Procyclical Leverage and Value-at-Risk

Tobias Adrian       Hyun Song Shin

September 2013
Figure 1. **Three modes of leveraging up:** Mode 1 is through an equity buyback through a debt issue. Mode 2 is through a dividend financed by asset sale. Mode 3 is through increased borrowing to fund new assets. In each case the grey area indicates balance sheet component that is held fixed.
Figure 2. Changes in debt and book equity to changes in assets of the US broker dealer sector (1990Q1 – 2012Q2) (Source: Federal Reserve Flow of Funds)
Sum of slopes

\[
\text{Sum of slopes} = \frac{Cov(\Delta A_t, \Delta D_t)}{Var(\Delta A_t)} + \frac{Cov(\Delta A_t, \Delta E_t)}{Var(\Delta A_t)} = \frac{Cov(\Delta A_t, \Delta D_t + \Delta E_t)}{Var(\Delta A_t)} = 1
\]
Figure 3. Leverage of US Securities broker dealer sector (Source: Federal Reserve Flow of Funds)
Figure 4. Scatter chart of broker-dealer leverage and lagged log VIX
Table 1. **Broker dealer leverage and VIX**. This table presents OLS regressions with broker dealer leverage as the dependent variable and the one-quarter lagged log VIX index as the explanatory variable. Column 2 includes the post-crisis dummy that takes the value 1 after 2007Q4 and zero otherwise.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>VIX(-1)</td>
<td>-5.797***</td>
<td>-3.100***</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.008]</td>
</tr>
<tr>
<td>Post-crisis dummy</td>
<td>-5.865***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>37.907***</td>
<td>31.188***</td>
</tr>
<tr>
<td></td>
<td>[0.000]</td>
<td>[0.000]</td>
</tr>
<tr>
<td>Observations</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.20</td>
<td>0.471</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.187</td>
<td>0.453</td>
</tr>
</tbody>
</table>
Credit Supply by Banks

• Book equity is given

• Leverage is determined by “risk conditions”. Can we be more precise?

• Credit supply is

\[ \text{Credit supply} = \text{Equity} \times \text{Leverage} \]

So theory of leverage is theory of credit supply

• \( \Rightarrow \) Credit supply (hence, capital flows, interbank runs) depend on risk conditions
Figure 5. BNP Paribas: annual change in assets, equity and debt (1999-2010) (Source: Bankscope)
Figure 6. Société Générale: annual change in assets, equity and debt (1999-2010) (Source: Bankscope)
Figure 7. Barclays: 2 year change in assets, equity and debt (1992-2010) (Source: Bankscope)
Figure 8. Société Générale: 2 year change in assets, equity and debt (1999-2010) (Source: Bankscope)
Book Equity or Market Cap?

• **Book equity**

  \[ \text{Value of bank’s portfolio of claims} - \text{value of liabilities} \]

  Example: repo haircut

• **Market capitalization**

  \[ \text{Discounted value of free cash flows} \]

  \[ \text{Marked-to-market value of book equity} \]
Assets or Enterprise Value?

Enterprise value is defined as

\[
\text{Enterprise value} = \text{market capitalization} + \text{debt}
\]

- Enterprise value addresses \textit{how much a bank is worth}

- Total assets address \textit{how much a bank lends}
Figure 9. The left panel shows the scatter chart of the asset-weighted growth in book leverage and total assets for the eight largest US broker dealers and banks. The right panel is the scatter for the asset-weighted growth in enterprise value leverage and enterprise value. Enterprise value is the sum of market capitalization and debt, and enterprise value leverage is the ratio of enterprise value to market capitalization. The dark dots are for 2007 - 2009. The eight institutions are Bank of America, Citibank, JP Morgan, Bear Stearns, Goldman Sachs, Lehman Brothers, Merrill Lynch and Morgan Stanley (Source: SEC 10Q filings). 
Figure 10. Goldman Sachs (2001Q3 - 2011Q1)
Goldman Sachs (1998Q2 - 2011Q2)

