Balance Sheet Capacity and Endogenous Risk

Jon Danielsson
London School of Economics

Hyun Song Shin
Princeton University

Jean-Pierre Zigrand
London School of Economics

CFTC Conference on Commodity Markets
Washington DC, August 25-26, 2011
Corporate Finance of Banking

<table>
<thead>
<tr>
<th>A</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
<td>Equity</td>
</tr>
<tr>
<td></td>
<td>Debt</td>
</tr>
</tbody>
</table>
Asset growth

Leverage growth

Slope = 1

0

A

B


Asset growth

Leverage growth

Increasing equity

Decreasing equity

Slope = 1

Leverage growth
Asset growth

Leverage growth

Slope = 1

Constant equity growth of \( g \)

0

Leverage growth

Constant equity line
Leverage and Total Assets Growth
Asset weighted, 1992Q3-2008Q1, Source: SEC
Balance Sheet Capacity

• Balance sheet capacity is capacity of banking (intermediary) sector to channel credit

• Balance sheet capacity is

   \[ \text{Equity} \times \text{“Permitted” Leverage} \]

• Our approach: intermediaries are subject to Value-at-Risk constraint:

   \[ \text{Equity} = \text{Total Assets} \times \text{VaR per dollar of assets} \]

   \[ \text{Leverage} = \frac{\text{Assets}}{\text{Equity}} = \frac{1}{\text{Unit VaR}} \]
Endogeneity of Volatility

- Volatility attributable (in part) to actions of economic agents
- Equilibrium volatility obtained as fixed point of mapping:

  \[ \text{perceived } \sigma \Rightarrow \text{true } \sigma \]
Volatility, Risk Premium and Balance Sheet Capacity

Solve simultaneously for volatility, risk premium and balance sheet capacity in closed form.
Preview of Closed Form Solution

Define

\[ \theta = \frac{\text{Size of long-only sector}}{\text{Banking sector equity}} \]

Equilibrium volatility \( \sigma \) is

\[ \sigma = \text{fundamental volatility} \times \theta \exp \{-\theta\} \times F(\theta) \]

where \( F(\theta) \) ensures

\[ \sigma \to \text{fundamental volatility} \quad \text{as} \quad \theta \to \infty \]
Volatility and Risk Premium
Excess Volatility or “Artificial” Compression of Risk
Model

- Time indexed by $t \in [0, \infty)$.

- $N > 0$ non-dividend paying risky securities (date $t$ price of $i$th risky security is $P_t^i$)

- One risk-free bond ($B_0 = 1, dB_t = r B_t dt$, with $r$ constant)

- Two types of traders
  - Active (risk-neutral, with VaR constraints) - banks
  - Passive (residual demand/supply curves) - households, value investors
Posit equilibrium of form:

\[
\begin{bmatrix}
\frac{dP_t^1}{P_t^1} \\
\vdots \\
\frac{dP_t^N}{P_t^N}
\end{bmatrix} = 
\begin{bmatrix}
\mu_t^1 \\
\vdots \\
\mu_t^N
\end{bmatrix} dt + 
\begin{bmatrix}
- \sigma_t^1 & - \\
- \sigma_t^N & -
\end{bmatrix} 
\begin{bmatrix}
dW_t^1 \\
\vdots \\
dW_t^N
\end{bmatrix}
\]

\{W_t^i\} independent Brownian motions (fundamental shocks enter via passive traders’ demands)

Scalars \{\mu_t^i\} and 1 \times N vectors \{\sigma_t^i\} are as yet undetermined coefficients to be solved in equilibrium

Solve for rational expectations equilibrium (REE) with respect to active traders’ beliefs
Portfolio Choice of Active Traders

- Risk-neutral traders
- (Ultra) short horizons
- Value-at-Risk (VaR) constraint: capital (i.e. equity) should be large enough to meet VaR

\( D_t^i \) is dollar holding of \( i \)th security at \( t \)

\( V_t \) is trader’s capital (notice no superscript for trader, due to aggregation result, to follow)

Balance sheet identity

\[ b_t B_t = V_t - \sum_i D_t^i \]
Evolution of capital

\[ dV_t = \left[ rV_t + D_t^\top (\mu_t - r) \right] dt + D_t^\top \sigma_t dW_t \]

\( D^\top \) is transpose of \( D \), \( \sigma_t \) is the \( N \times N \) diffusion matrix, \( r = (r, \ldots, r)^\top \).

