Leverage and Total Assets Growth
Asset weighted, 1992Q3-2008Q1, Source: SEC
Shifts in Leverage and Total Assets

Slope is approximately 1

$$\ln A_t - \ln A_{t-1} \simeq \delta + \ln \left( \frac{A_t}{E_t} \right) - \ln \left( \frac{A_{t-1}}{E_{t-1}} \right)$$

Suggests...

$$\ln E_t = \delta + \ln E_{t-1}$$

$$A_t = \lambda^{*} E_t$$

Equity seems to be forcing variable and $A_t$ is determined by realization of maximum allowed leverage $\lambda^{*}$

Stands usual way of thinking on its head (A first, then decide E and D).
Explaining Leverage

Value at risk (VaR) at confidence level $c$ relative to some base level $A_0$ is smallest non-negative number $V$ such that

$$\text{Prob}(A < A_0 - V) \leq 1 - c$$

Equity $E$ meets total value at risk

$$E = V = v \times A$$

$v$ is Unit VaR (Value-at-Risk per dollar of assets). Leverage $L$ satisfies

$$L \equiv \frac{A}{E} = \frac{1}{v}$$
Empirical implication:

\[ \ln L = - \ln \nu \]

so that

\[ \ln L_t - \ln L_{t-1} = - (\ln \nu_t - \ln \nu_{t-1}) \] (*)

Scatter chart of leverage changes against unit VaR changes should have slope \(-1\).

Evidence?
**Model**

Agency model in the spirit of Holmstrom and Tirole (1997)

- Moral hazard comes from choosing sub-optimally risky assets
- Temptation payoff is the additional option value of bad action
- Agent is the borrower (e.g. investment bank).
- Principal is creditor to the bank (e.g. money market fund)
Two dates: date 0 and date 1. Balance sheet at date 0 in market values:

\[
\begin{array}{|c|c|}
\hline
\text{Assets} & \text{Liabilities} \\
\hline
\text{Assets } A & \text{Debt } D \\
& \text{Equity } E \\
\hline
\end{array}
\]

Balance sheet in notional (face) values

\[
\begin{array}{|c|c|}
\hline
\text{Assets} & \text{Liabilities} \\
\hline
\text{Assets } A (1 + \bar{r}) & \text{Debt } \bar{D} \\
& \text{Equity } \bar{E} \\
\hline
\end{array}
\]

**Interpretation as repo transaction:** Sell $A$ worth of securities for $D$ at date 0, repurchase for $\bar{D}$ at date 1. Then, $A - D$ is “haircut” and $\bar{D}/D - 1$ is repo interest rate.
Solving for Leverage and Balance Sheet Size

Fix $E$.

Optimal contract maximizes agent’s payoff by choice of $A$, $D$ and $\bar{D}$ subject to (IC) and (IR).

Solve for

• Leverage $\lambda = A/E$

• Balance sheet size $A$
Overview of Solution

Creditor’s payoff has embedded short put option

Broker’s payoff has embedded long put option

• IC constraint ties down
  – strike price of put option
  – face value of debt

• IC and IR constraints together tie down
  – market value of debt
  – leverage
Net Payoffs

- Debt payoff
- Equity payoff
- Asset payoff

Net Payoffs
Moral Hazard

Choice between two types of securities. One dollar of good security has expected payoff

$$1 + r_H$$

density $f_H(\cdot)$. Bad security has expected payoff $1 + r_L$ with density $f_L(\cdot)$.

$$r_L < 0 < r_H$$

But bad security has higher upside potential. $F_H$ cuts $F_L$ precisely once from below.
Cumulative distribution functions

\[ F_L(z) \]

\[ F_H(z) \]
Creditor’s Payoff

Creditor’s (gross) payoff is portfolio consisting of

- cash of $\bar{D}$

- short put position on assets with strike $\bar{D}$.

\[
\bar{D} - A\pi_H (\bar{d}) \\
= A (\bar{d} - \pi_H (\bar{d}))
\]

where $\bar{d} = \bar{D} / A$

$\pi_H (\bar{d})$ is price of put on 1 dollar’s worth of assets, with strike $\bar{d}$.

(Merton (JF 1974))
Creditor’s net expected payoff is

\[ V = A \left( \bar{d} - d - \pi_H (\bar{d}) \right) \]

where \( d \equiv D/A \) is debt/asset ratio.