$y = -0.0264x + 1.642$

$R^2 = 0.9964$

Figure 11. Gold Sachs (1998Q2 - 2011Q2)
Figure 12. Morgan Stanley (2001Q3 - 2011Q1)
y = 0.0011x + 1.01

y = 0.9989x - 1.01

$R^2 = 0.9987$

Figure 13. Morgan Stanley (1996Q1 - 2011Q2)
Figure 14. Goldman Sachs (2001Q3 - 2011Q1)
Value-at-Risk

Value at risk (VaR) at confidence level $c$ relative to some base level $A_0$ is smallest non-negative number $V$ such that

$$\text{Prob}(A < A_0 - V) \leq 1 - c$$

Value-at-Risk Rule: Keep equity $E$ equal to Value-at-Risk:

$$E = V$$

Implication 1. $E/V$ should be constant over time, even though $v = V/A$ fluctuates with risk conditions

$$E = V = v \times A$$
Implication 2. Let $v$ be Unit VaR (Value-at-Risk per dollar of assets). Leverage $L$ satisfies

$$L \equiv \frac{A}{E} = \frac{1}{v}$$

Or

$$\ln L = -\ln v$$

Implication 3. Probability of default is constant over the cycle, no matter how tranquil or turbulent financial conditions are.

Corollary. Credit conditions are procyclical - excessively(?) lax during booms and excessively(?) tight during busts.
Figure 15. Scatter chart of 99% log unit Value-at-Risk and log leverage of five Wall Street investment banks (Source: SEC 10Q and 10K filings)
Figure 16. Scatter chart of 99% log unit Value-at-Risk and log leverage of Goldman Sachs and Morgan Stanley (Source: SEC 10Q and 10K filings)
Figure 17. **Risk Measures.** The figure plots the unit VaR and the implied volatility for the eight large commercial and investment banks. Both variables are standardized relative to the pre-crisis mean and standard deviation. The measures are the lagged weighted averages of the standardized variables across the eight banks, where the weights are lagged total assets. Unit VaR is the ratio of total VaR to total assets. Implied volatilities are from Bloomberg. The grey shaded area indicates $\pm 2$ standard deviations around zero.
Figure 18. **Risk and Balance Sheet Adjustment.** This figure plots the unit VaR together with the intermediary balance sheet adjustment variables VaR/Equity and Leverage. All variables are standardized relative to the pre-crisis mean and standard deviation. The measures are the value weighted averages of the standardized variables across the eight banks, where the weights are lagged total assets. Unit VaR is the ratio of total VaR to total assets. VaR/Equity is the ratio of total VaR to total book equity. Leverage is the ratio of total assets to total book equity. The grey shaded area indicates \( \pm 2 \) standard deviations around zero.
**Cyclical Variation in Lending**

- Variation in lending depends on variation of (book) leverage
- How does leverage vary over the cycle?
  - Haircut in repo is funded by borrower’s (book) equity
  - Suppose assets are fully marked to market (e.g. traded securities)
  - Marked-to-market (book) leverage is given by

\[
\frac{1}{\text{haircut}}
\]

What determines book leverage? Equivalently, what determines haircut?
Model

Two dates: date 0 and date 1. Balance sheet at date 0 in market values:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets $A$</td>
<td>Debt $D$</td>
</tr>
<tr>
<td>Equity $E$</td>
<td></td>
</tr>
</tbody>
</table>

Balance sheet in notional (face) values

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets $A (1 + \bar{r})$</td>
<td>Debt $\bar{D}$</td>
</tr>
<tr>
<td>Equity $\bar{E}$</td>
<td></td>
</tr>
</tbody>
</table>

**Interpretation as repo:** Sell $A$ worth of securities for $D$ at date 0, repurchase for $\bar{D}$ at date 1.

$A - D \ (= E)$ is **haircut**, $\bar{D} - D$ is **interest payment on repo**
Figure 19. (Net) payoffs of investor (equity holder) and creditor (debt holder)
Repo Haircuts as Solution to Moral Hazard Problem

