Scalarization Method and Expected Multi-Utility Representations

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Abstract

We characterize the class of (possibly incomplete) preference relations over lotteries that can be represented by a compact set of (continuous) expected utility functions which preserve both indifferences and strict preferences. This finding contrasts with the representation theorem of Dubra, Maccheroni and Ok (2004) which either delivers a noncompact set of utility functions, or at least one function that does not respect strict preferences (unless the completeness axiom holds). For a preference relation of the sort we consider, our representation theorem reduces the problem of recovering the induced choice correspondence over convex sets of lotteries to a scalar-valued, parametric optimization exercise. Several applications of this observation are presented. Most notably, in an otherwise standard game with incomplete preferences, the collection of *pure* strategy equilibria that one can find using this scalarization method is found to correspond to a refinement of the notion of Nash equilibrium that requires the (deterministic) action of each player be undominated by any *mixed* strategy that she can follow, given others' actions. Along similar lines, we also obtain a refinement of Walrasian equilibrium and provide an equilibrium existence theorem assuming a continuum of traders.

Keywords: Incomplete preferences; expected multi-utility; Nash equilibrium; Walrasian equilibrium; nonbinary choice; social planning

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1. Introduction

Starting with Aumann (1962), early research on representation of incomplete preference relations under risk explored sufficient conditions that allow one to extend a preference relation by a *single* expected utility (EU) function. Put precisely, given a (possibly incomplete) preference relation $\succsim$, the purpose of a typical work in this early literature is to find an EU-function u that is $\succ$*-increasing* ($E_p(u) > E_q(u)$ whenever $p \succ q$) and that is *indifference preserving* ($E_p(u) = E_q(u)$ whenever $p \sim q$). As noted by Aumann, the main merit of this representation notion is that maximization of an $\succ$-increasing function over a choice set delivers a maximal lottery that can possibly be selected from that set by the decision maker defined by $\succsim$.

However, when studying economic phenomena related to indecisiveness, the researcher often needs to recover the choice correspondence induced by an incomplete preference relation in its entirety. Indeed, the best-known behavioral consequences of indecisiveness include (i) a certain degree of randomness in choices, which, as Mandler (2005) notes, may reflect itself with intransitivity of observed choice behavior; and (ii) the multiplicity of alternatives that might be chosen in a given situation, which is the focus of Rigotti and Shannon (2005) in their work on indeterminacy of equilibria in security markets. The study of how an agent might resolve her indecisiveness is a related area of research where the central goal is to determine a suitable procedure that describes the agent's behavior in choice problems which involve a multiplicity of maximal alternatives.[1] More generally, it has been recently observed that a variety of interesting behavioral phenomena can be explained by two-stage choice procedures where in the first stage the agent identifies a *collection* of maximal alternatives in a given choice set (with respect to an endogenously determined incomplete preference relation), and then makes her final choice among these maximal alternatives according to a secondary criterion.[2]

The problem of recovering the choice correspondence induced by an incomplete preference relation gave rise to the literature on multi-utility representations which provide a set of utility functions that fully characterize a given preference relation. In fact, it seems fair to argue that the virtue of such a representation theorem lies in its potential use as

[1]For example, Ok et al. (2009) propose such a procedural model of *attraction effect* which refers to the phenomenon in which, given a set of two feasible alternatives, the addition of a third alternative that is clearly inferior to one of the existing alternatives increases agent's tendency to choose the item that dominates the new alternative.

[2]Various reference-dependent choice models, for instance, *necessitate* the use of incomplete preferences in such a procedural context (Masatlioglu and Ok, 2005; Apesteguia and Ballester, 2009). Another example is the procedural model of Manzini and Mariotti (2007) that accounts for intransitive choice behavior. A longer list of indecisiveness-related phenomena includes preference for flexibility (Danan and Ziegelmeyer, 2006), preference for commitment (Danan et al., 2009), and several implications for political games (Roemer, 1999; Levy, 2004).

an analytical tool that can facilitate the exercise of identifying the choice correspondence associated with a preference relation which satisfies certain behavioral axioms. The performance of a representation theorem in this regard depends, in turn, on the properties of the set of utility functions that it delivers.

In this paper, assuming a compact prize space, we provide necessary-sufficient axioms on a preference relation $\succsim$ over lotteries that allow one to represent $\succsim$ by a compact set of $\succ$-increasing and indifference preserving (continuous) EU-functions.[3] By a well-known "theorem of alternative," one can then show that, given such a preference relation and a representing set of utility indices U, an element of a *convex* set K of lotteries is maximal in K if and only if it maximizes over K the expectation of a weighted average of some utility indices in U. Thus, for a preference relation of the sort that we consider, our representation theorem reduces the problem of recovering the associated choice correspondence over convex sets of lotteries to a *scalar-valued*, parametric optimization exercise.

The axioms that we use in this representation theorem are surprisingly weak. Other than the independence axiom, all we require is the openness of (strict) upper and lower contour sets of the preference relation under consideration, and a further mild continuity axiom.

Though we focus on the same structural framework as Dubra, Maccheroni and Ok (2004) (henceforth, DMO), our representation theorem is logically distinct from theirs in that, under the axioms of DMO, a preorder *cannot* be characterized by a compact set of (strictly) increasing functions unless it is actually complete (or trivial).[4,5] Put differently,

[3]A suitable normalization condition ensures the uniqueness of the representing set of utility functions up to closed convex hull.

[4]The reason is that, as we just noted, this sort of a representation requires open contour sets while DMO focus on closed preorders. In turn, according to a fundamental result by Schmeidler (1971), an incomplete and nontrivial preorder on a connected domain *cannot* satisfy *both* of these continuity conditions. We will elaborate on this matter in the next section. (A preorder is *trivial* if it declares maximal all alternatives.)

[5]Bewley's (1986) seminal work in the Anscombe-Aumann framework also employs the open-continuity axiom and delivers increasing functions as the present paper. Though Bewley's original approach proved particularly useful in applications (see, e.g., Rigotti and Shannon, 2005), in the subsequent theoretical work attention shifted to closed preorders. To our knowledge, the only exception is Manzini and Moriotti (2008). Their representation is based on utility intervals (instead of a set of utility functions) and requires some independence assumptions which differ significantly from those used in the present paper and the rest of the multi-utility literature (see Footnote 23 below). The works in this literature that focus on closed preorders include Ghirardato et al. (2003) in a Savagean framework; Gilboa et al. (forthcoming) in the Anscombe-Aumann framework; Evren and Ok (2010) in the ordinal framework; Baucells and Shapley (2008) where a convex subset of a Euclidean space is chosen as the domain of preferences; and a paper of the present author which provides negative and positive results on DMO type representations over noncompact domains (see Evren, 2008). It is also worth noting that in the Anscombe-Aumann framework, the focus of the literature has been "indecisiveness in beliefs," rather than "indecisiveness in tastes" which is the subject of DMO and the present paper. Ok et al. (2008) combine indecisiveness in tastes with indecisiveness in beliefs, albeit under the closedness assumption.

assuming away the case of a complete preference relation, a set of EU-functions that one can ever find using DMO theorem will either be noncompact or contain (at least) one function that yields the same expected utility for two lotteries one of which dominating the other. This, in turn, implies that under the axioms of DMO, the conclusion of the aforementioned scalarization result will typically fail. The trouble is that maximization of nonincreasing EU-functions over a choice set will deliver nonmaximal lotteries, and if one maximizes only the increasing EU-functions, typically it will not be possible to identify all maximal lotteries. In particular, we will present in this paper an example of a (nontrivial) preorder of the type considered by DMO that admits a plethora of lotteries which are not dominated by any other lottery and which do *not maximize any nonconstant* EU-function. In the same example, it is also true that the elements of a dense subset of maximal lotteries do *not maximize any increasing* EU-function.

In fact, given a preorder of the type considered by DMO, their representation theorem transforms the problem of identifying the induced choice correspondence to a *vector-valued* optimization exercise that is equivalent to the problem of finding the (strong) "Pareto-frontier" of a utility possibility set. Moreover, this utility possibility set that one has to deal with typically consists of *infinite dimensional* utility vectors even when there are only finitely many riskless prizes.[6] It seems to us that the standard tools of economists are much more suitable for the class of aforementioned scalar optimization problems.

While the present paper is mainly motivated by this tractability concern, it is also possible to draw a conceptual line between our representation theorem and that of DMO. More specifically, our theorem can be seen as a *multi-self* representation of a decision maker, for (at least on convex sets of lotteries) there is a one to one correspondence between the utility functions delivered by the theorem and different patterns of choice behavior that the decision maker might actually follow. By contrast, the behavior of an agent who can be described à la DMO is analogous to that of a *coalition of distinct individuals* who respect the Pareto rule.[7]

In this paper, we also discuss several applications of our representation theorem to individual choice theory, consumer theory, game theory and social choice theory. In some of these applications, we use the scalarization method to obtain characterizations of some (new or known) solution concepts. Most notably, in an otherwise standard game with incomplete preferences, the collection of *pure* strategy equilibria that one can find using the

[6]Behavioral axioms that ensure representability of a preorder over lotteries by a finite set of EU-functions are unknown and likely to be rather restrictive. As a side payoff of our representation theorem, in Appendix B of this paper we will provide axiomatic foundations of preorders that can be represented à la DMO by a *compact* set of EU-functions (each increasing in at least one common direction).

[7]Obviously, in both cases the corresponding multi-person interpretation refers to a set of agents who respect the completeness axiom.

scalarization method corresponds to a refinement of the notion of Nash equilibrium that requires the (deterministic) action of each player be undominated by any *mixed* strategy that she can follow, given others' actions.[8] Along similar lines, we propose and characterize a refinement of the notion of Walrasian equilibrium that requires the (deterministic) consumption choice of each agent be undominated by any random mixture of consumption bundles that she can afford. As a simple corollary, we will also establish the existence of such a refined equilibrium assuming a continuum of traders and incomplete preferences.

2. Notation and Terminology

Given a compact metric space Y, we denote by $\mathbf{C}(Y)$ the Banach space of continuous, real functions on Y endowed with the sup-norm $\|\cdot\|_\infty$. In turn, $\Delta(Y)$ stands for the set of all (Borel) probability measures on Y, and $ca(Y)$ for the space of signed measures on Y. We equip $ca(Y)$ with the usual setwise algebraic operations and weak*-topology, which is the coarsest topology that makes continuous every functional of the form $\eta \to E_\eta(u) := \int_Y u d\eta$ with $u \in \mathbf{C}(Y)$. As is well-known, the induced topology on $\Delta(Y)$, the so called "topology of weak-convergence," can be metrized by the Prokhorov metric. In what follows, topological concepts regarding subsets and elements of $\Delta(Y)$ will refer to this relative topology. We will sometimes write $E(\eta, u)$ instead of $E_\eta(u)$.

Following the standard conventions, by a **binary relation** $\mathcal{R}$ on a set $\mathfrak{A}$ we mean a subset of $\mathfrak{A}^2$, and often write $a\mathcal{R}b$ instead of $(a, b) \in \mathcal{R}$. If $\mathfrak{A}$ is a topological space, when we say that $\mathcal{R}$ is **closed** or **open**, we will be referring to the product topology. As usual, a **preorder** refers to a reflexive and transitive binary relation, which is said to be a **partial order** if it is also antisymmetric. If $\mathcal{R}$ is a preorder on $\mathfrak{A}$, given any $K \subseteq \mathfrak{A}$, we say that a point $a \in K$ is **$\mathcal{R}$-maximal** in K if there does not exist $b \in K$ such that $b\mathcal{R}a$ and not $a\mathcal{R}b$.

Throughout the paper, X stands for a compact metric space of riskless prizes, and $\Delta(X)$ for the set of lotteries. In some part of our analysis, we take as the primitive a preorder $\succsim$ on $\Delta(X)$, which is interpreted as the **preference relation** of a decision maker. When we follow this approach, we denote by $\succ$ and $\sim$ the **asymmetric** and **symmetric** parts of $\succsim$, respectively, which are defined as usual: $p \succ q$ iff $p \succsim q$ and not $q \succsim p$; in turn,

[8] As Nehring (1997) and Heller (2010) note, for the case of an incomplete preference relation, a lottery over two prizes x, y may dominate another prize z even if x, y and z are pairwise incomparable. More generally, given a (predetermined) *nonconvex* set K of feasible lotteries, the maximality of a lottery in K may not be sufficient to "justify" the choice of that lottery, for a random choice over the feasible lotteries may render better off the decision maker in question. In a recent application of the present paper, Heller (2010) provides choice-theoretic foundations of behavior that complies with this stronger notion of rationality, and derives a representation result by utilizing our main findings. A more detailed discussion of the choice-theoretic implications of our results can be found in Section 6.1 below.

$p \sim q$ iff $p \succsim q$ and $q \succsim p$. The **incomplete part** of $\succsim$, denoted $\bowtie$, is defined by $p \bowtie q$ iff neither $p \succsim q$ nor $q \succsim p$. When $p \bowtie q$, we say that p and q are **$\succsim$-incomparable**, meaning that the decision maker is indecisive between p and q. The preference relation $\succsim$ is said to be **complete** if $\bowtie = \emptyset$, and **incomplete** otherwise. In turn, we say that $\succsim$ is **nontrivial** if $p \succ q$ for some p, q in $\Delta(X)$. As usual, the **open-continuity property** refers to the requirement that the sets $\{p \in \Delta(X) : p \succ q\}$ and $\{p \in \Delta(X) : q \succ p\}$ be open in $\Delta(X)$ for each $q \in \Delta(X)$.

It will often be convenient to focus on a transitive and irreflexive binary relation $\succ$ on $\Delta(X)$ which will be interpreted as a **strict preference relation.** When such a relation $\succ$ is taken as the primitive, incompleteness of the agent's (weak) preference relation can be deduced from the lack of negative-transitivity of $\succ$.

3. Scalarization Method and Representations Notions

In his seminal work, Aumann (1962) proposed representing a preference relation $\succsim$ on $\Delta(X)$ by an expected utility index $u \in \mathbf{C}(X)$ that is **$\succ$-increasing** ($E_p(u) > E_q(u)$ whenever $p \succ q$) and that is **indifference preserving** ($E_p(u) = E_q(u)$ whenever $p \sim q$). In what follows, we will refer to such a function u as an **Aumann utility** for $\succsim$. As we noted earlier, the appeal of this notion of representation mainly stems from the fact that a lottery which maximizes the expectation of an $\succ$-increasing function over a set of lotteries is guaranteed to be a $\succsim$-maximal element of that set.

On the other hand, the exercise of finding a single Aumann utility for a preference relation is of limited use, for such a function simply *extends* the relation in question to a complete preorder, but does not *characterize* it. In particular, this approach ceases to be useful when one wishes to understand among which sorts of alternatives the decision maker in question is indecisive, or to determine the associated choice correspondence in its entirety.

To overcome this difficulty, DMO identified necessary-sufficient conditions which allow one to find a set of functions $U \subseteq \mathbf{C}(X)$ such that, for every p, q in $\Delta(X)$,

$$p \succsim q \quad \text{if and only if} \quad E_p(u) \geq E_q(u) \text{ for every } u \in U. \tag{1}$$

When viewed as an analytical tool, this representation transforms the problem of preference maximization to a vector-valued optimization exercise. Specifically, given $\succsim$ and U as above, an element p of a set $K \subseteq \Delta(X)$ is $\succsim$-maximal in K if and only if the utility vector $(E_p(u))_{u\in U}$ is a $\geq$-maximal[9] element of the following set:

$$\left\{(E_q(u))_{u\in U} : q \in K\right\} \subseteq \mathbb{R}^U.$$

As the set U in (1) is typically infinite, it appears that this sort of a vector-valued optimization problem can be extremely tedious, even when the prize space X is finite.[10] To demonstrate how elusive such an exercise can be, it may suffice here to note that this sort of an optimization problem is equivalent to identifying the Pareto-frontier of the utility possibility set in an infinite society.

Similar observations led optimization theorists to search for conditions that might allow one transform a given multi-objective optimization problem to a scalar-valued, parametric optimization exercise which produces, at least approximately, the same solutions as the original problem.[11] With regard to this scalarization issue, in the present framework the best-case scenario is a one-to-one correspondence between the $\succsim$-maximal elements and the maximizers of the representing set of utility functions:

$$\mathcal{M}(\succsim, K) = \bigcup_{u\in U} \arg\max_{q\in K} E_q(u), \tag{2}$$

where $\mathcal{M}(\succsim, K) := \{p \in K : \text{there does not exist } q \in K \text{ such that } q \succ p\}$.

The main finding of the present paper is an expected multi-utility representation theorem which ensures that the equality (2) holds whenever K is a convex subset of $\Delta(X)$ (Theorem 3 below). This result characterizes the class of preference relations $\succsim$ that can be represented by a compact and convex set U of Aumann utilities as follows: For every p, q in $\Delta(X)$,

$$\begin{aligned} p \succ q \quad &\text{if and only if} \quad E_p(u) > E_q(u) \text{ for every } u \in U, \\ p \sim q \quad &\text{if and only if} \quad E_p(u) = E_q(u) \text{ for every } u \in U. \end{aligned}$$

If one assumes that the set K consists of two lotteries p, q, or equals the closed line segment between these two lotteries, the right side of (2) would be contained in the left side only if $p \succ q$ implies $E_p(u) > E_q(u)$. Put differently, a representation theorem can be compatible with the scalarization method that we shall utilize in the present paper, only if it delivers $\succ$-increasing functions, as in our representation theorem. By contrast, the set

[9] $\geq$ stands for the usual partial order on $\mathbb{R}^U$.

[10] Let us note that when X is finite, $\left\{(E_q(u))_{u\in U} : q \in K\right\}$ is contained in a finite dimensional subspace of $\mathbb{R}^U$, but the order structure of this subspace (as determined by $\geq$) can be much more complicated than the usual order structure of a Euclidean space, unless one can choose the set U to be finite.

[11] See, e.g., Ehrgott (2005) and references therein.

U in DMO representation (1) may contain functions that are not $\succ$-increasing.[12] In fact, if one requires U to be compact (in particular, finite) this is *necessarily* the case unless $\succsim$ is complete or trivial.

