VOLUME LXI, No. 2 JANUARY 16, 1964

THE JOURNAL OF PHILOSOPHY

IMPLICIT DEFINITION SUSTAINED *

THE characterization of axioms as implicit definitions can be found as far back as 1818, in Gergonne,[1] and it was still vigorous thirty years ago. What is exasperating about the doctrine is its facility, or cheapness, as a way of endowing statements with the security of analytic truths without ever having to show that they follow from definitions properly so called, definitions with eliminable definienda.

Russell gave the doctrine its due, I felt, though he did not mention it by name, when he wrote in 1919 that "the method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil." [2]

I am shocked now to find that the view of axioms as implicit definitions can be defended, and with a literalness that its own proponents can scarcely have thought possible. To discharge this somber duty is the purpose of the present note.

Briefly, the point is that there is a mechanical routine whereby, given an assortment of interpreted undefined predicates 'F_1', 'F_2', ..., 'F_n' governed by a true axiom or a finite list of such, we can switch to a new and equally economical set of undefined predicates and define 'F_1', ..., 'F_n' in terms of them, plus auxiliary arithmetical notations, in such a way that the old axioms become true by arithmetic. The predicates 'F_1', ..., 'F_n' do not become predicates of numbers, but continue under the definitions to be true of precisely the things that they were true of under their original interpretations.

This will not surprise readers who encountered the central idea in a 1940 paper by Goodman and me.[3] The link with the doctrine of implicit definition is an added thought, but the technical point

* For helpful remarks on a first draft I am indebted to Burton Dreben and Dagfinn Follesdal.

[1] José Diez Gergonne, "Essai sur la théorie des définitions," *Annales de mathématique pure et appliquée*, **9** (1818–19): 1–35, especially p. 23.

[2] Bertrand Russell, *Introduction to Mathematical Philosophy*, New York and London, 1919, p. 71.

[3] W. V. Quine and Nelson Goodman, "Elimination of Extra-logical Postulates," *Journal of Symbolic Logic*, **5** (1940), pp. 104–109.

itself, as formulated in the foregoing paragraph, merely improves on our 1940 result in these three ways: it is geared to predicates instead of singular terms, thus conforming to a more modern theory of theories; it draws specifically on arithmetic, in fact elementary number theory, rather than on set theory generally; and it assures a mechanical routine for finding the definitions.

This last improvement depends on a strengthened form of Löwenheim's theorem given by Hilbert and Bernays in 1939. Löwenheim's theorem dates from 1915 and says that every satisfiable schema that can be written in the notation of the logic of quantification can be satisfied by an interpretation in the universe of natural numbers. The strengthened version in Hilbert and Bernays specifies the interpretation in arithmetical notation. Hilbert and Bernays show how, given any schema in the notation of the logic of quantification, to find arithmetical predicates (better: open sentences of elementary number theory) which, when adopted as interpretations of the predicate letters of the schema, will make the schema come out true if it was satisfiable.[4]

In the remaining pages we shall see how, granted the ability thus conferred by Hilbert and Bernays, we can convert axioms to definitions as promised above. Imagine an interpreted deductive theory θ that presupposes elementary logic and treats of some extra-logical subject matter, say chemistry. Suppose it set forth in the standard way using primitive predicates 'F_1', ..., 'F_n', truth functions, quantifiers, and general variables. The exclusion of singular terms, function signs, and multiple sorts of variables is no real restriction, for these accessories are reducible to the narrower basis in familiar ways.

By a slight and innocuous reinterpretation, the range of values of the variables of a theory can be extended to take in any desired supplementary objects. We just pick one of the values originally available, say *a*, and then extend the original interpretation of each predicate by counting true of the supplementary objects whatever was true of *a*, and false of them what was false of *a*. The new objects thus enter undetectably, indiscriminable from *a*. This maneuver, which I shall call *hidden inflation,* is not new.[5]

In particular, then, let us understand the variables of θ as ranging not just over physical objects or other special objects of

[4] David Hilbert and Paul Bernays, *Grundlagen der Mathematik,* vol. 2, Berlin, 1939, p. 253. For exposition and additional references see my "Interpretations of Sets of Conditions," *Journal of Symbolic Logic,* **19** (1954), pp. 97–102, especially pp. 101 f.

[5] See, e.g., David Hilbert and Wilhelm Ackermann, *Grundzüge der theoretischen Logik,* Berlin, 1938, p. 92.

chemistry, but over the natural numbers too—all these things being pooled in a single universe of discourse. For the natural numbers, if they were not there, could always be incorporated by hidden inflation.