- Investor (agent) with endowment $E$
  - Investor borrows from creditor (principal) via repo
  - Investor has limited liability and engage in risk-shifting (take suboptimal investment with both large upside and large downside)

- Creditor imposes haircut to prevent excessive risk-taking
  - Similar to Holmstrom and Tirole (QJE 1997), but temptation payoff to bad action is the additional option value of bad action
  - Option value depends on financial conditions
  - Therefore, haircuts (and hence leverage) varies over the cycle
Overview of Solution

Optimal contract maximizes agent’s expected payoff by choice of $A$, $D$ and $\bar{D}$ subject to (IC) and (IR).

Creditor’s payoff has embedded short put option

Investor’s payoff has embedded long put option

- IC constraint ties down
  - strike price of put option
  - face value of debt

- IC and IR constraints together tie down
  - market value of debt
  - leverage
Moral Hazard

Choice between two actions - hold “good” security, hold “bad” security. One dollar of good security has expected payoff

\[ 1 + r_H \]

outcome density \( f_H(\cdot) \). Bad security has expected payoff \( 1 + r_L \) with density \( f_L(\cdot) \).

\[ r_L \leq r_H \]

But bad security has higher upside potential. \( F_H \) cuts \( F_L \) precisely once from below.
Cumulative distribution functions

\( F_L(z) \)

\( F_H(z) \)

\( z^* \)

\( 1 + \bar{r} \)
Creditor’s Payoff

Creditor’s (gross) payoff is portfolio consisting of

- cash of $\tilde{D}$
- short put position on assets with strike $\tilde{D}$.

\[
\tilde{D} - A\pi_H (\bar{d}) \\
= A(\bar{d} - \pi_H (\bar{d}))
\]

where $\bar{d} \equiv \tilde{D}/A$

$\pi_H (\bar{d})$ is price of put on 1 dollar’s worth of assets, with strike $\bar{d}$
Creditor’s net expected payoff is

\[ A (\bar{d} - d - \pi_H (\bar{d})) \]

where \( d \equiv D / A \). Creditor is promised \( \bar{d} - d \) of interest payment per dollar of assets but has sold a put option of value \( \pi_H (\bar{d}) \) per dollar of assets.

Creditor’s participation constraint ensures that the net payoff is non-negative

\[ \bar{d} - d - \pi_H (\bar{d}) \geq 0 \quad \text{(IR)} \]
Equity Holder’s Payoff

The investor is the residual claimant.

- Net expected payoff of the investor is

\[ U(A) = Ar - A(\bar{d} - d - \pi(\bar{d})) \]
\[ = A(r - (\bar{d} - d) + \pi(\bar{d})) \]

Investor’s expected payoff consists of (i) expected payoff from the security \( Ar \) (ii) minus promised interest payment \( A(\bar{d} - d) \) (iii) plus value of put option worth \( A \times \pi(\bar{d}) \)

- Sum of creditor’s (net) payoff and investor’s (net) payoff is \( Ar \), which is the surplus from principal-agent relationship
Investor’s payoff under $H$ is

$$A \left( r_H - \bar{d} + d + \pi_H (\bar{d}) \right)$$

Investor’s payoff under $L$ is $A \left( r_L - \bar{d} + d + \pi_L (\bar{d}) \right)$.

Incentive compatibility constraint is

$$r_H - r_L \geq \pi_L (\bar{d}) - \pi_H (\bar{d})$$

$$= \Delta \pi (\bar{d}) \quad \text{(IC)}$$

where $\Delta \pi (\bar{d}) \equiv \pi_L (\bar{d}) - \pi_H (\bar{d})$

(analogous to private benefit $B$ in Holmstrom-Tirole model).
Options and Outcome Distribution

Outcome space is $\{0, 1, 2, \cdots, Z\}$.

