

bust, it could raise additional equity capital by inviting new owners to take a stake in the bank, or asking the existing owners to put up more money to recapitalise the bank. Provided that the situation does not deteriorate too rapidly, there may still be time to recover the situation by taking corrective action. Here is where the time horizon of Value-at-Risk comes into play. The fact that value at risk is measured with a fixed horizon means that the time dimension can be set in an appropriate way for the particular case under consideration. The idea is that the time horizon in the definition of VaR is long enough so that

- corrective measures can be taken to rectify the problem (e.g. through recapitalisation, rescue or reorganisation)
- the horizon is also appropriate for the degree of illiquidity of the assets held by the firm
- confidence level chosen based on ease with which potential investors can be recruited in the recapitalisation

For these reasons, Value-at-Risk is a better measure of risk when it is interpreted as a buffer against possible failure, and has played an important role in the internal governance of banks and other financial firms, especially those that employ leverage. The concept of Value-at-Risk has been adopted widely, both by the private sector and regulators, and is the bedrock of the capital regulations adopted by Basel regulations. The 1996 Market Risk Amendment of the original 1988 Basel Accord is based on the notion of value at risk, and the Basel II regulations have further built on the notion of value at risk.

## 2.1 Portfolio Choice under VaR Constraint

There are consequences for portfolio decisions when an investor uses Value-at-Risk to manage risk. What is notable is the way that the attitude to risk varies as underlying conditions vary. We first examine the portfolio decision taken in isolation, before looking into how such individual decisions impact on the overall price of risk, which is the topic of Chapter 3.

An investor forms a portfolio consisting of two assets - a risky security and cash. The price of the risky security at date  $t$  is denoted  $p_t$ , and the

number of units of the risky security held by the investor is denoted by  $y_t$ . The holding of cash at date  $t$  is denoted by  $c_t$ . The price of the risky security next period (at date  $t + 1$ ) is denoted by  $p_{t+1}$ , and is uncertain when viewed from date  $t$ . Denote by  $\tilde{r}_{t+1}$  the return from date  $t$  to date  $t + 1$  on the risky security. Then

$$p_{t+1} = (1 + \tilde{r}_{t+1}) p_t \quad (2.4)$$

Let us make the simplifying assumption for now that  $\tilde{r}_{t+1}$  is independent across dates, and is identically distributed at all dates with mean  $\mu > 0$  and variance  $\sigma^2$ .

The capital of the investor at date  $t$  is denoted by  $e_t$ . The capital is the investor's net worth or equity that is allocated to the two assets in the portfolio. We will use the terms *capital*, *net worth* and *equity* interchangeably in these lectures.

The investor is free to allocate his capital between the risky asset and the risk-free asset, but is not limited to holding positive quantities of both. For instance, the investor is free to borrow cash and use the proceeds to buy up more of the risky security. In that case, the holding of cash  $c_t$  will be a negative number, and the investor incurs a debt of  $-c_t$  that needs to be repaid. For now, let us assume that the risk-free interest rate on cash is zero, and that the investor can borrow or lend any amount at this risk-free interest rate. We will comment later on how our conclusions are affected (if at all) if we were to relax these assumptions.

The investor who borrows in order to buy more of the risky asset has a balance sheet at date  $t$  which could be depicted as follows. The investor incurs debt of  $-c_t > 0$  and buys risky securities worth the sum of his own capital  $e_t$  and the borrowed money  $-c_t$ .

Assets	Liabilities
Securities $p_t y_t$	Equity $e_t$ Debt $-c_t$

(2.5)

The investor is leveraged, in the sense that the total assets on the balance sheet is larger than the size of his capital. Leverage is defined as the ratio of total assets to equity. The leverage of this investor is given by

$$\frac{p_t y_t}{e_t} \quad (2.6)$$

Such a balance sheet indicates the stance of an investor who is optimistic about the risky security. The greater is the borrowing  $-c_t$ , the more aggressive is the investor in backing the risky security.

In contrast, we could imagine an investor who is very pessimistic about the prospects of the risky security, and who takes a short position in it. In other words, the investor borrows some units of the risky security, sells it in the market, and keeps the proceeds of the sale in the form of cash. By having a short position in the risky security, the holding  $y_t$  is a negative number. The balance sheet of this pessimistic investor can be depicted as follows.

