is no sequence that satisfies C; and $= 0$ otherwise.

This is again exactly similar to (4-2) since satisfaction by s can be represented by a map into $\{0, 1\}$, with &, $\vee$, $\supset$, $\sim$ treated in the Boolean way. We also note that $P(E\,|\,t) = 1$ iff s_o satisfies E, hence if we can only verify that P obeys QVIII to QX, then our completeness proof will be finished.

For QVIII suppose that the first sequence in J which satisfies B also satisfies $(x)A$. Then it must satisfy $(b/x)A$, so it is *a fortiori* the first which satisfies B & $(b/x)A$. Both sides of QVIII then equal 1. Suppose secondly that it does not satisfy $(x)A$; then either it does not satisfy $(b/x)A$, or else the first one to satisfy B & $(b/x)A$ does not satisfy $(x)A$; in either case, both sides of the equation are zero. Finally suppose no sequence satisfies B. Then also no sequence satisfies B & $(b/x)A$; so both sides equal 1 again.

QIX needs no argument. For QX, suppose that $P((x)(x/b)A\,|B) \neq 1$. Then some sequences satisfy B, and the first of these, s, does not satisfy $(x)(x/b)A$. Now b may appear in B, but we note that if b' is foreign to both A and B, then $(x/b)A = (x/b')(b'/b)A$. In that case, if s' is like s except perhaps at b', it will satisfy B; but one such fails to satisfy $(b'/b)A$ because s does not satisfy $(x)(x/b')(b'/b)A$. Let s'' be the first to satisfy B which does not satisfy $(b'/b)A$; then it is the first to satisfy B & $\sim (b'/b)A$. In view of this, $P((b'/b)A\,|B$ & $\sim (b'/b)A) = 0$; thus QX holds.

7. SUBORDINATION: TOWARD MORE REASONABLE VALUATION CLASSES

What more can we expect, besides soundness, strong completeness, and non-triviality results? I gave one answer when I listed the Basic Lemma in section 3 as a guide to the selection of postulates. It ruled out, for example, such a replacement for QX as

(7-1) If $P(A \mid B \,\&\, \cdot) = 1$ and b does not occur in B, then $P((x)(x/b)A \mid B) = 1$,

which might look attractive for other reasons. For suppose that $P(E \mid \cdot) \neq 1$; then P could satisfy (7-1) (for all sentences A, B, terms b) while P^E does not, because $P^E(E \mid \cdot) = 1$, while $P^E((x)(x/b)E \mid t) = P((x)(x/b)E \mid E)$ need not be 1 at all.

But even with our present treatment, there is something odd. Let b_1, $b_2, \ldots$ be all the individual constants in our language, and suppose we could conditionalize on all of $F_1 = (b_1/b)E$, $F_2 = (b_2/b)E$, ... Presumably the result would be a function P' which is the limit of $P^{F_1 \,\&\, \ldots \,\&\, F_n}$, as n goes to infinity, if this can be done at all. But it is easily seen that although, if P obeys QX, then so does each new function $P^{F_1 \,\&\, \ldots \,\&\, F_n}$, that limit P' may not. For perhaps $P(F_1 \,\&\, \ldots F_n \mid t)$ was greater than zero but less than 1 for each number n, while $P((x)(x/b)E \mid t)$ was zero; in that case $P'((x)(x/b)E \mid t)$ is also zero.

In other words we have failures of an infinitary version of the Basic Lemma. Those failures would come to haunt us if we extended SY to infinitary conjunction. And if they do not haunt us now, it is therefore only because of the poverty of our syntax. Any virtue which depends on that poverty, is no virtue at all.

To see how we can ameliorate this situation, let us see how we can change the classes CQ(I–n) without upsetting logical soundness and strong completeness results. Clearly if we merely enlarge such a class, no completeness result is lost. And if we add new members P such that if A implies B in all members of the original class, then also A implies B in P, it will follow that no soundness results are lost, assuming that the logical sequence relation is finitary and $A_1, \ldots, A_n$ imply B only if $(A_1 \,\&\, \ldots A_n)$ does.

Finally we note a distinct difference between QV and QX on the one hand, and on the others. The former may be called *global* postulates, the latter *local*. The former are *directly* concerned with the *a priori*, and the latter at most indirecly. The local postulates will not affect validity of infinitary arguments. We arrive therefore at the following concepts:

(7-2) If P is a real binary valuation of SY, and W a class of real binary valuations, defined on syntaxes containing SY, we call P *subordinate* to W (briefly $W \rightrightarrows P$) exactly if for all A and B in SY, if A implies B in each member of W, then A implies B in P.