$p(s)$ is price of Arrow-Debreu contingent claim at outcome $s$
Put option with strike price $z$ has price

$$p(z - 1) + 2p(z - 2) + 3p(z - 3) + \cdots +zp(0)$$

First difference in option prices is c.d.f.

$$\pi(z) - \pi(z - 1) = p(z - 1) + 2p(z - 2) + 3p(z - 3) + \cdots +zp(0)$$

$$-p(z - 2) - 2p(z - 3) - \cdots - (z - 1)p(0)$$

$$= p(z - 1) + p(z - 2) + p(z - 3) + \cdots + p(0)$$

Second difference in option prices is (state price) density

In general, state price density is second derivative of option price w.r.t. strike price (Breeden and Litzenberger (Journal of Finance, 1978))
Figure 20. Temptation payoff $\Delta \pi(z)$ given by the difference in option values between $L$ and $H$ when default occurs at $z$. 
Leverage Constraint

Assume (IC) binds. Then optimal $\bar{d}^*$ satisfies:

$$\Delta \pi (\bar{d}) = r_H - r_L$$

$\bar{d}^*$ is mixes notional and market values. This is natural, since option specifies strike price in terms of notional value.

From participation constraint,

$$d^* = \bar{d}^* - \pi_H (\bar{d}^*)$$

$$= \int_0^{1+\bar{r}} \min \{\bar{d}^*, s\} f_H (s) \, ds$$

gives debt ratio in market values.
Recap of Solution

Optimal contract maximizes agent’s expected payoff by choice of $A$, $D$ and $ar{D}$ subject to (IC) and (IR).

- **IC constraint ties down**
  - strike price of put option
  - face value of debt $\bar{D}$

- **IC and IR constraints together tie down**
  - interest payment $\bar{D} - D$
  - market value of debt $D$
  - leverage $\lambda = A/D$


**Solving for Balance Sheet Size**

Bank equity holder’s expected payoff under the optimal contract is:

\[
U(A) \equiv A \left( r_H - (\dstar - d^*) + \pi_H (\dstar) \right)
\]

Expression inside the brackets is strictly positive, since the equity holder extracts the full surplus.

Equity holder’s payoff is strictly increasing in \( A \). For

\[
\lambda^* \equiv \frac{1}{1 - d^*}
\]

We have

\[
A = \lambda^* E
\]
Background on Extreme Value Theory

**Fisher-Tippett Theorem.**\(^1\) Suppose \(X_1, \cdots, X_n\) are i.i.d. with maximum realization \(M_n\). If there are \((a_n), (b_n)\) such that

\[
\lim_{n \to \infty} \Pr \left( \frac{M_n - b_n}{a_n} \leq z \right) = G(z)
\]

for non-degenerate \(G(z)\), then

\[
G(z) = \exp \left\{ - \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-\frac{1}{\xi}} \right\}
\]

for parameters \(\xi, \theta \in \mathbb{R}, \sigma > 0\) where \(1 + \xi \left( \frac{z - \theta}{\sigma} \right) > 0\)

Background on Extreme Value Theory

The Fisher-Tippett Theorem is the analogue of the central limit theorem, but for extreme realizations of i.i.d. draws rather than sums.

The distribution $G(\cdot)$ is the Generalized Extreme Value (GEV) distribution.

GEV encompasses the Fréchet ($\xi > 0$), Weibull ($\xi < 0$) and Gumbel ($\xi = 0$) distributions.

Case of $\xi = 0$ is the limit $\xi \to 0$. 
Outcome Distributions for Good and Bad Actions

Suppose that

\[
F_H(z) = \exp \left\{ - (1 + \xi \left( \frac{z - \theta}{\sigma} \right))^{-\frac{1}{\xi}} \right\}
\]

\[
F_L(z) = \exp \left\{ - \left( 1 + \xi \left( \frac{z - (\theta - k)}{m\sigma} \right) \right)^{-\frac{1}{\xi}} \right\}
\]

where \( k > 0 \) and \( m > 1 \).

