

Balance Sheet Capacity and Endogenous Risk

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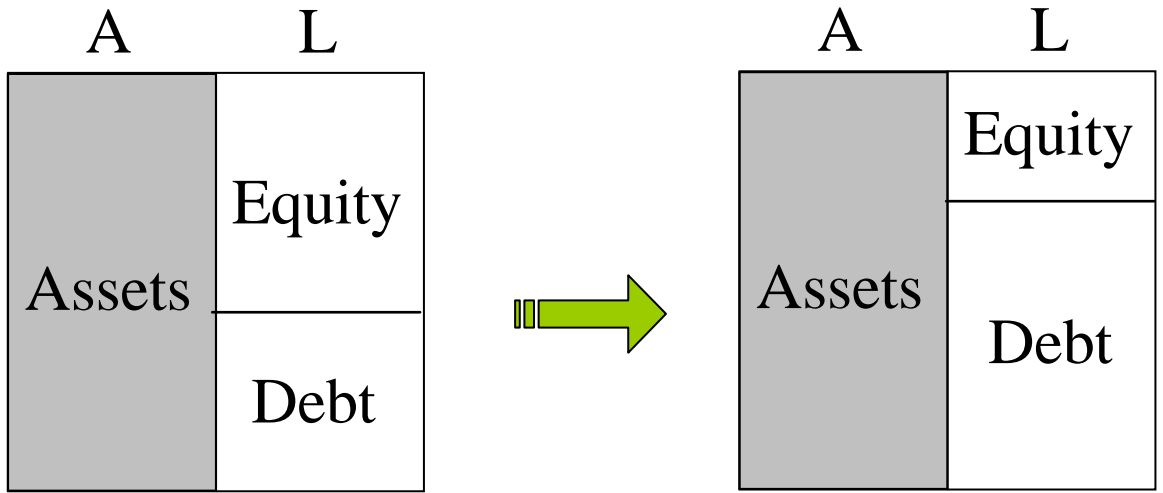
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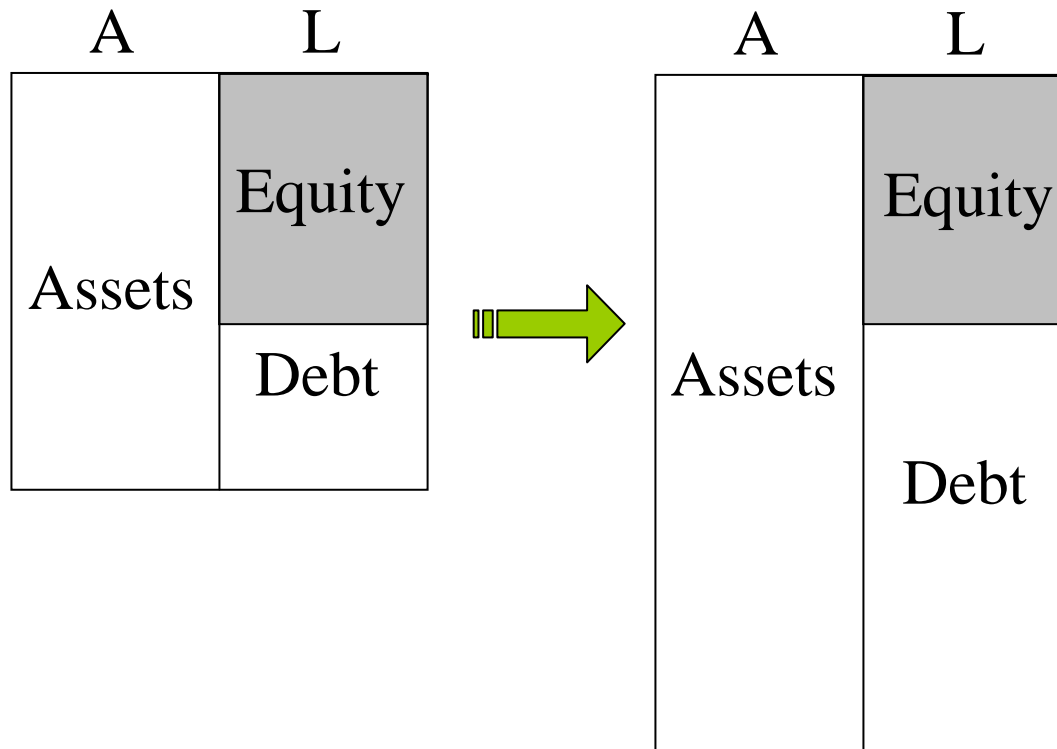
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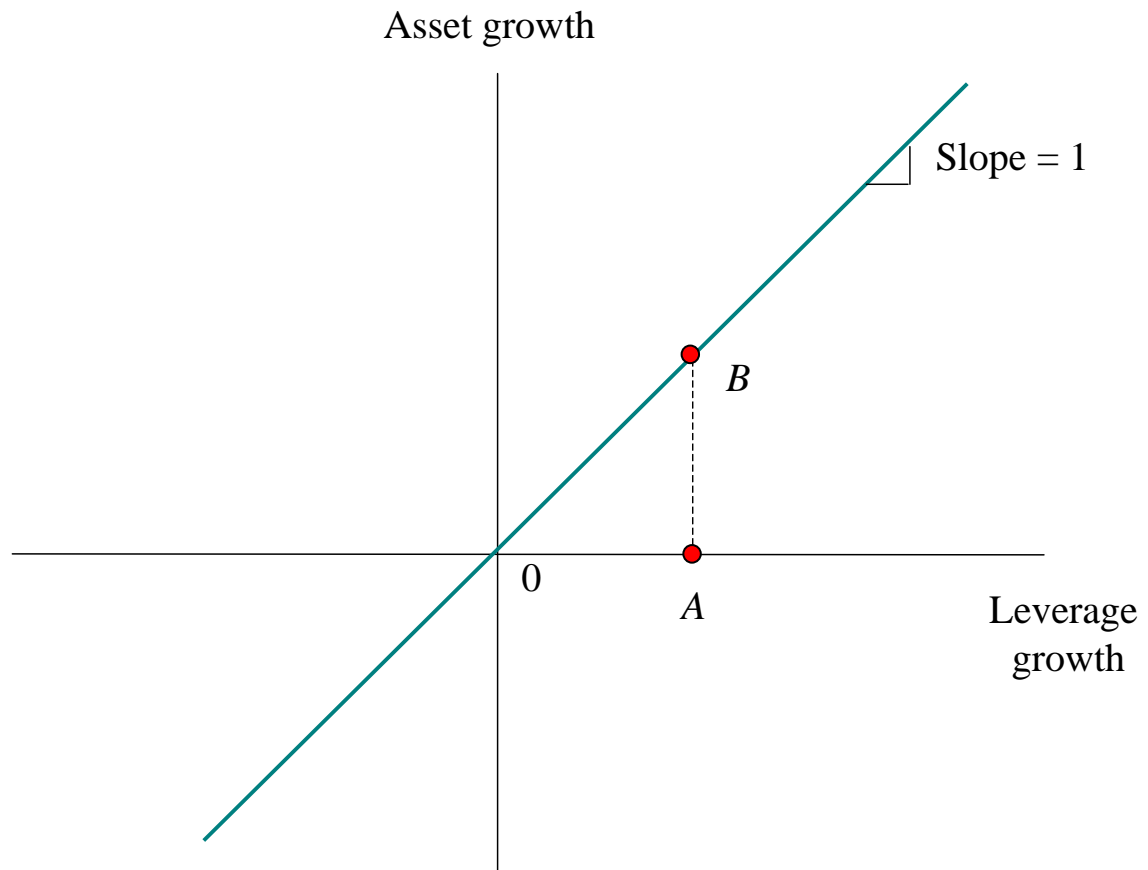
CFTC Conference on Commodity Markets
Washington DC, August 25-26, 2011

