Turbulent flow simulations for wall-bounded flows

Reynolds-averaged Navier Stokes and Large Eddy Simulations

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AST559: Turbulence and Nonlinear Processes in Fluids and Plasmas

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Reynolds averaged Navier-Stokes (RANS) derivation

- *Reynolds averaging:* Replace terms in conservation equations with an average plus fluctuation, $u = \overline{u} + u'$, and average the entire expression.
- Define the (finite) time-averaged quantity $\overline{u}$

$$\overline{u} \equiv \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} u \, dt \tag{1}$$

- In the governing equations, replace terms with averages plus fluctuations, e.g. $u = \overline{u} + u'$, and note some relations:

$$\overline{f'} = 0 \qquad \overline{\overline{f}g} = \overline{f}\,\overline{g} \qquad \overline{f+g} = \overline{f} + \overline{g} \qquad \overline{f'f'} \neq 0 \qquad \overline{\frac{\partial u}{\partial t}} = \frac{\partial \overline{u}}{\partial t} \tag{2}$$

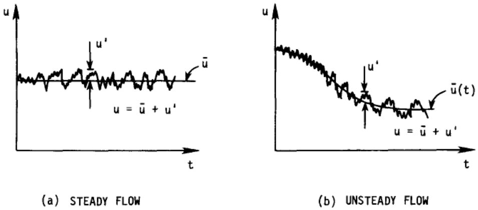

Figure: Averaged and fluctuating quantities, Tannehill et al.

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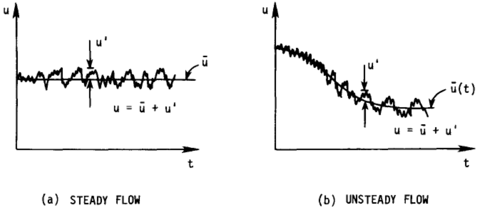

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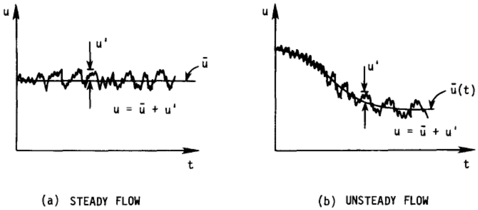

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- Incompressible continuity equation:

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial(\overline{u_i} + u_i')}{\partial x_i} = 0 \tag{3}$$

- Reynolds averaging:

$$\overline{\frac{\partial u_i}{\partial x_i}} = \frac{\overline{\partial(\overline{u_i} + u_i')}}{\partial x_i} = \frac{\partial \overline{u_i}}{\partial x_i} + \frac{\partial \overline{u_i'}}{\partial x_i} = \frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{4}$$

- Subtracting (**??**) from (**??**) shows the instaneous, average, and fluctuating velocity components all satisfy continuity:

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \frac{\partial \overline{u_i}}{\partial x_i} = 0 \qquad \frac{\partial u_i'}{\partial x_i} = 0 \tag{5}$$

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$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \quad (6)$$

- Applying the same process to the momentum equation:

$$\overline{\frac{\partial(\overline{u_i} + u_i')}{\partial t}} + \overline{(\overline{u_j} + u_j')\frac{\partial(\overline{u_i} + u_i')}{\partial x_j}} = -\frac{1}{\rho}\overline{\frac{\partial(\overline{p} + p')}{\partial x_i}} + \nu \overline{\frac{\partial^2(\overline{u_i} + u_i')}{\partial x_j^2}} \quad (7)$$

- The advective term, using some of the previous identities:

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- Averaged linear terms are simpler: $\overline{\frac{\partial(\overline{u_i}+u_i')}{\partial t}} = \frac{\partial \overline{u_i}}{\partial t}$, $\overline{\frac{\partial^2(\overline{u_i}+u_i')}{\partial x_j^2}} = \frac{\partial^2 \overline{u_i}}{\partial x_j^2}$.