The location parameter \( \theta \) can be given a cyclical interpretation (high \( \theta \) for booms, low \( \theta \) for busts)

The outcome distribution association with bad action is shifted to the left by \( k \) and has a scaling parameter \( \sigma' = m\sigma > \sigma \)
Figure 21. Distribution functions $F_L$ and $F_H$ for $\xi = 0.1$, $\theta = 0.5$, $\sigma = 0.1$, $k = 0.05$, and $m = 1.4$. 
Main Result

Suppose that $F_H(z)$ and $F_L(z)$ are given as above.

Let $\alpha(\theta)$ be the probability that the investor defaults at date 1 implied by contracting solution as a function of the location parameter $\theta$.

**Theorem 1.** $\alpha(\theta)$ is a constant function.

In other words, the probability of default by intermediary does not vary over the cycle.

Equivalently, intermediary follows the Value-at-Risk Rule.
Key step in the proof is that (see appendix):

\[ \Delta \pi (\theta) = \sigma \times \int t^{-(1+\xi)}e^{-t} dt \]

1

\[ m^{\frac{1}{\xi}}\left( (\ln(1/\alpha))^{-\xi} + \xi \frac{k}{\sigma} \right)^{-\frac{1}{\xi}} \]

The temptation payoff \( \Delta \pi (\theta) \) is a function of \( \theta \) solely through its effect on the probability of default \( \alpha \).
Special Case

Consider the case where $\xi = -1$ and $m \to 1$.

\[
F_H(z) = \exp \left\{ \frac{z - \theta}{\sigma} - 1 \right\}
\]

\[
F_L(z) = \exp \left\{ \frac{z - (\theta - k)}{\sigma} - 1 \right\}
\]

\[
\frac{F_L(z; \theta)}{F_H(z; \theta)} = e^{\frac{k}{\sigma}} > 1
\]
Then

$$\Delta \pi (z; \theta) = \int_{-\infty}^{z} (F_L (s; \theta) - F_H (s; \theta)) \, ds$$

$$= \left( e^{\frac{k}{\sigma}} - 1 \right) \int_{-\infty}^{z} F_H (s; \theta) \, ds$$

$$= \left( e^{\frac{k}{\sigma}} - 1 \right) \cdot \sigma F_H (z; \theta)$$

(2)

From the IC constraint,

$$\Delta \pi (\bar{d}^*; \theta) = r_H - r_L,$$

so

$$\left( e^{\frac{k}{\sigma}} - 1 \right) \cdot \sigma F_H (\bar{d}^*; \theta) = r_H - r_L$$

(3)
From (2) and (3), the probability of default $\alpha(\theta)$ is given by

$$\alpha(\theta) = F_H(\overline{d}^*; \theta)$$

$$= \frac{r_H - r_L}{\sigma \left( e^{\frac{k}{\sigma}} - 1 \right)}$$

which does not depend on $\theta$

Solution for $\alpha$ endogenizes the Value-at-Risk confidence level $c = 1 - \alpha$ in terms of parameters of moral hazard problem
Figure 22. Invariance of the probability of default $\alpha$. As the location parameter shifts from $\theta$ to $\hat{\theta}$, the bank’s leverage falls leaving default probability unchanged at $\alpha$. 
Cyclical Property of Leverage

The probability of default is

\[ \alpha (\theta) = F_H (\bar{d}^*; \theta) = \exp \left\{ \frac{\bar{d}^* - \theta}{\sigma} - 1 \right\} \]

Meanwhile, we have

\[ \alpha (\theta) = F_H (\tilde{d}^*; \theta) = \frac{r_H - r_L}{\sigma \left( e^{k/\sigma} - 1 \right)} \]

Solving for \( \bar{d}^* \),

\[ \bar{d}^* = \theta + \sigma + \sigma \ln \left( \frac{r_H - r_L}{\sigma \left( e^{k/\sigma} - 1 \right)} \right) \]

Leverage is increasing in \( \theta \). Leverage is procyclical.
Two Features of Exponential Tail

