Procyclical Leverage and Value-at-Risk

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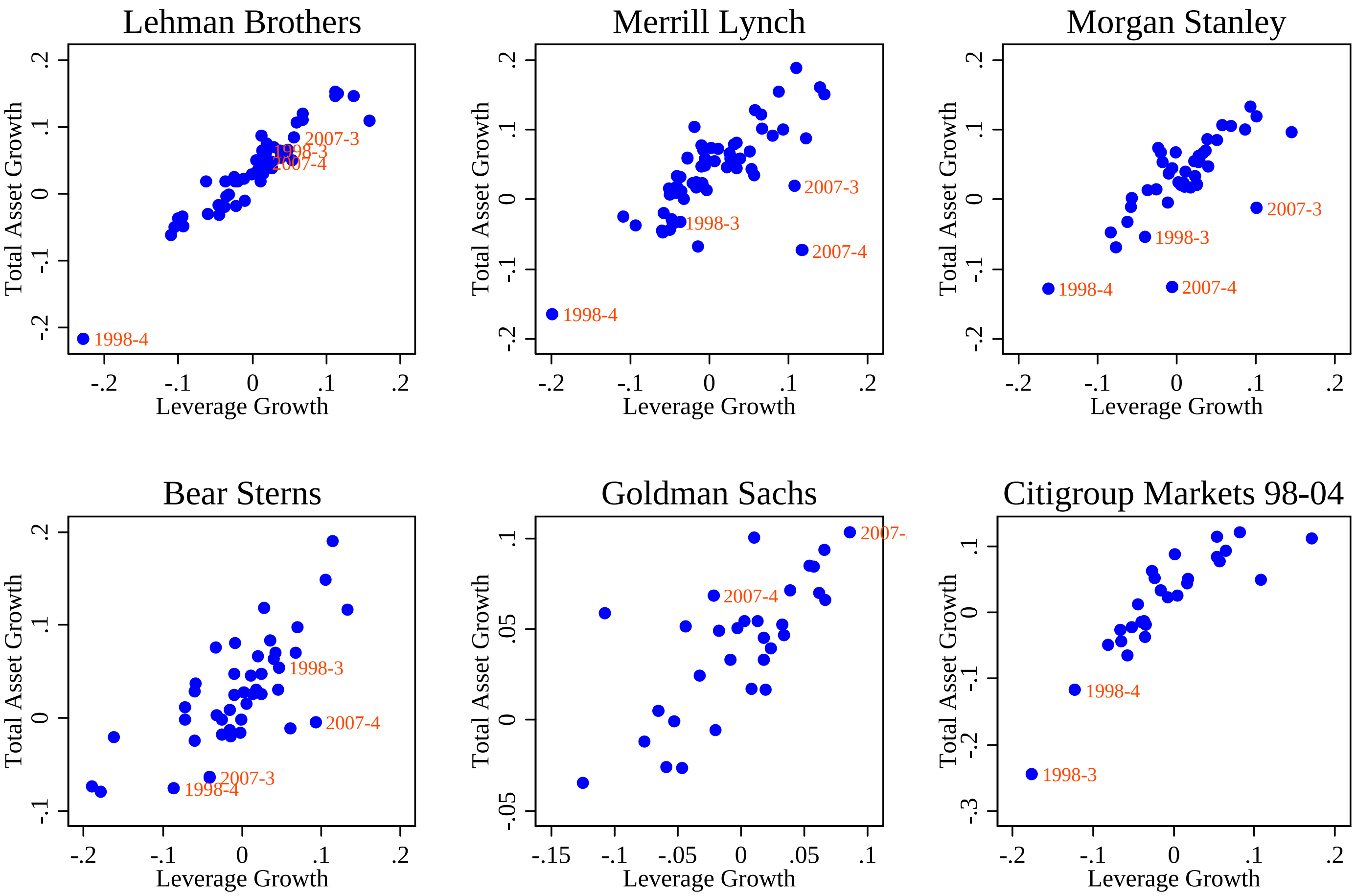
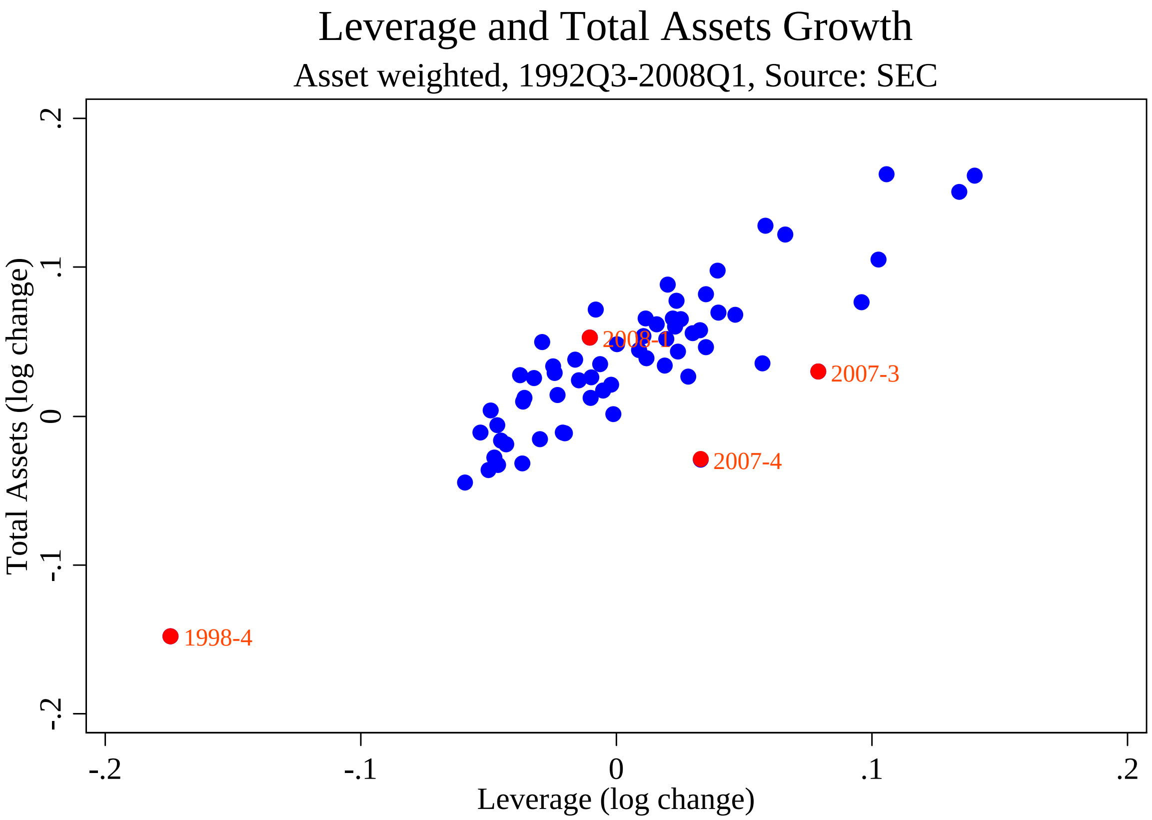
Total Assets and Leverage