Expected capital gain:

\[ E_t[dV_t] = \left[ rV_t + D_t^\top (\mu_t - r) \right] dt \tag{1} \]

Variance of capital:

\[ \text{Var}_t(dV_t) = D_t^\top \sigma_t \sigma_t^\top D_t dt \tag{2} \]

Trader maximizes (1) subject to VaR constraint, where VaR is \( \alpha \) times forward-looking standard deviation of return on equity.
Assuming trader is solvent \((V_t > 0)\) maximization problem is

\[
\max_{D_t} rV_t + D_t^\top (\mu_t - r) \quad \text{subject to} \quad \alpha \sqrt{D_t^\top \Sigma_t D_t} \leq V_t
\]

First-order condition

\[
\mu_t - r = \alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t \Sigma_t D_t
\]

where \(\gamma_t\) is Lagrange multiplier for VaR constraint, and \(\Sigma_t := \sigma_t \sigma_t^\top\).

\[
D_t = \frac{1}{\alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t} \Sigma_t^{-1} (\mu_t - r)
\]
Constraint binds due to risk-neutrality

\[ V_t = \alpha \sqrt{D_t^T \Sigma_t D_t} \]  \hspace{1cm} (3)

Therefore

\[ D_t = \frac{V_t}{\alpha^2 \gamma_t} \Sigma_t^{-1} (\mu_t - r) \]

“As if” preferences. Optimal portfolio is similar to mean-variance optimal portfolio where the Lagrange multiplier \( \gamma_t \) appears like a risk-aversion coefficient.

Substitute into (3) to solve for Lagrange multiplier

\[ \gamma_t = \frac{\sqrt{\xi_t}}{\alpha} \]
where

\[ \xi_t := (\mu_t - r)^\top \Sigma_t^{-1} (\mu_t - r) \]

Lagrange multiplier \( \gamma_t \) is

- proportional to generalized Sharpe ratio \( \sqrt{\xi} \)
- does not depend directly on equity \( V_t \)

**Interpretation.** Additional unit of capital relaxes VaR constraint by multiple \( \alpha \) of standard deviation, raising expected return by risk-premium on the portfolio per unit of standard deviation
Finally, solve for optimal portfolio:

\[ D_t = \frac{V_t}{\alpha \sqrt{\xi_t}} \Sigma_t^{-1}(\mu_t - r) \]

Optimal portfolio is homogeneous of degree one in equity \( V_t \)

**Aggregation result.** Portfolio depends on \( V_t \), total capital of active trading (banking?) sector, not on profile of individual equity capital.

⇒ Take aggregate capital, \( V_t \), as state variable
Closing the Model

Passive traders in aggregate have vector-valued exogenous demand schedule for the risky assets, $y_t = (y_t^1, \ldots, y_t^N)$ where

$$y_t = \sum_t^{-1} \left[ \begin{array}{c} \delta^1 (z_t^1 - \ln P_t^1) \\ \vdots \\ \delta^N (z_t^N - \ln P_t^N) \end{array} \right]$$

$\delta^i$ is scaling parameter for slope of the demand curve

$z_t^i$ is positive demand shock for asset $i$

$$dz_t^i = r^* dt + \eta \sigma_t^i dW_t$$
The market-clearing condition \( D_t + y_t = 0 \) gives

\[
\frac{V_t}{\alpha \sqrt{\xi_t}}(\mu_t - r) + \left[ \begin{array}{c} \delta^1 (z^1_t - \ln P^1_t) \\ \vdots \\ \delta^N (z^N_t - \ln P^N_t) \end{array} \right] = 0
\]

Equilibrium prices are

\[
P_t^i = \exp \left( \frac{V_t}{\alpha \delta^i \sqrt{\xi_t}}(\mu_t^i - r) + z_t^i \right); \quad i = 1, \ldots, N
\]
Single Risky Asset

Look for equilibrium of form:

\[
\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t \tag{4}
\]