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Reynolds averaged Navier-Stokes (RANS) - Closure

- Writing the time-mean terms $\overline{u} \doteq u$ for clarity, the Reynolds-averaged NS equation is:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j} - \overline{u_i' u_j'}\right) \tag{9}$$

- Note that the averaged equations of motion look identical to the original equations, but with an additional *Reynolds stress* $-\rho\overline{u_i' u_j'}$.
  - The same procedure is followed for compressible NS and energy conservation.
- The closure problem: the symmetric Reynolds stress tensor has introduced 6 new terms (or 1 for isotropic fluctuations):

$$\begin{bmatrix} -\rho\overline{u'u'} & -\rho\overline{u'v'} & -\rho\overline{u'w'} \\ -\rho\overline{u'v'} & -\rho\overline{v'v'} & -\rho\overline{v'w'} \\ -\rho\overline{u'w'} & -\rho\overline{v'w'} & -\rho\overline{w'w'} \end{bmatrix} \tag{10}$$

- 10 unknowns in 4 equations. *Turbulence modeling* is used to compute this stress in terms of the known flow variables.
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Reynolds averaged Navier-Stokes (RANS) - eddy viscosity

- Most common turbulence models use the *Boussinesq (1877) eddy viscosity hypothesis:*

$$-\rho\overline{u_i'u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho\delta_{ij}k \quad (11)$$

- where $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the mean strain tensor, $k = \frac{\overline{u_i'u_i'}}{2}$ is the turbulent kinetic energy, and μ_t is the "turbulent" or "eddy" viscosity.
- **Physical justification:** In laminar flows, energy dissipation and momentum transport normal to streamlines are governed by viscosity, so we can represent turbulent effects on the mean flow as an increased viscosity.
- By assuming the Reynolds stress is proportional to the mean strain, incompressible Navier-Stokes can be written

$$\begin{aligned}\rho\frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu S_{ij}\right) - \rho\overline{u_i'u_j'} \\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2(\mu+\mu_t)S_{ij}\right) - \frac{2}{3}\rho\delta_{ik}k\end{aligned} \quad (12)$$

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Reynolds averaged Navier-Stokes (RANS) - eddy viscosity

Three common categories of method for computing μ_t and k in RANS: $0-$equation, $1-$equation, and $2-$equation models.

- $0-$equation, or algebraic:
  - From dimensional analysis (Prandtl "mixing length", 1925), the turbulent viscosity can be modeled by a characteristic velocity and length,
    $$\mu_t = C_\mu \rho u_t L \tag{13}$$
    with C_μ a problem-dependent constant and L is a prescribed length based on grid location, i.e. distance from the wall.
  - Characteristic velocity can be computed as $u_t = L|\frac{\partial u}{\partial y}|$ in 2D.
  - Baldwin-Lomax (1978) is a prototypical algebraic model, which works well for thin attached boundary layers.

Reynolds averaged Navier-Stokes (RANS) - eddy viscosity

$1-$equation turbulence models

- Prandtl (1945) used the mean turbulent kinetic energy as the velocity scale:

$$\mu_T = C_k \rho L \overline{k}^{1/2} \tag{14}$$

  requiring the calculation of $\overline{k}$.

- Subtracting the transport equation for the mean flow from the total flow gives an equation of motion for the turbulent velocity u_i'. Then multiplying this by u_i' and averaging gives the mean turbulent kinetic energy equation (after lots of algebra):

$$\rho \frac{D\overline{k}}{Dt} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \overline{k}}{\partial x_j} - \frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right) - \rho\overline{u_i' u_j'}\frac{\partial u_i}{\partial x_j} - \mu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}\right) \tag{15}$$

- Unsurpringly, this brought up a new closure problem. Modeled as a gradient diffusion:

$$-\frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'} = \frac{\mu_t}{Pr_k}\frac{\partial \overline{k}}{\partial x_j} \tag{16}$$

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- The last term

$$-\mu \left(\overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \right) \equiv \rho \epsilon \quad (17)$$

is the irreversible dissipation rate of turbulent kinetic energy lost to heat. Applying the same process to the thermal energy equation necessarily results in a term of opposite sign.

- Again from dimensionality, the dissipation rate is $\epsilon = C_d \overline{k}^{3/2}/L$:

$$\rho \frac{D\overline{k}}{Dt} = \frac{\partial}{\partial x_j} \left[(\mu + \mu_t/Pr_k) \frac{\partial \overline{k}}{\partial x_j} \right] + \left(2\mu_t S_{ij} - \frac{2}{3} \rho \overline{k} \delta_{ij} \right) \frac{\partial u_i}{\partial x_j} - C_d \rho \overline{k}^{3/2}/L \quad (18)$$

- The right hand side terms are (1) diffusion of k, (2) generation of k, and (3) dissipation of k.
- Again the mixing length L needs to be specified algebraically, and C_d is an a priori closure coefficient.