- Exponential tail means that the ratios of c.d.f.s is constant:

\[
\int_{-\infty}^{z} (F_L (s; \theta) - F_H (s; \theta)) \, ds = \left( e^{\frac{k}{\sigma}} - 1 \right) \int_{-\infty}^{z} F_H (s; \theta) \, ds
\]

- Exponential tail means that integral of c.d.f. (to get option value) is the just the c.d.f. itself times \( \sigma \)

\[
\int_{-\infty}^{z} F_H (s; \theta) \, ds = \sigma F_H (z; \theta)
\]
Vasicek (2002) extension of Merton (1974) to allow for many borrowers $j$

Figure 23. Value of projects of local borrowers and default probability
\[ V_T = V_0 \exp \left\{ \left( \mu - \frac{s^2}{2} \right) T + s\sqrt{T}W_j \right\} \]

\[
\text{Prob} (V_T < F) = \Phi \left( \frac{- \ln (V_0/F) + \left( \mu - \frac{s^2}{2} \right) T}{s\sqrt{T}} \right) = \Phi (-d_j)
\]

Vasicek (2002):

\[ W_j = \sqrt{\rho Y} + \sqrt{1 - \rho}X_j \]

Borrower \( j \) repays when \( \hat{Z}_j > 0 \), where \( \Phi(\cdot) \) is standard normal cdf

\[ \hat{Z}_j = -\Phi^{-1} (\varepsilon) + \sqrt{\rho Y} + \sqrt{1 - \rho}X_j \]

\[
\text{Pr} \left( \hat{Z}_j < 0 \right) = \text{Pr} \left( \sqrt{\rho Y} + \sqrt{1 - \rho}X_j < \Phi^{-1} (\varepsilon) \right) = \Phi (\Phi^{-1} (\varepsilon)) = \varepsilon
\]
Notation for Bank Balance Sheet
Bank diversifies away idiosyncratic risk

Conditional on $Y$, defaults are independent.

Diversify across many borrowers and eliminate idiosyncratic risk. Realized value of assets is function of $Y$ only

$$w(Y) \equiv (1 + r) C \cdot \Pr \left( \hat{Z}_j \geq 0 | Y \right)$$

$$= (1 + r) C \cdot \Pr \left( \sqrt{\rho} Y + \sqrt{1 - \rho} X_j \geq \Phi^{-1} (\varepsilon) | Y \right)$$

$$= (1 + r) C \cdot \Phi \left( \frac{Y \sqrt{\rho} - \Phi^{-1} (\varepsilon)}{\sqrt{1 - \rho}} \right) \quad (\ast)$$
Figure 24. The two charts plot the densities over realized assets when \( C (1 + r) = 1 \). The left hand charts plots the density over asset realizations of the bank when \( \rho = 0.1 \) and \( \varepsilon \) is varied from 0.1 to 0.3. The right hand chart plots the asset realization density when \( \varepsilon = 0.2 \) and \( \rho \) varies from 0.01 to 0.3.
Contracting Problem between Bank and Depositors

Bank chooses portfolio of loans between:

- Good portfolio has probability of default $\varepsilon > 0$, $\rho_G = 0$. Outcome distribution (normalized) is

$$F_G(z) = \begin{cases} 
0 & \text{if } z < 1 - \varepsilon \\
1 & \text{if } z \geq 1 - \varepsilon 
\end{cases} \quad (4)$$

- Bad portfolio has probability of default $\varepsilon + k$, with $k > 0$, and $\rho_B = \rho > 0$. Outcome distribution is

$$F_B(z) = \Phi \left( \frac{\Phi^{-1}(\varepsilon + k) + \sqrt{1 - \rho} \Phi^{-1}(z)}{\sqrt{\rho}} \right) \quad (5)$$
**Bank default point** $\varphi$

Define $\varphi = (1 + f) L / (1 + r) C$.