Assets	Liabilities	(2.7)
Cash $c_t$	Equity $e_t$ Securities $-p_t y_t$	

This investor's short position in the risky securities enters on the liabilities side of the balance sheet, since the investor has incurred the liability of buying back the securities at a future date to redeem the short position. Since  $y_t < 0$ , the liability  $-p_t y_t$  is recorded as a positive number on the liabilities side of the balance sheet. On the asset side, the investor holds cash only. Of course, even though the investor only holds cash on the asset side, this doesn't mean that the investor's position is risk free. The problem is that the investor's *liabilities* are uncertain, so that the investor's net worth  $e_t$  (which is what ultimately the investor cares about) is highly variable. The pessimistic investor is also leveraged, in the sense that the ratio of assets to equity is bigger than one. The leverage of the pessimistic investor with balance sheet (2.7) is

$$\frac{e_t - p_t y_t}{e_t} \tag{2.8}$$

Finally, we can consider an investor who holds positive quantities of both the risky security and cash, and funds the positive holding of both from his capital. The balance sheet that corresponds to such a position is

Assets	Liabilities	(2.9)
Cash $c_t$ Securities $p_t y_t$	Equity $e_t$	

Such an asset portfolio would be typical of a “long only” investor such as a pension fund or mutual fund who allocates a given amount of wealth  $e_t$  between cash and securities. Such an investor is not leveraged, in that the ratio of total assets to equity is 1.

We can see from (2.5), (2.7) and (2.9) that whatever form the balance sheet takes, the following balance sheet identity holds at every date  $t$ , which ties down the two sides of the balance sheet.

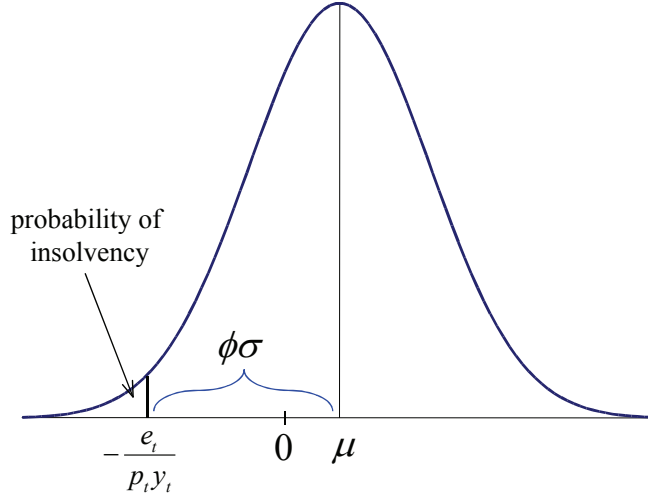
$$p_t y_t + c_t = e_t \tag{2.10}$$

At the time that the investor chooses how much of the risky security  $y_t$  to hold he knows the current price  $p_t$ , and his own equity  $e_t$ . Once  $y_t$  has been chosen, the new price  $p_{t+1}$  is drawn by Nature. If this new price is higher than the old price and  $y_t > 0$ , the investor makes a gain. If the new price is lower, the investor makes a loss. The value of equity that the investor takes into the next period’s portfolio choice problem reflects the gain or loss. The new value of capital  $e_{t+1}$  is a function of the new realised price  $p_{t+1}$ , and satisfies:

$$\begin{aligned} e_{t+1} &= p_{t+1} y_t + c_t \\ &= p_{t+1} y_t + e_t - p_t y_t \\ &= (p_{t+1} - p_t) y_t + e_t \\ &= [(1 + \tilde{r}_{t+1}) p_t - p_t] y_t + e_t \\ &= \tilde{r}_{t+1} p_t y_t + e_t \end{aligned} \tag{2.11}$$

Since the interest rate on cash is zero in our set-up, the new equity value reflects any gains or losses on the holding of the risky security. Of course, when viewed from date  $t$ , this future capital value is uncertain, since  $\tilde{r}_{t+1}$  is uncertain.

Let us now consider how risk management enters into portfolio choice by considering the decision problem of an investor who faces a Value-at-Risk constraint. The investor is risk-neutral otherwise. Consider the case where  $\mu > 0$ , so that the expected return on the risky security is higher than that on cash. For the risk-neutral investor, holding the risky security is preferred. Without any constraint on the size of the portfolio, the choice problem will be ill-defined, since the investor would wish to hold a larger and larger position in the risky security. However, the risk constraint will put a bound on the size of the risky security holding.