Observation 1. *If U and $\succsim$ satisfy* (1) *for every p, q in $\Delta(X)$, and if U is a compact subset of $\mathbf{C}(X)$ that consists of $\succ$-increasing functions, then the preorder $\succsim$ is either complete or trivial.*

In the Appendix, we will deduce Observation 1 from Schmeidler's (1971) theorem which shows that on a connected set a nontrivial preorder that satisfies the open-continuity property cannot be closed unless it is actually complete. To gain intuition, let us offer here an alternate proof for the case of a set of the form $U = \{u, v\} \subseteq \mathbf{C}(X)$. Suppose $\succsim$ is nontrivial, and let r, w be two lotteries such that $r \succ w$. Then, if both u and v are $\succ$-increasing we must have $E_r(u) > E_w(u)$ and $E_r(v) > E_w(v)$. Let p and q be arbitrary lotteries. If we can show that

$$E_p(u) = (>) E_q(u) \quad \text{imply} \quad E_p(v) = (>) E_q(v),$$

we can deduce the desired conclusion from the uniqueness result of the classical expected utility theory. To this end, first suppose $E_p(u) = E_q(u)$ and $E_p(v) \neq E_q(v)$, say $E_p(v) > E_q(v)$. Then, if DMO representation holds, we cannot have $q \succsim p$. As u is $\succ$-increasing, we can also rule out the case $p \succ q$. It follows that $p \bowtie q$. For DMO representation to hold, $\succsim$ must be closed, and hence, $\bowtie$ must be open. Thus, there exists a sufficiently small $\alpha \in (0, 1)$, such that $\alpha r + (1 - \alpha)p \bowtie \alpha w + (1 - \alpha)q$. But this is a contradiction, as we clearly have $E_{\alpha r+(1-\alpha)p}(f) > E_{\alpha w+(1-\alpha)q}(f)$ for $f = u, v$, implying that $\alpha r + (1 - \alpha)p \succ \alpha w + (1 - \alpha)q$. To complete the proof, suppose now $E_p(u) > E_q(u)$. Then, clearly, there is a number $\beta \in (0, 1)$ such that $E_{\beta w+(1-\beta)p}(u) = E_{\beta r+(1-\beta)q}(u)$. As we have just seen, this implies $E_{\beta w+(1-\beta)p}(v) = E_{\beta r+(1-\beta)q}(v)$, and hence, $E_p(v) > E_q(v)$, as sought.

3.1. Examples

For further motivation, we shall now present a few examples that demonstrate the difficulties of DMO approach with regard to the scalarization issue. First of all, as we already noted, if U represents $\succsim$ in the sense of DMO, the equality (2) will typically fail. One may therefore think of utilizing some approximation methods.

[12] However, any DMO type preorder admits an Aumann utility. In fact, as shown by DMO, given a countable dense subset U_0 of a set U as in (1), any continuous function on X which can be written as a positive-weighted sum of elements of U_0 is an Aumann utility for $\succsim$. It is also true that the set of all such functions would also represent $\succsim$ in the sense of (1), but typically this set will not be closed.

Following earlier works on Pareto type, vector-valued optimization problems, one approach to this approximation issue is to seek for inequalities of the following form:

$$\bigcup_{u\in U_{\mathcal{A}}} \arg\max_{q\in K} E_q(u) \subseteq \mathcal{M}(\succsim, K) \subseteq \bigcup_{u\in U} \arg\max_{q\in K} E_q(u), \tag{3}$$

where $U_{\mathcal{A}}$ stands for the set of $\succ$-increasing functions in U. Of course, if one chooses the set U to be large enough to contain a constant function, the second inclusion in the above expression is not informative, for any lottery maximizes a constant function. This raises the following question:

Q1. *Does there exist a nontrivial preorder of the type considered by DMO that admits maximal lotteries on a (compact, convex) set which maximize on that set only a constant EU-function?*

As we shall shortly see, the answer to this question is affirmative.

An alternate approach that one might follow is to maximize only $\succ$-increasing functions and utilize a topological method to approximate the set of $\succsim$-maximal elements in a given choice set. Indeed, for a DMO type partial order $\succsim$ on $\Delta(X)$ and a compact, convex $K \subseteq \Delta(X)$, the set $\mathcal{M}_{\mathcal{A}}(\succsim, K)$ is dense in $\mathcal{M}(\succsim, K)$, where we denote by $\mathcal{M}_{\mathcal{A}}(\succsim, K)$ the union of all sets of the form $\arg\max_{q\in K} E_q(u)$ for some u in $\mathbf{C}(X)$ that is $\succ$-increasing.[13] However, in principle, the set $\mathcal{M}(\succsim, K) \setminus \mathcal{M}_{\mathcal{A}}(\succsim, K)$ may also be topologically large. Thus, in a given choice problem associated with a DMO type preorder, it may be impossible to recover a topologically large set of potential choices by simply maximizing Aumann utilities. In particular, $\mathcal{M}(\succsim, K) \setminus \mathcal{M}_{\mathcal{A}}(\succsim, K)$ may be a dense subset of K.[14] When this is the case, in order to recover the set $\mathcal{M}(\succsim, K) \setminus \mathcal{M}_{\mathcal{A}}(\succsim, K)$, if one applies the closure operator to $\mathcal{M}_{\mathcal{A}}(\succsim, K)$, one would also end up with every lottery in $K \setminus \mathcal{M}(\succsim, K)$, which are not likely to be selected from K by the decision maker in question. In summary, the following question also seems to be of interest:

Q2. *Does there exist a nontrivial partial order $\succsim$ of the type considered by DMO such that $\mathcal{M}(\succsim, \Delta(X)) \setminus \mathcal{M}_{\mathcal{A}}(\succsim, \Delta(X))$ is a dense subset of $\Delta(X)$?*

Let us now construct a nontrivial, DMO type partial order which provides positive

[13] Makarov and Rachovski (1996) investigate this approximation problem for the case of an affine partial order on a topological vector space. The said observation on DMO type partial orders is an immediate consequence of their Corollary 4.4. To the best of our knowledge, it is an open question if the antisymmetry requirement is dispensable.

[14] Here, an implicit difficulty is the fact that $\mathcal{M}(\succsim, K)$ need not be a (relatively) closed subset of K. As we will see in Section 5.1, such anomalies do not occur for preference relations that we will characterize in our main representation theorems.

answers to both Q1 and Q2. Put $X := [0,1]$,

$$\hat{U} := \{u \in \mathbf{C}(X) : u(0) = 0,\ u(1) = 1 \text{ and } \|u\|_\infty \leq 2\},$$

and let $\succsim^\wedge$ be the preorder on $\Delta(X)$ induced by $\hat{U}$ via the rule (1). The next claim lists the interesting properties of $\succsim^\wedge$.

Observation 2.(i) *For any $p \in \Delta(X)$ and $\alpha \in (1/2, 1]$, we have $\delta_1 \succ^\wedge \alpha\delta_0 + (1-\alpha)p$.*[15]
(ii) *Any lottery r on X with $r(\{0\}) = 0$ is $\succsim^\wedge$-maximal on $\Delta(X)$.*
(iii) *In particular, if $r(\{0\}) = 0$ and $r(I) > 0$ for every nondegenerate interval I in X, then $r \in \mathcal{M}(\succsim^\wedge, \Delta(X))$. But whenever such an r belongs to* $\arg\max_{q \in \Delta(X)} E_q(u)$ *for some $u \in \mathbf{C}(X)$, then u is a constant function.*
(iv) *Moreover, the set $\mathcal{M}(\succsim^\wedge, \Delta(X)) \backslash \mathcal{M}_{\mathcal{A}}(\succsim^\wedge, \Delta(X))$ is dense in $\Delta(X)$. (Hence, the set $\mathcal{M}_{\mathcal{A}}(\succsim^\wedge, \Delta(X))$ is also dense in $\Delta(X)$, as $\succsim^\wedge$ is a partial order.)*

Part (i) of this observation shows that $\Delta(X) \backslash \mathcal{M}(\succsim^\wedge, \Delta(X))$ is a substantially large set.[16] However, it follows from part (iv) that the set $\mathcal{M}(\succsim^\wedge, \Delta(X)) \backslash \mathcal{M}_{\mathcal{A}}(\succsim^\wedge, \Delta(X))$ is topologically so large that if one wishes to recover this set by applying the closure operator to $\mathcal{M}_{\mathcal{A}}(\succsim^\wedge, \Delta(X))$, it is inevitable to cover every "bad" lottery in $\Delta(X) \backslash \mathcal{M}(\succsim^\wedge, \Delta(X))$ as well. To see why $\mathcal{M}(\succsim^\wedge, \Delta(X)) \backslash \mathcal{M}_{\mathcal{A}}(\succsim^\wedge, \Delta(X))$ is dense in $\Delta(X)$, we simply note that any neighborhood of a lottery on $[0,1]$ contains a lottery r such that $r(\{0\}) = 0$ and $r(I) > 0$ for every nondegenerate interval I in $[0,1]$ that contains 0 or 1.[17] Assuming (ii), this proves our claim, for if such a lottery r maximizes $E(\cdot, u)$ on $\Delta([0,1])$ for some $u \in \mathbf{C}([0,1])$, then $u(0) = u(1)$, though we have $\delta_1 \succ^\wedge \delta_0$.

Part (iii) of Observation 2 is a straightforward consequence of (ii), which we prove in the Appendix. For the present, it should be noted that the set $\hat{U}$ is closed, convex and bounded, but it does not contain a constant function. Hence, it follows from part (iii) that the *maximization of a set U of utility functions delivered by DMO representation may not allow us to recover $\mathcal{M}(\succsim, K)$, not even up to the inclusion relation given by* (3) *provided that one focuses on nonconstant utility functions.*

In this example, part of the trouble is caused by the lack of compactness of the set $\hat{U}$. Indeed, it is not difficult to verify the following claim:

[15] Throughout the paper, δ_x stands for the degenerate lottery supported at $x \in X$.

[16] In fact, for any finite dimensional, convex subset Δ_0 of $\Delta(X)$ which contains δ_0, the set $\Delta_0 \backslash \mathcal{M}(\succsim^\wedge, \Delta(X))$ has interior points relative to the affine hull of Δ_0. (But as X is infinite in this example, the relative interior of $\Delta(X)$ (with respect to its affine hull) is empty.)

[17] To construct such an r that approximates a given lottery p on $[0,1]$, if $p(\{0\}) > 0$ we can transfer this mass to a sequence (x_k) that converges to 0 and that is contained in an arbitrarily small neighborhood of 0. Similarly, an arbitrarily small mass from the support of p can be transferred to points that are arbitrarily close to 1 (and to 0 if we already have $p(\{0\}) = 0$).

Observation 3. *If U and $\succsim$ satisfy* (1) *for every p, q in $\Delta(X)$, and if U is a compact and convex subset of $\mathbf{C}(X)$, then* (3) *holds.*[18]

As a side payoff of our main representation theorems, in Appendix B of this paper we will provide axiomatic foundations of preorders that can be represented à la DMO by a *compact* (and convex) set of utility functions (each increasing in a common direction). In light of Observation 3, this subclass of DMO type preorders seem to be better-behaved. However, still, maximization of a set of functions U as in Observation 3 may not allow us to recover the set $\mathcal{M}(\succsim, K)$ precisely, for both of the inclusions in (3) may indeed be proper inclusions. Let us now demonstrate this point within the context of the classical consumer theory.

Example 1. Put

$$X := \left\{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 4\right\} \quad \text{and} \quad U := \{u, v\},$$

where, for every $x \in X$,[19]

$$u(x) := (x_1 + x_2)^{1/2} (x_3)^{1/2} \quad \text{and} \quad v(x) := 2 (x_1)^{1/2} (x_2)^{1/2}.$$

It is clear that

$$\begin{aligned} \arg\max_{x \in X} u(x) &= \{x \in X : x_1 + x_2 = 2,\ x_3 = 2\}, \quad \text{and} \\ \arg\max_{q \in \Delta(X)} E_q(u) &= \left\{q \in \Delta(X) : q\left(\arg\max_{x \in X} u(x)\right) = 1\right\}. \end{aligned}$$

However, $x^* := (1, 1, 2)$ is the unique maximizer of v on $\arg\max_{x \in X} u(x)$, implying that the lottery δ_{x^*} is the only element of $\arg\max_{q \in \Delta(X)} E_q(u)$ that is $\succsim$-maximal on $\Delta(X)$, where $\succsim$ is the preorder on $\Delta(X)$ induced by U via the rule (1). Hence, $\arg\max_{q \in \Delta(X)} E_q(u)$ is not contained in $\mathcal{M}(\succsim, \Delta(X))$.

Moreover, δ_{x^*} does not maximize the expectation of any $\succ$-increasing function $f \in \mathbf{C}(X)$ on $\Delta(X)$. To see this, take any such f. Since U is normalized in the sense that $u(x^*) = v(x^*) = 2$ and $u(\mathbf{0}) = v(\mathbf{0}) = 0$, where $\mathbf{0} := (0, 0, 0)$, by normalizing f accordingly we can assume that $f \in \text{co}(U)$ (for more on this argument, see the proof of Theorem 2 below).[20] In fact, since neither u nor v are $\succ$-increasing, we can write $f = \alpha u + (1 - \alpha) v$ for some $\alpha \in (0, 1)$. It easily follows that $\frac{\partial f}{\partial x_1}(x^*) > \frac{\partial f}{\partial x_3}(x^*)$, and hence, $x^* \notin \arg\max_{x \in X} f(x)$.

[18]Bade (2005) proves similar results for the case of a polyhedral set U.

[19]When X is a subset of a Euclidean space, we denote by x_i the ith coordinate of a vector $x \in X$.

[20]By co we mean the convex hull operator, and $\overline{\text{co}}$ stands for the closed-convex hull operator.

□

Finally, we remark that when the set X is finite we can precisely recover $\mathcal{M}(\succsim, \Delta(X))$ by maximizing all Aumann utilities for a DMO type preorder $\succsim$, but this observation (which is due to Aumann (1962, 1964)) cannot be generalized to the case of an arbitrary compact convex set $K \subseteq \Delta(X)$. Suppose, for instance, that X consists of three alternatives, and consider a closed ball B in the interior of $\Delta(X)$. Let $u \in \mathbb{R}^3$ be a nonconstant utility vector, and denote by p the (unique) maximizer of $E(\cdot, u)$ on B. Now, we can pick a $q \in \Delta(X)$ such that $E(q, u) = E(p, u)$ and $q \neq p$. Put $v := p - q$,[21] $U := \{u, v\}$ and $K := \text{co}(\{q\} \cup \{r \in B : E(r, v) \geq E(p, v)\})$. Then, both p and q maximize $E(\cdot, u)$ on K, but only p is a maximal element of K with respect to the DMO type preorder $\succsim'$ induced by U (since $p \succ' q$). Moreover, there does not exist an $\succ'$-increasing $f \in \mathbb{R}^3$ such that $p \in \arg\max_{r \in K} E(r, f)$. (Hence, the conclusion of Example 1 applies to the set K as it is.) As Figure 1 illustrates, this scenario simply replicates a well-known problem related to the identification of the Pareto frontier of a utility possibility set contained in a Euclidean space.

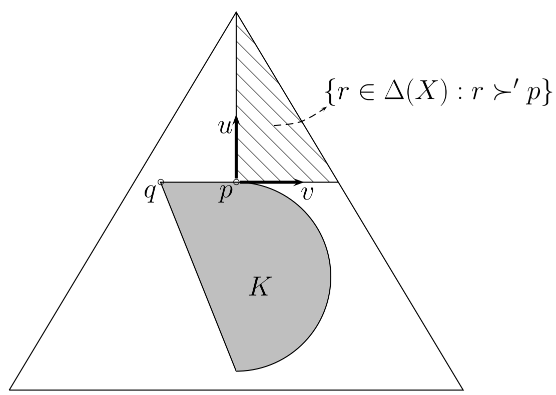

Figure 1

4. Representation Theorems

As before, X stands for a compact metric space of riskless prizes. We first focus on a binary relation $\succ$ on $\Delta(X)$ that is understood as a strict preference relation. We will later extend our model to distinguish between the notions of indifference and indecisiveness.

We say that $\succ$ is an *open-continuous strict preference relation* if it satisfies the following axioms.

[21] As usual, we identify $\Delta(X)$ with the unit simplex in $\mathbb{R}^3$.

Open-Continuity. For every p, q in $\Delta(X)$, whenever $p \succ q$ there exist a neighborhood N_p of p and a neighborhood N_q of q such that $N_p \succ q$ and $p \succ N_q$.[22]

Independence. For every p, q, r in $\Delta(X)$ and $\alpha \in (0, 1)$,

$$p \succ q \quad \text{if and only if} \quad \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r.$$

Strict Preorder. $\succ$ is irreflexive and transitive.

Nontriviality. $p^\bullet \succ q^\bullet$ for some $p^\bullet, q^\bullet$ in $\Delta(X)$.

Remark 1. Let $\succ$ be an open-continuous strict preference relation. Then $\succ$ is *open* in $\Delta(X)^2$. Indeed, $p \succ q$ implies $p \succ \frac{1}{2}p + \frac{1}{2}q \succ q$ by the independence axiom, and applying the open-continuity axiom to the pairs $\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$ and $\left(\frac{1}{2}p + \frac{1}{2}q, q\right)$ yields, by transitivity of $\succ$, a neighborhood N_p of p and a neighborhood N_q of q such that $r \succ w$ for every $(r, w) \in N_p \times N_q$. Moreover, $\succ$ is also *asymmetric*, for $p \succ q$ and $q \succ p$ would imply $p \succ p$ by transitivity, which contradicts irreflexivity of $\succ$.

The next theorem shows that an open-continuous strict preference relation $\succ$ can be characterized by a compact set of $\succ$-increasing functions.

Theorem 1. *Let X be a compact metric space. A binary relation $\succ$ on $\Delta(X)$ is an open-continuous strict preference relation if and only if there exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*

(i) *For every p, q in $\Delta(X)$,*

$$p \succ q \quad \textit{if and only if} \quad E_p(u) > E_q(u) \textit{ for every } u \in U.$$

(ii) $E_{p^\bullet}(u) > E_{q^\bullet}(u)$ *for every $u \in U$ and some $p^\bullet, q^\bullet$ in $\Delta(X)$.*

If $\succ$ admits a set $U \subseteq \mathbf{C}(X)$ as in Theorem 1, we will say that U is a **utility set** for $\succ$. When proving this theorem, we will see that, in fact, given any pair of lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$, we can find a utility set U such that $E_{p^\bullet}(u) = 1$ and $E_{q^\bullet}(u) = 0$ for every $u \in U$. We will refer to such a set U as a $(p^\bullet, q^\bullet)$**-normalized utility set** for $\succ$, or simply as a **normalized utility set** if the choice of a particular pair $(p^\bullet, q^\bullet)$ is immaterial. In turn, given any nonempty, compact $U \subseteq \mathbf{C}(X)$, by $\succ_U$ we will denote the binary relation on $\Delta(X)$ defined by U as in part (i) of Theorem 1.