(7-3) $CQ^*(\text{I}-n)$ is the set of real binary valuations P such that $X \rightrightarrows P$ for some subset X of $CQ(\text{I}-n)$ and such that P obeys the *local* postulates among QI–Qn.

This idea has several consequences for our preceding results.

(7-4) If P is in $CQ^*(\text{I}-n)$, with n = VI, VII, X, and $Y = \langle B_1, B_2, \ldots \rangle$ and the function P^Y:
$$P^Y(A \mid C) = \underset{n \to \infty}{\text{limit}}\ P(A \mid C \,\&\, B_1 \,\&\, \ldots \,\&\, B_n)$$
exists, then it is also in $CQ^*(\text{I}-n)$.

To prove (7-4) we first note that the Basic Lemma holds in these cases, for the simple reason that if $W \rightrightarrows P$ then $W \rightrightarrows P^A$; and we know from previous considerations that P^A satisfies the local postulates if P does.

So, defining P^n to be $P^{B_1 \,\&\, \ldots \,\&\, B_n}$, we see that any class W in $CQ(\text{I}-n)$ such that $W \rightrightarrows P$, is also such that $W \rightrightarrows P^n$ for all n. But if A implies B in all P^n, then $P(A \mid C \,\&\, B_1 \,\&\, \ldots \,\&\, B_n) \leqslant P(B \mid C \,\&\, B_1 \,\&\, \ldots \,\&\, B_n)$ for all n, so also the limit of the first sequence is no greater than that of the second: A implies B in P^Y. Thus $W \rightrightarrows P^Y$.

We must finally check that P^Y obeys the local postulates which all members of W obey. In some cases this is obvious: since $A \,\&\, B$ implies $B \,\&\, A$ in P^Y, and vice versa, QII must hold. Similarly for QVI. Since $P^n(t \mid A) = P^n(A \mid B \,\&\, A) = 1$ for all n, and $0 \leqslant P^n(A \mid B) \leqslant 1$ for all n, P^Y obeys the first part of QI. Suppose now that $P^Y(\cdot \mid C) \neq 1$ so let $P^Y(B \mid C) \neq 1$, i.e. $\underset{n \to \infty}{\text{limit}}\ P^n(B \mid C) \neq 1$. But $P^n(f \mid C) \leqslant P^n(B \mid C)$ for all n; therefore $P^Y(f \mid C) \neq 1$. It follows that for every n there is a $q > n$ such that $P^q(f \mid C) \neq 1$. But if it is not 1 then it must be zero; hence for each n there is a $q > n$ such that $P^q(f \mid C) = 0$; thus $P^Y(f \mid C) = 0$.

For QIII and QVIII, we can appeal to the fact that *limit* $(y \cdot z)$ = *limit* y · *limit* z; and for QIV to the fact that *limit* $(y + z)$ = *limit* y · *limit* z; similarly for QVII. This ends the proof.

It may be as well to give a concrete example at this point of a function in $CQ^*(\text{I}-\text{X})$. To show that the fact that $Fb_1, Fb_2, \ldots$ are all *a priori* in P does not guarantee that $(x)Fx$ is, let us recall the construction in the completeness proof in section 6. In that construction, take X to be $\{Fb_1, Fb_2, \ldots\}$, where $b_1, b_2, \ldots$ are all the constants, and B the sentence $(x)Fx$. The first sequence s_o in J satisfied all of X but not B; keep it there but well-order J in such a way that all the sequences which satisfy X come first. The constructed function P is such that if $X \cup \{C\}$ is satisfiable in the model at all, then the first sequence which satisfies C, also satisfies X; and of course we still have $P((x)Fx \mid t) = 0$. If we now let P^n be P conditioned on $Fb_1 \; \& \ldots \& \; Fb_n$, we see the tail-end of J being lopped of bit by bit. Because we know the "insides" of the construction of P, we have a very straightforward way of conditioning P on all of X: we delete all members of J which do not satisfy X. Let the result be J', and let the function constructed from J' in the way that P was constructed from J, be P'. Suppose now that for given E and C, E does not imply C in P'. Then the first sequence in J' which satisfies E exists and does not satisfy C; it must be in J, and must be the first in J to satisfy E, because J' is an initial segment of J. Thus E does not imply C in P. Hence $P \rightrightarrows P'$. So we see that P' is in CQ^*(I–X). Yet $Fb_1, Fb_2, \ldots$ are all *a priori* in P', and $P'((x)Fx \mid t)$ is zero.