Let the axioms of θ be finite in number, and true. Think of '$A(F_1, \ldots, F_n)$' as abbreviating the conjunction of them all. This is a schema of the logic of quantification if we forget the chemical interpretations of 'F_1', ..., 'F_n'; and it is a satisfiable one, since under the chemical interpretations it was true. So by Hilbert and Bernays's method we can find predicates in elementary number theory, abbreviated say as 'K_1', ..., 'K_n', such that $A(K_1, \ldots, K_n)$. Nor must the quantified variables thereupon be narrowed in range to the natural numbers; by hidden inflation we can still let them range over the whole universe of θ, numerical and otherwise.

Adopt, next, a new interpreted theory, having again the same inclusive universe of discourse as θ. Give it primitive predicates sufficient for elementary number theory, and in addition give it the primitive predicates 'G_1', ..., 'G_n', subject to the same chemical interpretations that 'F_1', ..., 'F_n' enjoyed in θ. But give the new theory no axioms involving 'G_1', ..., 'G_n'. Now in the new theory let us introduce 'F_1', ..., 'F_n' as defined predicates, as follows. For each i, explain '$F_i(x_1, \ldots, x_j)$' (with the appropriate number j of places) as short for:

$$A(G_1, \ldots, G_n) \cdot G_i(x_1, \ldots, x_j) \cdot\!\vee\!\cdot \sim A(G_1, \ldots, G_n) \cdot K_i(x_1, \ldots, x_j)$$

It will be recalled that the axioms of θ are chemically true. Hence also, as a chemical matter of unaxiomatized fact, $A(G_1, \ldots, G_n)$. Therefore the above definition makes 'F_i' in fact coextensive with 'G_i'. Therefore it agrees with the chemical interpretation of 'F_i' in θ, for each i. Yet, under the above definition, '$A(F_1, \ldots, F_n)$' is logically deducible from just the arithmetical truth '$A(K_1, \ldots, K_n)$'. (Proof: 'F_1', ..., 'F_n' are equated by the definition to 'K_1', ..., 'K_n' unless $A(G_1, \ldots, G_n)$, and in this event they are equated to 'G_1', ..., 'G_n', so that again $A(F_1, \ldots, F_n)$.)

The shift of system was of course farcical. We merely rewrote the primitive predicates of θ as new letters, keeping the old chemical interpretations, and then pleonastically defined the old predicate letters anew in terms of these so that their chemical interpretations were again preserved (extensionally anyway). Yet the erstwhile chemical axioms of θ became, under this definitional hocus pocus, arithmetically true.

I do not speak of arithmetical demonstrability, for a question there arises of choosing among incomplete systems of number theory. I speak of arithmetical truth.

The doctrine that axioms are implicit definitions thus gains support. If axioms are satisfiable at all, they can be viewed as a shorthand instruction to adopt definitions as above, rendering one's theory true by arithmetic. And, if the axioms were true on a literal reading, the interpretation of their predicates remains undisturbed.

The doctrine of implicit definition has been deplored as a too facile way of making any desired truth analytic: just call it an axiom. Now we see that such claims to analyticity are every bit as firm as can be made for sentences whose truth follows by definition from arithmetic. So much the worse, surely, for the notion of analyticity.

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COMMENTS AND CRITICISM

AYER ON NEGATION

In his article, "Negation,"* A. J. Ayer makes the difference between affirmation and negation turn on the matter of how specific a term is. "Among complementary pairs of singular referential statements it may happen that one member of a pair has a higher . . . degree of specificity than the other. In that case the more specific statement may be said to be affirmative and the less specific to be negative" (813, 814). The statement that an object is blue is apparently more specific than the statement that it is not blue. However, there are several objections to this. Ayer himself and his reviewers[1] mention two anomalies. First, for some contradictory pairs of statements neither seems to be more specific. The statements that Mt. Everest is the highest mountain in the world and that it is not seem to Ayer to be equally specific. Second, some apparent negatives come out as affirmatives on this basis. Ayer's example here is "the statement that an object is colored will have to count as negative: for it is less specific than the statement that the object is colorless" (814).

A *third* objection might be raised, and it seems to eliminate Professor Ayer's proposal. The decision on specificity always presupposes a definite, perhaps even a finite or denumerable, uni-

* In this JOURNAL, **49**, 26 (Dec. 18, 1952): 797–815.

[1] Baylis in the *Journal of Symbolic Logic*, **20**, 1 (1955): 58–59. A number of people have commented on Ayer's proposal. For the purposes of this note I will not refer to this literature.