Definition 1. Let $A$ and $B$ be sets. A function $f$ from $A$ to $B$ is a relation on $A \times B$ such that: for each $x \in A$, there is a unique $y \in B$ such that $\langle x, y \rangle \in f$. In this case, we can write $f(x)$ to denote that unique element of $B$ to which $x$ bears the relation. We sometimes use $f : A \to B$ to denote a function from $A$ to $B$.

Definition 2. $f$ is said to be one-to-one or injective just in case:

$$\forall x \forall y [(f(x) = f(y)) \implies x = y]$$

Definition 3. $f$ is said to be onto or surjective just in case:

$$(\forall y \in B)(\exists x \in A)[f(x) = y]$$

Definition 4. A sentential syntax consists of:

1. Vocabulary

   (a) A set $\text{At}$ of atomic sentences (we usually denote these by lower-case letters, starting with $p$);
   (b) For each $n = 1, 2, \ldots$, a set $C_n$ of $n$-ary connectives;
   (c) Punctuation marks “)” and “(”;
   (d) The preceding three sets are disjoint.

2. The set $\text{Snt}$ of sentences is defined inductively:

   (a) All elements of $\text{At}$ are in $\text{Snt}$;
   (b) If $\varphi$ is an $n$-ary connective and $a_1, \ldots, a_n \in \text{Snt}$, then $\varphi(a_1, \ldots, a_n) \in \text{Snt}$;
   (c) Nothing is in $\text{Snt}$ except things generated by the preceding two clauses.
We usually denote sentences by lowercase letters \(a, b, c, d\). So, note: \(a\) could denote the sentence \(p\) or the sentence \(p \land q\), etc.. But \(p\) is a sentence — it does not denote some other sentence.

So, a syntax is said to be a “triple” \(\langle \text{At}, C, \text{Snt} \rangle\), where \(C = \bigcup_n C_n\).

**Definition 5.** A *valuation* of a syntax \(\langle \text{At}, C, \text{Snt} \rangle\) is a function from \(\text{Snt}\) into some value set \(V\).

**Definition 6.** A *language* \(L\) consists of:

1. A syntax \(\langle \text{At}, C, \text{Snt} \rangle\);

2. A semantics, which includes:
   
   (a) A set \(\text{Val}(L)\) of valuations (called the *admissible valuations* of \(L\));
   
   (b) A relation “true” between admissible valuations and sentences (called *satisfaction* or *making true*).

**Definition 7.** A sentence \(a\) of \(L\) is *valid*, written \(\models a\), just in case:

\[
\forall \omega \in \text{Val}(L)[\omega \text{ satisfies } a]
\]

**Definition 8.** A sentence \(A\) of \(L\) is *satisfiable* just in case:

\[
\exists \omega \in \text{Val}(L)[\omega \text{ satisfies } a]
\]

**Definition 9.** \(a\) implies \(b\) in \(L\), written \(a \models b\), just in case:

\[
\forall \omega \in \text{Val}(L)[\omega \text{ satisfies } a \Rightarrow \omega \text{ satisfies } b]
\]

**Definition 10.** A subset \(X\) of sentences is *satisfiable* in \(L\) just in case:

\[
(\exists \omega \in \text{Val}(L))(\forall a \in X)[\omega \text{ satisfies } a]
\]

**Definition 11.** A subset \(X\) of sentences *implies* a sentence \(a\), written \(X \models a\), just in case:

\[
(\forall \omega \in \text{Val}(L))[\omega \text{ satisfies } X \Rightarrow \omega \text{ satisfies } a]
\]

**Definition 12.** A *logical system* for a language \(L\) is a relation \(\vdash\) of “derivability from” (note: in reverse order!) between elements of \(\text{Snt}\) and finite subsets of \(\text{Snt}\). So, \(a_1, \ldots, a_n \vdash b_1\) means: “there is a proof with \(a_1, \ldots, a_n\) as premises, and \(b\) as a conclusion.” This relation should, at the very least, be “recursively enumerable.”
Definition 13. A sentence $a$ is a theorem of the logical system $K$ iff $\vdash a$.

Definition 14. $\vdash$ is statement sound relative to $\models$:

$$\vdash a \implies \models a$$

Definition 15. $\vdash$ is statement complete relative to $\models$:

$$\vdash a \iff \models a$$

Definition 16. $\vdash$ is argument sound relative to $\models$:

$$X \vdash a \implies X \models a, \text{ for finite } X.$$

Definition 17. $\vdash$ is argument complete relative to $\models$:

$$X \vdash a \iff X \models a, \text{ for finite } X.$$

Definition 18. $\vdash$ is strongly complete relative to $\models$:

$$\models a \implies [(\exists b_1, \ldots, b_n \in X) \text{ s.t. } b_1, \ldots, b_n \vdash a]$$

Definition 19. Let $\omega : \text{Snt} \rightarrow V$ be a valuation. A base for $\omega$ consists of operators $\mathbb{C}$ on $V$ and a function $\odot : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\omega(\varphi(a_1, \ldots, a_n)) = \odot_\varphi(\omega(a_1), \ldots, \omega(a_n)),$$

for all connectives $\varphi \in \mathbb{C}$.

Definition 20. A language $\mathcal{L}$ is said to be compact if for any set $X$ of sentences: if all finite subsets of $X$ are satisfiable, then $X$ is satisfiable.