It is important to note that if U is a utility set for $\succ$, so is any closed subset V of $\mathbf{C}(X)$ such that $\overline{\mathrm{co}}\,(V) = \overline{\mathrm{co}}\,(U)$. By the uniqueness result of DMO, it can be shown that the

[22]For any nonempty set $N \subseteq \Delta(X)$ and $r \in \Delta(X)$, by $N \succ r$ we mean that $w \succ r$ for every $w \in N$. The expression $r \succ N$ is understood analogously.

converse is also true if one focuses on normalized utility sets:

Theorem 2. *Let $U \subseteq \mathbf{C}(X)$ be a $(p^\bullet, q^\bullet)$-normalized utility set for an open-continuous strict preference relation. Then $V \subseteq \mathbf{C}(X)$ is another such set if and only if V is closed and $\overline{\text{co}}\,(V) = \overline{\text{co}}\,(U)$.*

Theorem 2 shows that a $(p^\bullet, q^\bullet)$-normalized utility set is *unique* up to closed-convex hull. An immediate implication is that, depending on the choice of $(p^\bullet, q^\bullet)$, there exists a unique, *convex* $(p^\bullet, q^\bullet)$-normalized utility set. It is also clear that, in fact, this is the *largest* $(p^\bullet, q^\bullet)$-normalized utility set. Moreover, by the Krein-Milman theorem, taking the closure of the set of extreme points of this largest set gives us the *smallest* $(p^\bullet, q^\bullet)$-normalized utility set. The next observation highlights these points.

Observation 4. *Let $\succ$ be an open-continuous strict preference relation, and pick any two lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$. Then, there exist largest and smallest $(p^\bullet, q^\bullet)$-normalized utility sets, U_+ and U_-, respectively. Here, $U_+ = \overline{\text{co}}\,(U_-)$ and U_- is the closure of the set of extreme points of U_+.*

In passing, a few remarks on the proof of the "only if" part of Theorem 1 are in order. First of all, it should be noted that when the prize space X is finite, the proof is relatively simpler, for then the preference cone $\mathcal{C} := \{\gamma(p-q) : p \succ q \text{ and } \gamma > 0\}$ turns out to be open (in its span), and standard duality arguments suffice to obtain a closed and bounded utility set, which will also be compact. In general, however, the set $\mathcal{C}$ is only boundedly open, i.e., for any (total variation) norm-bounded subset B of the span of $\mathcal{C}$, the set $B \cap \mathcal{C}$ is relatively weak*-open in B. We provide two different methods that overcome this difficulty. The initial step in the first method is to show that the closure $\succsim^*$ of an open-continuous strict preference relation $\succ$ is a DMO type preorder. (The main challenge in this direction is to verify transitivity of $\succsim^*$.) After completing this task, using the open-continuity axiom and the findings of DMO, we show that, in fact, $\succsim^*$ admits a nonempty compact set $U \subseteq \mathbf{C}(X)$ that represents $\succsim^*$ in the sense of DMO and that satisfies condition (ii) of Theorem 1 for some predetermined lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$. Finally, by using the fact that $\mathcal{C}$ is boundedly open, we show that such a set U must also satisfy condition (i) of Theorem 1. Alternatively, one can focus on a finer topology which declares $\mathcal{C}$ open and which produces the same set of continuous linear functionals as the weak*-topology. This allows one to proceed, roughly speaking, as one would do in the case of a finite prize space.

4.1. Extension to Preorders

We now consider a binary relation $\succsim$ on $\Delta(X)$ that is interpreted as a (weak) preference relation. (As usual, we will denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$, respectively.) Here, our purpose is to give a suitable extension of Theorem 1 that allows one to distinguish between the notions of indifference and indecisiveness embodied in $\succsim$. To this end, we will employ the following axioms.

Indifference Independence (II). For every p, q, r in $\Delta(X)$ and $\alpha \in (0,1)$,

$$p \sim q \quad \text{implies} \quad \alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r.$$ [23]

Symmetric Algebraic-Closedness (SAC). For every p, q, r, w in $\Delta(X)$ with $r \succ w$,

$$\left.\begin{array}{ll} \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)w & \\ \text{and} & \text{for all } \alpha \in (0,1) \\ \alpha q + (1-\alpha)r \succ \alpha p + (1-\alpha)w & \end{array}\right\} \quad \text{imply} \quad p \sim q.$$

II is motivated exactly as in the standard independence axiom. SAC, on the other hand, amounts to saying that, given any ρ^1, ρ^2 in $\{p, q\}$, if $\alpha\rho^1 + (1-\alpha)r$ is strictly preferred to $\alpha\rho^2 + (1-\alpha)w$ regardless of how large α might be and regardless of whether ρ^1 or ρ^2 equals p or q, then p must be indifferent to q. The reader will note that if the set $\{\alpha \in [0,1] : \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)w\}$ were closed for any lotteries p, q, r, w, then SAC would trivially hold.[24] (In particular, every DMO type preorder satisfies SAC.) This observation also clarifies the intuition behind the term "SAC."

The following theorem is our main result, which completes the task of characterizing the class of preorders that can fully be described by a compact set of Aumann utilities.

Theorem 3. *Let X be a compact metric space. For a binary relation $\succsim$ on $\Delta(X)$ the following two statements are equivalent.*

(i) *$\succsim$ is a preorder that satisfies* II *and* SAC, *and $\succ$ is an open-continuous strict preference relation.*

(ii) *There exists a utility set $U \subseteq \mathbf{C}(X)$ for $\succ$ such that, for every p, q in $\Delta(X)$,*

$$p \sim q \quad \textit{if and only if} \quad E_p(u) = E_q(u) \textit{ for every } u \in U. \tag{4}$$

Moreover, upon normalization, the set U is unique up to closed convex hull.

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[23]By contrast to the interval representation of Manzini and Mariotti (2008), our multi-utility approach does not require independence axioms on $\succsim$-incomparable lotteries. More specifically, an important difference between the two models is that in the present approach we allow for the existence of pairwise $\succsim$-incomparable lotteries p, q, r such that $\frac{1}{2}p + \frac{1}{2}q \succ r$. Such situations may give rise to nonbinariness of choice behavior induced by an incomplete preference relation, which has attracted considerable attention in the literature. (For more on this, see Section 6.1 below.) Furthermore, the representation of Manzini and Mariotti implies that for any p, q, r with $p \succ q$, the independence property $\alpha r + (1-\alpha)p \succ \alpha r + (1-\alpha)q$ will *typically fail* for large $\alpha \in (0,1)$.

[24]The said closedness property, however, is too strong for our purposes by Schmeidler's (1971) theorem.

Remark 2. As we have seen in Observation 1, the representation theorem of DMO is logically distinct from Theorem 3.

While we are interested in Theorem 3 mainly for tractability concerns, the conceptual content of this result, as a multi-self representation, is also remarkable. Though the term "multi-self representation" is used in the literature in several different meanings, in the present context it seems reasonable to view a function $u \in \mathbf{C}(X)$ as a description of a possible self of the agent defined by $\succsim$ if, in principle, the agent defined by $\succsim$ might behave as if her choices are guided by maximization of $E(\cdot, u)$.[25] In formal terms, this amounts to requiring that maximization of $E(\cdot, u)$ over any set $K \subseteq \Delta(X)$ should return $\succsim$-maximal elements of K. In this precise sense, Theorem 3 is a multi-self representation result thanks to the fact that it delivers $\succ$-increasing functions.

It is also worth noting that, given a set U as in Theorem 3, whenever $E_p(u) = E_q(u)$ for some $u \in U$ we cannot have $p \succ q$ (as each function in U is $\succ$-increasing). This is a logical requirement for the validity of the above interpretation. Indeed, whenever $E_p(u) = E_q(u)$ for some $u \in U$, it would follow that a "self" of the agent may choose q when p is available, while $p \succ q$ would imply that the agent "herself" would never behave in the same way. Put differently, in the present model, whenever $E_p(u) = E_q(u)$ for some $u \in U$, the agent defined by $\succsim$ may choose either alternative from the set $\{p, q\}$.[26]

By contrast, given a nonempty set $U \subseteq \mathbf{C}(X)$ that represents a preorder $\succsim^*$ in the sense of DMO, the choice behavior induced by $\succsim^*$ is analogous to that of a coalition of distinct individuals, as defined by U, who respect the Pareto rule:

$$p \succ^* q \quad \text{iff} \quad E_p(u) \geq E_q(u) \text{ for all } u \in U \text{ and } E_p(v) > E_q(v) \text{ for some } v \in U.$$

Hence, a typical function $u \in U$ may not bear sufficient information to determine a $\succsim^*$-consistent choice among two lotteries p and q, as we may have $E_p(u) = E_q(u)$ even if $p \succ^* q$.

5. More on Scalarization and Maximal Elements

In this section, we will show that recovering the choice correspondence associated with a preference relation can be reduced to a scalar-valued, parametric optimization exercise provided that the strict part of the preference relation satisfies hypotheses of Theorem 1.

[25] The agent defined by $\succsim$ refers to a decision maker who might select a lottery from a choice set $K \subseteq \Delta(X)$ if and only if that lottery is a $\succsim$-maximal element of K. (In Section 6.1, we will discuss an alternate choice behavior that is consistent with a given preference relation in a stronger sense.)

[26] In particular, p and q are $\succsim$-incomparable whenever $E_p(u) = E_q(u)$ and $E_p(v) > E_q(v)$ for some u, v in U.

We will then investigate some further desirable properties that are peculiar to choice correspondences associated with such preference relations.

In the remainder of the paper, the symmetric part of the preferences of the decision maker in question will be irrelevant for our purposes. Hence, we simply focus on a strict preference relation $\succ$ on $\Delta(X)$, and modify our notation and terminology in an obvious way. For example, $\mathcal{M}(\succ, K)$ denotes the set of $\succ$**-maximal** elements of a set $K \subseteq \Delta(X)$; that is $\mathcal{M}(\succ, K) := \{p \in K : \text{there does not exist } q \in K \text{ such that } q \succ p\}$, which is interpreted as the set of lotteries that can possibly be chosen from K by the decision maker. Moreover, throughout this section, without further mention we assume that X is a compact metric space.

As we noted several times, it is plain that maximization of the expectation of an $\succ$-increasing function on a set $K \subseteq \Delta(X)$ would deliver a $\succ$-maximal element of K. A more interesting question is the converse: Given a utility set U for $\succ$, is it true that each element of $\mathcal{M}(\succ, K)$ maximizes $E(\cdot, u)$ on K for some $u \in U$? The next proposition shows that the answer is affirmative if K is convex and if one focuses on a convex utility set.

Proposition 1. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U \subseteq \mathbf{C}(X)$ be a utility set for $\succ$. Then, for any convex subset K of $\Delta(X)$,*

$$\mathcal{M}(\succ, K) = \bigcup_{v \in \overline{\text{co}}(U)} \arg\max_{q \in K} E_q(v).$$

In particular, if U is a convex utility set, $\mathcal{M}(\succ, K) = \bigcup_{u \in U} \arg\max_{q \in K} E_q(u)$.

Remark 3. Using the terminology of the previous section, Proposition 1 simply says that the agent defined by $\succ$ may choose a lottery from a convex subset of $\Delta(X)$ if and only if this is consistent with the behavior of a self of the agent as defined by a function in a convex utility set.

On occasion, it may be of interest to focus on a smaller utility set U, and express every function in $\overline{\text{co}}(U)$ as a weighted average of functions in U. This can easily be done thanks to compactness of U and a (primitive) version of Choquet's theorem which guarantees that a point l of a locally convex topological vector space $\mathcal{L}$ belongs to the closed-convex hull of a compact subset H of $\mathcal{L}$ if and only if there is a (countably additive, Borel) probability φ on the set H such that $\mathfrak{T}(l) = \int_H \mathfrak{T}(h) d\varphi(h)$ for every continuous, linear functional $\mathfrak{T}$ on $\mathcal{L}$.[27] Since (by the Riesz representation theorem) a continuous linear functional on $\mathbf{C}(X)$ is none but a function of the form $v \to E_\eta(v)$ for some $\eta \in ca(X)$, it readily follows that a

[27] See Phelps (2001, Proposition 1.2, p. 4).

continuous, real function v on X belongs to $\overline{\text{co}}\,(U)$ if and only if there exists a $\varphi \in \Delta(U)$ such that $E_q(v) = \int_U E_q(u)d\varphi(u)$ for every $q \in \Delta(X)$.[28] Hence, *Proposition 1 is equivalent to the following:*

Proposition 1'. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U \subseteq \mathbf{C}(X)$ be a utility set for $\succ$. Then, for any convex subset K of $\Delta(X)$, we have*

$$\mathcal{M}(\succ, K) = \bigcup_{\varphi \in \Delta(U)} \arg\max_{q \in K} \int_U E_q(u)d\varphi(u).$$

Given the definition of a utility set, Proposition 1' is simply a version of a theorem of alternative due to Fan et al. (1957). In passing, we sketch the argument for the sake of completeness.

Proof of Proposition 1'. Since the other inclusion is trivial, suffices to show that $\mathcal{M}(\succ, K) \subseteq \bigcup_{\varphi \in \Delta(U)} \arg\max_{q \in K} \int_U E_q(u)d\varphi(u)$. Let $q^* \in \mathcal{M}(\succ, K)$, and note that for each $q \in K$, the function $u \to E_{q-q^*}(u)$ is continuous on U. Since $q \to E_{q-q^*}(\cdot)$ is an affine operator, convexity of the set K implies that $\widetilde{K} := \{E_{q-q^*}(\cdot) : q \in K\} \subseteq \mathbf{C}(U)$ is also convex. Moreover, by $\succ$-maximality of q^* on K, we have $\widetilde{K} \cap \mathbf{C}(U)_{++} = \varnothing$ where $\mathbf{C}(U)_{++} := \{f \in \mathbf{C}(U) : f(u) > 0 \text{ for every } u \in U\}$. Since $\mathbf{C}(U)_{++}$ is an open convex cone,[29] by standard separation and duality arguments we conclude that there exists a $\varphi \in \Delta(U)$ such that $\int_U f(u)d\varphi(u) \leq 0$ for every $f \in \widetilde{K}$. $\square$

5.1. Continuity Properties and Connectedness of $\mathcal{M}(\succ, K)$

Our next task will be to establish upper hemicontinuity of the choice correspondence induced by an open-continuous strict preference relation. Given a sequence (K_n) of subsets of $\Delta(X)$, we define $\liminf K_n := \{\lim p_n : (p_n) \text{ converges and } p_n \in K_n \text{ for every } n\}$, and $\limsup K_n := \bigcup \liminf K'_n$ where the union is taken over the collection of all subsequences of (K_n) with a generic member (K'_n). When $\liminf K_n = K = \limsup K_n$, the set K is said to be Kuratowski limit of (K_n). Since $\Delta(X)$ is compact, on the collection of nonempty closed subsets of $\Delta(X)$ (denoted as $\mathcal{K}$), the notion of Kuratowski convergence coincides with convergence in the Hausdorff metric, d_H.

Upper hemicontinuity of a choice correspondence induced by a strict preference relation demands, in fact, nothing more than openness of that relation:

[28] It can be shown that, given a compact set $U \subseteq \mathbf{C}(X)$ and any $\varphi \in \Delta(U)$, the system of equalities $E_q(v) = \int_U E_q(u)d\varphi(u)$ $(q \in \Delta(X))$ has a unique solution $v^* \in \mathbf{C}(X)$ which is defined by $v^*(x) := \int_U u(x)d\varphi(u)$ for $x \in X$.

[29] Throughout the paper, by a *convex cone* we mean a convex subset of a vector space that is closed under positive scalar multiplication.

Observation 5. *Let* $\succ \subseteq \Delta(X)^2$ *be relatively open. Then:*
(i) *For any* $K \subseteq \Delta(X)$*, the set* $\mathcal{M}(\succ, K)$ *is relatively closed in* K*.*
(ii) *Given a sequence* (K_n) *of subsets of* $\Delta(X)$*, we have*

$$\liminf \mathcal{M}(\succ, K_n) \subseteq \mathcal{M}(\succ, \limsup K_n).$$

In particular, for any $K \subseteq \limsup K_n$*,*

$$K \cap \liminf \mathcal{M}(\succ, K_n) \subseteq \mathcal{M}(\succ, K).$$

That is, for any convergent sequence (p_n) *with* $p_n \in \mathcal{M}(\succ, K_n)$ *for every* n*, whenever* $\lim p_n$ *belongs to* K *it also belongs to* $\mathcal{M}(\succ, K)$*.*
(iii) *The correspondence* $K \rightrightarrows \mathcal{M}(\succ, K)$ *is upper hemicontinuous on the metric space* $(\mathcal{K}, d_H)$*.*

Here, the critical observation is (ii). Indeed, (i) is a trivial consequence of (ii), and (iii) also follows immediately because $K \rightrightarrows \mathcal{M}(\succ, K)$, when considered as a correspondence from $(\mathcal{K}, d_H)$ into $\Delta(X)$, has a closed graph by (ii), and its range is compact. On the other hand, (ii) readily follows from definitions: If $q \succ \lim p_n$ for a lottery q and a convergent sequence $(p_n) \in \mathcal{M}(\succ, K_1) \times \mathcal{M}(\succ, K_2) \times \cdots$, then q cannot belong to $\limsup K_n$, for otherwise openness of $\succ$ would imply that $q_n \succ p_n$ for some large n and $q_n \in K_n$.

In contrast to the conclusions of Observation 5, for a DMO type preorder $\succsim^*$, the set $\mathcal{M}(\succsim^*, K)$ need not be closed, even if $K \subseteq \Delta(X)$ is compact and convex. While Observation 2 already demonstrates this point, since the Pareto frontier of a compact convex set in a Euclidean space may not be closed,[30] using the aforementioned analogy one can also provide finite dimensional examples in the same direction. Moreover, typically, the correspondence $\mathcal{M}(\succsim^*, \cdot)$ is not upper hemicontinuous. In Figure 2, for example, the increasing sequence of closed convex sets (K_n) converges to K_∞. But with $U := \{u, v\}$, the lottery p is the unique maximal element of K_∞ with respect to the DMO type preorder $\succsim^*$ induced by U, while the lottery q belongs to $\mathcal{M}(\succsim^*, K_n)$ for all n.

We will close this section with the proof of the following result.

Observation 6. *If* $\succ$ *is an open-continuous strict preference relation on* $\Delta(X)$*, and if* K *is a closed and convex subset of* $\Delta(X)$*, then* $\mathcal{M}(\succ, K)$ *is a connected set.*

Thanks to Proposition 1, this observation is easily proved by adapting to the present setting an argument that is well-known in the literature on multi-criteria optimization.

[30] See, e.g., Arrow et al. (1953, Section 3).