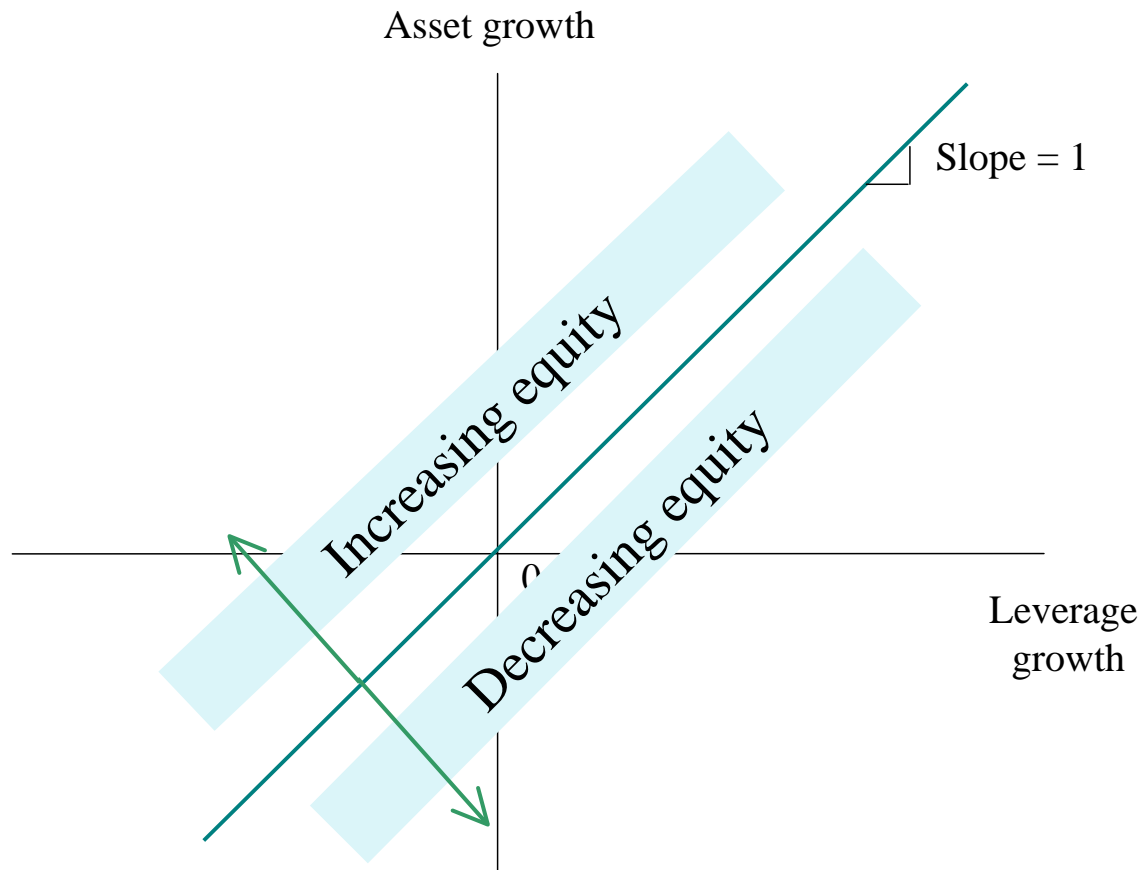
Corporate Finance of Banking

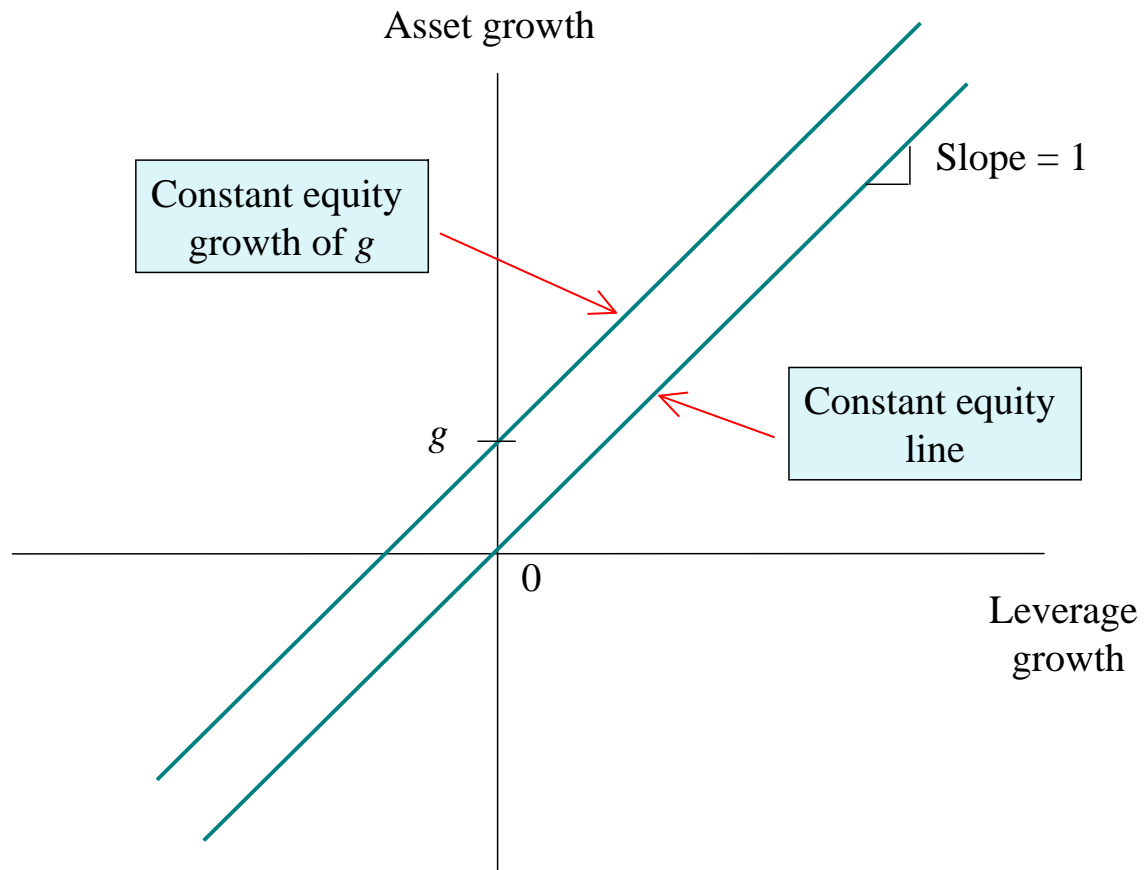
A	L
	Equity
Assets	Debt



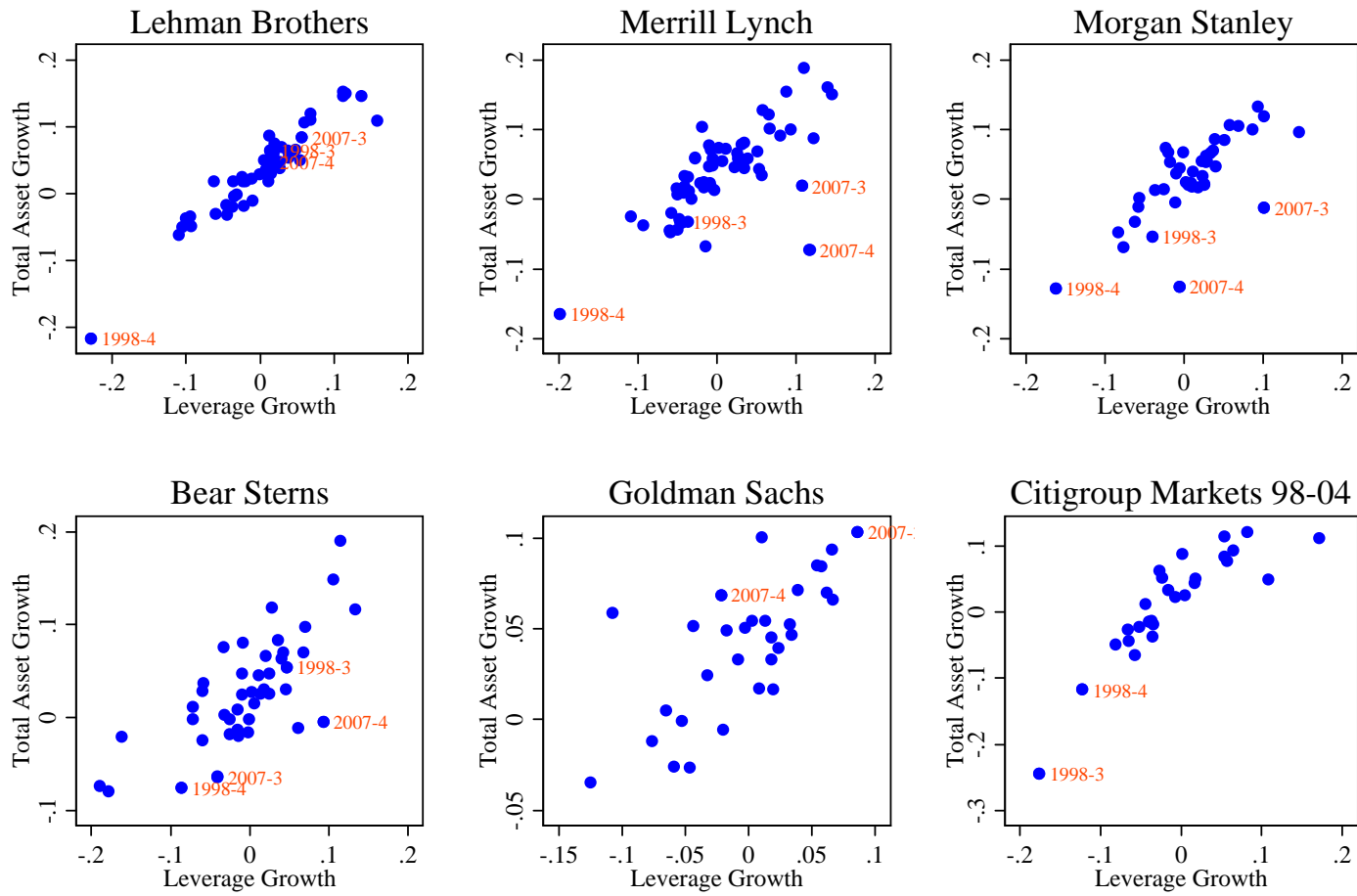




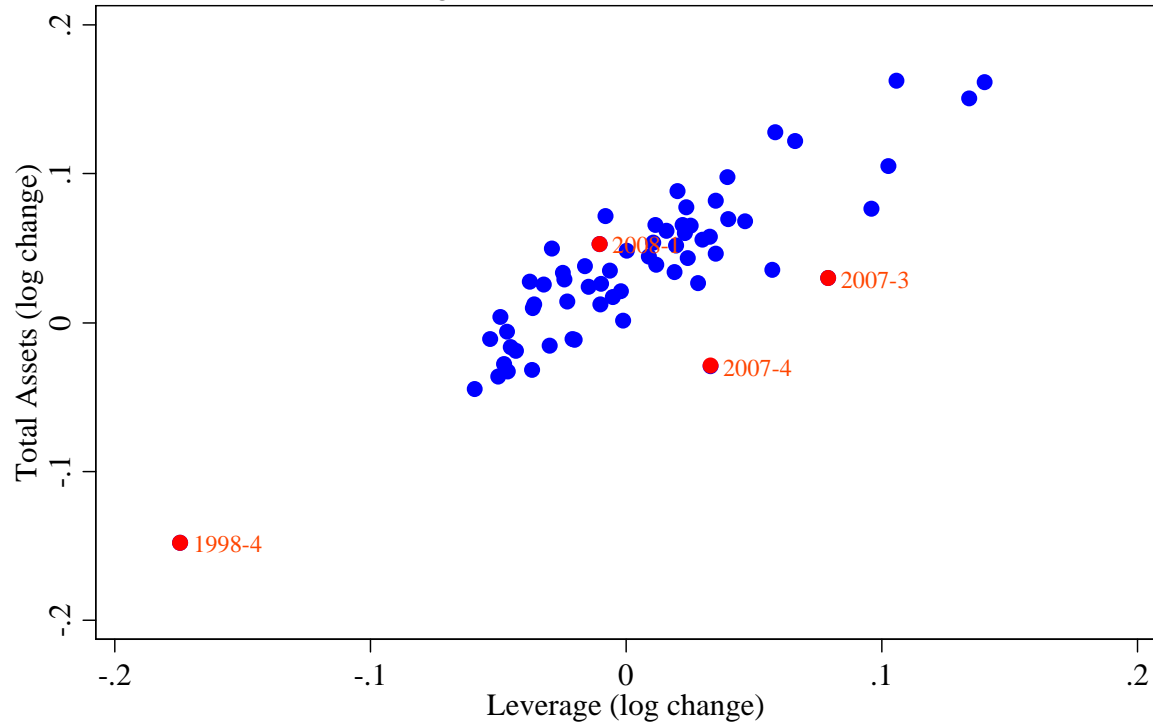




Total Assets and Leverage



Leverage and Total Assets Growth
Asset weighted, 1992Q3-2008Q1, Source: SEC



Balance Sheet Capacity

- Balance sheet capacity is capacity of banking (intermediary) sector to channel credit
- Balance sheet capacity is

$$\text{Equity} \times \text{“Permitted” Leverage}$$

- Our approach: intermediaries are subject to Value-at-Risk constraint:

$$\text{Equity} = \text{Total Assets} \times \text{VaR per dollar of assets}$$