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$$\rho\frac{D\overline{k}}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \mu_t/Pr_k\right)\frac{\partial \overline{k}}{\partial x_j}\right] + \left(2\mu_t S_{ij} - \frac{2}{3}\rho\overline{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - C_d\rho\overline{k}^{3/2}/L \tag{18}$$

- The right hand side terms are (1) diffusion of k, (2) generation of k, and (3) dissipation of k.
- Again the mixing length L needs to be specified algebraically, and C_d is an a priori closure coefficient.

Reynolds averaged Navier-Stokes (RANS) - eddy viscosity

- One-equation models don't necessarily have to solve a transport equation for k ("Prandtl's one-equation model")
- Spalart-Allmaras (1992) is a popular model in CFD, solving a transport equation for the eddy viscosity itself (with *six* closure coefficients!).
- Generally the one-equation models perform considerably better than the algebraic for separated flows, but inferior to the two-equation models.
- A canonical two-equation model, $k-\epsilon$, solves the previous k transport PDE and one for the dissipation rate ϵ:

$$\begin{aligned}
\rho\frac{D\overline{k}}{Dt} &= \frac{\partial}{\partial x_j}\left[\left(\mu+\mu_t/Pr_k\right)\frac{\partial\overline{k}}{\partial x_j}\right]+\left(2\mu_t S_{ij}-\frac{2}{3}\rho\overline{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j}-\rho\epsilon \\
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- Also popular are $k-\omega$ models, where $\omega=\overline{k}/\nu_t$ is a "specific dissipation rate." The two-equation models in general perform reasonably well in separated flows.

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Reynolds averaged Navier-Stokes (RANS) - eddy viscosity

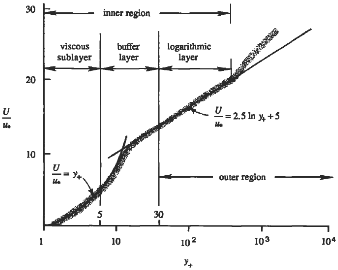
Law of the wall

- The **law of the wall** (von Karman 1930, Coles 1956) consists of two self-similar regions and a buffer between then.
- Dimension wall variables are

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \qquad y^+ = \frac{y u_\tau}{\nu} \qquad u^+ = \frac{u}{u_\tau} \tag{20}$$

- The innermost layer ($y^+ < 3 - 5$) is called the *viscous* or *laminar* sublayer. It assumed dominated by viscous effects, giving a linear velocity profile very near the wall.

$$\mu \frac{du}{dy} = \tau_w \qquad u = \frac{y \tau_w}{\mu} \qquad u^+ = y^+ \tag{21}$$



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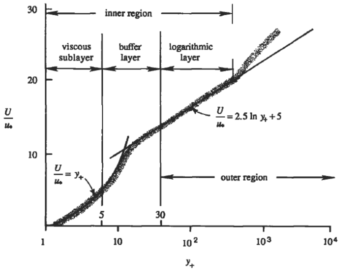

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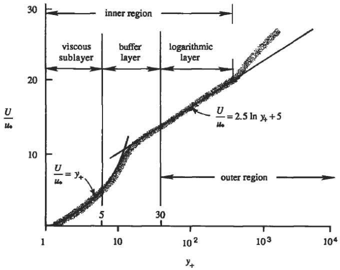

Law of the wall

- This layer is *very thin*. Resolving this would require at least a few wall-normal grid points in the laminar sublayer.
- **Example:** A Cessna 172 (small prop aircraft) has $Re \approx 6,000,000$ and a chord of $c \approx 1.5m$. This gives $y^+ = 1$ at $0.006mm$.
- Fully resolving this would require a prohibitive number of grid points, and the self-similarity of the laminar region can be exploited by using *wall functions*.
- In general, the first grid point is placed in the log region ($y^+ \approx 20$, $y \approx 0.1mm$)

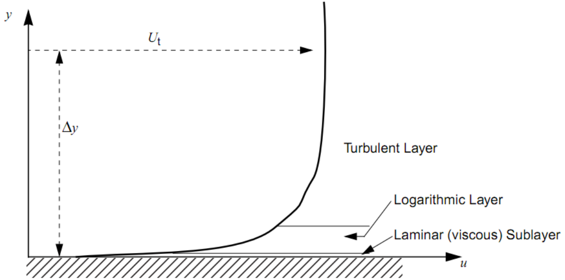

Figure: Turbulent boundary layer scale (CFX user manual)