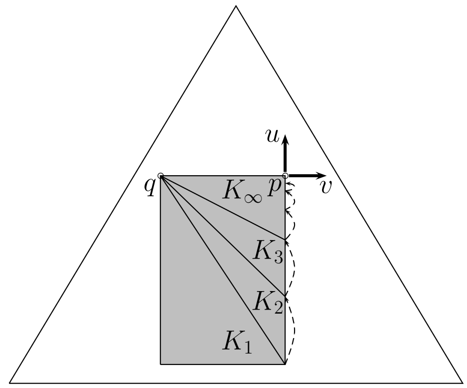

Figure 2

Proof of Observation 6. Let U be a convex utility set for $\succ$, and assume by contradiction that there are two disjoint, nonempty, closed sets $\mathcal{M}_1, \mathcal{M}_2$ such that $\mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}(\succ, K)$. Then, for each $u \in U$ we have $\arg\max_{q \in K} E_q(u) \subseteq \mathcal{M}_1$ or $\arg\max_{q \in K} E_q(u) \subseteq \mathcal{M}_2$, for $\arg\max_{q \in K} E_q(u)$ is a connected (in fact, convex) subset of $\mathcal{M}_1 \cup \mathcal{M}_2$. It follows that $U = U_1 \cup U_2$ where $U_i := \{u \in U : \arg\max_{q \in K} E_q(u) \subseteq \mathcal{M}_i\}$ for $i = 1, 2$. The sets U_1 and U_2 are *disjoint* since $\arg\max_{q \in K} E_q(u)$ is nonempty for each $u \in U$ by compactness of K. Moreover, by Proposition 1, for any $p \in \mathcal{M}_i$ we have $p \in \arg\max_{q \in K} E_q(u)$ for some $u \in U$ which implies in fact that $\arg\max_{q \in K} E_q(u) \subseteq \mathcal{M}_i$. Thus, U_1 and U_2 are *nonempty*, and as can easily be seen they are also *closed*. These conclusions contradict convexity of U.[31]

6. Applications

6.1. Incomplete Preferences and Nonbinary Choice Behavior

Let $\succ$ represent the strict preference relation of an agent who has to choose a lottery from a set $K \subseteq \Delta(X)$. Following the traditional practice, so far we have assumed that such an agent might choose any element of $\mathcal{M}(\succ, K)$. However, analogously to the use of a mixed strategy in a game-theoretic framework, in principle, our agent can condition her choice from the set K to the outcome of a random experiment such as flipping a coin or rolling a die. Considering any such randomization device that she could possibly use, we can thus say that, effectively, the choice set available to our agent is equal to $\text{co}(K)$, or even more generally, to $\overline{\text{co}}(K)$ (provided that one also allows for the use of randomization

[31]It seems to be a nontrivial problem to determine whether the analogue of Observation 6 holds for an arbitrary DMO type preorder. However, one can prove a positive result for the case of an *antisymmetric* DMO type preorder using the aforementioned finding of Makarov and Rachovski (1996) (see Footnote 13).

devices that can return infinitely many outcomes).[32] Upon relaxation of the completeness axiom, this observation becomes material, for $\succ$-maximality of a lottery in K does not guarantee its $\succ$-maximality in $\mathrm{co}\,(K)$, implying that the agent may have a reason to avoid choosing some elements of $\mathcal{M}\,(\succ, K)$. That indecisiveness may give rise to such nonbinary choice behavior has been widely recognized following the seminal work of Nehring (1997).[33]

For instance, let us consider the following adaptation of Nehring's Example 1.

Example 2. Let $X := \{x, y, z\}$ and pick any number $\varepsilon \in (0, 1/2)$. Consider the open-continuous strict preference relation $\succ_U$ on $\Delta(X)$ induced by the set $U := \{u, v\}$ where u and v are the real functions on X defined as in the following table:

	u	v
x	1	0
y	ε	ε
z	0	1

Then, obviously, δ_y is $\succ_U$-maximal in $\{\delta_x, \delta_y, \delta_z\}$, but we have $\frac{1}{2}\delta_x + \frac{1}{2}\delta_z \succ_U \delta_y$. $\square$

One can think of various real-life choice situations in concert with this example. Suppose, for example, that x, y and z are three different restaurants. While x and z are specialized in vegetarian and meat dishes, respectively, y offers both types of dishes, but at a lower quality. Our decision maker is supposed to make a reservation for two in one of these restaurants, but she does not know the preferences of her guest, who may or may not like meat. Then, in the former case, our decision maker may rank the restaurants according to v, while the relevant ranking may be as in u in the latter case.

More generally, Example 2 is intimately linked with *extremeness seeking* which refers to a tendency to opt for extreme alternatives in choice problems where feasible alternatives vary in multiple dimensions. Gourville and Soman (2007) provide empirical evidence which shows that extremeness seeking is rather common when alternatives differ in non-compensatory, not so easily comparable attributes, such as the features of a high-priced, fully-loaded model of a car versus those of a mid-priced, average model or a low-priced, basic model. Example 2 is in concert with this phenomenon. Indeed, x and z in this example can be considered as extreme alternatives that perform very well with respect to one of u or v, and very poorly with respect to the other function. Moreover, the observation

[32] Of course, we view the convex combination $\alpha_1 p_1 + \cdots + \alpha_n p_n$ as a compound lottery that yields the lottery p_i with probability α_i. After all, the usual justification of the independence axiom relies on this interpretation. (By Choquet's theorem, one can similarly interpret the elements of $\overline{\mathrm{co}}\,(K)$ provided that $K \subseteq \Delta(X)$ is compact.)

[33] A recent discussion of the literature on nonbinary choice behavior can be found in Alcantud (2006).

$\frac{1}{2}\delta_x + \frac{1}{2}\delta_z \succ_U \delta_y$ suggests that, when faced with the choice set $\{x, y, z\}$, the decision maker in question would be better off if she randomly selects x or z with equal probabilities, instead of opting for the mild alternative y (despite the fact that neither x nor z is better than y).[34]

In light of these observations, it seems to be in order to replace the traditional binary approach with the alternate model that declares rational to select a lottery p from the choice set K if and only if $p \in K \cap \mathcal{M}(\succ, \overline{\mathrm{co}}(K))$. Recently, Heller (2010) provided necessary-sufficient conditions that allow one to rationalize a choice correspondence $\mathfrak{C}$ in this stronger sense. In fact, Heller studies the class of choice correspondences $\mathfrak{C}$ that admit an open-continuous strict preference relation $\succ$ such that $\mathfrak{C}(K) = K \cap \mathcal{M}(\succ, \overline{\mathrm{co}}(K))$ for every $K \in \mathcal{K}$. By combining Theorem 1 with the following observation, one obtains Heller's representation for such choice correspondences:

Corollary 1. *Suppose that $\succ$ is an open-continuous strict preference relation on $\Delta(X)$ for a compact metric space X, and let $U \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ$. Then, for every nonempty $K \subseteq \Delta(X)$,*

$$\bigcup_{u \in U} \arg\max_{q \in K} E_q(u) = K \cap \mathcal{M}(\succ, \mathrm{co}(K)) = K \cap \mathcal{M}(\succ, \overline{\mathrm{co}}(K)). \tag{5}$$

This corollary is an obvious consequence of Proposition 1, and hence, we omit its proof.[35]

Remark 4. As we have seen in Section 3, excluding some special situations, for the case of a DMO type preorder, one cannot obtain an analogue of the first equality in (5) (even if one focuses on a compact and convex set K). The second equality in (5) may also fail. For instance, if we let K_0 be the convex set obtained by removing the half-open line segment $(q, p]$ from the set K in Figure 1, then q is a $\succsim'$-maximal element of K_0, but we have $p \succ' q$ and $p \in \mathrm{cl}(K_0) = K$. (When the set X is infinite, one can find more interesting examples in the same direction. For example, in this case, the set of all lotteries with finite support is not closed, and hence, a lottery p which is maximal on this set with respect to a DMO type preorder may not be maximal on $\Delta(X)$.)

[34]As Gourville and Soman (2007) discuss in detail, the opposite (i.e., extremeness averse) behavior is predominant in choice problems where feasible alternatives vary in easily measurable and comparable attributes, such as size/quantity and price. Thus, in a broader sense, extremeness seeking seems to be a consequence of incomparability of alternatives.

[35]It follows from Theorem 1 and Corollary 1 that given a choice correspondence $\mathfrak{C}$ that satisfies Heller's (2010) axioms, one can find a compact, convex set $U \subseteq \mathbf{C}(X)$ such that $\mathfrak{C}(K) = \bigcup_{u \in U} \arg\max_{q \in K} E_q(u)$ for every $K \in \mathcal{K}$; which is the content of Heller's representation result. In an ordinal framework with finitely many alternatives, Eliaz, Richter and Rubinstein (forthcoming) prove an analogous representation which characterizes a decision maker who selects the feasible alternatives that are deemed best by (at least) one of *two* utility functions.

In the remainder of this subsection, we assume that X is a convex subset of a vector space. The next observation shows that if $\succ$ admits a utility set that consists of concave functions, then the traditional, binary approach coincides with the present approach in every choice problem where the set of feasible alternatives is a convex set of riskless prizes.[36] Put differently, under the said conditions, the scalarization method can also be used to identify the maximal elements of a convex set of riskless prizes. (For a set $D \subseteq X$, we define $K_D := \{\delta_x : x \in D\}$.)

Corollary 2. *Let X be a compact metric space which is also a convex subset of a vector space. Consider an open-continuous strict preference relation $\succ$ on $\Delta(X)$ which admits a convex utility set $U \subseteq \mathbf{C}(X)$ that consists of concave functions. Then, for every convex $D \subseteq X$ and $x \in D$, the following four conditions are equivalent:* **(i)** $x \in \bigcup_{u \in U} \arg\max_{y \in D} u(y)$; **(ii)** $\delta_x \in \mathcal{M}(\succ, \mathrm{co}(K_D))$; **(iii)** $\delta_x \in \mathcal{M}(\succ, \overline{\mathrm{co}}(K_D))$; **(iv)** $\delta_x \in \mathcal{M}(\succ, K_D)$.

Proof. The equivalence of conditions (i)-(iii) is a trivial consequence of Corollary 1. What needs to be shown is that $\delta_x \in \mathcal{M}(\succ, K_D)$ implies $\delta_x \in \mathcal{M}(\succ, \mathrm{co}(K_D))$. To this end, let us assume by contradiction that we have $\alpha_1\delta_{x_1} + \cdots + \alpha_n\delta_{x_n} \succ \delta_x$ for some $\{x_1, ..., x_n\} \subseteq D$ and $\{\alpha_1, ..., \alpha_n\} \subseteq [0, 1]$ with $\Sigma_{i=1}^n \alpha_i = 1$. Then, by definition of U and concavity of functions in U, we must also have $u(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 u(x_1) + \cdots + \alpha_n u(x_n) > u(x)$ for each $u \in U$. But if $D \subseteq X$ is convex, the riskless prize $y := \alpha_1 x_1 + \cdots + \alpha_n x_n$ belongs to D, and the definition of U implies $\delta_y \succ \delta_x$, as we sought.[37] $\square$

In passing, along the usual lines, we characterize concavity of functions in a utility set.

Remark 5. Given a set X as in Corollary 2, let $U \subseteq \mathbf{C}(X)$ be a utility set for an open-continuous strict preference relation $\succ$ on $\Delta(X)$. Then, U consists of concave functions if and only if for every x, y in X, α, β in $(0, 1)$ and p, q in $\Delta(X)$, we have

$$\beta\delta_{\alpha x + (1-\alpha)y} + (1 - \beta)p \succ \beta(\alpha\delta_x + (1 - \alpha)\delta_y) + (1 - \beta)q$$

whenever $p \succ q$. (We omit the simple proof of this assertion.)

[36] Needless to say, even if $D \subseteq X$ is convex, the set of degenerate lotteries supported at elements of D will be a nonconvex subset of $\Delta(X)$ whenever $|D| \geq 2$.

[37] In his work on consumer theory without the completeness axiom, Nascimento (2009) notes the equivalence of (i) and (iv) in an ordinal setup. The focus of Nascimento is a DMO type preorder induced by a compact set of utility functions. While Theorem B in Appendix B of this paper facilitates Nascimento's approach, his findings provide further instances of the uses of our representation theorems, for the desirable properties of the induced demand correspondence follow from some assumptions that allow Nascimento to conclude that the strict part of the preference relation that he studies behaves *as if* it is the restriction of an open-continuous strict preference relation to deterministic alternatives.

6.2. *A Refinement of the Notion of Walrasian Equilibrium*

The need to relax the completeness axiom has been long recognized in the literature on general equilibrium theory (see, e.g., Schmeidler, 1969; Mas-Colell, 1974). However, to the best of our knowledge, none of the available models accounts for the nonbinary choice behavior that may arise as a consequence of incomparability of alternatives, as we discussed in the previous subsection. While Corollary 2 and Remark 5 provide sufficient conditions that make redundant the modification of the traditional approach that we will propose below, from a foundational point of view, concavity of utility functions over consumption bundles seems to be a rather strong requirement. In particular, a long tradition that models the notion of perfect competition using a continuum of agents avoids assuming convexity of preferences[38] over consumption bundles (cf. Aumann, 1966; Schmeidler, 1969), which is, of course, a weaker property than concavity of utility functions.[39] Following this tradition, we will focus here on an exchange economy with a continuum of agents.

For further motivation, let us first show that upon relaxation of the completeness axiom, even with Cobb-Douglas utility functions, a random mixture of (deterministic) consumption bundles available to a consumer may be preferable to a bundle that is, in fact, undominated by any other available bundle.

Example 3. Put

$$X := \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 20\} \quad \text{and} \quad U := \{u, v\},$$

where, for every $x \in X$,

$$u(x) := (x_1)^4 \, x_2 \quad \text{and} \quad v(x) := (x_1)^{0.1} \, (x_2)^{0.9} \, .$$

Then u attains its maximum on X at the point $x^* := (16, 4)$, and v at $z^* := (2, 18)$. Moreover, since both u and v are $>$-increasing and strictly quasiconcave on $\{x \in X : x_1 \neq 0 \neq x_2\}$, for any y in the line segment $[x^*, z^*]$ and any $x \in X \backslash \{y\}$, we have either $u(y) > u(x)$ or $v(y) > v(x)$. It follows that δ_y is $\succ_U$-maximal in K_X for each $y \in [x^*, z^*]$. But with $y := (5, 15)$ and $p := \frac{1}{10}\delta_{x^*} + \frac{9}{10}\delta_{z^*}$, it is easily verified that $p \succ_U \delta_y$. (In fact, $E_p(u) = 26,473.6 > u(y) = 9,375$ and $E_p(v) = 13.464 > v(y) = 13.439$.) $\square$

We now consider an exchange economy. To compactify the domain of preferences, we

[38] This well-known concept can be adapted to the present setting as follows: If X is a convex set of consumption bundles, then $\succ \subseteq \Delta(X)^2$ is *convex on* X if for every x, y in X, α, β in $(0, 1)$ and p, q in $\Delta(X)$, we have $\beta\delta_{\alpha x+(1-\alpha)y} + (1-\beta)p \succ \beta\delta_y + (1-\beta)q$ whenever $\delta_x \succ \delta_y$ and $p \succ q$.

[39] In fact, convexity of preferences over consumption bundles would not suffice for the conclusion of Corollary 2. (See Example 3 below.)

assume that each agent's consumption of a given commodity cannot exceed a sufficiently large number $b > 0$. Thus, the commodity space is given by $X := \{x \in \mathbb{R}^{\mathfrak{n}}_{+} : x \leq \mathbf{b}\}$ where $\mathbf{b}$ is the $\mathfrak{n}$-vector $(b, ..., b)$. (Similarly, $\mathbf{0}$ stands for the origin of $\mathbb{R}^{\mathfrak{n}}$.) The strict preference relation of a consumer t is denoted by $\succ_t$. We suppose that $\succ_t$ is defined on $\Delta(X)$ (instead of X) which is a central departure from the classical approach.

The consumer space is identified with a set T, a σ-algebra Σ of subsets of T, and a measure ℓ on Σ with $\ell(T) < \infty$. As usual, we assume that the measure space (T, Σ, ℓ) is **complete**, that is, any subset of a ℓ-null member of Σ belongs to Σ as well. Each consumer t is endowed with a bundle $e(t) \in X$ such that $e : t \to e(t)$ is a (Σ-Borel) measurable map. Given a measurable function $g := (g_1, ..., g_{\mathfrak{n}})$ that maps T into X, we set $\int g d\ell := \left(\int_T g_1(t) d\ell(t), ..., \int_T g_{\mathfrak{n}}(t) d\ell(t)\right)$ which is well defined by compactness of X. Such a function g is referred to as an **allocation**. An allocation g is said to be **feasible** if $\int g d\ell \leq \int e d\ell$.

In the remainder of this subsection, we will not distinguish between a point $x \in X$ and the lottery δ_x.

As usual, a **Walrasian equilibrium** refers to a price vector $\phi \in \mathbb{R}^{\mathfrak{n}}_{+} \backslash \{\mathbf{0}\}$ and a feasible allocation g such that, for ℓ-almost every $t \in T$, the budget set $B_t(\phi) := \{x \in X : \phi x \leq \phi e(t)\}$ does not contain any element x with $x \succ_t g(t)$. Our focus will be the following refinement of this traditional solution concept, which requires the consumption choice of an agent be undominated not only by the consumption bundles in her budget set, but also by any random mixture of such bundles.

Definition 1. A Walrasian equilibrium (ϕ, g) is **randomization proof** if, for ℓ-almost every $t \in T$, there does not exist $p \in \Delta(X)$ such that $p \succ_t g(t)$ and $p(B_t(\phi)) = 1$.

$\mathcal{WE}\left((\succ_t)_{t \in T}, e\right)$ (resp. $\mathcal{RPE}\left((\succ_t)_{t \in T}, e\right)$) will denote the set of Walrasian (resp. randomization proof) equilibria. When each $\succ_t$ is induced by a utility function $u_t \in \mathbf{C}(X)$, we will write $\mathcal{WE}\left((u_t)_{t \in T}, e\right)$ instead of $\mathcal{WE}\left((\succ_t)_{t \in T}, e\right)$.

Since the set $\{p \in \Delta(X) : p(B_t(\phi)) = 1\}$ equals $\overline{\text{co}}\left(K_{B_t(\phi)}\right)$, as an immediate consequence of Corollary 1 (and the axiom of choice), we obtain the following characterization of randomization proof equilibria.

Corollary 3. *For each $t \in T$, suppose that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U_t \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ_t$. Then*

$$\mathcal{RPE}\left((\succ_t)_{t \in T}, e\right) = \bigcup \mathcal{WE}\left((u_t)_{t \in T}, e\right), \tag{6}$$

where the union is taken over $(u_t)_{t \in T}$ such that $u_t \in U_t$ for each $t \in T$.