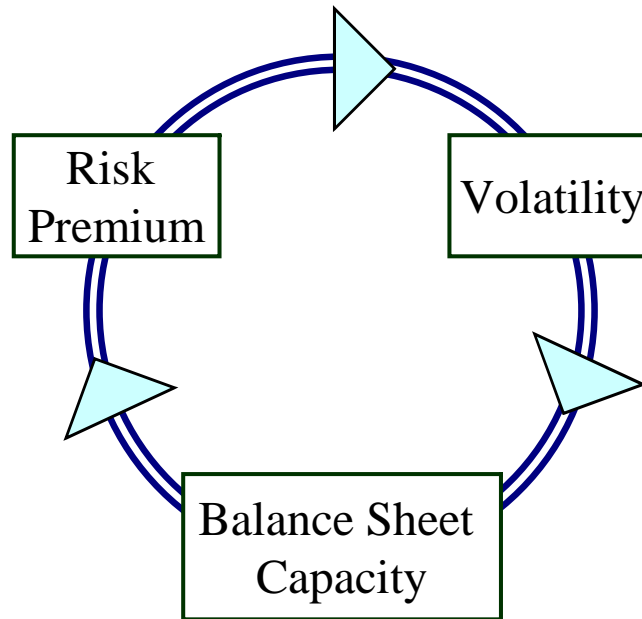
$$\text{Leverage} = \frac{\text{Assets}}{\text{Equity}} = \frac{1}{\text{Unit VaR}}$$

Endogeneity of Volatility

- Volatility attributable (in part) to actions of economic agents
- Equilibrium volatility obtained as fixed point of mapping:

perceived $\sigma \Rightarrow$ true σ

Volatility, Risk Premium and Balance Sheet Capacity



Solve simultaneously for **volatility**, **risk premium** and **balance sheet capacity** in closed form.

Preview of Closed Form Solution

Define

$$\theta = \frac{\text{Size of long-only sector}}{\text{Banking sector equity}}$$