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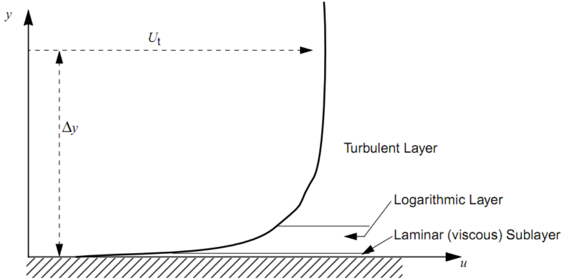

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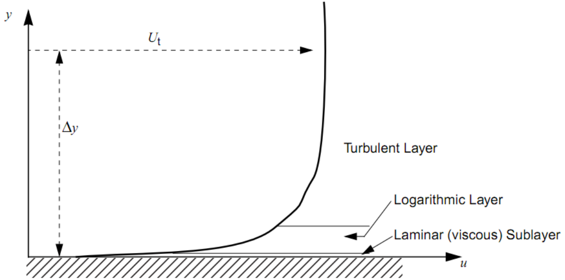

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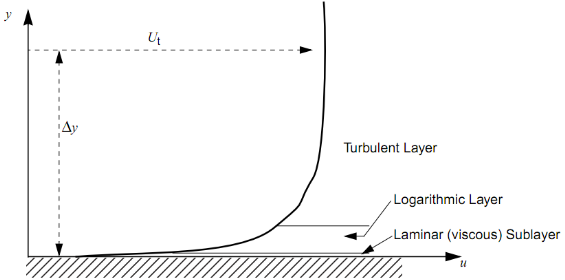

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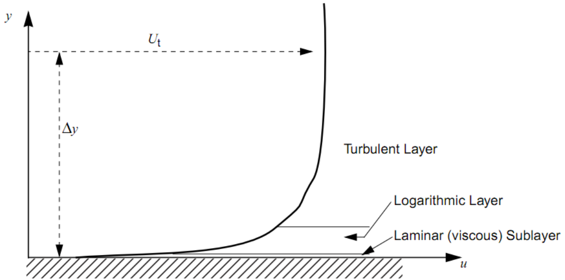
Law of the wall - wall modeling

- Wall models have to provide some transition/blending between the turbulent outer flow regime and inner layer, and impose boundary conditions on the turbulence model used.
  - e.g., wall functions for the $k-\omega$ model impose (Menter 2001)

$$\frac{\partial k}{\partial n}=0 \qquad \frac{\partial \omega}{\partial n}=0 \qquad k_w=\frac{u_\tau^2}{\sqrt{C_\mu y_w}} \qquad \omega_w=\frac{\sqrt{k_w}}{C_\mu^{1/4}\kappa y} \tag{22}$$



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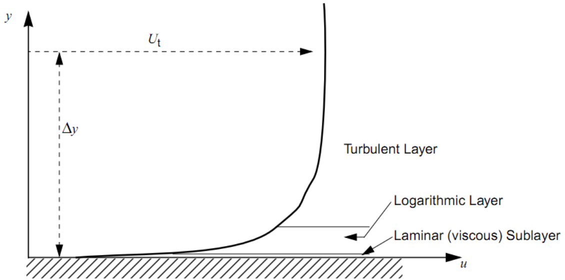

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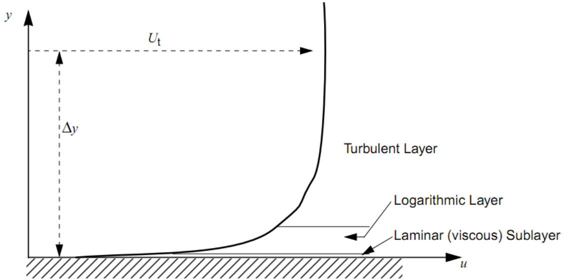
Law of the wall - wall modeling

- The choice of turbulence model and wall function can produce dramatic differences when predicting complex flows (compared next slides).
- In a region of separated flow, all wall models are questionable. The "least wrong" method is generally chosen by comparing to a similar experimental result.
- Wall models can be avoided by a grid down to $y^+ = 1$ to resolve the full boundary layer, but this is often impractical (Re dependent).