Figure 2.3: Probability density of  $\tilde{r}_{t+1}$ 

The investor has a period-by-period decision problem in which the objective at date  $t$  is to maximise the expected return on his capital from date  $t$  to date  $t + 1$ , subject only to the constraint that the risk is kept to some acceptable limit. The “acceptable” level of risk is expressed as a Value-at-Risk constraint on the probability of insolvency next period. The investor goes bust when the capital next period falls below zero. Let the confidence level associated with the VaR constraint be  $\alpha$ . At each date, the investor must ensure that the probability that he will become insolvent next period is at most  $1 - \alpha$ .

The investor becomes insolvent if  $e_{t+1} \leq 0$ . From equation (2.11), this happens when the return on the risky security is sufficiently bad so that  $\tilde{r}_{t+1} p_t y_t + e_t \leq 0$ , or

$$\tilde{r}_{t+1} \leq -\frac{e_t}{p_t y_t} \quad (2.12)$$

Figure 2.3 illustrates the argument. The smaller is the initial equity level  $e_t$  or the larger is the initial holding  $y_t$ , the greater is the chance of going bust. Let  $\phi$  be defined as the constant for which we have

$$\text{Prob} (\tilde{r}_{t+1} \leq \mu - \phi\sigma) = 1 - \alpha \quad (2.13)$$

In other words,  $\phi\sigma$  is the Value-at-Risk for the risky return  $\tilde{r}_{t+1}$  at the confidence level  $\alpha$  relative to the mean return  $\mu$ . Then, by choosing the size of

the holding of the risky asset  $y_t$ , the investor can ensure that the probability of his becoming insolvent next period is kept at most  $1 - \alpha$ . From Figure 2.3 we see that the probability of insolvency is exactly  $1 - \alpha$  when

$$\mu + \frac{e_t}{p_t y_t} = \phi \sigma \quad (2.14)$$

Solving for the dollar value of the risky security position, we have

$$p_t y_t = \frac{e_t}{\phi \sigma - \mu} \quad (2.15)$$

The investor cannot hold any more than this amount of the risky security, since then the probability of insolvency rises above the threshold value  $1 - \alpha$ , thereby violating his Value-at-Risk constraint.

Will the investor hold any less than the amount in (2.15)? No, since we are considering the case where  $\mu > 0$  so that from (2.11), we have

$$E(e_{t+1}) = \mu p_t y_t + e_t \quad (2.16)$$

The expected equity value next period is strictly increasing in  $y_t$ , so that the investor wishes to hold as much of the risky security as is permitted by his Value-at-Risk constraint. This is a consequence of the fact that the return on the risky security is strictly higher than that on cash. The upshot is that the investor's holding of the risky security is given exactly by (2.15).

## 2.2 Upward-Sloping Demand Reactions

Another perspective on the investor's decision is to consider the leverage maintained on the balance sheet. From equation (2.15) we see that the investor's leverage is given by

$$L = \frac{p_t y_t}{e_t} = \frac{1}{\phi \sigma - \mu} \quad (2.17)$$

Given our assumption of constant  $\mu$  and  $\sigma$ , leverage is also constant. Therefore, one way we can characterise the investor's portfolio decision is one of maintaining constant leverage in the face of price changes. However, leveraging targeting entails upward-sloping demand responses and downward-sloping supply responses - that is, the investor will buy more of the risky

security if its price rises, and sells some of the risky security if the price falls. Such price responses provide the pre-conditions for the amplification of shocks.

In order to appreciate the consequences of a leverage target on the demand and supply responses to price changes, let us first consider a simple numerical example of an investor who aims to maintain a *constant* leverage of 10. The initial balance sheet is as follows. The investor holds 100 dollars worth of securities, and has funded this holding with debt worth 90.

Assets	Liabilities
Securities, 100	Equity, 10
	Debt, 90

Assume that the price of debt is approximately constant for small changes in the price of the securities, so that the burden of adjustment falls on the equity. Suppose the price of securities increases by 1% so that the dollar holding of the securities rises to 101.