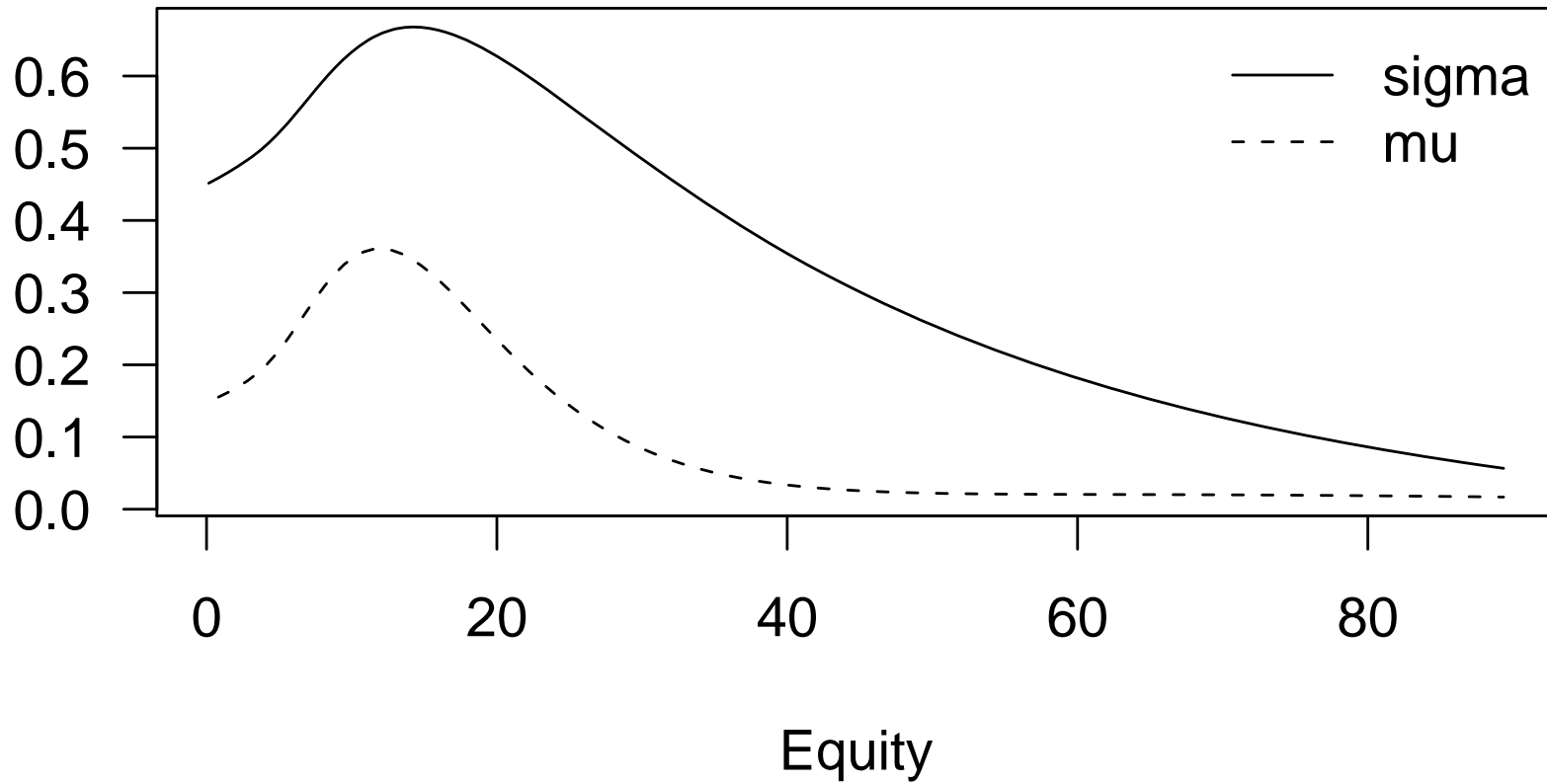
Equilibrium volatility σ is

$$\sigma = \text{fundamental volatility} \times \theta \exp\{-\theta\} \times F(\theta)$$

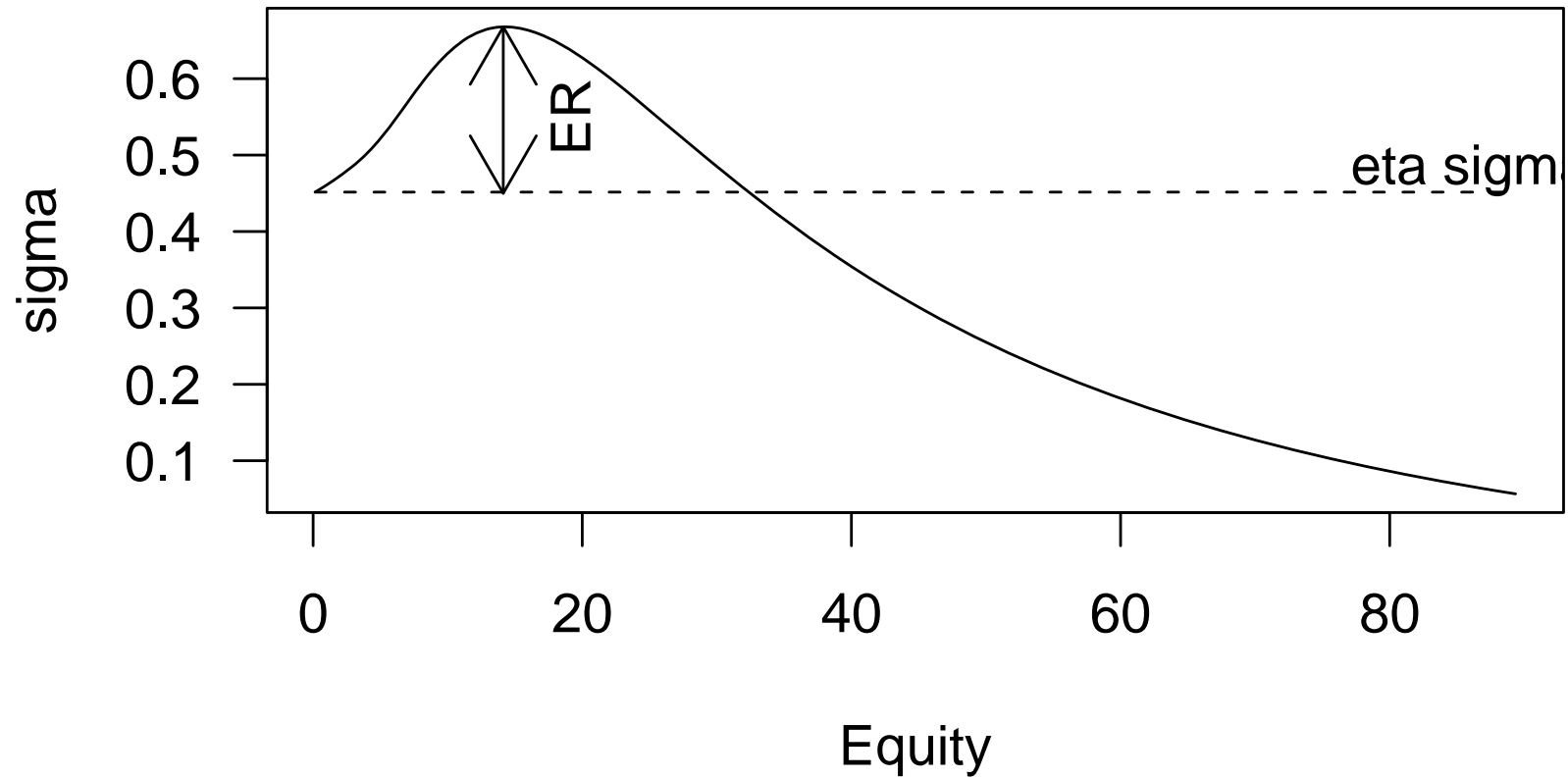
where $F(\theta)$ ensures

$$\sigma \rightarrow \text{fundamental volatility} \quad \text{as } \theta \rightarrow \infty$$

Volatility and Risk Premium



Excess Volatility or “Artificial” Compression of Risk



Model

- Time indexed by $t \in [0, \infty)$.
- $N > 0$ non-dividend paying risky securities (date t price of i th risky security is P_t^i)
- One risk-free bond ($B_0 = 1$, $dB_t = rB_t dt$, with r constant)
- Two types of traders
 - Active (risk-neutral, with VaR constraints) - banks
 - Passive (residual demand/supply curves) - households, value investors

Posit equilibrium of form:

$$\begin{bmatrix} dP_t^1/P_t^1 \\ \vdots \\ dP_t^N/P_t^N \end{bmatrix} = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^N \end{bmatrix} dt + \begin{bmatrix} - & \sigma_t^1 & - \\ - & \sigma_t^N & - \end{bmatrix} \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^N \end{bmatrix}$$

$\{W_t^i\}$ independent Brownian motions (fundamental shocks enter via passive traders' demands)

Scalars $\{\mu_t^i\}$ and $1 \times N$ vectors $\{\sigma_t^i\}$ are as yet undetermined coefficients to be solved in equilibrium

Solve for rational expectations equilibrium (REE) with respect to active traders' beliefs

Portfolio Choice of Active Traders

- Risk-neutral traders
- (Ultra) short horizons
- Value-at-Risk (VaR) constraint: capital (i.e. equity) should be large enough to meet VaR

D_t^i is dollar holding of i th security at t

V_t is trader's capital (notice no superscript for trader, due to aggregation result, to follow)

Balance sheet identity

$$b_t B_t = V_t - \sum_i D_t^i$$

Evolution of capital

$$dV_t = [rV_t + D_t^\top (\mu_t - r)] dt + D_t^\top \sigma_t dW_t$$

D^\top is transpose of D , σ_t is the $N \times N$ diffusion matrix, $r = (r, \dots, r)^\top$.