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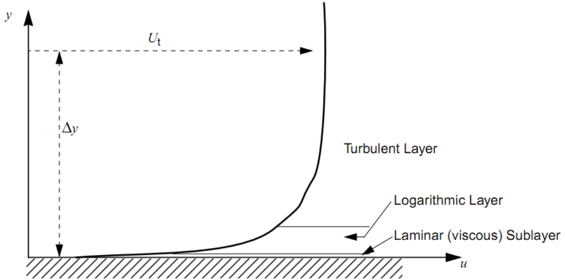

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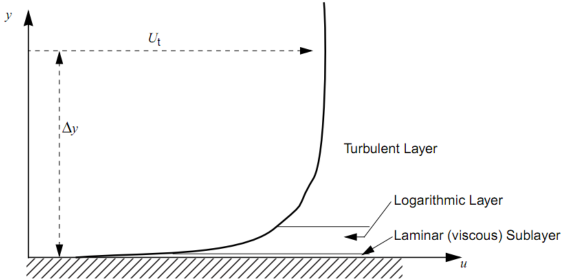

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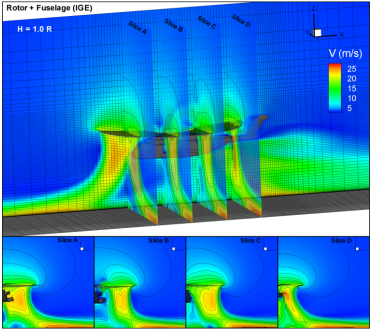
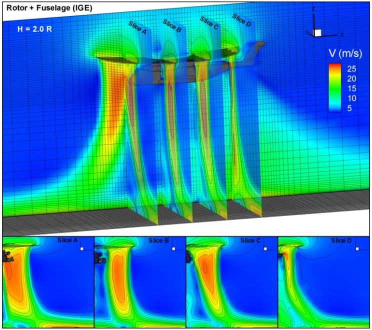
Reynolds averaged Navier-Stokes (RANS) - comparisons

Rotorcraft in ground effect (Ghosh et al 2008), $k - \epsilon$ turbulence model.



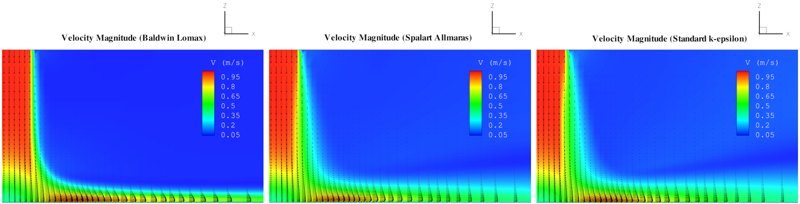
(a) Hover at $r/R = 1$



(b) Hover at $r/R = 2$

Reynolds averaged Navier-Stokes (RANS) - comparisons

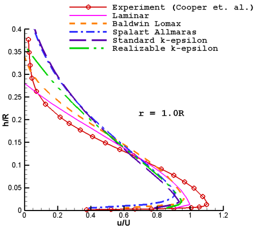
The quantities interest here were the rotor loads, and ground shear stress. The model is chosen by comparison with an impinging jet test case (Ghosh 2008):



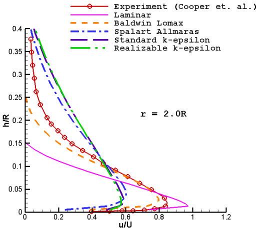
(c) Baldwin-Lomax (d) Spalart-Allmaras (e) $k - \epsilon$

Reynolds averaged Navier-Stokes (RANS) - comparisons

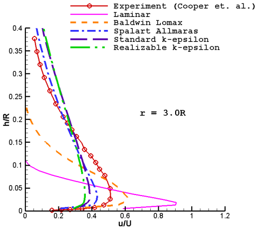
Impinging jet test case (Ghosh 2008) - Velocity profile at increasing radius from jet



(f) $r = 1R$



(g) $r = 2R$

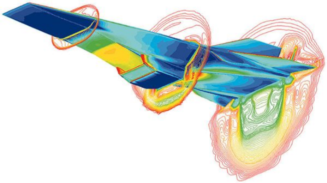


(h) $r = 3R$

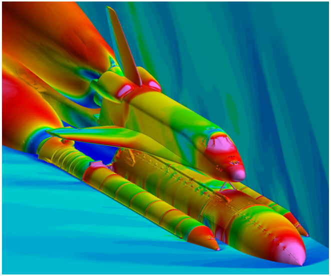
Figure: Note this was using a very coarse mesh, exaggerating the effects of different turbulence models.