Assets	Liabilities
Securities, 101	Equity, 11
	Debt, 90

At this new higher price, the equity rises to 11, so that leverage then falls to  $101/11 = 9.18$ . This is because the equity rises by a much larger percentage rate (10%) due to the leverage. At the higher level of equity, the investor can restore leverage by taking on additional debt of  $D$  to purchase  $D$  worth of securities on the asset side so that

$$\frac{\text{assets}}{\text{equity}} = \frac{101 + D}{11} = 10$$

The solution is  $D = 9$ . The investor takes on additional debt worth 9, and with this money purchases securities worth 9. Thus, an increase in the price of the security of 1 leads to an increased holding worth 9. The demand response is upward-sloping. After the purchase, leverage is now back up to 10.

Assets	Liabilities
Securities, 110	Equity, 11
	Debt, 99

The mechanism works in reverse, too. Suppose there is shock to the securities price so that the value of security holdings falls to 109. On the liabilities side, it is equity that bears the burden of adjustment, since the value of debt stays approximately constant.

Assets	Liabilities
Securities, 109	Equity, 10
	Debt, 99

Leverage is now too high ( $109/10 = 10.9$ ). The investor can adjust down his leverage by selling securities worth 9, and paying down 9 worth of debt. Thus, a *fall* in the price of securities of leads to *sales* of securities. The supply response is downward-sloping. The new balance sheet then looks as follows. The balance sheet is now back to where it started before the price changes. Leverage is back down to the target level of 10.

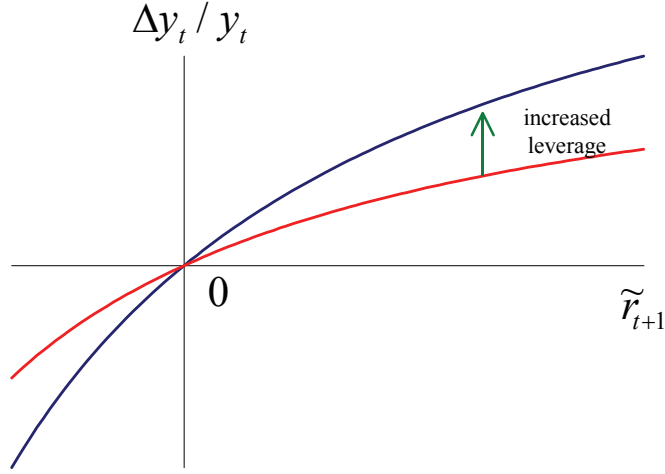
Assets	Liabilities
Securities, 100	Equity, 10
	Debt, 90

In contrast to the textbook demand and supply responses to price changes, we see that investors with Value-at-Risk constraints exhibit perverse demand and supply responses where higher prices lead to purchases and lower prices lead to sales. To see how the magnitudes relate to the leverage targeted by the investor, note from (2.11) and (2.15) that the proportional change in equity can be written as

$$\frac{e_{t+1} - e_t}{e_t} = \tilde{r}_{t+1} \frac{p_t y_t}{e_t} = \tilde{r}_{t+1} \cdot L \quad (2.18)$$

while the proportional change in total assets as a consequence of the price change (but before the portfolio adjustment) is

$$\frac{p_{t+1} y_t - p_t y_t}{p_t y_t} = \tilde{r}_{t+1} \quad (2.19)$$

Figure 2.4: Upward-sloping demand response to  $\tilde{r}_{t+1}$ 

Comparing (2.18) and (2.19), we see that for a leveraged investor, equity rises  $L$ -times faster than total assets. The price response of the investor can be obtained by tracking the new quantity  $y_{t+1}$ . Since the investor maintains constant leverage  $L$ , we have

$$\frac{p_t y_t}{e_t} = \frac{p_{t+1} y_{t+1}}{e_{t+1}} = L \quad (2.20)$$

Hence

$$\frac{y_{t+1}}{y_t} = \frac{e_{t+1}/e_t}{p_{t+1}/p_t} = \frac{1 + \tilde{r}_{t+1} \cdot L}{1 + \tilde{r}_{t+1}} \quad (2.21)$$

The proportional increase in the holding of the risky security can be expressed as a function of the return on the risky asset  $\tilde{r}_{t+1}$  and the degree of leverage  $L$ .

$$\frac{y_{t+1} - y_t}{y_t} = \frac{\tilde{r}_{t+1}}{1 + \tilde{r}_{t+1}} \cdot (L - 1) \quad (2.22)$$

The price response is upward-sloping in the return  $\tilde{r}_{t+1}$ , and is illustrated in Figure 2.4. The higher is the target leverage  $L$  maintained by the investor, the steeper is the demand response to price changes.