The classical result of Schmeidler (1969) establishes the existence of a Walrasian equilibrium in an economy with a continuum of traders and incomplete preferences. We will now strengthen this conclusion by proving the existence of a randomization proof equilibrium in the present setup. To this end, we need the following additional assumptions.

(A1) The measure space (T, Σ, ℓ) is nonatomic, that is, for every $T_0 \in \Sigma$ with $\ell(T_0) > 0$ there exists a set $T_1 \in \Sigma$ such that $0 < \ell(T_1) < \ell(T_0)$.

(A2) $t \rightrightarrows \succ_t$ is a graph-measurable correspondence from T into $\Delta(X)^2$, that is, its graph $G_\succ := \{(t, p, q) \in T \times \Delta(X)^2 : p \succ_t q\}$ belongs to the product σ-algebra $\Sigma \otimes \mathfrak{B}\left(\Delta(X)^2\right)$.[40]

(A3) $e_i(t) > 0$ for every $i = 1, ..., \mathfrak{n}$ and $t \in T$.

(A4) For each $t \in T$, $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$ such that $\mathbf{b} \succ_t \mathbf{0}$.

(A1) formalizes the notion of perfect competition as suggested by Aumann (1966) and Schmeidler (1969), among others. Measurability assumptions of the sort (A2) are also standard in this strand of literature. A merit of the present cardinal approach is that graph-measurability of the preference correspondence is equivalent to graph-measurability of a utility correspondence obtained upon a natural choice of normalization:

Lemma 1. *Suppose* (A4) *holds. For each $t \in T$, let $U_{+,t}$ stand for the unique, convex $(\mathbf{b}, \mathbf{0})$-normalized utility set for $\succ_t$. Then,* (A2) *holds if and only if $G_U := \{(t, u) \in T \times \mathbf{C}(X) : u \in U_{+,t}\}$ belongs to $\Sigma \otimes \mathfrak{B}\left(\mathbf{C}(X)\right)$.*

The proof of Lemma 1 can be found in Appendix A. We are now ready to state our existence result which is a simple consequence of our previous findings and known existence results for the case of complete preferences.

Corollary 4. $\mathcal{RPE}\left((\succ_t)_{t \in T}, e\right)$ *is nonempty whenever* (A1)-(A4) *hold.*

Proof. Let $U : t \rightrightarrows U_{+,t}$ be the correspondence defined as in Lemma 1. Since U is a graph-measurable, nonempty valued correspondence from the complete measure space (T, Σ, ℓ) into separable, complete metric space $\mathbf{C}(X)$, Aumann's measurable selection theorem implies that there exists a Σ-$\mathfrak{B}\left(\mathbf{C}(X)\right)$ measurable function $t \to u_t$ such that $u_t \in U_{+,t}$ for every $t \in T$ (Aliprantis and Border, 1999, Theorem 17.25, p.574). From measurability of $t \to u_t$ and the existence theorem of Khan and Yannelis (1991), it easily follows that

[40] Given a topological space Y, the product σ-algebra $\Sigma \otimes \mathfrak{B}(Y)$ refers to the smallest σ-algebra of subsets of $T \times Y$ that contains $\{T' \times Y' : T' \in \Sigma \text{ and } Y' \in \mathfrak{B}(Y)\}$ where $\mathfrak{B}(Y)$ is the Borel σ-algebra on Y.

$\mathcal{WE}\left((u_t)_{t\in T}, e\right) \neq \varnothing$.[41] Thus, we obtain the desired conclusion by Corollary 3.[42] $\square$

Remark 6. Following known approximation arguments, the conclusion of Corollary 4 can be extended to include the case $X := \mathbb{R}^{\mathfrak{n}}_+$. Specifically, suppose that for each $t \in T$ and $b > 0$, the restriction of $\succ_t$ to lotteries over the compact box $\{x \in \mathbb{R}^{\mathfrak{n}}_+ : x \leq \mathbf{b}\}$ is an open-continuous strict preference relation. Assuming $e(T)$ is a bounded subset of $\mathbb{R}^{\mathfrak{n}}_+$, we can find a number $a > 0$ such that, for every $t \in T$ and $b \in \mathbb{N}$, the set $X_b(t) := \{x \in \mathbb{R}^{\mathfrak{n}}_+ : x \leq (\Sigma_{i=1}^{\mathfrak{n}} e_i(t))\mathbf{b}\}$ is contained in $X'_b := \{x \in \mathbb{R}^{\mathfrak{n}}_+ : x \leq a\mathbf{b}\}$. Then, upon modifying the statement of Lemma 1 so that X'_b takes the role of X (and $a\mathbf{b}$ that of $\mathbf{b}$), the existence result of Khan and Yannelis (1991) (which allows for t dependent consumption sets) would enable us to show that the economy induced by the consumption set correspondence $t \rightrightarrows X_b(t)$ has a randomization proof equilibrium (ϕ_b, g_b), for each $b \in \mathbb{N}$. By normalizing and passing to a subsequence if necessary, one can assume that (ϕ_b) converges. Since (ϕ_b, g_b) is also a Walrasian equilibrium of the corresponding economy, from Schmeidler's (1969, Section 4) findings it immediately follows that (ϕ_b) must actually be converging to a strictly positive vector if all commodities are desirable in the sense that $x > y$ implies $x \succ_t y$, for each $t \in T$ and x, y in $\mathbb{R}^{\mathfrak{n}}_+$. Finally, given such a sequence (ϕ_b), it is a simple exercise to show that for all sufficiently large b, we have $\{x \in \mathbb{R}^{\mathfrak{n}}_+ : \phi_b x \leq \phi_b e(t)\} \subseteq X_b(t)$ for each $t \in T$, implying that (ϕ_b, g_b) is a randomization proof equilibrium in the economy where consumption sets equal $\mathbb{R}^{\mathfrak{n}}_+$.

Remark 7. Assuming (A2) and (A4), by a measurable selection argument it can be shown that in equation (6), we can restrict our attention to union over measurable selections of the correspondence $t \rightrightarrows U_{+,t}$. Put differently, the method that we used when proving Corollary 4 gives us a generic element of $\mathcal{RPE}\left((\succ_t)_{t\in T}, e\right)$. Of course, such an equivalence result would not hold for the case of a DMO type preorder. However, one could still prove an analogue of Corollary 4 by finding an Aumann utility u_t for each $t \in T$ such that $t \to u_t$

[41] While Khan and Yannelis (1991) assume convexity of preferences over consumption bundles, as they discuss in detail, this assumption becomes redundant when there are finitely many commodities and the consumer space is nonatomic (as we assume here). The key observation that allows us to utilize their existence theorem is that if $t \to u_t$ is Σ-$\mathfrak{B}(\mathbf{C}(X))$ measurable, then the induced weak preference correspondence on X^2 is graph-measurable; that is, $\{(t, x, y) \in T \times X^2 : u_t(x) \geq u_t(y)\}$ belongs to $\Sigma \otimes \mathfrak{B}\left(X^2\right)$. (This assertion is analogous to the "if" part of Lemma 1, and its proof is a routine exercise.) Continuity of u_t on X, the definition of X, and the assumption (A3) immediately imply that the economy $((T, \Sigma, \ell), (u_t)_{t\in T}, e, X)$ also satisfies the remaining hypotheses of Khan and Yannelis.

[42] Rustichini and Yannelis (1991) prove a generalization of Schmeidler's (1969) result where agents' consumption sets are weakly compact subsets of a separable Banach space. Their key assumption formalizes the idea that there are "many more agents than commodities," and allows them to relax the convexity requirement of Khan and Yannelis (1991), even when there are infinitely many commodities. Extending Corollary 4 to the framework considered by Rustichini and Yannelis, therefore, is a routine exercise. (I am grateful to Nicholas C. Yannelis for calling my attention to this point.)

is a measurable function.

6.3. On Weak Pareto Optimality and Social Planing with Incompletely Known Preferences

We now consider a finite society $T := \{1, ..., \mathcal{T}\}$, and denote by X a compact metric space of social alternatives. Each agent t is assumed to have a strict preference relation $\succ_t$ on $\Delta(X)$. The weak Pareto order $\succ^{\text{wp}}$ is then defined as, for every p, r in $\Delta(X)$,

$$p \succ^{\text{wp}} r \quad \text{if and only if} \quad p \succ_t r \text{ for every } t = 1, ..., \mathcal{T}.$$

The following result is a Negishi-type characterization of weak Pareto optimality that does not require completeness of agents' preferences.

Corollary 5. *For each $t = 1, ..., \mathcal{T}$, suppose that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U_t \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ_t$. Then, for every convex $K \subseteq \Delta(X)$,*

$$\mathcal{M}(\succ^{\text{wp}}, K) = \bigcup \arg\max_{q \in K} E_q(\alpha_1 u_1 + \cdots + \alpha_{\mathcal{T}} u_{\mathcal{T}}),$$

where the union is taken over $(\alpha_t, u_t)_{t \in T} \in \mathbb{R}_+^{\mathcal{T}} \times \mathbf{C}(X)^{\mathcal{T}}$ such that $\sum_{t=1}^{\mathcal{T}} \alpha_t = 1$ and $u_t \in U_t$ for every $t = 1, ..., \mathcal{T}$.

Proof. It is clear that $\succ^{\text{wp}} = \succ_U$ where $U := \bigcup_{t=1}^{\mathcal{T}} U_t$. Moreover, co (U) is a compact set that consists of all functions of the form $\alpha_1 u_1 + \cdots + \alpha_{\mathcal{T}} u_{\mathcal{T}}$ for some $(\alpha_t, u_t)_{t \in T} \in \mathbb{R}_+^{\mathcal{T}} \times \mathbf{C}(X)^{\mathcal{T}}$ such that $\sum_{t=1}^{\mathcal{T}} \alpha_t = 1$ and $u_t \in U_t$ for every $t = 1, ..., \mathcal{T}$. Thus, the proof follows from Proposition 1.[43] $\square$

Alternatively, in Corollary 5, we can consider $\succ_t$ as a binary relation that represents the incomplete knowledge of a social planner about the strict part of preferences of agent t, which may themselves be complete. When viewed from this perspective, Corollary 5 resembles the efficiency theorems of McLennan (2002) and Carroll (forthcoming). The present approach, however, differs from theirs in many respects. First, we do not directly assume that planner's knowledge about a given agent can be summarized by a set of utility functions. Rather, we derive this conclusion from the properties of the binary relations that model planner's knowledge. Second, we allow X to be infinite and do not restrict our attention to the grand set $K = \Delta(X)$. Moreover, the notion of optimality considered by McLennan and Carroll is stronger than weak Pareto optimality, and hence, requires

[43] More precisely, we have in mind the obvious generalization of Proposition 1 that also applies to a *possibly trivial* strict preference relation $\succ$ such that $\succ = \succ_U$ for some nonempty, compact $U \subseteq \mathbf{C}(X)$.

different analytical tools.[44]

6.4. On Nash Equilibria of Games with Incomplete Preferences

Let us consider a finite set of players $T := \{1, ..., \mathcal{T}\}$, and denote by t and i generic players. X_t stands for the set of pure strategies available to player t, which is assumed to be a compact metric space. Thus, the set $X := X_1 \times \cdots \times X_\mathcal{T}$ of pure strategy profiles is also a compact, metrizable space. Each player t has a strict preference relation $\succ_t$ on the set $\Delta(X)$. Given a generic element $\mathbf{p} := (p_1, ..., p_\mathcal{T})$ of $\mathbf{\Delta} := \Delta(X_1) \times \cdots \times \Delta(X_\mathcal{T})$, the product probability $\mathbf{p}^{\otimes}$ is the unique element of $\Delta(X)$ which satisfies $\mathbf{p}^{\otimes}(X_1' \times \cdots \times X_\mathcal{T}') = \prod_{t=1}^{\mathcal{T}} p_t(X_t')$ for every $(X_1', ..., X_\mathcal{T}') \in \mathfrak{B}(X_1) \times \cdots \times \mathfrak{B}(X_\mathcal{T})$. It is important to note that for each $f \in \mathbf{C}(X)$, the real function $\mathbf{p} \to \mathfrak{E}(f, \mathbf{p}) := \int_X f d\mathbf{p}^{\otimes}$ is continuous on $\mathbf{\Delta}$ (with respect to the product topology) (see, e.g., Glycopantis and Muir, 2000).

We will denote by $\mathbf{K}$ a generic set of the form $\mathbf{K} = K_1 \times \cdots \times K_\mathcal{T}$ for some $K_t \subseteq \Delta(X_t)$, $t \in T$. A **Nash equilibrium for the K-restricted game** is a strategy profile $\mathbf{p}$ in $\mathbf{K}$ such that $\mathbf{p}^{\otimes} \in \mathcal{M}\left(\succ_t, \left\{(q_t, p_{-t})^{\otimes} : q_t \in K_t\right\}\right)$ for each player t, where $p_{-t} := (p_i)_{i \in T \setminus \{t\}}$. The set of all such equilibria will be denoted by $\mathcal{NE}\left((\succ_t)_{t \in T}, \mathbf{K}\right)$. When each $\succ_t$ is induced by a utility function $u_t \in \mathbf{C}(X)$ (via the rule $p \succ_t q$ iff $\int_X u_t dp > \int_X u_t dq$) we will write $\mathcal{NE}\left((u_t)_{t \in T}, \mathbf{K}\right)$ instead of $\mathcal{NE}\left((\succ_t)_{t \in T}, \mathbf{K}\right)$.

It is clear that whenever $K_t \subseteq \Delta(X_t)$ is convex, the set $\{(q_t, p_{-t})^{\otimes} : q_t \in K_t\}$ is a convex subset of $\Delta(X)$ for each $\mathbf{p} \in \mathbf{\Delta}$. Thus, as an obvious consequence of Proposition 1, we obtain the following characterization of Nash equilibria for convex games.

Corollary 6. *For each $t \in T$, suppose that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and let $U_t \subseteq \mathbf{C}(X)$ be a convex utility set for $\succ_t$. If $K_t \subseteq \Delta(X_t)$ is convex for each $t \in T$, then*

$$\mathcal{NE}\left((\succ_t)_{t \in T}, \mathbf{K}\right) = \bigcup \mathcal{NE}\left((u_t)_{t \in T}, \mathbf{K}\right),$$

where the union is taken over $(u_t)_{t \in T}$ such that $u_t \in U_t$ for each $t \in T$.

As we have seen in Section 6.1, even with a single player, an equilibrium in pure strategies may not be an equilibrium when the use of mixed strategies are allowed. The next result is a game-theoretic version of Corollary 1, which shows that the scalarization method characterizes pure strategy equilibria that survive upon the introduction of mixed strategies. (We define $\mathbf{K}_X := \{(\delta_{x_t})_{t \in T} : (x_t)_{t \in T} \in X\}$.)

[44] Specifically, given $U_1, ..., U_\mathcal{T} \subseteq \mathbf{C}(X)$, a lottery p dominates a lottery r in the sense of McLennan and Carroll if there exists an agent t such that $E_p(u) > E_r(u)$ for every $u \in U_t$, and $E_p(u) \geq E_r(u)$ for every $u \in U_i$ and $i \in T \setminus \{t\}$.

Corollary 7. *For each $t \in T$, let $\succ_t$ and U_t be as in Corollary 6. Then:*

$$\mathbf{K}_X \cap \mathcal{NE}\left((\succ_t)_{t\in T}, \mathbf{\Delta}\right) = \bigcup \mathcal{NE}\left((u_t)_{t\in T}, \mathbf{K}_X\right), \tag{7}$$

where the union is taken over $(u_t)_{t\in T}$ such that $u_t \in U_t$ for each $t \in T$.

Proof. That the left side of (7) is contained in the right side is an immediate consequence of Corollary 6 and definitions. To prove the other inclusion, first note that if a point $x_t \in X_t$ maximizes the function $u_t(\cdot, x_{-t})$ on X_t for some $x_{-t} \in \times_{i\neq t} X_i$ and $u_t \in \mathbf{C}(X)$, then $u_t(x_t, x_{-t}) \geq \int_{X_t} u_t(\cdot, x_{-t}) dq_t = \mathfrak{E}\left(u_t, (q_t, (\delta_{x_i})_{i\neq t})\right)$ for every $q_t \in \Delta(X_t)$. Thus, the right side of (7) is contained in $\bigcup \mathcal{NE}\left((u_t)_{t\in T}, \mathbf{\Delta}\right)$. Hence, the proof follows from Corollary 6. $\square$

The notion of an ε-equilibrium can be adapted to the present setting as follows.

Definition 2. Let $(\succ_t)_{t\in T}$ and $(U_t)_{t\in T}$ be as in Corollary 6. For any number $\varepsilon \geq 0$ and sets $K_t \subseteq \Delta(X_t)$ ($t \in T$), a strategy profile $\mathbf{p}$ in $\mathbf{K}$ is an **$(\varepsilon, \mathbf{U})$-equilibrium for the K-restricted game** if there exists $(u_t)_{t\in T} \in \mathbf{U} := U_1 \times \cdots \times U_{\mathcal{T}}$ such that, for each $t \in T$,

$$\mathfrak{E}\left(u_t, \mathbf{p}\right) \geq \mathfrak{E}\left(u_t, (q_t, p_{-t})\right) - \varepsilon \quad \text{for every } q_t \in K_t.$$

As Fudenberg and Levine (1986) note, restricting the set of strategies available to players sometimes produces more tractable games. Hence, the following continuity result seems to be of interest.

Corollary 8. *Let $(\succ_t)_{t\in T}$ and $(U_t)_{t\in T}$ be as in Corollary 6. For each $t \in T$, consider a convex set $K_t \subseteq \Delta(X_t)$, and let (K_t^n) be a sequence of subsets of K_t. Suppose that the sets $\mathbf{K}^n := K_1^n \times \cdots \times K_{\mathcal{T}}^n$ converge to $\mathbf{K} := K_1 \times \cdots \times K_{\mathcal{T}}$ in the sense that for each $\mathbf{p} \in \mathbf{K}$, we have $\mathbf{p} = \lim \mathbf{p}^n$ for a sequence $(\mathbf{p}^n) \in \mathbf{K}^1 \times \mathbf{K}^2 \times \cdots$.[45] Then, a strategy profile $\mathbf{p}$ belongs to $\mathcal{NE}\left((\succ_t)_{t\in T}, \mathbf{K}\right)$ if and only if there exist a sequence $(\mathbf{p}^n) \in \mathbf{K}^1 \times \mathbf{K}^2 \times \cdots$ and a sequence $(\varepsilon^n) \in \mathbb{R}_+^\infty$ such that:* **(i)** $\mathbf{p}^n \to \mathbf{p}$; **(ii)** $\varepsilon^n \to 0$; and **(iii)** $\mathbf{p}^n$ *is an $(\varepsilon^n, \mathbf{U})$-equilibrium for the $\mathbf{K}^n$-restricted game, for each $n \in \mathbb{N}$.*

Proof. To prove the "if" part suppose that $\mathbf{p}$, $(\mathbf{p}^n)$ and (ε^n) satisfy (i)-(iii). Then, for each n, there exists $(u_t^n)_{t\in T} \in \mathbf{U}$ such that

$$\mathfrak{E}\left(u_t^n, \mathbf{p}^n\right) \geq \mathfrak{E}\left(u_t^n, \left(q_t^n, p_{-t}^n\right)\right) - \varepsilon^n \quad \text{for every } q_t^n \in K_t^n \text{ and } t \in T. \tag{8}$$

[45] Since $\mathbf{K}^n \subseteq \mathbf{K}$ for each n, when $\mathbf{K}$ is closed, this is equivalent to saying that $\mathbf{K}$ is the Kuratowski limit of $(\mathbf{K}^n)$.