We are able to claim a similar improvement in our relations with intuitionistic logic. Let us go back to the construction in its completeness proof in section 5, and suppose that B is a specific sentence of initial morphology M which is not deducible from X; then there is already a prime theory T in the class $Z(0)$ which does not contain B. Let that theory be placed first in the well-ordering of $Z(0)$. (Note that it may for instance include the negations of all the "foreign" atomic sentences $A_1, A_2, \ldots.$) Thus the constructed function P, and its restriction P' to M are such that they assign *one* to each member A of X conditional on t, and zero to B conditional on t.

But P' is in CQ^*(I–VI). It is defined on M, and so cannot violate QI–IV or QVI, since P obeys those (the *local* postulates). Finally it is clear that if E and C belong to M, and E does not imply C in P', then neither does it in P; hence $P \rightrightarrows P'$.

So we can state the stronger strong completeness result: if B is not deducible from X in intuitionistic propositional logic, then there is a function P in CQ^*(I–VI), defined on the minimal (&, ∨, ⊃) syntax containing X and B such that X does not imply B in P.

The subordination relationship, a generalization of that described in the Basic Lemma, allows us therefore to define more reasonable classes of probability functions than the postulates alone. But the Basic Lemma, and closure under subordination for the classes of admissible valuations, should be facts explained by the right semantics; not imposed on it. To that extent, the mystery remains. We will see in Part II that the problems we have just discussed are in fact representative of a cluster of deep problems that already exist in CQ(I–III).

8. A FINAL WORD ON QUANTIFIERS

My postulate QX is still "substitutional", in that it makes explicit reference to the set of constants of the syntax. Stated informally it says in part that if Ft' is *a priori* certain for each constant t' then so is $(x)Fx$. The use of constants and avoidance of assignments to open formulas is of course inessential; we could have decreed instead that if Fy is *a priori* for each variable y, then so is $(x)Fx$. The constants play the role here of 'substitutive variables' (Curry) or 'individual parameters' (Thomason), not of names. It is therefore natural to stipulate that there are infinitely many.

The substitutional character of QX means that from a model-theoretic point of view it must eventually turn out to have a derivative status. Its correctness must be something that is explicable on more fundamental grounds; and it can in any case be correct only for a restricted class of probability functions. Such restriction we have already found; QX characterizes CQ(I–X), and not the more natural, reasonable class CQ^*(I–X).

At this point we may not be too far removed from a genuine model-theoretic point of view. For in the standard model theory of quantificational logic we can also see a special place occupied by those valuations which are the complete stock of valuations used by the substitution interpretation. Given a domain D, an interpretation function I, and a countable sequence s in D, let us write $s(b_n)$ for the n^{th} member of s (where b_n is on n^{th} constant). The associated valuation $v(s)$ assigns *True* to A if s satisfies A (in model $M = \langle D, I\rangle$) and *False* otherwise. Let us call sequence s and its associated valuation $v(s)$ *proper* exactly if for each variable x and sentence B, s satisfies $(x)(x/t)B$ if and only if it satisfies all the sentences $(t'/t)B$, for each constant t'. We know very well that quantificational logic is not compact if we restrict the class of admissible valuations to those associated with proper sequences. But for each sequence s' in the domain we can find a function f of the set of constants into itself, and a proper sequence s such that $s'(b_n) = s(f(b_n))$ for each number n. This observation is readily suggested by reflection on the use of term extensions in Henkin's proof of strong completeness. So all the sequences (and their associated valuations) can be regarded as 'manufactured from' proper sequences. Indeed, this fact is the heart of Henkin's proof.

So it is possible, if somewhat *outré* to think of standard semantics as beginning with substitutional semantics (the class of valuations that can be associated with proper sequences), and then liberalizing that by admitting also valuations related in certain way to those in the original class. At that point, only a Gestalt switch is needed to place us in the world of model theory proper. But to make sense of this world we will need a direct characterization of models, yielding a probabilistic analogue of the standard truth-conditional semantics. This is the task of Part III of this three-part paper.

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NOTES

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[1] See bibliography: Field, Harper, Leblanc, Morgan.

[2] In Putnam (1978); see also Nola's review.

[3] See further the quote from Harman at the beginning of Field's article, the book by Ellis, my review of the latter (1980a) and my (1980b).

[4] van Fraassen (1976) or (1979), section 7.

[5] Birkhoff (1967), Chapter XII, Section 3.

[6] Balbes and Dwinger, page 185.

[7] Thomason (1978).

[8] Fitting, Chapter Six, section 5.

[9] Leblanc (1979).

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