$\varphi$ is (1) (normalized) notional debt of bank and (2) strike price of embedded put option from limited liability (Merton (1974))

Bank chooses $C$ to maximize marked-to-market equity $E(\hat{w}) - [\varphi - \pi(\varphi)]$ subject to IC constraint:

$$E_G(\hat{w}) - [\varphi - \pi_G(\varphi)] \geq E_B(\hat{w}) - [\varphi - \pi_B(\varphi)] \quad (6)$$

**Lemma 2.** *There is a unique solution $\varphi^*$, where $\varphi^* < 1 - \varepsilon$.***

**Corollary 3.** *Bank borrows at the risk-free rate*
State price density is second derivative of the option price with respect to its strike price (Breeden and Litzenberger (1978)). Given risk-neutrality,

$$\Delta \pi (\varphi) = \int_0^{\varphi} [F_B (s) - F_G (s)] \, ds$$

or

$$\Delta \pi (\varphi) = \begin{cases} \int_0^{\varphi} F_B (s) \, ds & \text{if } \varphi < 1 - \varepsilon \\ \int_0^{1-\varepsilon} F_B (s) \, ds - \int_0^{\varphi} [1 - F_B (s)] \, ds & \text{if } \varphi \geq 1 - \varepsilon \end{cases}$$  \hspace{1cm} (7)$$
Thus $\Delta \pi (\varphi)$ is single-peaked, reaching its maximum at $\varphi = 1 - \varepsilon$.

\[
\int_0^1 [F_B(s) - F_G(s)] \, ds = \int_0^1 [1 - F_G(s)] \, ds - \int_0^1 [1 - F_B(s)] \, ds
\]

\[
= E_G(\hat{w}) - E_B(\hat{w}) = k
\]

(8)

so $\Delta \pi (\varphi)$ approaches $k$ from above as $\varphi \to 1$.

$\varphi < 1$ for any bank with positive notional equity. So, we have a unique solution to $\Delta \pi (\varphi) = k$ where the solution is in the range where $\Delta \pi (\varphi)$ is increasing.

Therefore $\varphi^* < 1 - \varepsilon$. 
Supply of Credit by Bank

Credit supply $C$ obtained from $\varphi^* = (1 + f) \frac{L}{(1 + r)} C$ and balance sheet identity $C = E + L$

$$C = \frac{E}{1 - \frac{1+r}{1+f} \cdot \varphi^*}$$

Funding rate $f$ is risk-free rate

This is a special case of the Value-at-Risk (VaR) rule, where the probability of default of the bank is kept constant at zero.
Alternative Modeling Approaches

- VaR constraint is imposed by the creditors (as here).
  - Haircuts in repo contracts

- VaR constraint is self-imposed by equity holders who maximize long-term payoff
  - Future payoffs are attainable only if the bank remains solvent
  - Trade off the increased short term payoff from large balance sheet against greater probability of default
Consider the Generalized Extreme Value (GEV) distribution given by

\[ G(z) = \exp \left\{ - \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-\frac{1}{\xi}} \right\} \]

We would like the expression for the integral of the GEV distribution. As a first step, consider the incomplete gamma function \( \Gamma(x, y) \) defined as

\[ \Gamma(x, y) \equiv \int_y^\infty t^{x-1}e^{-t} \, dt \]
where

\[ x = -\xi \]

\[ y = \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-\frac{1}{\xi}} \]

Note that

\[ \frac{dy}{dz} = -\frac{1}{\xi} \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-\frac{1}{\xi}-1} \cdot \frac{\xi}{\sigma} \]

\[ = -\frac{1}{\sigma} \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-\frac{1}{\xi}} \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-1} \]

\[ = -\frac{1}{\sigma} y^{\xi+1} \]
Hence

\[
\frac{d}{dz} \Gamma(x, y) = \frac{d}{dy} \Gamma(x, y) \cdot \frac{dy}{dz}
\]

\[
= -y^{x-1}e^{-y} \cdot \left( -\frac{1}{\sigma} y^{\xi+1} \right)
\]

\[
= y^{-\xi-1}e^{-y} \cdot \frac{1}{\sigma} y^{\xi+1}
\]

\[
= \frac{1}{\sigma} e^{-y}
\]

\[
= \frac{1}{\sigma} G(z)
\]  