Since $\mathbf{U}$ is compact, by passing to a subsequence if necessary, we can assume that $(u_t^n)_{t\in T}$ converges to a vector of utility functions $(u_t)_{t\in T} \in \mathbf{U}$ in the product of sup-norm topologies.

Fix any $\gamma > 0$, $\mathbf{q} \in \mathbf{K}$ and $t \in T$. By hypothesis, there exists a sequence $(\mathbf{q}^n) \in \mathbf{K}^1 \times \mathbf{K}^2 \times \cdots$ such that $\mathbf{q}^n \to \mathbf{q}$.

As we noted earlier, for each $f \in \mathbf{C}(X)$ the function $\mathfrak{E}(f, \cdot)$ is continuous on $\mathbf{\Delta}$. Hence,

$$\mathfrak{E}(u_t, \mathbf{p}^n) - \mathfrak{E}(u_t, \mathbf{p}) \leq \gamma \quad \text{for all sufficiently large } n. \tag{9}$$

Similarly,

$$\mathfrak{E}(u_t, (q_t, p_{-t})) - \mathfrak{E}\left(u_t, \left(q_t^n, p_{-t}^n\right)\right) \leq \gamma \quad \text{for all sufficiently large } n. \tag{10}$$

Moreover, $\|f - h\|_\infty \leq \gamma$ implies $|\mathfrak{E}(f, \mathbf{r}) - \mathfrak{E}(h, \mathbf{r})| = |\mathfrak{E}(f - h, \mathbf{r})| \leq \gamma$ for any f, h in $\mathbf{C}(X)$ and $\mathbf{r} \in \mathbf{K}$. It follows that

$$\mathfrak{E}(u_t^n, \mathbf{p}^n) - \mathfrak{E}(u_t, \mathbf{p}^n) \leq \gamma \quad \text{for all sufficiently large } n. \tag{11}$$

Similarly,

$$\mathfrak{E}\left(u_t, \left(q_t^n, p_{-t}^n\right)\right) - \mathfrak{E}\left(u_t^n, \left(q_t^n, p_{-t}^n\right)\right) \leq \gamma \quad \text{for all sufficiently large } n. \tag{12}$$

Combining (8)-(12) yields $\mathfrak{E}(u_t, \mathbf{p}) \geq \mathfrak{E}(u_t, (q_t, p_{-t})) - \varepsilon^n - 4\gamma$ for all sufficiently large n. Since $\varepsilon^n \to 0$, in view of arbitrariness of γ, it follows that $\mathfrak{E}(u_t, \mathbf{p}) \geq \mathfrak{E}(u_t, (q_t, p_{-t}))$. As $\mathbf{q} \in \mathbf{K}$ and $t \in T$ are also arbitrarily chosen, we conclude that $\mathbf{p} \in \mathcal{NE}\left((u_t)_{t\in T}, \mathbf{K}\right)$. By Corollary 6, this proves the "if" part.

Conversely, suppose $\mathbf{p} \in \mathcal{NE}\left((\succ_t)_{t\in T}, \mathbf{K}\right)$. Then, Corollary 6 implies that for some $(u_t)_{t\in T} \in \mathbf{U}$, we have $\mathbf{p} \in \mathcal{NE}\left((u_t)_{t\in T}, \mathbf{K}\right)$. Pick a sequence $(\mathbf{p}^n) \in \mathbf{K}^1 \times \mathbf{K}^2 \times \cdots$ such that $\mathbf{p}^n \to \mathbf{p}$. Fix a player $t \in T$. Since the function $\mathfrak{E}(u_t, \cdot)$ is continuous on the compact space $\mathbf{\Delta}$, it is in fact uniformly continuous. Thus, the sequence $\gamma_t^n := \sup\{\left|\mathfrak{E}(u_t, (q_t, p_{-t})) - \mathfrak{E}\left(u_t, \left(q_t, p_{-t}^n\right)\right)\right| : q_t \in \Delta(X_t)\}$ tends to 0 as $n \to \infty$. It follows that the sequence $\varepsilon_t^n := \gamma_t^n + |\mathfrak{E}(u_t, \mathbf{p}^n) - \mathfrak{E}(u_t, \mathbf{p})|$ also tends to 0.

Next, we observe that since $\mathbf{p}$ belongs to $\mathcal{NE}\left((u_t)_{t\in T}, \mathbf{K}\right)$, for each $t \in T$, $q_t \in K_t$ and $n \in \mathbb{N}$ we have $\mathfrak{E}(u_t, \mathbf{p}^n) \geq \mathfrak{E}\left(u_t, \left(q_t, p_{-t}^n\right)\right) - \varepsilon_t^n$. As $K_t^n \subseteq K_t$ for each $t \in T$ and $n \in \mathbb{N}$, upon setting $\varepsilon^n := \max\{\varepsilon_t^n : t \in T\}$, we conclude that $\mathbf{p}^n$ is an $(\varepsilon^n, \mathbf{U}_+)$-equilibrium for the $\mathbf{K}^n$-restricted game, for each $n \in \mathbb{N}$. $\square$

It should be noted that the "only if" part of Corollary 8 could also be proved by combining Corollary 6 with the limit theorem of Fudenberg and Levine (1986) for games with complete preferences. However, the "if" part of Corollary 8 differs substantially from the corresponding finding of Fudenberg and Levine, for it forces us to deal with utility functions that vary with n.

The proof of Corollary 8 makes it transparent that pure strategy equilibria that can be identified with the scalarization method satisfy an analogous continuity property. By combining this observation with Corollary 7, we obtain a further characterization of pure strategy equilibria that survive upon the introduction of mixed strategies:

Corollary 9. *Let $(\succ_t)_{t\in T}$ and $(U_t)_{t\in T}$ be as in Corollary 6. For each $t \in T$, let (X_t^n) be a sequence of subsets of X_t. Suppose that the sets $X^n := X_1^n \times \cdots \times X_{\mathcal{T}}^n$ converge to X in the sense that for each $x \in X$, we have $x = \lim x^n$ for a sequence $(x^n) \in X^1 \times X^2 \times \cdots$. Then, a strategy profile $\mathbf{p}$ belongs to $\mathbf{K}_X \cap \mathcal{NE}\left((\succ_t)_{t\in T}, \mathbf{\Delta}\right)$ if and only if there exist a sequence $(\mathbf{p}^n) \in \mathbf{K}_{X^1} \times \mathbf{K}_{X^2} \times \cdots$ and a sequence $(\varepsilon^n) \in \mathbb{R}_+^\infty$ such that:* **(i)** *$\mathbf{p}^n \to \mathbf{p}$;* **(ii)** *$\varepsilon^n \to 0$; and* **(iii)** *$\mathbf{p}^n$ is an $(\varepsilon^n, \mathbf{U})$-equilibrium for the $\mathbf{K}_{X^n}$-restricted game, for each $n \in \mathbb{N}$.*[46]

Remark 8. The results of the present subsection would not hold if players' preferences were defined à la DMO. In particular, the approximation idea above would entail an additional difficulty caused by the fact that a rich set of Aumann utilities that one may want to focus on need not be compact.

We conclude with an example that illustrates the contents of Corollaries 8 and 9.

Example 4. Consider an infinitely repeated game, where $A := A_1 \times \cdots \times A_{\mathcal{T}}$ is the set of action profiles in the stage game. We suppose that A_t is finite and nonempty for each player t. For every integer $n > 1$ and every $t \in T$, we let $\mathcal{F}_t^n$ denote the set of all functions that map A^{n-1} into A_t, and put $\mathcal{F}_t^1 := A_t$. Thus, a pure strategy for player t is a sequence $x_t := (x_t^1, x_t^2, ...) \in \mathcal{F}_t^1 \times \mathcal{F}_t^2 \times \cdots$. It is clear that (the product topology of) $X_t := \mathcal{F}_t^1 \times \mathcal{F}_t^2 \times \cdots$ is compact and metrizable.

A pure strategy profile $x := (x_1, ..., x_{\mathcal{T}}) \in X := X_1 \times \cdots \times X_{\mathcal{T}}$ induces an outcome path $\mathbf{a}(x) := (a^n(x)) \in A^\infty$ which is inductively defined by $a^1(x) := (x_1^1, ..., x_{\mathcal{T}}^1)$ and $a^n(x) := \left(x_1^n\left(a^1(x), ..., a^{n-1}(x)\right), ..., x_{\mathcal{T}}^n\left(a^1(x), ..., a^{n-1}(x)\right)\right)$ for $n > 1$.

Each player t has a strict preference relation $\succ_t'$ on $\Delta(A^\infty)$. This induces a strict preference relation $\succ_t$ on $\Delta(X)$ as follows: For every p, q in $\Delta(X)$,

$$p \succ_t q \quad \text{if and only if} \quad p_{\mathbf{a}} \succ_t' q_{\mathbf{a}},$$

where $r_{\mathbf{a}}(A') := r(\mathbf{a}^{-1}(A'))$ for every $A' \in \mathfrak{B}(A^\infty)$ and $r \in \Delta(X)$. It is not difficult to

[46] As usual, $\mathbf{K}_{X^n} := \{(\delta_{x_t})_{t\in T} : (x_t)_{t\in T} \in X^n\}$ for each n.

verify that $\mathbf{a}(\cdot)$ is a continuous map from X into A^{∞}.[47] Thus, $r_{\mathbf{a}}$ is a well defined element of $\Delta(A^{\infty})$ for every $r \in \Delta(X)$.

For each player t, assume that $\succ_t'$ is an open-continuous strict preference relation on $\Delta(A^{\infty})$, and let $V_{+,t} \subseteq \mathbf{C}(A^{\infty})$ be a convex utility set for $\succ_t'$. Since $v \to v \circ \mathbf{a}$ is an affine and continuous map from $\mathbf{C}(A^{\infty})$ into $\mathbf{C}(X)$, the set $U_t := \{v \circ \mathbf{a} : v \in V_{+,t}\} \subseteq \mathbf{C}(X)$ is also convex and compact. Moreover, by changing variables, we can write $\int_{A^{\infty}} v dr_{\mathbf{a}} = \int_X v \circ \mathbf{a} dr$ for any $v \in \mathbf{C}(A^{\infty})$ and $r \in \Delta(X)$. By construction, it follows that $\succ_t$ is an open-continuous strict preference relation on $\Delta(X)$, and U_t is a convex utility set for $\succ_t$.[48]

Finally, fix a point $(a_1, ..., a_{\mathcal{T}}) \in A$ and put $X_t^n := \{x_t \in X_t : x_t^l \equiv a_t \text{ for every } l \geq n\}$ for each n and t. Then, the sets $X^n := X_1^n \times \cdots \times X_{\mathcal{T}}^n$ converge to X in the sense of Corollary 9. Moreover, if we let $K_t^n := \{p_t \in \Delta(X_t) : p_t(X_t^n) = 1\}$ for each n and t, the sets $\mathbf{K}^n := K_1^n \times \cdots \times K_{\mathcal{T}}^n$ converge to $\boldsymbol{\Delta} := \Delta(X_1) \times \cdots \times \Delta(X_{\mathcal{T}})$ in the sense of Corollary 8. Hence, we can approximate both $\mathcal{NE}\left((\succ_t)_{t\in T}, \boldsymbol{\Delta}\right)$ and $\mathbf{K}_X \cap \mathcal{NE}\left((\succ_t)_{t\in T}, \boldsymbol{\Delta}\right)$ with $(\varepsilon, \mathbf{U})$-equilibria of the corresponding finite horizon games.

Appendix A. Omitted Proofs

Proof of Observation 1. If U and $\succsim$ satisfy (1) for every p, q in $\Delta(X)$, and if U consists of $\succ$-increasing functions, then it readily follows that, for every p, q in $\Delta(X)$,

$$p \succ q \quad \text{if and only if} \quad E_p(u) - E_q(u) > 0 \text{ for every } u \in U.$$

Thus, if U is a compact subset of $\mathbf{C}(X)$, then $\succ$ must satisfy the open-continuity property as we shall see in Appendix A1. Hence, the proof follows from Schmeidler's (1971) theorem. $\square$

Proof of Observation 2. Part (i) of this observation is obvious. For part (ii), let $r \in \Delta(X)$ be such that $r(\{0\}) = 0$. To prove that r is $\succsim^{\wedge}$-maximal, take any $q \in \Delta(X)$ with $q \neq r$. Then, there is a Borel set $X_0 \subseteq X$ such that $r(X_0) > q(X_0)$.

First assume $r(\{1\}) \leq q(\{1\})$. Then, as we also have $r(\{0\}) \leq q(\{0\})$, it follows that $r(X_0 \backslash \{0, 1\}) > q(X_0 \backslash \{0, 1\})$. Hence, by normality of countably additive measures on a metric space, there exists a closed set F contained in $X_0 \backslash \{0, 1\}$ such that $r(F) > q(F)$ (see Aliprantis and Border, 1999, Theorem 17.24, p. 574).

[47]To prove continuity of $\mathbf{a}(\cdot)$, it suffices to note that for each $x \in X$ and $n \in \mathbb{N}$, there exists a neighborhood O of x such that $y \in O$ implies $y_t^l = x_t^l$ for every $t \in T$ and positive integer $l \leq n$, so that $a^l(y) = a^l(x)$ for every such l.

[48]As a minor point, let us note that nontriviality of $\succ_t$ follows from surjectivity of $r \to r_{\mathbf{a}}$. (In turn, that this map is surjective can be proved by noting that $\{r_{\mathbf{a}} : r \in \Delta(X)\}$ is a closed, convex subset of $\Delta(A^{\infty})$ which contains every degenerate lottery.)

For each $\varepsilon > 0$, let $\mathcal{B}_\varepsilon := \{x \in X : |x-y| < \varepsilon \text{ for some } y \in F \cup \{0\}\}$. Note that by Tietze extension theorem, there exists a function $u_\varepsilon \in \hat{U}$ such that, for any $x \in [0,1]$,

$$u_\varepsilon(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in \{1\} \cup (X \backslash \mathcal{B}_\varepsilon), \\ 2 & \text{if } x \in F. \end{cases}$$

It is plain that, for every $p \in \Delta(X)$,

$$\lim_{\varepsilon \to 0} \int_{\mathcal{B}_\varepsilon \backslash F} u_\varepsilon dp = \lim_{\varepsilon \to 0} \left(u_\varepsilon(0) p(\{0\}) + \int_{\mathcal{B}_\varepsilon \backslash (F \cup \{0\})} u_\varepsilon dp \right) = 0.$$

Hence,

$$\begin{aligned} \lim_{\varepsilon \to 0} E_q(u_\varepsilon) &= \lim_{\varepsilon \to 0} \int_{F \cup (X \backslash \mathcal{B}_\varepsilon)} u_\varepsilon dq = 2q(F) + q\left(X \backslash \left(F \cup \{0\}\right)\right) \leq q(F) + 1, \\ \lim_{\varepsilon \to 0} E_r(u_\varepsilon) &= \lim_{\varepsilon \to 0} \int_{F \cup (X \backslash \mathcal{B}_\varepsilon)} u_\varepsilon dr = 2r(F) + r\left(X \backslash \left(F \cup \{0\}\right)\right) = r(F) + 1. \end{aligned}$$

It follows that $E_r(u_\varepsilon) > E_q(u_\varepsilon)$ for all sufficiently small ε.

Suppose now $r(\{1\}) > q(\{1\})$. For each $\varepsilon \in (0,1)$, pick any $v_\varepsilon \in \hat{U}$ such that $v_\varepsilon(x) = 0$ for $x \in [0, 1-\varepsilon]$. Then, as $v_\varepsilon(1) = 1$ for every $\varepsilon \in (0,1)$, we obviously have $\lim_{\varepsilon \to 0} E_r(v_\varepsilon) = r(\{1\})$ and $\lim_{\varepsilon \to 0} E_q(v_\varepsilon) = q(\{1\})$, implying that $E_r(v_\varepsilon) > E_q(v_\varepsilon)$ for all sufficiently small ε. This completes the proof of (ii), which obviously implies (iii) as well.

That the first conclusion in part (iv) follows from (ii) is shown in text. In turn, the proof of antisymmetry of $\succsim^\wedge$ resembles the proof of part (ii), and hence, we omit it. $\square$

Proof of Observation 3 is implicit in our discussion of Propositions 1 and 1'.

A.1. First Proof of Theorem 1

Throughout the remainder of this appendix, we will sometimes write $\widetilde{u}(\eta)$ instead of $E_\eta(u)$.

First, we need to show that if condition (i) of Theorem 1 holds for a compact set $U \subseteq \mathbf{C}(X)$, then the open-continuity axiom must also hold. To this end, let U be such a set, and take any two lotteries p, q with $p \succ q$. Continuity of the real function $u \to E_p(u) - E_q(u)$ on $\mathbf{C}(X)$ implies that there is a positive number γ such that $E_p(u) - E_q(u) > \gamma$ for every $u \in U$. Moreover, since $u \to \widetilde{u}$ is a continuous map from $\mathbf{C}(X)$ to $\mathbf{C}(\Delta(X))$, from compactness of U it follows that the set $\{\widetilde{u} : u \in U\} \subseteq \mathbf{C}(\Delta(X))$ is also compact. By Arzelà-Ascoli theorem (see, e.g., Dunford and Schwartz, 1958, Theorem IV.6.7, p. 266), we thus conclude that there is a neighborhood N of p such that $\widetilde{u}(p') - \widetilde{u}(p) > -\gamma/2$ for every $p' \in N$ and $u \in U$. But for any such p', we have $\widetilde{u}(p') - \widetilde{u}(q) > \gamma/2$ for all $u \in U$, and hence, $p' \succ q$ by condition (i). Thus, the set $\{p \in \Delta(X) : p \succ q\}$ is open, and similarly, so is the set $\{p \in \Delta(X) : q \succ p\}$. This verifies that $\succ$ satisfies the open-continuity axiom. The remainder of the proof of the "if" part of the theorem is trivial.

