Practice Final Exam (Note: The actual final will be slightly shorter.)

Please make sure that you have all three pages of the exam. Write your name, preceptor's name, and pledge on the exam booklet. You have three hours to complete the exam.

Short answer

1. (2 pts.) State the Existential Elimination (EE) rule, along with each of its restrictions.
2. (3 pts.) True or False (explain your answer by appealing to any relevant definitions or metatheorems): If $A_1, \ldots, A_n$ are inconsistent predicate logic sentences, then there is a correctly written proof whose premises are $A_1, \ldots, A_n$ and whose conclusion is $P \wedge \neg P$.
3. (2 pts.) Complete the following sentence: Predicate logic sentences A and B are inconsistent just in case ... (Note: Please give the *semantic* definition that uses the concept of an "interpretation.")
4. (3 pts.) Grade the following proof. Locate any errors and explain what went wrong. If corrected, does the conclusion still follow by another line of reasoning?

1	(1)	$P \vee Q$	A
2	(2)	P	A
3	(3)	Q	A
2,3	(4)	$P \wedge Q$	2,3 $\wedge$I
2,3	(5)	P	4 $\wedge$E
1	(6)	P	1,2,2,3,5 $\vee$E

5. (3 pts.) Grade the following proof as above.

1	(1)	$\neg P$	A
2	(2)	$\exists x(Fx \wedge \neg Fx)$	A
2	(3)	$\neg\neg P$	1,2 RAA
2	(4)	P	3 DN
	(5)	$\exists x(Fx \wedge \neg Fx) \rightarrow P$	2,4 CP

6. (2 pts.) A "bad line" in a proof is a line where the sentence on the right is not a logical consequence of its dependencies. Identify all the bad lines in the previous two proofs.

Translation

(2 pts. each) Translate the following sentences into predicate logic notation. You may use the equals sign "=" as well as the following relation symbols:

$Mx \equiv x$ is male $Pxy \equiv x$ is a parent of y $Axy \equiv x$ adores y

(The domain of quantification is persons — you do not need a predicate symbol for "is a person." For the purposes of this problem, a "child" is anyone who has a parent.)

1. Every man who has a son adores him.
2. Every man who has a daughter adores his daughter's mother.
3. Everybody adores their own grandchildren.
4. Every woman adores her brothers' children.
5. No man adores children unless he has his own.
6. Someone has no more than two children.

Proofs and Counterexamples

1. (4 pts.) Prove the following tautology using only basic rules of inference.

$$\vdash \neg(P \to Q) \leftrightarrow (P \land \neg Q)$$

2. (4 pts.) Prove the validity of the following argument using only basic rules of inference.

$$\exists x(Fx \land \forall y(Gy \to Rxy)), \forall x(Fx \to \forall y(Hy \to \neg Rxy)) \vdash \forall x(Gx \to \neg Hx)$$

3. Consider the sentence "$\forall x \forall y[Qxy \leftrightarrow \forall z(Rzx \to Rzy)]$".
   (a) (4 pts.) Show by giving a proof that this sentence implies "$\forall x Qxx$". You are free to use substitution instances of any theorem from *propositional* logic.
   (b) (3 pts.) Give an interpretation that shows that the sentence does not imply "$\forall x \forall y(Qxy \to Qyx)$".
   (c) (7 pts.) The sentence implies one of (i) and (ii) but not the other; give a proof to show the implication in the one case, and give an interpretation to show the lack of implication in the other:
   (i) $\exists y \forall x Rxy \to \exists y \forall x Qxy$
   (ii) $\exists y \forall x Qxy \to \exists y \forall x Rxy$

4. (4 pts.) Determine whether or not the following argument is valid. If the argument is invalid, provide a counterexample interpretation. If the argument is valid, explain how you know that there is no counterexample.

$$\exists x Gx \lor \neg \forall x Fx \ , \ \neg \forall x \neg Fx \to \neg \forall x Fx \ \vdash \ \exists x Fx \to \exists x Gx$$

Metatheory

For the following problems, please give rigorous (but informal) arguments.

1. (4 pts.) Use proof by induction to show that the connective "$\lor$" is not by itself complete relative to truth functions with two inputs (i.e. there is a truth-function with two inputs

that cannot be expressed using only "$\vee$"). [Hint: Let Σ be the set of sentences defined inductively by: (1) P and Q are in Σ, and (2) if X and Y are in Σ, then $X \vee Y$ is in Σ. Show that every sentence in Σ has "T" on the first row of its truth table.]

2. (4 pts.) State precisely what it means to say that the predicate logic inference rules are "sound" and "complete." (i.e. state the soundness and completeness theorems for the predicate calculus.) Prove the soundness of Reductio ad Absurdum (RAA).

3. (4 pts.) Let's say that a "schminterpretation" of a predicate logic sentence is an interpretation whose domain of quantification has *at most two elements*; and let's say that a "schmautology" is a sentence that is true relative to all schminterpretations. Give an example of a schmautology that is not a tautology. Do not use the equality symbol "=".

Extra Credit

Recall that a *pure monadic sentence* is a sentence in the language of predicate logic containing single quantifier (either universal or existential) and only 1-place predicates (i.e. no relations).

1. Prove that there is no pure monadic sentence that is true relative to all and only those interpretations whose domains have exactly n elements. [Hint: Show that if a pure monadic sentence A is true relative to an interpretation of size n (i.e., an interpretation whose domain has n elements), and $n < m$, then A is true relative to an interpretation of size m.]
2. Prove that if a pure monadic sentence is consistent, then it is true relative to some interpretation whose domain has a *finite* number of elements. [Hint: You may assume that "Algorithm B" is a reliable test of consistency for pure monadic sentences.]
3. Prove that the set $\{\neg, \leftrightarrow\}$ of connectives is not complete relative to truth-functions of two variables.