\(\mu_t\) and \(\sigma_t\) are undetermined coefficients to be solved in equilibrium, \(W_t\) is standard (scalar) Brownian motion. Equivalently,

\[
d\ln P_t = \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t \tag{5}
\]

“Seeds” of fundamental shocks given by exogenous shocks to passive traders’ demands:

\[
dz_t = r^* dt + \eta \sigma_z dW_t, \text{ for known constants } \eta > 0, \sigma_z > 0
\]
Start with market-clearing price with VaR-constrained traders

\[ P_t = \exp \left( z_t + \frac{\sigma_t V_t}{\alpha \delta} \right) \]

Take logs and apply Ito’s Lemma

\[
d \ln P_t = d \left( z_t + \frac{\sigma_t V_t}{\alpha \delta} \right) = r^* dt + \eta \sigma_z dW_t + \frac{1}{\alpha \delta} (\sigma_t dV_t + V_t d\sigma_t + dV_t d\sigma_t) \tag{6}
\]

- Unpack \( dV_t \) and \( d\sigma_t \), and substitute back into (6)
- Compare coefficients with (5)
Step 1. Unpack $dV_t$ as an Ito process

\[
dV_t = \left[rV_t + D_t(\mu_t - r)\right]dt + D_t\sigma_t dW_t
\]

\[
= \left[rV_t + \frac{V_t(\mu_t - r)}{\alpha\sigma_t}\right]dt + \frac{V_t}{\alpha}dW_t
\]

Key step is the simplification allowed by binding VaR constraint:

\[
\alpha\sigma_tD_t = V_t
\]
Step 2. Unpack $d\sigma_t$ as an Ito process

From Itô’s Lemma on $\sigma(V_t)$,

$$d\sigma_t = \frac{\partial \sigma}{\partial V_t} dV_t + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} (dV_t)^2$$

$$= \left\{ \frac{\partial \sigma}{\partial V_t} \left[ rV_t + \frac{V_t(\mu_t - r)}{\alpha \sigma_t} \right] + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} \left( \frac{V_t}{\alpha} \right)^2 \right\} dt + \frac{\partial \sigma}{\partial V_t} \frac{V_t}{\alpha} dW_t \quad (7)$$

where we substitute in $dV_t$ and where $(dV_t)^2 = \left( \frac{V_t}{\alpha} \right)^2 dt$
Substitute everything back into (6) and re-arrange:

\[ d \ln P_t = (\text{drift term}) \, dt + \left[ \eta \sigma_z + \frac{1}{\alpha \delta} \left( \sigma_t \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right) \right] dW_t \quad (*) \]

By hypothesis,

\[ d \ln P_t = \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) \, dt + \sigma_t dW_t \quad (**) \]

Comparing coefficients between (*) and (**),

\[ \sigma(V_t) = \eta \sigma_z + \frac{1}{\alpha \delta} \left( \sigma_t \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right) \]
giving ordinary differential equation:

$$V_t^2 \frac{\partial \sigma}{\partial V_t} = \alpha^2 \delta (\sigma_t - \eta \sigma_z) - V_t \sigma_t$$

Generic solution:

$$\sigma(V_t) = \frac{1}{V_t} e^{-\frac{\alpha^2 \delta}{V_t}} \left[ c - \alpha^2 \delta \eta \sigma_z \int_{-\frac{\alpha^2 \delta}{V_t}}^{\infty} \frac{e^{-u}}{u} du \right]$$

where $c$ is a constant of integration, and $\text{Ei}(z)$ function ("exponential integral" function) $- \int_{-z}^{\infty} \frac{e^{-u}}{u} du$ is defined provided $z \neq 0$. 
Constant of integration $c$ can be tied down as follows.

