PHI 340 Handout

September 29, 2005

Lemma 1. *If $\mathcal{X} \subseteq \mathcal{Y}$ and ω is a valuation that satisfies $\mathcal{Y}$, then ω also satisfies $\mathcal{X}$.*

This is, um, obvious. But it's not an axiom; so if it's true, we should be able to prove it.

Proof. Suppose that $\mathcal{X} \subseteq \mathcal{Y}$, and that ω satisfies $\mathcal{Y}$. The latter statement means: ω satisfies every sentence in $\mathcal{Y}$. Let A be an arbitrary sentence in $\mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{Y}$, A is also in $\mathcal{Y}$. Thus, ω satisfies A. Since A was an arbitrary sentence of $\mathcal{X}$, ω satisfies every sentence in $\mathcal{X}$; i.e. ω satisfies $\mathcal{X}$. □

Recall that we have defined "$\mathcal{X} \models A$" to mean: any valuation that satisfies $\mathcal{X}$ (i.e. satisfies all sentences in $\mathcal{X}$) also satisfies A.

Lemma 2. *If $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \models A$ then $\mathcal{Y} \models A$.*

Sketch of proof. We're trying to prove a conditional; so assume the antecedent, and try to derive the consequent. The antecedent is the conjunction of the following two statements:

(A1) $\mathcal{X} \subseteq \mathcal{Y}$.

(A2) $\mathcal{X} \models A$.

The consequent is the following statement:

(C) $\mathcal{Y} \models A$.

If we plug in the definitions of all of the relevant terms, we have:

(A1) $\forall B(B \in \mathcal{X} \Rightarrow B \in \mathcal{Y})$.

(A2) $\forall \omega(\omega \text{ satisfies } \mathcal{X} \Rightarrow \omega \text{ satisfies } A)$.

(C) $\forall \omega(\omega \text{ satisfies } \mathcal{Y} \Rightarrow \omega \text{ satisfies } A)$.

The consequent (C) is a universal statement (about valuations). So, we fix an arbitrary valuation ω, and we try to prove the instantiated statement:

ω satisfies $\mathcal{Y} \Rightarrow \omega$ satisfies A.

So, again, assume the antecedent and try to prove the consequent. The antecedent means that ω satisfies every sentence in $\mathcal{Y}$. Since $\mathcal{X} \subseteq \mathcal{Y}$, ω satisfies every sentence in $\mathcal{X}$. Since $\mathcal{X} \models A$, ω satisfies A. ETC ... □

[Note: I have slipped into the practice — abjured in PHI 201 — of using the same symbol "ω" both for a variable and for an arbitrary name. But this abuse is justified by the need to avoid notation inflation.]

Definition. If $\mathcal{X}$ is a set of sentences, then $Cn(\mathcal{X}) = \{A : X \models A\}$.

Definition. Let $\mathcal{X}$ be a set of sentences. We say that $\mathcal{X}$ is a *deductive system* if $\mathcal{X} = Cn(\mathcal{X})$.

Proposition 1. *If $\mathcal{X} \subseteq \mathcal{Y}$ then $Cn(\mathcal{X}) \subseteq Cn(\mathcal{Y})$.*

Sketch of proof. Suppose that $\mathcal{X} \subseteq \mathcal{Y}$. Let A be an arbitrary sentence in $Cn(\mathcal{X})$. By definition, $A \in \{B : X \models B\}$, i.e. $X \models A$. By the previous lemma, $\mathcal{Y} \models A$. Thus, $A \in Cn(\mathcal{Y})$. Since A was an arbitrary element of $Cn(\mathcal{X})$, we have $Cn(\mathcal{X}) \subseteq Cn(\mathcal{Y})$. □