Participation constraint is

\[ \bar{d} - d - \pi_H (\bar{d}) \geq 0 \] (IR)
Equity Holder’s Payoff

The bank’s equity holder is residual claimant.

Net expected payoff is

\[
U(A) = Ar - A(\bar{d} - d - \pi(\bar{d})) \\
= A(r - \bar{d} + d + \pi(\bar{d}))
\]

The equity holder’s stake is a portfolio consisting of

- put option on the assets of the bank with strike price \(\bar{D}\)
- risky asset with expected payoff \(A(r - \bar{d} + d)\)
When the asset portfolio consists of the good asset

\[ A \left( r_H - \bar{d} + d + \pi_H (\bar{d}) \right) \]

with bad assets, it is \( A \left( r_L - \bar{d} + d + \pi_L (\bar{d}) \right) \). Incentive compatibility constraint

\[ r_H - r_L \geq \pi_L (\bar{d}) - \pi_H (\bar{d}) \]

\[ = \Delta \pi (\bar{d}) \quad (IC) \]

where \( \Delta \pi (\bar{d}) \equiv \pi_L (\bar{d}) - \pi_H (\bar{d}) \)

(analogous to private benefit \( B \) in Holmstrom-Tirole model).

**Lemma 1.** \( \Delta \pi (z) \) is a single-peaked function of \( z \), and is maximized at the value of \( z \) where \( F_H \) cuts \( F_L \) from below.
Outcome space is \{0, 1, 2, \cdots, Z\}.

\( p(s) \) is price Arrow-Debreu contingent claim at outcome \( s \).
Put option with strike price \( z \) has price

\[
p(z - 1) + 2p(z - 2) + 3p(z - 3) + \cdots + zp(0)
\]

First difference in option prices is

\[
\pi(z) - \pi(z - 1) = p(z - 1) + 2p(z - 2) + 3p(z - 3) + \cdots + zp(0)
\]

\[
- p(z - 2) - 2p(z - 3) - \cdots - (z - 1)p(0)
\]

\[
= p(z - 1) + p(z - 2) + p(z - 3) + \cdots + p(0)
\]

State price density is second difference in option price with respect to the strike price.

(Breeden and Litzenberger (1978))
Leverage Constraint

Assume (IC) binds. Then optimal \( \bar{d}^* \) is smallest solution to the equation:

\[
\Delta \pi (\bar{d}) = r_H - r_L
\]

\( \bar{d}^* \) is mixes notional and market values. This is natural, since option specifies strike price in terms of notional value.

From participation constraint,

\[
d^* = \bar{d}^* - \pi_H (\bar{d}^*)
\]

\[
= \int_0^{1+\bar{r}} \min \{ \bar{d}^*, s \} f_H (s) \, ds
\]

gives debt ratio in market values.
Solving for Balance Sheet Size

Bank equity holder’s expected payoff under the optimal contract is:

\[ U(A) \equiv A \left( r_H - \bar{d}^* + d^* + \pi_H \left( \bar{d}^* \right) \right) \]

Expression inside the brackets is strictly positive, since the equity holder extracts the full surplus.

Equity holder’s payoff is strictly increasing in \( A \). For

\[ \lambda^* \equiv \frac{1}{1 - d^*} \]

We have

\[ A = \lambda^* E \]
Comparative Statics

Outcome densities parameterized by $\sigma$

Higher $\sigma$ indicating mean preserving spreads.

$\pi_H (z, \sigma)$ is value of put option parameterized by $\sigma$.

$$\sigma' > \sigma \Rightarrow \pi_H (z, \sigma') > \pi_H (z, \sigma)$$
Proposition 2. If $\Delta \pi (z, \sigma)$ is increasing in $\sigma$, then both $\bar{d}^*$ and $d^*$ are decreasing in $\sigma$.

Step 1. Show $\bar{d}^*$ is decreasing in $\sigma$ by IC constraint, and use $\Delta \pi (z, \sigma)$ is increasing in $\sigma$.

From (IC) and scalability,

$$\Delta \pi (\bar{d}^*(\sigma), \sigma) = r_H - r_L$$

LHS is increasing in both arguments.

Hence, $\bar{d}^*(\sigma)$ is decreasing in $\sigma$. 
Step 2.