Expected capital gain:

$$E_t[dV_t] = [rV_t + D_t^\top (\mu_t - r)]dt \quad (1)$$

Variance of capital:

$$\text{Var}_t(dV_t) = D_t^\top \sigma_t \sigma_t^\top D_t dt \quad (2)$$

Trader maximizes (1) subject to VaR constraint, where VaR is α times forward-looking standard deviation of return on equity.

Assuming trader is solvent ($V_t > 0$) maximization problem is

$$\max_{D_t} rV_t + D_t^\top (\mu_t - r) \quad \text{subject to} \quad \alpha \sqrt{D_t^\top \sigma_t \sigma_t^\top D_t} \leq V_t$$

First-order condition

$$\mu_t - r = \alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t \Sigma_t D_t$$

where γ_t is Lagrange multiplier for VaR constraint, and $\Sigma_t := \sigma_t \sigma_t^\top$.

$$D_t = \frac{1}{\alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t} \Sigma_t^{-1} (\mu_t - r)$$

Constraint binds due to risk-neutrality

$$V_t = \alpha \sqrt{D_t^\top \Sigma_t D_t} \quad (3)$$

Therefore

$$D_t = \frac{V_t}{\alpha^2 \gamma_t} \Sigma_t^{-1} (\mu_t - r)$$

“As if” preferences. Optimal portfolio is similar to mean-variance optimal portfolio where the Lagrange multiplier γ_t appears like a risk-aversion coefficient.

Substitute into (3) to solve for Lagrange multiplier

$$\gamma_t = \frac{\sqrt{\xi_t}}{\alpha}$$

where

$$\xi_t := (\mu_t - r)^\top \Sigma_t^{-1} (\mu_t - r)$$

Lagrange multiplier γ_t is

- proportional to generalized Sharpe ratio $\sqrt{\xi}$
- does not depend directly on equity V_t

Interpretation. Additional unit of capital relaxes VaR constraint by multiple α of standard deviation, raising expected return by risk-premium on the portfolio per unit of standard deviation

Finally, solve for optimal portfolio:

$$D_t = \frac{V_t}{\alpha \sqrt{\xi_t}} \Sigma_t^{-1} (\mu_t - r)$$

Optimal portfolio is homogeneous of degree one in equity V_t

Aggregation result. Portfolio depends on V_t , total capital of active trading (banking?) sector, not on profile of individual equity capital.

⇒ Take aggregate capital, V_t , as state variable

Closing the Model

Passive traders in aggregate have vector-valued exogenous demand schedule for the risky assets, $y_t = (y_t^1, \dots, y_t^N)$ where

$$y_t = \Sigma_t^{-1} \begin{bmatrix} \delta^1 (z_t^1 - \ln P_t^1) \\ \vdots \\ \delta^N (z_t^N - \ln P_t^N) \end{bmatrix}$$

δ^i is scaling parameter for slope of the demand curve

z_t^i is positive demand shock for asset i

$$dz_t^i = r^* dt + \eta \sigma_z^i dW_t$$

The market-clearing condition $D_t + y_t = 0$ gives

$$\frac{V_t}{\alpha\sqrt{\xi_t}}(\mu_t - r) + \begin{bmatrix} \delta^1 (z_t^1 - \ln P_t^1) \\ \vdots \\ \delta^N (z_t^N - \ln P_t^N) \end{bmatrix} = 0$$

Equilibrium prices are

$$P_t^i = \exp \left(\frac{V_t}{\alpha\delta^i\sqrt{\xi_t}}(\mu_t^i - r) + z_t^i \right); \quad i = 1, \dots, N$$

Single Risky Asset

Look for equilibrium of form:

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t \quad (4)$$

μ_t and σ_t are undetermined coefficients to be solved in equilibrium, W_t is standard (scalar) Brownian motion. Equivalently,

$$d \ln P_t = \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t \quad (5)$$

“Seeds” of fundamental shocks given by exogenous shocks to passive traders’ demands:

$$dz_t = r^* dt + \eta \sigma_z dW_t, \text{ for known constants } \eta > 0, \sigma_z > 0$$

Start with market-clearing price with VaR-constrained traders

$$P_t = \exp \left(z_t + \frac{\sigma_t V_t}{\alpha \delta} \right)$$

Take logs and apply Ito's Lemma

$$\begin{aligned} d \ln P_t &= d \left(z_t + \frac{\sigma_t V_t}{\alpha \delta} \right) \\ &= r^* dt + \eta \sigma_z dW_t + \frac{1}{\alpha \delta} (\sigma_t dV_t + V_t d\sigma_t + dV_t d\sigma_t) \end{aligned} \quad (6)$$

- Unpack dV_t and $d\sigma_t$, and substitute back into (6)
- Compare coefficients with (5)

Step 1. Unpack dV_t as an Ito process

$$\begin{aligned}dV_t &= [rV_t + D_t(\mu_t - r)]dt + D_t\sigma_t dW_t \\ &= \left[rV_t + \frac{V_t(\mu_t - r)}{\alpha\sigma_t} \right] dt + \frac{V_t}{\alpha} dW_t\end{aligned}$$

Key step is the simplification allowed by binding VaR constraint:

$$\alpha\sigma_t D_t = V_t$$

Step 2. Unpack $d\sigma_t$ as an Ito process

From Itô's Lemma on $\sigma(V_t)$,

$$\begin{aligned} d\sigma_t &= \frac{\partial \sigma}{\partial V_t} dV_t + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} (dV_t)^2 \\ &= \left\{ \frac{\partial \sigma}{\partial V_t} \left[rV_t + \frac{V_t(\mu_t - r)}{\alpha \sigma_t} \right] + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} \left(\frac{V_t}{\alpha} \right)^2 \right\} dt + \frac{\partial \sigma}{\partial V_t} \frac{V_t}{\alpha} dW_t \quad (7) \end{aligned}$$

where we substitute in dV_t and where $(dV_t)^2 = \left(\frac{V_t}{\alpha}\right)^2 dt$

Substitute everything back into (6) and re-arrange:

$$d \ln P_t = (\text{drift term}) dt + \left[\eta \sigma_z + \frac{1}{\alpha \delta} \left(\sigma_t \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right) \right] dW_t \quad (*)$$

By hypothesis,

$$d \ln P_t = \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t \quad (**)$$

Comparing coefficients between (*) and (**),

$$\sigma(V_t) = \eta \sigma_z + \frac{1}{\alpha \delta} \left(\sigma_t \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right)$$

giving ordinary differential equation:

$$V_t^2 \frac{\partial \sigma}{\partial V_t} = \alpha^2 \delta (\sigma_t - \eta \sigma_z) - V_t \sigma_t$$

Generic solution:

$$\sigma(V_t) = \frac{1}{V_t} e^{-\frac{\alpha^2 \delta}{V_t}} \left[c - \alpha^2 \delta \eta \sigma_z \int_{-\frac{\alpha^2 \delta}{V_t}}^{\infty} \frac{e^{-u}}{u} du \right]$$

where c is a constant of integration, and $\text{Ei}(z)$ function (“exponential integral” function) $-\int_{-z}^{\infty} \frac{e^{-u}}{u} du$ is defined provided $z \neq 0$.

