Lemmon Style Proofs for S5

The elimination rule for □, and the introduction rule for ◊ are obvious ("Stage 1 Rules").

**Definition.** The *scope* of an instance of a connective is the smallest WFF in which that instance of the connective occurs.

**Definition.** Let A be a sentence of modal propositional logic. We say that A is *fully modalized* just in case each atomic sentence in A is within the scope of some modal connective.

A line of a Lemmon style proof looks like this:

\[ \Delta_n \quad (n) \ A \quad \text{citation} \]

This is equivalent to the metalanguage sentence "\( \Delta_n \vdash A \)". The citation is just for the sake of the reader of the proof, so s/he understands why you believe that \( \Delta_n \vdash A \).

**□ Introduction**

Suppose that A occurs on a line n with dependency set \( \Delta_n \). If each sentence in \( \Delta_n \) is fully modalized, then we can write □A on a subsequent line with dependency set \( \Delta_n \). We cite "\( n \ □ I \)".

**◊ Elimination**

Suppose that:
1. ♦A occurs on line $i$ with dependency set $\Delta_i$;

2. A occurs on line $j$ with dependency set $\Delta_j = \{j\}$, i.e. A is an assumption on line $j$;

3. $B$ occurs on line $k$ with dependency set $\Delta_k$, and
   
   (a) $B$ is fully modalized;
   
   (b) each sentence in $\Delta_k - \{j\}$ is fully modalized.

Then we can write $B$ on a subsequent line with dependency set $\Delta_i \cup (\Delta_k - \{j\})$. We cite “$i, j, k$ ♦E”.