Consider limit case $\delta \to 0$ where “fundamental shocks” are shut down

As $\delta \to 0$

$$\sigma (V_t) \to \frac{c}{V_t}$$

**Restriction.** In the absence of fundamental shocks, volatility is zero

This implies $c = 0$ so that

$$\sigma (V_t) = \eta \sigma z \frac{\alpha^2 \delta}{V_t} \exp \left\{ -\frac{\alpha^2 \delta}{V_t} \right\} \times \text{Ei} \left( \frac{\alpha^2 \delta}{V_t} \right)$$
Higher risk premium
higher risk
greater "as if" risk aversion
Risk and Return

![Graph showing risk and return relationship with equity on the horizontal axis and risk and return on the vertical axis. The graph includes a red line labeled \( \sigma \) for volatility and a blue line labeled \( \mu \) for mean return. There is text noting a loss of balance sheet capacity and equity down with VaR up.](image)
Risk and Return

Equity

σ
μ

Loss of balance sheet capacity equity down VaR up

Procylical leverage
Leverage under P

Asset Growth

Leverage growth
Risk and Return

Expected return to dollar of bank capital is sharply higher when risk premium goes up (risk premium compensates for risk aversion).
Risk and Return

Shadow value of bank capital turns up as leverage increases.
Equilibrium drift $\mu_t$

$$
\mu_t = r + \frac{\sigma_t}{2\alpha\eta\sigma_z} \left\{ 2\alpha (r^* - r) + \alpha\sigma^2_t - \eta\sigma_z + (\sigma_t - \eta\sigma_z) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\}
$$

Sharpe ratio:

$$
\frac{\mu_t - r}{\sigma_t} = \frac{1}{2\alpha\eta\sigma_z} \left\{ 2\alpha (r^* - r) + \alpha\sigma^2_t - \eta\sigma_z + (\sigma_t - \eta\sigma_z) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\}
$$
Many Risky Assets

Special case of $N$ risky securities case can be solved using ODE solution from the single risky asset case.

Assumption (Symmetry)

1. Diffusion matrix for $z$ is $\tilde{\sigma}_z I_N$ where $\tilde{\sigma}_z > 0$ is a scalar and $I_N$ is the $n \times n$ identity matrix.

2. $\delta^i = \delta$ for all $i$.

Let $\sigma^{ij}_t$ be coefficient giving effect of change in demand shock of $j$th security on price of $i$th security.
From assumption of symmetry, we only need to solve for one diffusion variable, $\sigma_{ti}^{ii} = \sigma_{t}^{11}$, since for $i \neq j$ the cross effects are tied down by $\sigma_{t}^{ij} = \sigma_{t}^{12} = \sigma_{t}^{11} - \eta \tilde{\sigma}_z$.

Define $x_t \equiv x(V_t)$ the solution to the ODE for single risky asset with $\eta$ replaced by $\frac{\eta}{N}$.

**Proposition 1.** Assume (S). The following is a REE.

The REE diffusion coefficients are $\sigma_{t}^{ii} = x_t + \frac{N-1}{N} \eta \tilde{\sigma}_z$, and for $i \neq j$, $\sigma_{t}^{ij} = x_t - \frac{1}{N} \eta \tilde{\sigma}_z$. Also, $\Sigma_{t}^{ii} = \text{Var}_t(\text{return on security } i) = \eta^2 \tilde{\sigma}_z^2 + \frac{1}{N} \left( N^2 x_t^2 - \eta^2 \tilde{\sigma}_z^2 \right)$, and for $i \neq j$, $\Sigma_{t}^{ij} = \text{Cov}_t(\text{return on security } i, \text{return on security } j) = \frac{1}{N} \left( N^2 x_t^2 - \eta^2 \tilde{\sigma}_z^2 \right)$ and $\text{Corr}_t(\text{return on security } i, \text{return on security } j) = \frac{N x_t^2 - (N-1) \eta^2 \tilde{\sigma}_z^2}{N x_t^2 + (N-1) \eta^2 \tilde{\sigma}_z^2}$.

Risky holdings are $D_t^i = \frac{V_t}{\alpha N^{3/2} x_t}$.
The risk-reward relationship is given by

\[
\frac{\mu_t^i - r}{x_t} = \frac{1}{2\alpha \frac{\eta}{N} \tilde{\sigma}_z} \left\{ \alpha \left( x_t + \frac{N - 1}{N} \eta \tilde{\sigma}_z \right)^2 - \frac{\eta}{\sqrt{N}} \tilde{\sigma}_z + \sqrt{N} \left( x_t - \frac{\eta}{N} \tilde{\sigma}_z \right) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\}
\]

(8)

The intuition and form of the drift term is very similar to the \( N = 1 \) case and reduces to it if \( N \) is set equal to 1.