Constant of integration c can be tied down as follows.

Consider limit case $\delta \rightarrow 0$ where “fundamental shocks” are shut down

As $\delta \rightarrow 0$

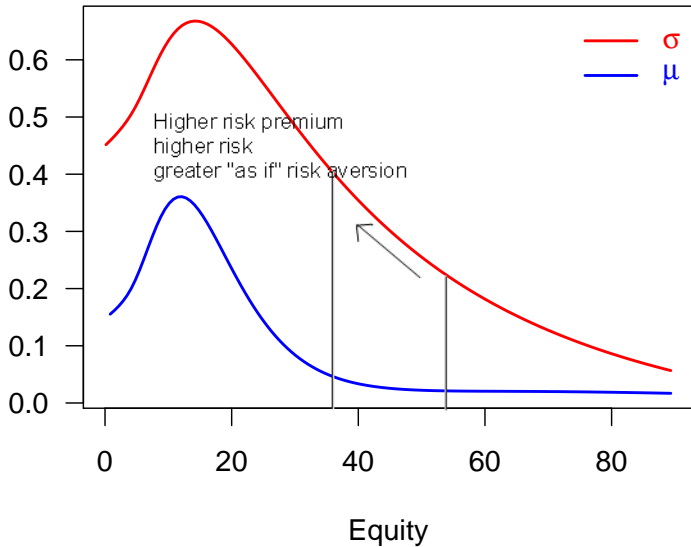
$$\sigma(V_t) \rightarrow \frac{c}{V_t}$$

Restriction. In the absence of fundamental shocks, volatility is zero

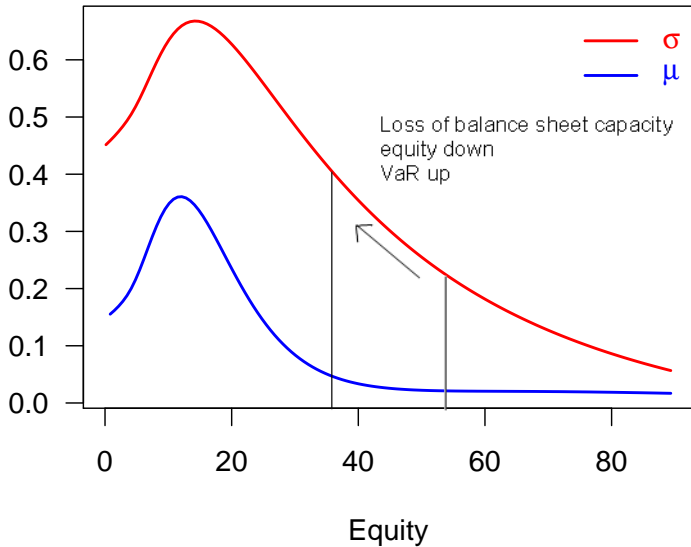
This implies $c = 0$ so that

$$\sigma(V_t) = \eta \sigma_z \frac{\alpha^2 \delta}{V_t} \exp \left\{ -\frac{\alpha^2 \delta}{V_t} \right\} \times \text{Ei} \left(\frac{\alpha^2 \delta}{V_t} \right)$$

