There is no invariant, four-dimensional stuff

Hans Halvorson

February 9, 2020

Many people say that Einstein's special theory of relativity (STR) favors a four-dimensional ontology. Some philosophers have expressed this view via the following precise thesis:

> Four-dimensional stuff is invariant, whereas three-dimensional stuff is not.

For example, Yuri Balashov claims that "an object viewed as a 4d being is relativistically invariant in a sense in which its 3d parts are not" (1999, p 659).[1] Similarly, Thomas Sattig says that "there is a permanent shape standing behind the different three-dimensional shapes of the object, namely, an invariant four-dimensional shape, rendering the various three-dimensional shapes different perspectival representations of the single invariant shape" (2015, p 220). Finally, Thomas Hofweber and Marc Lange argue against Kit Fine's fragmentalist interpretation of STR on the basis that "the spacetime interval, as a frame-invariant fact, is the reality, whereas the facts related by the coordinate transformations are frame-dependent facts and hence are appearances of that reality" (2017, p 876).

In this note, I show that these claims are false. First I show the precise sense in which there are no invariant four-dimensional objects. Then I argue that there are no frame-invariant facts.

A four-dimensional object is represented by a region of Minkowski spacetime. Typically one thinks of a "spacetime worm" that represents the region swept out over time by a spatially extended object. The following result shows that no non-trivial subset of Minkowski spacetime is Lorentz invariant, *a fortiori* spacetime worms are not Lorentz invariant.

[1] Balashov's claim was contested by Davidson (2013), who argues that 4d objects themselves fail to be relativistically invariant. However, Balashov (2014) and Calosi (2015) argue that Davidson's conclusion and the reasoning behind it are in error. I show here that Davidson's conclusion is correct.

Proposition. *Let O be a region in Minkowski spacetime M that is invariant under all Lorentz transformations. Then O is either M or the empty set.*

This result follows from the fact that the Lorentz group acts transitively on Minkowski spacetime. In particular:

Lemma. *For any two points $p, q \in M$, there are Lorentz transformations L_1 and L_2 such that $L_1 p = L_2 q$.*

Sketch of proof. Suppose first that p and q are spacelike related. In this case, we let L_2 be the identity. Let $\overline{pq}$ be the spacelike line segment connecting p and q, and let m be its midpoint. Now let ℓ be a (timelike) line passing through m and orthogonal to $\overline{pq}$. If one follows ℓ backwards in time, eventually there is a point $s \in \ell$ that is timelike related to both p and q; and there is a Lorentz boost L_1 based at s such that $L_1 p = q$.

Now for the general case, for any two points $p, q \in M$, there is a point $r \in M$ that is spacelike related to both p and q. By the previous argument, there are Lorentz transformations L_1 and L_2 such that $L_1 p = r$ and $L_2 q = r$ □

Thus, there is nothing in Minkowski spacetime that could be called a physical object — whether three or four dimensional — and that is relativistically invariant.

We now turn our attention to facts. Hofweber and Lange claim that the spacetime interval is an invariant fact. But depending on how we disambiguate "spacetime interval," either it's not a fact, or it's not invariant. The Minkowski metric is a function η from pairs of points in Minkowski spacetime to real numbers. By "spacetime interval," Hofweber and Lange could either mean the function η, or they could mean a statement $\ulcorner \eta(p,q) = \lambda \urcorner$, where p, q are points of Minkowski spacetime, and λ is a real number. In the first case, η is invariant, but it cannot plausibly be conceived of as a fact. In the second case, $\ulcorner \eta(p,q) = \lambda \urcorner$ is a fact, but it is not invariant. In particular, applying a Lorentz transformation L to this fact yields $\ulcorner \eta(Lp, Lq) = \lambda \urcorner$, which is true (if the first is true), but is a different fact. The first fact is about the points p, q and the second fact is about the points Lp, Lq.

Let's try to be charitable. What Hofweber and Lange probably mean is that the spacetime distance between events is something that all observers can agree upon. But what does it mean to say that all observers can agree on this fact? We have already seen that it does *not* mean that the fact is invariant. What's more, a fact doesn't have to be invariant for all observers to agree upon it. For example, all observers can agree upon the spatial distance between two events, relative to a spacelike hypersurface Σ that contains both events. Perhaps, though, the former fact is somehow intrinsic to Minkowski spacetime, i.e. it's a relation that the events bear to each other merely in virtue of their being events in Minkowski spacetime. (One might be tempted to suggest that it's a "frame-independent fact".) But here is another relation that these events bear to each other merely in virtue of their being events in Minkowski spacetime: they are contained in the spacelike hypersurface Σ, and they have a certain spatial distance within this hypersurface.

In conclusion, it's simply not true that invariance features of Minkowski spacetime favor a four-dimensional ontology over a three-dimensional ontology.

Acknowledgment: Thanks to David Malament for the lemma and its proof.

References

Balashov, Y. (1999). Relativistic objects. *Noûs 33*(4), 644–662.

Balashov, Y. (2014). On the invariance and intrinsicality of four-dimensional shapes in Special Relativity. *Analysis 74*(4), 608–612.

Calosi, C. (2015). The relativistic invariance of 4d-shapes. *Studies in History and Philosophy of Modern Physics 50*, 1–4.

Davidson, M. (2013). Special relativity and the intrinsicality of shape. *Analysis 74*(1), 57–58.

Hofweber, T. and M. Lange (2017). Fine's fragmentalist interpretation of special relativity. *Noûs 51*(4), 871–883.

Sattig, T. (2015). *The Double Lives of Objects: An Essay in the Metaphysics of the Ordinary World.* Oxford University Press.