To prove the "only if" part, let $\succ$ be an open-continuous strict preference relation on $\Delta(X)$. Put $\mathcal{C} := \{\gamma(p - q) : p \succ q,\ \gamma > 0\}$ and let $\mathcal{S}$ stand for the span of $\Delta(X) - \Delta(X)$. It is worth noting that, by Jordan decomposition theorem, we have $\mathcal{S} = \{\eta \in ca(X) : \eta(X) = 0\}$. The proof of the next claim is a routine exercise, and hence, omitted.

Claim 1. *$\mathcal{C}$ is a convex cone such that for every p, q in $\Delta(X)$, we have $p \succ q$ if and only if $p - q \in \mathcal{C}$.*

For every positive real number λ, we denote by B_λ the closed λ-ball in $\mathcal{S}$, that is $B_\lambda := \{\eta \in \mathcal{S} : \|\eta\| \leq \lambda\}$ where $\|\cdot\|$ is the total variation norm. The next claim will prove useful in what follows.

Claim 2. *For any $\lambda > 0$, the set $B_\lambda \cap \mathcal{C}$ is relatively weak*-open in B_λ.*

Proof. Since $(ca(X), \|\cdot\|)$ is isometrically isomorphic to the topological dual of the separable Banach space $\mathbf{C}(X)$, the weak*-topology of B_λ is metrizable (see, e.g., Dunford and Schwartz, 1958, Theorem V.5.1, p. 426). Let σ stand for a compatible metric. Suppose by contradiction that $B_\lambda \cap \mathcal{C}$ is not open in B_λ. Then there exists a point $\mu \in B_\lambda \cap \mathcal{C}$ such that, for every natural number n, we have $\sigma(\mu, \mu_n) < 1/n$ for some $\mu_n \in B_\lambda \backslash \mathcal{C}$. Note that $\mu \neq 0$ since $\succ$ is irreflexive. So, by passing to a subsequence if necessary, we can assume for every n that $\mu_n \neq 0$, for the sequence (μ_n) converges to μ. Thus, for each n, by Jordan decomposition theorem we can write $\mu_n = \gamma_n(p_n - q_n)$ for some mutually singular p_n, q_n in $\Delta(X)$ and $\gamma_n > 0$. By mutual singularity we have $\|p_n - q_n\| = 2$ for every n, and hence, $\gamma_n \leq \lambda/2$. Since $\Delta(X)$ is compact and (γ_n) is bounded, it follows that there is an increasing self-map $k \to n_k$ on $\mathbb{N}$ such that $(\gamma_{n_k}), (p_{n_k})$ and (q_{n_k}) are convergent subsequences. Let the corresponding limits be γ, p and q, respectively. Since $\gamma_{n_k}(p_{n_k} - q_{n_k}) = \mu_{n_k}$ converges to μ as $k \to \infty$, clearly, we must have $\gamma(p - q) = \mu$. It follows that $\gamma > 0$ and $p - q = \mu/\gamma \in \mathcal{C}$. So, by Claim 1, we have $p \succ q$. Since $\succ$ is an open subset of $\Delta(X)^2$ as we have shown in Remark 1, by definitions of p and q we conclude that $p_{n_k} \succ q_{n_k}$ for all large k, implying that μ_{n_k} belongs to $\mathcal{C}$, a contradiction.[49] $\square$

[49]The reader will notice in these arguments a few similarities with the proof of Claim 1 of DMO, which shows that for any preorder $\succsim^*$ on $\Delta(X)$ that satisfies their axioms, the set $\{\gamma(p - q) : p \succsim^* q,\ \gamma > 0\}$ is weak*-closed. The most important difference between the two exercises is that, Claim 1 of DMO implicitly benefits from a version of Krein-Šmulian theorem which ensures that a convex subset $\mathcal{K}$ of $\mathcal{S}$ is weak*-closed if $\mathcal{K} \cap B_\lambda$ is weak*-closed for every $\lambda > 0$ (see Appendix A.2 below). In our case, the analogue of

The focus of our first method of proof is the closure of $\succ$, which we denote by $\succsim^*$. So, for any two lotteries p, q, we have $p \succsim^* q$ if and only if there exist two sequences $(p_n), (q_n)$ in $\Delta(X)$ such that $p = \lim p_n$, $q = \lim q_n$ and $p_n \succ q_n$ for every n. We also fix two lotteries $p^\bullet, q^\bullet$ with $p^\bullet \succ q^\bullet$ and set $\eta^\bullet := p^\bullet - q^\bullet$. The following claim will allow us to benefit from the findings of DMO.

Claim 3. *$\succsim^*$ is a closed preorder on $\Delta(X)$ such that, for every p, q, r in $\Delta(X)$ and $\alpha \in (0, 1)$,*

$$p \succsim^* q \quad \textit{implies} \quad \alpha p + (1-\alpha)\, r \succsim^* \alpha q + (1-\alpha)\, r. \tag{13}$$

Proof. $\succsim^*$ is closed by definition. Moreover, for every $p \in \Delta(X)$ and $\alpha \in (0, 1)$, we have $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)p$ by the independence axiom. Passing to limit as $\alpha \to 0$ yields $p \succsim^* p$. So, $\succsim^*$ is reflexive.

To verify (13), take any lotteries p, q, r with $p \succsim^* q$ and any $\alpha \in (0, 1)$. Then there exist two sequences $(p_n), (q_n)$ in $\Delta(X)$ such that $p = \lim p_n$, $q = \lim q_n$ and $p_n \succ q_n$ for every n. Now, by the independence axiom, we have $\alpha p_n + (1-\alpha)r \succ \alpha q_n + (1-\alpha)r$ for every n. Since $\alpha p_n + (1-\alpha)r \to \alpha p + (1-\alpha)r$ and $\alpha q_n + (1-\alpha)r \to \alpha q + (1-\alpha)r$ as $n \to \infty$, it follows that $\alpha p + (1-\alpha)r \succsim^* \alpha q + (1-\alpha)r$, as we seek.

When establishing transitivity of $\succsim^*$ we will benefit from the following fact: For any $\alpha \in (0, 1)$,

$$p \succsim^* q \quad \text{implies} \quad \alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)q. \tag{14}$$

To prove (14), consider any such α, p, q, and let the sequences $(p_n), (q_n)$ be as in the previous paragraph. Put $\mu := p - q$ and $\mu_n := p_n - q_n$ for every n. Since $\Delta(X)$ is norm-bounded, we can pick a number $\lambda > 0$ such that $\left\{ \frac{1-\alpha}{\alpha}(\mu - \mu_n) : n \in \mathbb{N} \right\} \cup \{\eta^\bullet\} \subseteq B_{\lambda/2}$. By Claim 2, the set $B_\lambda \cap \mathcal{C}$ is open in B_λ. As $\eta^\bullet \in B_\lambda \cap \mathcal{C}$, it thus follows that there is a weak*-open neighborhood $W \subseteq ca(X)$ of the origin such that $(\eta^\bullet + W) \cap B_\lambda \subseteq \mathcal{C}$. Moreover, since $\mu_n \to \mu$ by definitions of p and q, for a sufficiently large n the point μ_n belongs to $\mu - \frac{\alpha}{1-\alpha} W$, i.e., we have $\mu_n = \mu - \frac{\alpha}{1-\alpha} w$ for some $w \in W$. From the choice of λ it follows that $\|w\| = \left\| \frac{1-\alpha}{\alpha}(\mu - \mu_n) \right\| \leq \lambda/2$ and $\|\eta^\bullet + w\| \leq \lambda$. The latter inequality implies that $\eta^\bullet + w$ is in $\mathcal{C}$. Since $\mathcal{C}$ is a convex set that also contains μ_n, we conclude that $\alpha(\eta^\bullet + w) + (1-\alpha)\mu_n$

this property does not hold; i.e., Claim 2 above does not imply that $\mathcal{C}$ is weak*-open in $\mathcal{S}$. The reason is that when X is infinite, there does not exist a nonempty weak*-open set in $ca(X)$ that is norm-bounded. Hence, there are unbounded nets in $ca(X)$ that weak*-converge to the origin (as it is the case for any linear topology which is coarser than the norm-topology).

belongs to $\mathcal{C}$ as well. Now, we note that

$$\begin{aligned}\alpha(\eta^\bullet + w) + (1-\alpha)\mu_n &= \alpha\left(\eta^\bullet + \frac{1-\alpha}{\alpha}(\mu - \mu_n)\right) + (1-\alpha)\mu_n \\ &= \alpha\eta^\bullet + (1-\alpha)\mu \\ &= (\alpha p^\bullet + (1-\alpha)p) - (\alpha q^\bullet + (1-\alpha)q)\,.\end{aligned}$$

Thus, by Claim 1, we see that $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)q$, as we claimed.

Finally, to show that $\succsim^*$ is transitive, consider lotteries p, q, r such that $p \succsim^* q$ and $q \succsim^* r$. Then, there exist two sequences $(q_n), (r_n)$ in $\Delta(X)$ such that $q = \lim q_n$, $r = \lim r_n$ and $q_n \succ r_n$ for every n. Moreover, for any fixed $\alpha \in (0,1)$, we have $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)q$ by (14). Since $\alpha q^\bullet + (1-\alpha)q_n \to \alpha q^\bullet + (1-\alpha)q$, it follows from the open-continuity axiom that $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)q_n$ for all sufficiently large n. Furthermore, by the independence axiom, we have $\alpha q^\bullet + (1-\alpha)q_n \succ \alpha q^\bullet + (1-\alpha)r_n$ for every n. Transitivity of $\succ$ therefore implies that $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)r_n$ for all sufficiently large n. Passing to limit as $n \to \infty$ gives $\alpha p^\bullet + (1-\alpha)p \succsim^* \alpha q^\bullet + (1-\alpha)r$. Since $\succsim^*$ is closed, passing to limit as $\alpha \to 0$ yields the desired conclusion: $p \succsim^* r$. □

By Claim 3, $\succsim^*$ satisfies all axioms of DMO. Hence, $\mathcal{C}^* := \{\gamma(p-q) : p \succsim^* q,\ \gamma > 0\}$ is a weak*-closed set by Claim 1 of DMO. As $\mathcal{C} \subseteq \mathcal{C}^*$, it immediately follows that the algebraic closure of $\mathcal{C}$, which we denote by $\mathrm{acl}\,(\mathcal{C})$, is contained in $\mathcal{C}^*$. Moreover, clearly, (14) and the definition of $\mathcal{C}^*$ imply the converse inclusion: $\mathcal{C}^* \subseteq \mathrm{acl}\,(\mathcal{C})$. Hence, we see that $\mathrm{acl}\,(\mathcal{C}) = \mathcal{C}^*$ is a weak*-closed set. Driving this conclusion is, in fact, the main purpose of Claim 3 in the present proof. Since Claim 2 obviously implies that $\mathcal{C}$ is algebraically open (relative to $\mathcal{S}$), the first proof of Theorem 1 will easily follow from the next claim which will be proved momentarily.[50]

Claim 4. *There exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) $\widetilde{u}(p^\bullet) = 1$ *and* $\widetilde{u}(q^\bullet) = 0$ *for every* $u \in U$;
(ii) $\mathcal{C}^* = \{\eta \in \mathcal{S} : \widetilde{u}(\eta) \geq 0$ *for every* $u \in U\}$.

We now show how the proof of Theorem 1 can be completed assuming Claim 4.

Claim 5. *Given a set U as in Claim 4, for every p, q in $\Delta(X)$, we have $p \succ q$ if and only if $\widetilde{u}(p) > \widetilde{u}(q)$ for every $u \in U$.*

Proof. Consider any two lotteries p, q, and put $\mu := p - q$. Suppose first that $\widetilde{u}(\mu) > 0$ for

[50]Though Claim 2 appears to be indispensable for our purposes, in Appendix A.2, we will be able to give a shorter proof that does not make use of Claim 3. To this end, we will focus on openness of $\mathcal{C}$ with respect to an alternate topology that is finer than weak*-topology but coarser than the algebraic topology.

every $u \in U$. Then, as U is compact, there exists a number $\beta > 0$ such that $\widetilde{u}(\mu) \geq \beta$ for every $u \in U$. By using boundedness of U and part (ii) of Claim 4 in an obvious way, we therefore see that μ is in the algebraic interior of $\mathcal{C}^*$. But, as we discussed before, the set $\mathcal{C}^*$ equals $\operatorname{acl}(\mathcal{C})$, and $\mathcal{C}$ is algebraically open. Thus, the algebraic interior of $\mathcal{C}^*$ coincides with $\mathcal{C}$, and hence, μ belongs to $\mathcal{C}$. This amounts to saying $p \succ q$, as we seek.

Conversely, suppose now that $p \succ q$ and take any $u \in U$. As $\mathcal{C}$ is algebraically open and $\mu \in \mathcal{C}$, it follows that $\mu - \alpha\eta^\bullet \in \mathcal{C}$ for some $\alpha > 0$. By Claim 4(ii), we therefore have $\widetilde{u}(\mu - \alpha\eta^\bullet) \geq 0$, i.e., $\widetilde{u}(\mu) \geq \alpha$. $\square$

We conclude with the proof of Claim 4.

Proof of Claim 4. Let us define $\mathfrak{G} := \{u \in \mathbf{C}(X) : \widetilde{u}(\eta) \geq 0 \text{ for every } \eta \in \mathcal{C}^*\}$, $U := \{u \in \mathfrak{G} : \widetilde{u}(p^\bullet) = 1,\ \widetilde{u}(q^\bullet) = 0\}$ and $\mathcal{C}^+ := \{\eta \in \mathcal{S} : \widetilde{u}(\eta) \geq 0 \text{ for every } u \in U\}$. Note that $\mathfrak{G}$ is closed, and as a closed subset of $\mathfrak{G}$, the set U is also closed. Hence, by the Arzelà-Ascoli theorem, to verify compactness of U it suffices to show that this set is bounded and equicontinuous.

Since the weak*-topology is coarser than the norm-topology of $ca(X)$, and since $\Delta(X)$ is a norm-bounded set, applying the open-continuity axiom to the lotteries $p^\bullet, q^\bullet$ yields an $\alpha \in (0,1)$, close enough to 1, such that $p^\bullet \succ \alpha q^\bullet + (1-\alpha)\,\Delta(X)$ and $\alpha p^\bullet + (1-\alpha)\,\Delta(X) \succ q^\bullet$. In particular, we have $p^\bullet \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$ and $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ q^\bullet$ for each $x \in X$. We thus see by definition of U that $\frac{1}{1-\alpha} \geq u(x) \geq \frac{-\alpha}{1-\alpha}$ for every $u \in U$ and $x \in X$. This shows that U is bounded.

Now let $x \in X$ and $\varepsilon > 0$. Pick an $\alpha \in (0,1)$ such that $\frac{\alpha}{1-\alpha} < \varepsilon$. Since $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$, clearly, the open-continuity axiom implies that there is a neighborhood $O \subseteq X$ of x such that $\alpha p^\bullet + (1-\alpha)\,\delta_z \succ \alpha q^\bullet + (1-\alpha)\,\delta_x$ and $\alpha p^\bullet + (1-\alpha)\,\delta_x \succ \alpha q^\bullet + (1-\alpha)\,\delta_z$ for every $z \in O$. It readily follows that $|u(x) - u(z)| \leq \frac{\alpha}{1-\alpha} < \varepsilon$ for every $z \in O$ and $u \in U$. Hence, U is also equicontinuous, as required.

What remains to show is that U is nonempty and that $\mathcal{C}^+ \subseteq \mathcal{C}^*$, for the converse inclusion is trivial. To this end, we first note that since $\mathcal{C}^*$ is a weak*-closed convex cone, by standard separation and duality arguments, for every $\eta \in \mathcal{S} \backslash \mathcal{C}^*$ we can find a function $u \in \mathfrak{G}$ such that $\widetilde{u}(\eta) < 0$.

We now show that $q^\bullet - p^\bullet$ does not belong to $\mathcal{C}^*$. Since $\succ$ is open in $\Delta(X)^2$, there exist open subsets $N_{p^\bullet}, N_{q^\bullet}$ of $\Delta(X)$ such that $(p^\bullet, q^\bullet) \in N_{p^\bullet} \times N_{q^\bullet} \subseteq \succ$. From asymmetry of $\succ$ it follows that $(N_{q^\bullet} \times N_{p^\bullet}) \cap \succ = \varnothing$. Since $N_{q^\bullet} \times N_{p^\bullet}$ is an open neighborhood of $(q^\bullet, p^\bullet)$, we conclude that $(q^\bullet, p^\bullet)$ does not belong to the closure of $\succ$, i.e., it is not true that $q^\bullet \succsim^* p^\bullet$. By Lemma 2 of DMO, this is equivalent to saying $q^\bullet - p^\bullet \notin \mathcal{C}^*$.

Hence, we have $\widetilde{u_0}(q^\bullet - p^\bullet) < 0$ for some $u_0 \in \mathfrak{G}$. Then $v_0 := \frac{1}{\widetilde{u_0}(p^\bullet - q^\bullet)}\,(u_0 - \widetilde{u_0}(q^\bullet)\mathbf{1}_X)$ also belongs to $\mathfrak{G}$. Moreover, as $\widetilde{v_0}(p^\bullet) = 1$ and $\widetilde{v_0}(q^\bullet) = 0$, the set U contains v_0, and is nonempty.

Finally, to show that $\mathcal{C}^+ \subseteq \mathcal{C}^*$, let $\eta \in \mathcal{S}\backslash\mathcal{C}^*$ and pick any $u \in \mathfrak{G}$ such that $\widetilde{u}(\eta) < 0$. Fix a sufficiently small $\alpha > 0$ which satisfies $\widetilde{u}(\eta) + \alpha\widetilde{u_0}(\eta) < 0$. Notice that $u_1 := u + \alpha u_0$ belongs to $\mathfrak{G}$. Moreover, $\widetilde{u_1}(q^\bullet - p^\bullet) < 0$, for $\widetilde{u}(q^\bullet - p^\bullet) \leq 0$ by definition of $\mathfrak{G}$. It follows that $v_1 := \frac{1}{\widetilde{u_1}(p^\bullet - q^\bullet)}\left(u_1 - \widetilde{u_1}(q^\bullet)\mathbf{1}_X\right)$ belongs to $\mathfrak{G}$ as well. In fact, v_1 is an element of U such that $\widetilde{v_1}(\eta) = \frac{\widetilde{u_1}(\eta)}{\widetilde{u_1}(p^\bullet - q^\bullet)} < 0$. Hence, $\eta \notin \mathcal{C}^+$, as we seek. $\square$

Next, we will present a shorter proof of Theorem 1 that benefits from some classical results in functional analysis instead of Claim 3. We believe, however, that Claim 3 may be of independent interest as it uncovers useful facts on the structure of open-continuous strict preference relations.