(9)
Hence the indefinite integral of $G(z)$ is:

$$
\int \exp \left\{ - (1 + \xi \left( \frac{z - \theta}{\sigma} \right))^{-1} \right\} \, dz = \sigma \Gamma(x, y) + \text{constant}
$$

Define $y(z, \theta, \sigma)$ as

$$
y(z, \theta, \sigma) \equiv \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-1}
$$

Let $\alpha$ be the probability of default for the bank. Then

$$
\alpha = \exp \left\{ - \left( 1 + \xi \left( \frac{\bar{d}^* - \theta}{\sigma} \right) \right)^{-1} \right\} = e^{-y(\bar{d}^*, \theta, \sigma)}
$$
so that \( y(d^*, \theta, \sigma) = -\ln \alpha = \ln (1/\alpha) \). Now, consider the expression for the temptation payoff for the bad action in the moral hazard problem for the bank. The temptation payoff to risk-shifting is the additional option value given by the bad action \( L \). The option values of \( L \) and \( H \) are given, respectively, by

\[
\begin{align*}
\int_0^{d^*} F_H(z, \theta) \, dz &= \int_{-\infty}^{d^*} \exp \left\{ - \left( 1 + \xi \left( \frac{z-\theta}{\sigma} \right) \right)^{-\frac{1}{\xi}} \right\} \, dz \\
\int_0^{d^*} F_L(z, \theta) \, dz &= \int_{-\infty}^{d^*} \exp \left\{ - \left( 1 + \xi \left( \frac{z-(\theta-k)}{m\sigma} \right) \right)^{-\frac{1}{\xi}} \right\} \, dz
\end{align*}
\]

where \( m > 1 \) and \( k > 0 \).

The temptation payoff in the moral hazard problem is given by
\[ \Delta \pi (\theta) = \int_0^{\bar{d}^*} (F_L (z, \theta) - F_H (z, \theta)) \, dz \]

\[ = \sigma \left[ \Gamma (-\xi, y (z, \theta - k, m\sigma)) \right]_{-\infty}^{\bar{d}^*} - \sigma \left[ \Gamma (-\xi, y (z, \theta, \sigma)) \right]_{-\infty}^{\bar{d}^*} \]

\[ = \sigma \int_{y(\bar{d}^*, \theta-k, m\sigma)}^{\infty} t^{-(1+\xi)} e^{-t} \, dt - \sigma \int_{y(\bar{d}^*, \theta, \sigma)}^{\infty} t^{-(1+\xi)} e^{-t} \, dt \]

\[ = \sigma \int_{y(\bar{d}^*, \theta-k, m\sigma)}^{\infty} t^{-(1+\xi)} e^{-t} \, dt \]

\[ (10) \]

Meanwhile, we know that \( y (\bar{d}^*, \theta, \sigma) = \ln (1/\alpha) \). Also, by re-arranging
the expression:

\[ y(d^*, \theta - k, m\sigma) = \left(1 + \xi \left(\frac{z - (\theta - k)}{m\sigma}\right)\right)^{-\frac{1}{\xi}} \]

we have

\[ y(d^*, \theta - k, m\sigma) = m^{\frac{1}{\xi}} \left((\ln(1/\alpha))^{-\xi} + \xi \frac{k}{\sigma}\right)^{-\frac{1}{\xi}} \]

Therefore,

\[ \Delta\pi(\theta) = \sigma \times \int_{0}^{\ln(1/\alpha)} t^{-(1+\xi)}e^{-t}dt \]

\[ = m^{\frac{1}{\xi}}((\ln(1/\alpha))^{-\xi} + \xi \frac{k}{\sigma})^{-\frac{1}{\xi}} \]

The temptation payoff \( \Delta\pi(\theta) \) is a function of \( \theta \) solely through its effect on the probability of default \( \alpha \).