From IR

\[ d^* = \bar{d}^* - \pi_H (\bar{d}^*) \]

\[ = \int_0^{1+\bar{\rho}} \min \{ \bar{d}^*, s \} f_H (s) \, ds \quad (1) \]

Since \( \bar{d}^* \) is decreasing in \( \sigma \), \( d^* \) is decreasing in \( \sigma \).
Value at Risk

Value at risk (VaR) is smallest non-negative number $V$ such that

$$\text{Prob} \left( A_1 < A - V \right) \leq 1 - c$$

Generalized extreme value distribution

$$G \left( z \right) = \exp \left\{ - \left( 1 + \xi \left( \frac{z - \theta}{\sigma} \right) \right)^{-1/\xi} \right\}$$

Parameter $\xi$ is any real number, support is $(-\infty, \theta - \sigma/\xi)$ when $\xi < 0$. 
Take $\xi = -1$, $\sigma = 1$. Family $\{G_L, G_H\}_\theta$ parametrized by $\theta$

$$G_L (z; \theta) = \exp \{z - \theta\} \quad \text{and} \quad G_H (z; \theta) = \exp \{z - k - \theta\}$$

where $k > 0$.

**Condition.** There is $\hat{z}$ such that for all $z \in (0, \hat{z})$

$$F_L (z; \theta) = G_L (z; \theta) \quad \text{and} \quad F_H (z; \theta) = G_H (z; \theta)$$

When $z = 0$

$$F_L (0; \theta) = \int_{-\infty}^{0} G_L (s; \theta) \, ds \quad \text{and} \quad F_H (0; \theta) = \int_{-\infty}^{0} G_H (s; \theta) \, ds$$

Let $\bar{d}^* (\theta)$ be value of $\bar{d}^*$ in contracting problem parameterized by $\theta$. 

31
Proposition 3. For all $\theta \in [\underline{\theta}, \bar{\theta}]$ suppose that $\bar{d}^*(\theta) < \hat{\zeta}$. Suppose also that $r_H - r_L$ stays constant to shifts in $\theta$. Finally, suppose that above condition holds. Then the probability that the bank defaults is constant over all optimal contracts parameterized by $\theta \in [\underline{\theta}, \bar{\theta}]$.

Corollary. Under the conditions of the proposition, assets of the bank are adjusted so that total Value-at-Risk is kept constant and set equal to $E$ in spite of shifts in parameter $\theta$. 

32
Value at Risk

$F_L(z; \hat{\theta})$

$F_H(z; \hat{\theta})$

$F_L(z; \theta)$

$F_H(z; \theta)$

$\pi_L - \pi_H$

$1 - c$

$\bar{d}_\hat{\theta}$

$\bar{d}_\theta$

$0$

$0$
Proof. For \( z \in (0, \hat{z}) \)

\[
\frac{F_L (z; \theta)}{F_H (z; \theta)} = \frac{G_L (z; \theta)}{G_H (z; \theta)} = e^k > 1
\]

Then

\[
\Delta \pi (z; \theta) = \int_0^z (F_L (s; \theta) - F_H (s; \theta)) \, ds = \int_{-\infty}^{\hat{z}} (G_L (s; \theta) - G_H (s; \theta)) \, ds
\]

\[
= (e^k - 1) \int_{-\infty}^{\hat{z}} G_H (s; \theta) \, ds = (e^k - 1) G_H (z; \theta)
\]

(2)
From the IC constraint,

\[ \Delta \pi (\bar{d}^*; \theta) = r_H - r_L, \]

so

\[ (e^k - 1) G_H (\bar{d}^*; \theta) = r_H - r_L \]  \hspace{1cm} (3)

From (2) and (3)

\[ G_H (\bar{d}^*; \theta) = \frac{r_H - r_L}{e^k - 1} = \text{constant} \]
Two Features of Exponential Tail

The result comes from two features:

• Exponential tail means that the ratios of c.d.f.s are constant:

\[
\int_{-\infty}^{z} (G_L (s; \theta) - G_H (s; \theta)) \, ds = (e^k - 1) \int_{-\infty}^{z} G_H (s; \theta) \, ds
\]

• Exponential tail means that \textit{integral} of c.d.f. (to get option value) is the just the c.d.f. itself

\[
(e^k - 1) \int_{-\infty}^{z} G_H (s; \theta) \, ds = (e^k - 1) G_H (z; \theta)
\]
Alternative Modeling Approaches

- VaR constraint is imposed by the creditors (as here).
  - Haircuts in repo contracts

- VaR constraint is self-imposed by equity holders who maximize long-term payoff
  - Future payoffs are attainable only if the bank remains solvent
  - Trade off the increased short term payoff from large balance sheet against greater probability of default