A.2. Second Proof of Theorem 1

The *bounded weak*-topology* on $ca(X)$, which we denote by τ, is the finest topology that coincides with the weak*-topology on every positive multiple of the unit ball of $ca(X)$; that is, on sets of the form $B_\lambda^\circ := \{\eta \in ca(X) : \|\eta\| \leq \lambda\}$ for $\lambda > 0$. Thus, a set $O \subseteq ca(X)$ is τ-open if and only if $O \cap B_\lambda^\circ$ is relatively weak*-open in B_λ° for every $\lambda > 0$, and a set $\mathcal{K} \subseteq ca(X)$ is τ-closed if and only if $\mathcal{K} \cap B_\lambda^\circ$ is weak*-closed for every $\lambda > 0$. It is a straightforward exercise to show that restricting τ to $\mathcal{S}$ gives rise to analogous rules: A set $O \subseteq \mathcal{S}$ is relatively τ-open in $\mathcal{S}$ if and only if $O \cap B_\lambda$ is relatively weak*-open in B_λ for every $\lambda > 0$, and a set $\mathcal{K} \subseteq \mathcal{S}$ is τ-closed if and only if $\mathcal{K} \cap B_\lambda$ is weak*-closed for every $\lambda > 0$.[51]

It is known that τ is a locally convex linear topology, and a linear functional on $ca(X)$ is τ-continuous if and only if it is weak*-continuous. These observations lead to Krein-Šmulian theorem: A convex subset of $ca(X)$ is τ-closed if and only if it is weak*-closed.[52] Thus, given an open-continuous strict preference relation $\succ$, the τ-closure of $\mathcal{C}$ coincides with its weak*-closure, $\mathrm{cl}\,(\mathcal{C})$.

Moreover, without making use of Claim 3, we can modify Claim 4 by writing $\mathrm{cl}\,(\mathcal{C})$ instead of $\mathcal{C}^*$. The only difference in the proof of this modified version is the verification of the claim $q^\bullet - p^\bullet := -\eta^\bullet \notin \mathrm{cl}\,(\mathcal{C})$. To prove this point, we first note that $\mathcal{C}$ is τ-open (in $\mathcal{S}$) by Claim 2, and hence, the set $\mathcal{C} - \frac{\eta^\bullet}{2}$ is a τ-open neighborhood of the origin. Thus, $-\eta^\bullet - \left(\mathcal{C} - \frac{\eta^\bullet}{2}\right)$ is a τ-open neighborhood of $-\eta^\bullet$. This set does not intersect $\mathcal{C}$, for otherwise we would have $-\eta^\bullet - \left(\mu_1 - \frac{\eta^\bullet}{2}\right) = \mu_2$ for some μ_1, μ_2 in $\mathcal{C}$, and this would imply

[51] Of course, weak*-closedness of $\mathcal{S}$ and B_λ° play an implicit role in these assertions. (In fact, as is well-known, B_λ° is weak*-compact by the Banach-Alaoglu theorem.)

[52] These results actually apply on the topological dual of any Banach space. For a detailed discussion, see Dunford and Schwartz (1957, Section V.5), in particular Corollary V.5.5, Theorems V.5.6 and V.5.7.

$-\eta^\bullet = 2(\mu_1+\mu_2) \in \mathcal{C}$, i.e., $q^\bullet \succ p^\bullet$, a contradiction to asymmetry of $\succ$. Since we have found a τ-open neighborhood of $-\eta^\bullet$ that does not intersect $\mathcal{C}$, we can conclude that $-\eta^\bullet$ does not belong to the τ-closure of $\mathcal{C}$ which coincides with $\mathrm{cl}(\mathcal{C})$. Hence, without using Claim 3, we proved that:

Claim 4'. *There exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) $\widetilde{u}(p^\bullet) = 1$ *and* $\widetilde{u}(q^\bullet) = 0$ *for every* $u \in U$;
(ii) $\mathrm{cl}(\mathcal{C}) = \{\eta \in \mathcal{S} : \widetilde{u}(\eta) \geq 0$ *for every* $u \in U\}$.

Now we complete the proof by modifying Claim 5 accordingly. Given a set U as in Claim 4', let the lotteries p, q and the number $\beta > 0$ be as in the proof of Claim 5; that is, assume $\widetilde{u}(p-q) \geq \beta$ for every $u \in U$. Now pick any $\alpha \in (0, \beta)$. We will show that $\mu := p-q$ belongs to the τ-interior of $\mathrm{cl}(\mathcal{C})$ (relative to $\mathcal{S}$). To this end, first note that $\mu + (\mathcal{C} - \alpha\eta^\bullet)$ is a τ-neighborhood of μ, and any element η of this set is of the form $\eta = \mu + (\mu_1 - \alpha\eta^\bullet)$ for some $\mu_1 \in \mathcal{C}$. From the properties of U, it thus follows that $\widetilde{u}(\eta) \geq \beta - \alpha > 0$ for every $\eta \in \mu + (\mathcal{C} - \alpha\eta^\bullet)$ and $u \in U$. Applying part (ii) of Claim 4' then yields $\mu + (\mathcal{C} - \alpha\eta^\bullet) \subseteq \mathrm{cl}(\mathcal{C})$. Hence, μ belongs to the τ-interior of $\mathrm{cl}(\mathcal{C})$, as we argued. Since $\mathcal{C}$ is a τ-open convex set, the τ-interior of the τ-closure of $\mathcal{C}$ equals $\mathcal{C}$. Thus, in fact, we have $\mu \in \mathcal{C}$. The remainder of the proof of Claim 5 applies as is.

A.3. Proof of Theorem 2

The "if" part of Theorem 2 is trivial. For the "only if" part, let U and V be $(p^\bullet, q^\bullet)$-normalized utility sets for an open-continuous strict preference relation $\succ$. We first need to show that, for every p, q in $\Delta(X)$,

$$\widetilde{u}(p) \geq \widetilde{u}(q) \quad \forall u \in U \quad \text{imply} \quad \widetilde{v}(p) \geq \widetilde{v}(q) \quad \forall v \in V.$$

For each $\alpha \in (0,1)$, by definition of U the former set of inequalities imply $\alpha p^\bullet + (1-\alpha)p \succ \alpha q^\bullet + (1-\alpha)q$, and hence, $\widetilde{v}(\alpha p^\bullet + (1-\alpha)p) > \widetilde{v}(\alpha q^\bullet + (1-\alpha)q)$ for $v \in V$, by definition of V. Passing to limit as $\alpha \to 0$ yields the desired conclusion: $\widetilde{v}(p) \geq \widetilde{v}(q)$ for every $v \in V$.

By the proof of the uniqueness result of DMO, it thus follows that V is contained in $\mathrm{cl}(\mathrm{cone}(U) + \{\beta \mathbf{1}_X : \beta \in \mathbb{R}\})$ where $\mathrm{cone}(U) \subseteq \mathbf{C}(X)$ is the smallest convex cone that contains U. Clearly, we can write $\mathrm{cone}(U) = \bigcup_{\gamma>0} \gamma\, \mathrm{co}(U)$. Hence, for each $v \in V$, there exist real sequences $(\beta_n), (\gamma_n)$ and a sequence (u_n) in $\mathrm{co}(U)$ such that $\gamma_n u_n + \beta_n \mathbf{1}_X \to v$. Since convergence in sup-norm implies weak-convergence, and since $\widetilde{u}_n(q^\bullet) = 0$ for every n, we then have $\lim \beta_n = \lim \beta_n \widetilde{\mathbf{1}_X}(q^\bullet) = \widetilde{v}(q^\bullet) := 0$. It follows that $\gamma_n u_n \to v$, and hence, $\lim \gamma_n = \lim \gamma_n \widetilde{u}_n(p^\bullet) = \widetilde{v}(p^\bullet) := 1$. These two observations, in turn, imply $u_n \to v$. Thus, $V \subseteq \overline{\mathrm{co}}(U)$, and we similarly have $U \subseteq \overline{\mathrm{co}}(V)$ so that $\overline{\mathrm{co}}(U) = \overline{\mathrm{co}}(V)$.

A.4. Proof of Theorem 3

Since the other implication can easily be verified, we shall show here that (i) implies (ii). Fix a preorder $\succsim$ on $\Delta(X)$ that satisfies II and SAC. Also assume that $\succ$ is an open-continuous strict preference relation, and let U be a utility set for $\succ$. To verify (4), pick any pair of lotteries p, q. As $\succ$ is nontrivial, there exists another pair r, w in $\Delta(X)$ with $r \succ w$.

Suppose first that $p \sim q$. Fix any $u \in U$. Then, for any $\alpha \in (0,1)$, by the independence axiom $\alpha p+(1-\alpha)r \succ \alpha p+(1-\alpha)w$, and $\alpha p+(1-\alpha)w \sim \alpha q+(1-\alpha)w$ by II. Transitivity of $\succsim$ therefore implies that $\alpha p+(1-\alpha)r \succ \alpha q+(1-\alpha)w$ for every $\alpha \in (0,1)$. By definition of U, we thus have $\widetilde{u}(\alpha p+(1-\alpha)r) > \widetilde{u}(\alpha q+(1-\alpha)w)$ for every $\alpha \in (0,1)$. Passing to limit as $\alpha \to 1$ yields $\widetilde{u}(p) \geq \widetilde{u}(q)$. Similarly, we also have $\widetilde{u}(p) \leq \widetilde{u}(q)$. Hence, we conclude that $\widetilde{u}(p) = \widetilde{u}(q)$ for every $u \in U$.

Conversely, assume now that $\widetilde{u}(p) = \widetilde{u}(q)$ for every $u \in U$. Then, since $\widetilde{u}(r) > \widetilde{u}(w)$, we have $\widetilde{u}(\alpha p+(1-\alpha)r) > \widetilde{u}(\alpha q+(1-\alpha)w)$ for every $u \in U$ and $\alpha \in (0,1)$. That is, $\alpha p+(1-\alpha)r \succ \alpha q+(1-\alpha)w$ for every $\alpha \in (0,1)$. Symmetrically, we also have $\alpha q+(1-\alpha)r \succ \alpha p+(1-\alpha)w$ for every $\alpha \in (0,1)$. Thus, SAC implies $p \sim q$, as required.

A.5. Proof of Lemma 1

To prove the "only if" part, assume $G_\succ := \{(t,p,q) \in T \times \Delta(X)^2 : p \succ_t q\}$ belongs to $\Sigma \otimes \mathfrak{B}(\Delta(X)^2)$. First, we note that for each (relatively) open subset $\mathcal{N}$ of $\Delta(X)^2$, we have $\{t \in T : \mathrm{cl}(\succ_t) \cap \mathcal{N} \neq \varnothing\} = \{t \in T : \succ_t \cap \mathcal{N} \neq \varnothing\}$, and the latter set equals $proj_T[G_\succ \cap (T \times \mathcal{N})] := \{t \in T : (t,p,q) \in G_\succ \cap (T \times \mathcal{N})$ for some $(p,q) \in \Delta(X)^2\}$. As $\Delta(X)^2$ is a separable metric space and (T, Σ, ℓ) is complete, it follows from the projection theorem that $\{t \in T : \mathrm{cl}(\succ_t) \cap \mathcal{N} \neq \varnothing\}$ belongs to Σ for every open $\mathcal{N} \subseteq \Delta(X)^2$ (Aliprantis and Border, 1999, Theorem 17.24, p. 574). That is, $t \rightrightarrows \mathrm{cl}(\succ_t)$ is a weakly measurable (nonempty valued) correspondence. Noting that $\Delta(X)^2$ is also complete as a metric space, we can therefore conclude by Castaing's representation theorem that there exists a sequence of Σ-$\mathfrak{B}(\Delta(X)^2)$ measurable functions $t \to (p_t^n, q_t^n)$ $(n \in \mathbb{N})$ such that, for each $t \in T$,

$$\mathrm{cl}(\succ_t) = \mathrm{cl}\left(\{(p_t^1, q_t^1), (p_t^2, q_t^2), ...\}\right) \tag{15}$$

(Aliprantis and Border, 1999, Corollary 17.14, p. 568).

We shall now show that

$$G_U = \bigcap\nolimits_{n=1}^{\infty} \Theta_n \cap (T \times \{u \in \mathbf{C}(X) \colon u(\mathbf{b}) = 1,\ u(\mathbf{0}) = 0\}), \tag{16}$$

where $\Theta_n := \{(t,u) \in T \times \mathbf{C}(X) \colon \widetilde{u}(p_t^n - q_t^n) \geq 0\}$ for each $n \in \mathbb{N}$. That G_U is contained in the right side of (16) is a simple consequence of definitions. To prove the converse inclusion, let (t,v) belong to the right side of (16). From the proof of Theorem 2, it is clear that for any p,q in $\Delta(X)$, whenever $\widetilde{u}(p-q) \geq 0$ for every $u \in U_{+,t}$, we have $(p,q) \in \mathrm{cl}\,(\succ_t)$. But since $\widetilde{v}(p_t^n - q_t^n) \geq 0$ for each $n \in \mathbb{N}$, by (15), we also have $\widetilde{v}(p-q) \geq 0$ for every $(p,q) \in \mathrm{cl}\,(\succ_t)$. Upon putting together these two observations, it follows that $\widetilde{v}(p-q) \geq 0$ whenever $\widetilde{u}(p-q) \geq 0$ for every $u \in U_{+,t}$. Hence, as in the proof of Theorem 2, we conclude that v belongs to $U_{+,t}$. This proves (16).

Now fix a natural number n, and note that $\Theta_n = \Upsilon^{-1}([0,\infty))$ where $\Upsilon : T \times \mathbf{C}(X) \to \mathbb{R}$ is defined by $\Upsilon(t,u) := \int_X u d(p_t^n - q_t^n)$. Since $t \to (p_t^n, q_t^n)$ is a measurable map from T into $\Delta(X)^2$, and since $(u,p,q) \to \int_X u d(p-q)$ is a continuous real map on $\mathbf{C}(X) \times \Delta(X)^2$, it is a routine exercise to verify that Υ is a $\Sigma \otimes \mathfrak{B}(\mathbf{C}(X))$-$\mathfrak{B}(\mathbb{R})$ measurable function (see, e.g., Aliprantis and Border, 1999, Lemmas 4.50 and 4.51, p. 151). Hence, Θ_n belongs to $\Sigma \otimes \mathfrak{B}(\mathbf{C}(X))$. As $\{u \in \mathbf{C}(X) \colon u(\mathbf{b}) = 1,\ u(\mathbf{0}) = 0\}$ is closed, in view of arbitrariness of n, it follows from (16) that G_U belongs to $\Sigma \otimes \mathfrak{B}(\mathbf{C}(X))$ as a countable intersection of members of $\Sigma \otimes \mathfrak{B}(\mathbf{C}(X))$.

Conversely, assume now that $G_U \in \Sigma \otimes \mathfrak{B}(\mathbf{C}(X))$. Then, applying Castaing's representation theorem to the correspondence $t \rightrightarrows U_{+,t}$ yields a sequence of Σ-$\mathfrak{B}(\mathbf{C}(X))$ measurable functions $t \to u_t^n$ $(n \in \mathbb{N})$ such that, for each $t \in T$,

$$U_{+,t} = \mathrm{cl}\left(\{u_t^1, u_t^2, ...\}\right).$$

Since $U_{+,t}$ is a (compact) utility set for $\succ_t$ it obviously follows that, for each $t \in T$ and p,q in $\Delta(X)$, we have $p \succ_t q$ if and only if there exists a $k \in \mathbb{N}$ such that $\int_X u_t^n d(p-q) \geq 1/k$ for every $n \in \mathbb{N}$. In other words, we have

$$G_\succ = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{(t,p,q) \in T \times \Delta(X)^2 \colon \int_X u_t^n d(p-q) \geq 1/k\right\}.$$

Since $t \to u_t^n$ is a measurable map from T into $\mathbf{C}(X)$, as in the first part of the proof it easily follows that the set $\left\{(t,p,q) \in T \times \Delta(X)^2 \colon \int_X u_t^n d(p-q) \geq 1/k\right\}$ belongs to $\Sigma \otimes \mathfrak{B}(\Delta(X)^2)$ for each k and n. As $\Sigma \otimes \mathfrak{B}(\Delta(X)^2)$ is closed under countable intersections and unions, we obtain the desired conclusion: $G_\succ \in \Sigma \otimes \mathfrak{B}(\Delta(X)^2)$.

Appendix B. A DMO Type Representation with a Compact set of Utility Functions

Consider the following axiom imposed on the asymmetric part of a preorder $\succsim^*$ on

$\Delta(X)$.

Compatibility with Strong Preference (CWSP). There exist $p^\bullet, q^\bullet$ in $\Delta(X)$ such that for each $r \in \Delta(X)$ and $\alpha \in (0, 1]$, we have $N_1 \succ^* \alpha q^\bullet + (1-\alpha)r$ and $\alpha p^\bullet + (1-\alpha)r \succ^* N_2$ for a neighborhood N_1 of $\alpha p^\bullet + (1-\alpha)r$ and a neighborhood N_2 of $\alpha q^\bullet + (1-\alpha)r$.

Intuitively, CWSP simply says that the open-continuity property holds on a pair of compound lotteries ρ^1, ρ^2 whenever ρ^1 is obtained from ρ^2 by shifting a positive weight from a "bad" lottery $q^\bullet$ to a "good" lottery $p^\bullet$. It is also clear that the term "compatibility with *strong* preference" refers to the obvious normative content of the open-continuity property.

The following DMO type representation theorem is a side payoff of our main findings.

Theorem B. *Let X be a compact metric space. A binary relation $\succsim^*$ on $\Delta(X)$ is a closed preorder that satisfies* CWSP *and the independence property* (13) *if, and only if, there exists a nonempty compact set $U \subseteq \mathbf{C}(X)$ such that:*
(i) *For every p, q in $\Delta(X)$, we have $p \succsim^* q$ if and only if $E_p(u) \geq E_q(u)$ for every $u \in U$.*
(ii) *$E_{p^\bullet}(u) > E_{q^\bullet}(u)$ for every $u \in U$ and some $p^\bullet, q^\bullet$ in $\Delta(X)$.*

In view of our previous discussions, the "if" part of Theorem B is obvious. Moreover, a brief examination of the proof of Claim 4 suffices to establish the "only if" part.[53] Needless to say, by normalizing the set U as in text, one also obtains a uniqueness result that is analogous to Theorem 2.

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[53]In fact, the proof of Theorem B is slightly shorter, as it is trivially true that $q^\bullet - p^\bullet$ does not belong to $\{\gamma(p-q) : p \succsim^* q,\ \gamma > 0\}$, where $p^\bullet$ and $q^\bullet$ are as posited by CWSP.

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