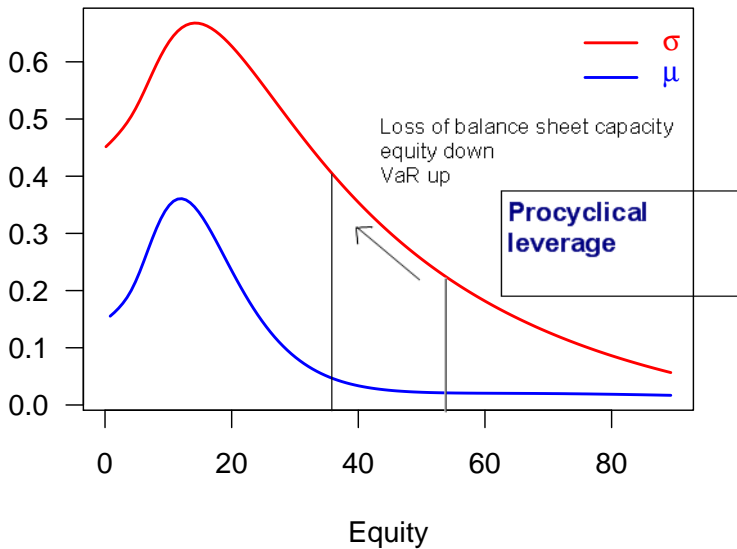
Risk and Return



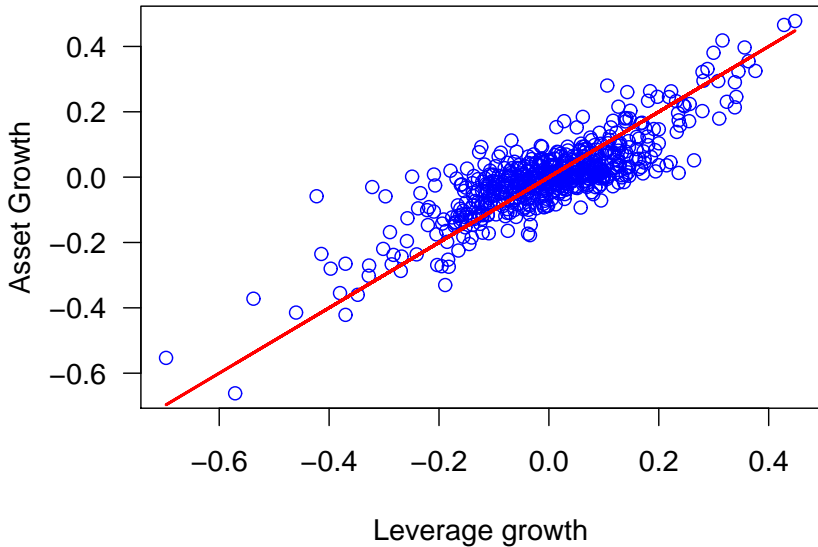
Risk and Return



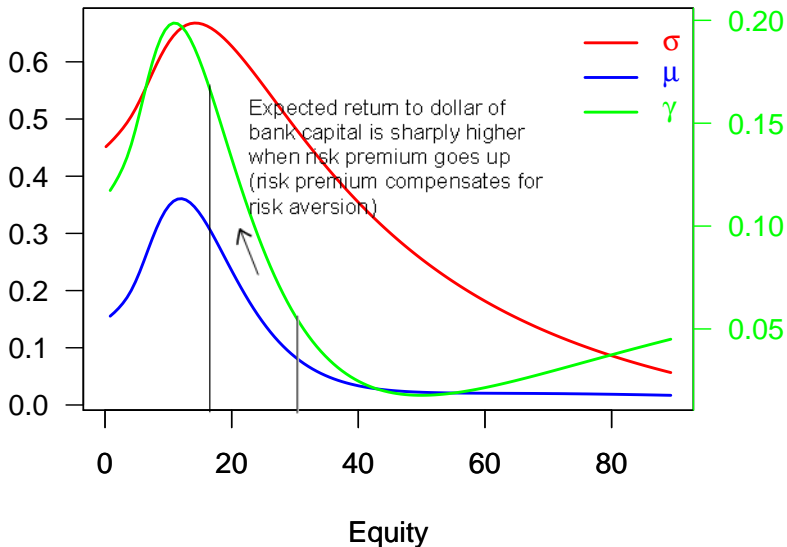
Risk and Return



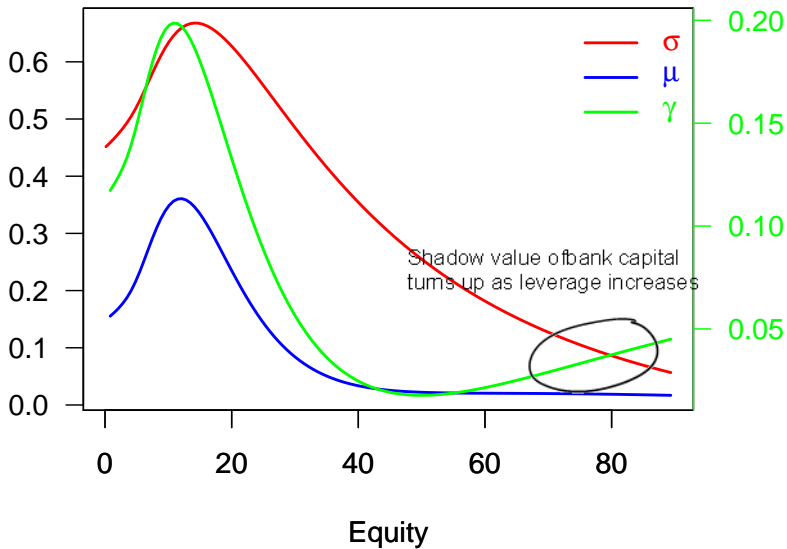
Leverage under P



Risk and Return



Risk and Return



Equilibrium drift μ_t

$$\mu_t = r + \frac{\sigma_t}{2\alpha\eta\sigma_z} \left\{ 2\alpha(r^* - r) + \alpha\sigma_t^2 - \eta\sigma_z + (\sigma_t - \eta\sigma_z) \left[2\alpha^2 r + \frac{\alpha^2\delta}{V_t} - 2 \right] \right\}$$

Sharpe ratio:

$$\frac{\mu_t - r}{\sigma_t} = \frac{1}{2\alpha\eta\sigma_z} \left\{ 2\alpha(r^* - r) + \alpha\sigma_t^2 - \eta\sigma_z + (\sigma_t - \eta\sigma_z) \left[2\alpha^2 r + \frac{\alpha^2\delta}{V_t} - 2 \right] \right\}$$

Many Risky Assets

Special case of N risky securities case can be solved using ODE solution from the single risky asset case.

Assumption (Symmetry)

1. Diffusion matrix for z is $\tilde{\sigma}_z I_N$ where $\tilde{\sigma}_z > 0$ is a scalar and I_N is the $n \times n$ identity matrix.
2. $\delta^i = \delta$ for all i .

Let σ_t^{ij} be coefficient giving effect of change in demand shock of j th security on price of i th security.

From assumption of symmetry, we only need to solve for one diffusion variable, $\sigma_t^{ii} = \sigma_t^{11}$, since for $i \neq j$ the cross effects are tied down by $\sigma_t^{ij} = \sigma_t^{12} = \sigma_t^{11} - \eta\tilde{\sigma}_z$.

Define $x_t \equiv x(V_t)$ the solution to the ODE for single risky asset with η replaced by $\frac{\eta}{N}$.

Proposition 1. *Assume (S). The following is a REE.*

The REE diffusion coefficients are $\sigma_t^{ii} = x_t + \frac{N-1}{N}\eta\tilde{\sigma}_z$, and for $i \neq j$, $\sigma_t^{ij} = x_t - \frac{1}{N}\eta\tilde{\sigma}_z$. Also, $\Sigma_t^{ii} = \text{Var}_t(\text{return on security } i) = \eta^2\tilde{\sigma}_z^2 + \frac{1}{N}(N^2x_t^2 - \eta^2\tilde{\sigma}_z^2)$, and for $i \neq j$, $\Sigma_t^{ij} = \text{Cov}_t(\text{return on security } i, \text{return on security } j) = \frac{1}{N}(N^2x_t^2 - \eta^2\tilde{\sigma}_z^2)$ and $\text{Corr}_t(\text{return on security } i, \text{return on security } j) = \frac{Nx_t^2 - \frac{1}{N}\eta^2\tilde{\sigma}_z^2}{Nx_t^2 + \frac{N-1}{N}\eta^2\tilde{\sigma}_z^2}$.

Risky holdings are $D_t^i = \frac{V_t}{\alpha N^{3/2}x_t}$.

The risk-reward relationship is given by

$$\frac{\mu_t^i - r}{x_t} = \frac{1}{2\alpha\frac{\eta}{N}\tilde{\sigma}_z} \left\{ \alpha \left(x_t + \frac{N-1}{N}\eta\tilde{\sigma}_z \right)^2 - \frac{\eta}{\sqrt{N}}\tilde{\sigma}_z + \sqrt{N} \left(x_t - \frac{\eta}{N}\tilde{\sigma}_z \right) \left[2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\} \quad (8)$$

The intuition and form of the drift term is very similar to the $N = 1$ case and reduces to it if N is set equal to 1.