Reynolds averaged Navier-Stokes (RANS) - Success

- Despite a lot of closure problems and empirically determined constants, RANS is the front-line fluid dynamic design and analysis tool.
- Provides accurate body forces on complex geometries for a wide range of flows, including high Re (10^6+) and high Mach regimes.



(a) Mach 7 X-43 Waverider (NASA)



(b) Shuttle pressure contours (NASA)

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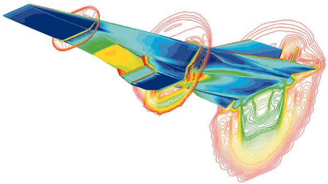

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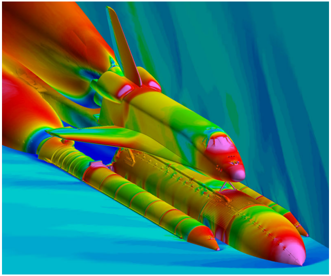

(d) Shuttle pressure contours (NASA)

Reynolds averaged Navier-Stokes (RANS)

Vertical axis wind turbine, unsteady moving grid, $Re \approx 4,000,000$, ”shear stress transport” variant of $k - \omega$.

(Vertical axis wind turbine)

Reynolds averaged Navier-Stokes (RANS)

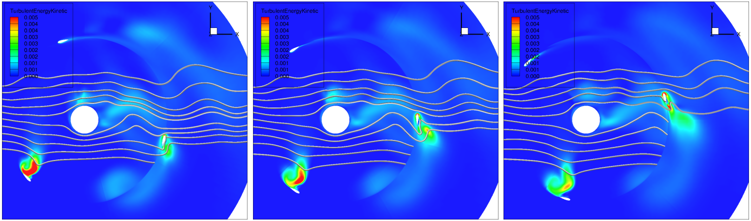

Figure: Turbulent kinetic energy contours, vertical axis wind turbine

Note the TKE field is ”blurred” over local regions here.

Large eddy simulation

- LES is based on the idea of self-similarity of small scale turbulent structures where dissipation occurs (K41) and thus can be modeled.
- Whereas RANS separates terms into time-averaged and fluctuating components, LES **filters** into a **resolved scale** and a **sub-grid scale**.
  - i.e., RANS is time averaged, LES spatially averaged.
- Define the resolved scale as a convolution integral with filter G

$$\overline{u_i}(\mathbf{x}) = \int G(\mathbf{x} - \boldsymbol{\xi}) u_i(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad (23)$$

- giving $u_i = \overline{u_i} + u_i'$
- The filter G has some associated physical width Δ (usually the grid cell size). Commonly a box filter or normalized Gaussian.

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Large eddy simulation

- Similar to RANS, substitute $u_i = \overline{u_i} + u_i'$ apply the spatial filter to the momentum equations:

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\overline{\partial u_i u_j}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} \quad (24)$$

- where again the nonlinear term presents the difficulty. This is split:

$$\begin{aligned} \overline{u_i u_j} &= \overline{(\overline{u}_i + u_i')(\overline{u}_j + u_j')} \\ &= \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u_j'} + \overline{u_i' \overline{u}_j} + \overline{u_i' u_j'} \\ \overline{u_i u_j} &= \overline{u}_i \overline{u}_j - \tau_{sgs} \\ -\tau_{sgs} &\equiv (\overline{\overline{u}_i \overline{u}_j} - \overline{u}_i \overline{u}_j) + (\overline{u_i' \overline{u}_j} + \overline{\overline{u}_i u_j'}) + (\overline{u_i' u_j'}) \\ \tau_{sgs} &= \overline{u}_i \overline{u}_j - \overline{u_i u_j} \end{aligned} \quad (25)$$

- giving an expression for the resolved field and additional sub-grid stress term

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \frac{1}{\rho}\frac{\partial \tau_{sgs}}{\partial x_j} \quad (26)$$

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Large eddy simulation - closure problem

- The closure problem for LES is then evaluating the subgrid-scale stress tensor:

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- *Note: if* $\overline{u}$ *represented a time-average, the first two terms would be* 0*, leaving only the Reynolds' stress.* $\overline{\overline{u}} \neq \overline{u}$ in the filtering approach.
- The earliest analysis of this describes the first term *Leonard stress* (after Leonard 1974), the second *cross-term stress* and the third *SGS Reynolds stress.*
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- Common LES closures include the simplest and first, Smagorinsky (1963) or Smagorinsky-Lilly model,

$$\begin{aligned} \tau_{sgs} &= -2\mu_{sgs} S_{ij} \\ \mu_{sgs} &= \rho (C_S \Delta)^2 \sqrt{S_{ij} S_{ij}} \\ 0.10 &< C_S < 0.24 \end{aligned} \tag{28}$$

  assumes the stresses also follow a gradient diffiusion similar to the early Prandtl (**??**) idea, and the methods can be written in the same eddy-viscosity form as seen in RANS ($\mu_{\text{effective}} = \mu_{\text{molecular}} + \mu_t$). Note the inherent isotropy present.
- Practical offshoot: methods/codes developed for RANS and laminar flow can be modified to perform LES, provided sufficient resolution of the large time and space scales.
- Fixing the Smagorinsky constant C_S necessarily has the effect of fixing constants in RANS; they lose generality and must be tuned to specific problems.
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Large eddy simulation - wall bounded flow

- A major problem is simulating the anisotropy of real boundary layers.
- Distinct streaks are formed with high aspect ratios $50:1$, requiring highly stretched grids.
- Treating these areas as isotropic results in overestimation of the stress and dissipation (Piomelli 1991). Wall functions are sometimes used in the same manner as RANS, but with the same drawbacks.
- A length scale Δ is needed, typically from the grid $\Delta = (\Delta_x \Delta_y \Delta_z)^{1/3}$, but this implies an isotropic subgrid form.

Figure: Boundary layer streaks and transition (Zake & Durbin 2005 JFM, direct numerical simulation up to $Re_x = 40000$)

Large eddy simulation - closure problem

- The current most popular LES approaches are variants of the dynamic SGS model (Germano 1990) to choose C_S based on the local flow solution.
- Basic procedure (via Ferziger & Peric):
  1. Compute the usual LES solution and filter to give the usual $\tau_{sgs} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$.
  2. Apply a larger filter ($\widehat{\Delta}$) to the same solution to get $T_{sgs} = \widehat{\overline{u_i u_j}} - \widehat{\overline{u}_i}\widehat{\overline{u}_j}$. The "sub-grid stress" found by this coarser filter should correspond to the smallest scale resolved structures of the original filter.
  3. Applying the coarser filter to the first and subtracting gives $L_{ij} = T_{sgs} - \widehat{\tau_{ij}}$ which gives an estimate of the "resolved" turbulent stresses.
  4. Comparing to the original Smagorinsky expression $\mu_{sgs} = \rho C_s^2 \Delta^2 |\overline{S}|$, and with a considerable amount of algebra, we recover an estimate of C_S based on a kind of local correction

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\begin{aligned}
L_{ij} &= -2 C_S M_{ij} \\
M_{ij} &\equiv (\widehat{\Delta})^2 |\widehat{\overline{S}}| \widehat{\overline{S}_{ij}} - (\Delta)^2 (\widehat{|\overline{S}| \overline{S}_{ij}}) \\
C_S^2 &= -\frac{L_{kl} \overline{S}_{kl}}{2 M_{mn} \overline{S}_{mn}}
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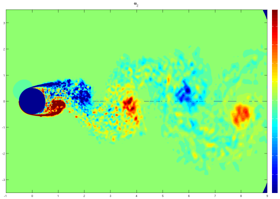
RANS vs LES vs DNS

Large eddy simulation of $Re = 10000$ cylinder, Mani et al, JFM (2009). This was a 50M cell 3D mesh, and took approximately 500 CPU hours for 10 shedding cycles.



(a) Vorticity contour

(b) Density gradient magnitude

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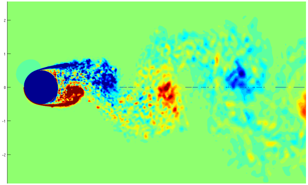
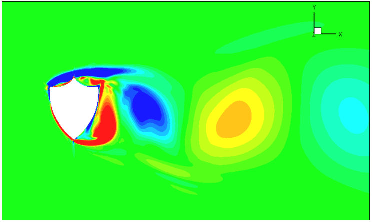


(c) LES, $Re = 10000$



(d) RANS, $Re = 2000$, approx. 1 CPU hour.

