PHI 340 Sample Solutions

September 24, 2005

1. *Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

   [Note: I will write the proof twice over. The second proof is much more detailed than necessary; but it makes the steps more explicit.]

   (a) Let $a \in A$. Since $A \subseteq B$, $a \in B$. Since $B \subseteq C$, $a \in C$. Thus, every element in A is also in C. That is, $A \subseteq C$.

   (b) Suppose that $A \subseteq B$ and $B \subseteq C$. That is, $\forall x[x \in A \Rightarrow x \in B]$ and $\forall x[x \in B \Rightarrow x \in C]$. We want to show that $A \subseteq C$, that is $\forall x[x \in A \Rightarrow x \in C]$. So, let a be an arbitrary object such that $a \in A$. Then by universal instantiation, $a \in A \Rightarrow a \in B$; and by modus ponens, $a \in B$. By universal instantiation again, $a \in B \Rightarrow a \in C$; and by modus ponens, $a \in C$. By conditional proof, $a \in A \Rightarrow a \in C$. But since a was an arbitrary object, we can apply universal generalization to conclude that $\forall x[x \in A \Rightarrow x \in C]$. That is, $A \subseteq C$.

2. *Show that:*

   $$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

   Proof. Let x be an arbitrary element of $(A \times B) \cup (C \times D)$. Thus, either $x \in A \times B$ or $x \in C \times D$. In the first case, $x = \langle a, b\rangle$ for some $a \in A$ and $b \in B$. But then $a \in A \cup C$ and $b \in B \cup D$. Therefore,

   $$\begin{aligned} x &= \langle a, b\rangle \\ &\in \{\langle y, z\rangle : y \in A \cup C, z \in B \cup D\} \\ &= (A \cup C) \times (B \cup D). \end{aligned}$$

   In the second case, $x = \langle c, d\rangle$ for some $c \in C$ and $d \in D$. But then $c \in A \cup C$ and $d \in B \cup D$. Therefore $x = \langle c, d\rangle \in (A \cup C) \times (B \cup D)$. In either case, $x \in (A \cup C) \times (B \cup D)$. Thus (by disjunction elimination)

$x \in (A \cup C) \times (B \cup D)$. Since x was an arbitrary element of $(A \times B) \cup (C \times D)$, we have $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. □

[Note: the lack of symmetry between my arguments the two cases $x \in A \times B$ and $x \in C \times D$ was not necessary. In the first case, I invoked the definition of the Cartesian product. In the second case, I tacitly used the definition of the Cartesian product without explicitly invoking it.]

3. *Show that $A \subseteq B$ iff $A \cap B = A$.*

Proof. We have to prove two conditionals.

("$\Rightarrow$") Suppose first that $A \subseteq B$. (Show that $A \cap B = A$.) Now clearly $A \cap B \subseteq A$. To show that $A \subseteq A \cap B$, let x be an arbitrary element of A. But we assumed that $A \subseteq B$, so $x \in B$. Thus, $x \in A \cap B$, and since x was an arbitrary element of A, $A \subseteq A \cap B$.

("$\Leftarrow$") Conversely, suppose that $A \cap B = A$. (Show that $A \subseteq B$.) Let x be an arbitrary element of A. Since $A \cap B = A$, $x \in A$ implies that $x \in A \cap B$. Therefore $x \in B$, and since x was an arbitrary element of A, $A \subseteq B$. □

4. *Show that $\bigcup \mathcal{P}(A) = A$.*

Proof. By the definition of $\bigcup$ and of $\mathcal{P}$, we have:

$$\begin{aligned} \bigcup \mathcal{P}(A) &= \{x : \exists S \in \mathcal{P}(A) \text{ such that } x \in S\} && (1) \\ &= \{x : \exists S \text{ such that } S \subseteq A, \text{ and } x \in S\}. && (2) \end{aligned}$$

We have to prove two inclusions: $\bigcup \mathcal{P}(A) \subseteq A$ and $A \subseteq \bigcup \mathcal{P}(A)$.

("$\subseteq$") Suppose that $a \in \bigcup \mathcal{P}(A)$. Then by Equation 2, there is a subset S of A such that $a \in S$. Thus $a \in A$, and since a was an arbitrary element of $\bigcup \mathcal{P}(A)$, $\bigcup \mathcal{P}(A) \subseteq A$.

("$\supseteq$") Suppose that $a \in A$. But $A \in \mathcal{P}(A)$, and so $\exists S \in \mathcal{P}(A)$ such that $a \in S$. In other words,

$$a \in \{x : \exists S \in \mathcal{P}(A) \text{ such that } x \in S\}.$$

Applying Equation 1, we conclude that $a \in \bigcup \mathcal{P}(A)$. Since a was an arbitrary element of A, $A \subseteq \bigcup \mathcal{P}(A)$. □