LES turbulence spectrum - periodic HIT box

LES is generally very successful in computing turbulent flows at a fraction of the expense of DNS, while capable of extending to much higher Re.

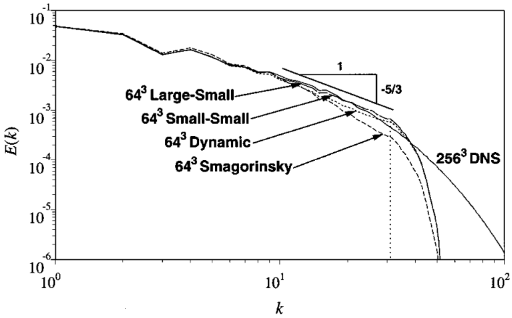
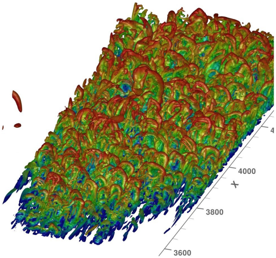


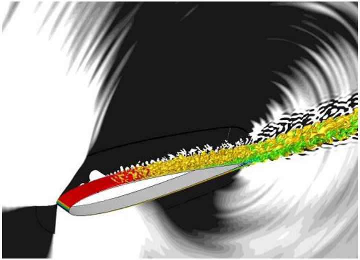
Figure: Turbulence spectrum in HIT, LES vs DNS (Hughes et al 2000).

LES vs DNS

- LES can't resolve fine details like the left figure..
- DNS requires an enormous amount of gridpoints $N \sim (100Re_L)^{9/4}$ (Tennekes/Lumley 1976).



(a) Hairpin vortices from DNS $Re_\theta \approx 1000$ (Moin et al 2010)



(b) LES of airfoil $Re = 408,000$, $\alpha = 5^\circ$ (Lele et al 2010)

- For the airfoil on the right, $(100 \times 408000)^{9/4} \times 8\text{vars} \times 8\text{bytes/var} \approx 7,000,000,000$ gigabytes.

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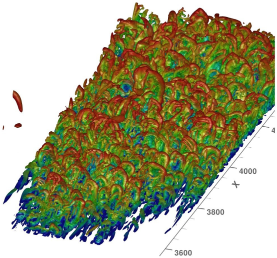
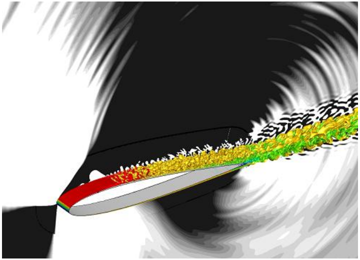


(c) Hairpin vortices from DNS $Re_\theta \approx 1000$ (Moin et al 2010)

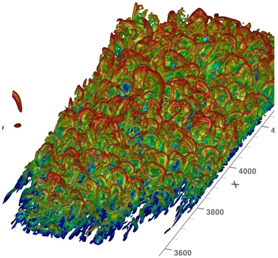


(d) LES of airfoil $Re = 408,000$, $\alpha = 5^\circ$ (Lele et al 2010)

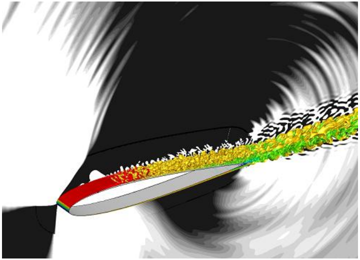
- For the airfoil on the right, $(100 \times 408000)^{9/4} \times$ 8vars $\times$ 8bytes/var $\approx 7,000,000,000$ gigabytes.

LES vs DNS

- LES can't resolve fine details like the left figure..
- DNS requires an enormous amount of gridpoints $N \sim (100Re_L)^{9/4}$ (Tennekes/Lumley 1976).



(e) Hairpin vortices from DNS $Re_\theta \approx 1000$ (Moin et al 2010)



(f) LES of airfoil $Re = 408,000$, $\alpha = 5^\circ$ (Lele et al 2010)

- For the airfoil on the right, $(100 \times 408000)^{9/4} \times 8$vars $\times$ 8bytes/var $\approx 7,000,000,000$ gigabytes.

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