Leverage and Total Assets Growth
Asset weighted, 1992Q3-2008Q1, Source: SEC
Total Assets (log change)
.2
.1
0
-.1
-.2
2008-1
2007-3
2007-4
1998-4
-.2
-.1
0
.1
.2
Leverage (log change)

Shifts in Leverage and Total Assets

Slope is approximately 1

$$\ln A_t - \ln A_{t-1} \simeq \delta + \ln\left(\frac{A_t}{E_t}\right) - \ln\left(\frac{A_{t-1}}{E_{t-1}}\right)$$

Suggests...

$$\begin{aligned} \ln E_t &= \delta + \ln E_{t-1} \\ A_t &= \lambda_t^* E_t \end{aligned}$$

Equity seems to be forcing variable and A_t is determined by realization of maximum allowed leverage λ^*

Stands usual way of thinking on its head (A first, then decide E and D).

Explaining Leverage

Value at risk (VaR) at confidence level c relative to some base level A_0 is smallest non-negative number V such that

$$\text{Prob}\left(A < A_0 - V\right) \leq 1 - c$$

Equity E meets total value at risk

$$E = V = v \times A$$

v is Unit VaR (Value-at-Risk per dollar of assets). Leverage L satisfies

$$L \equiv \frac{A}{E} = \frac{1}{v}$$

Empirical implication:

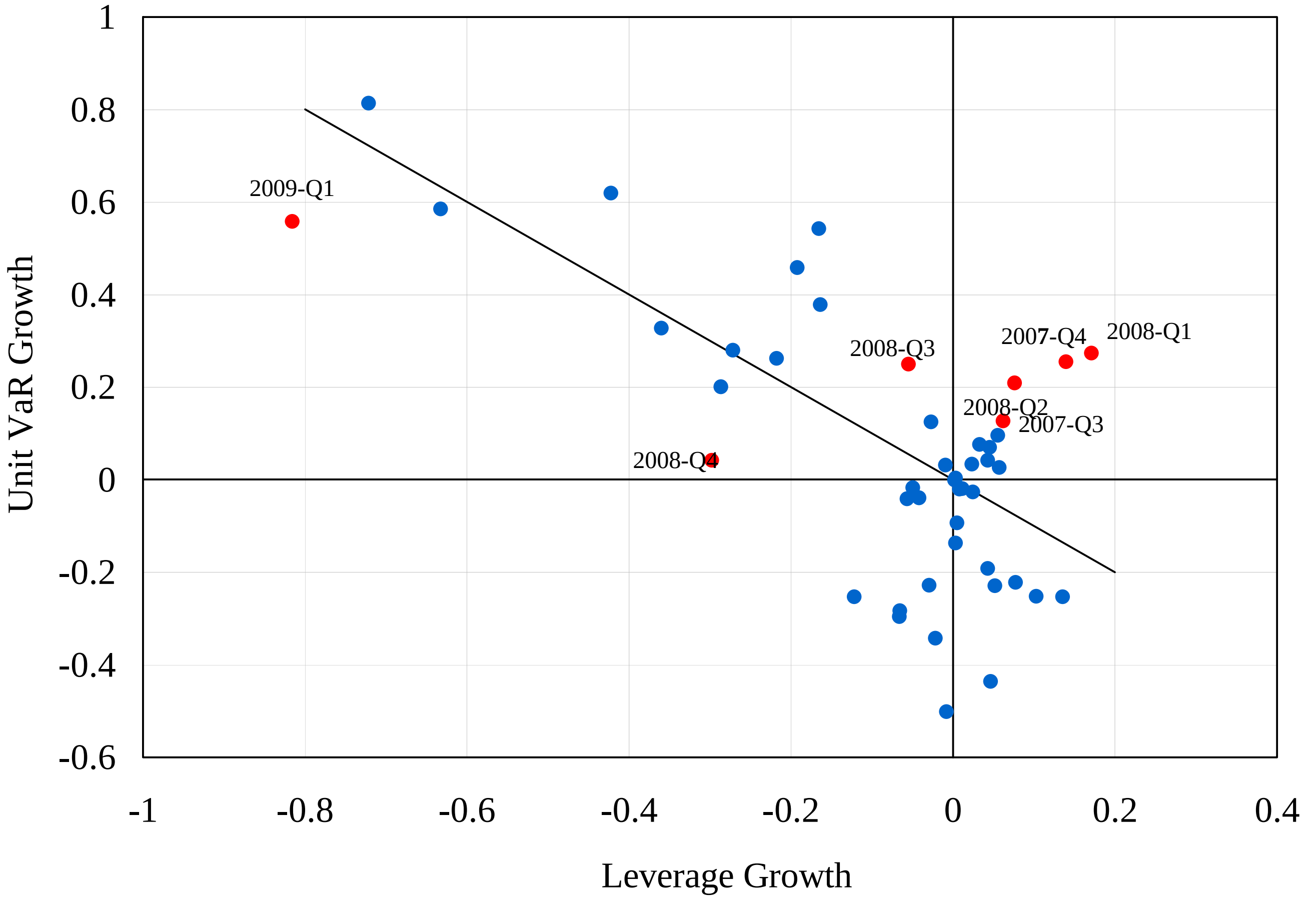
$$\ln L = -\ln v$$

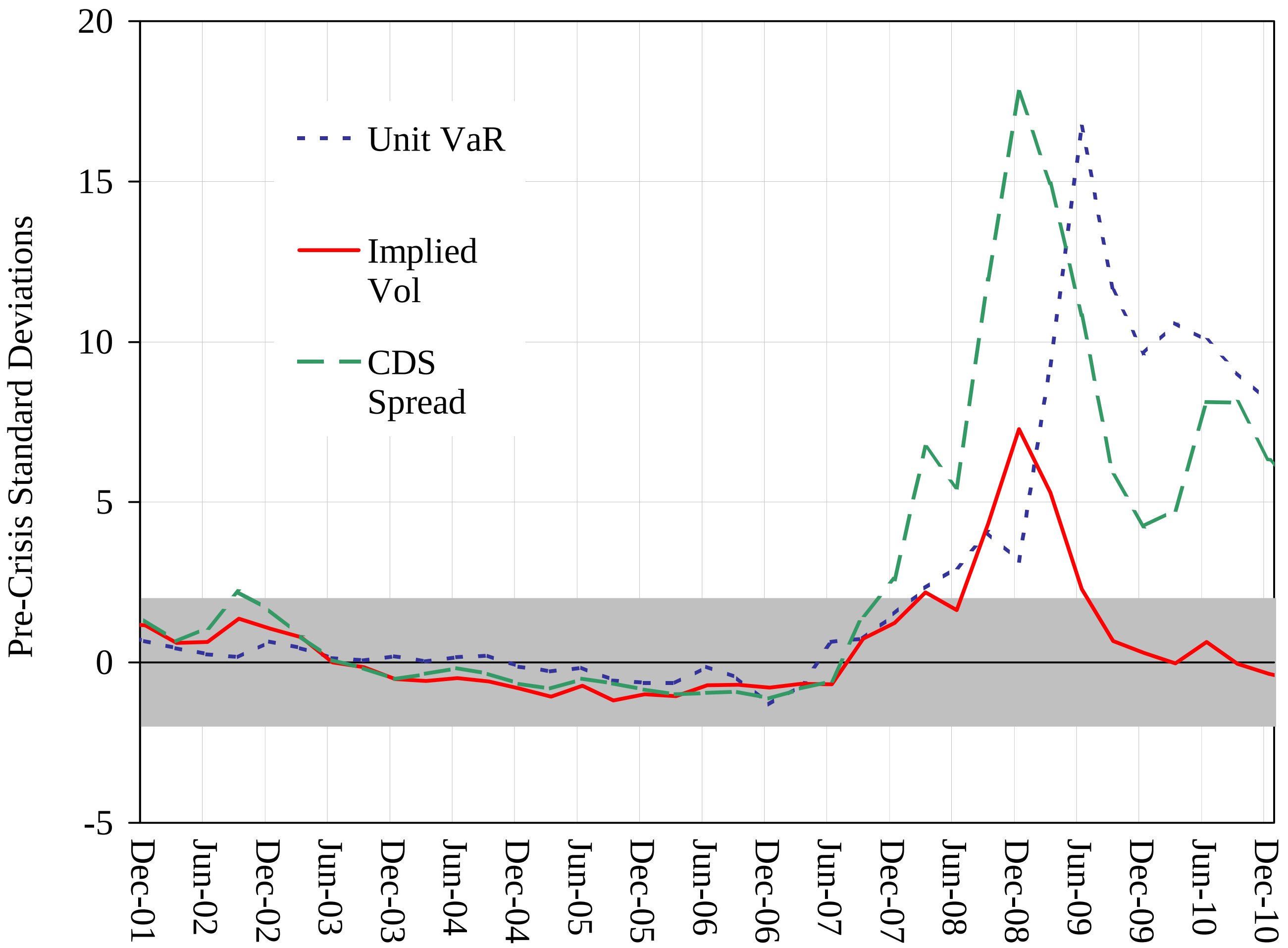
so that

$$\ln L_t - \ln L_{t-1} = -\left(\ln v_t - \ln v_{t-1}\right) \qquad (*)$$

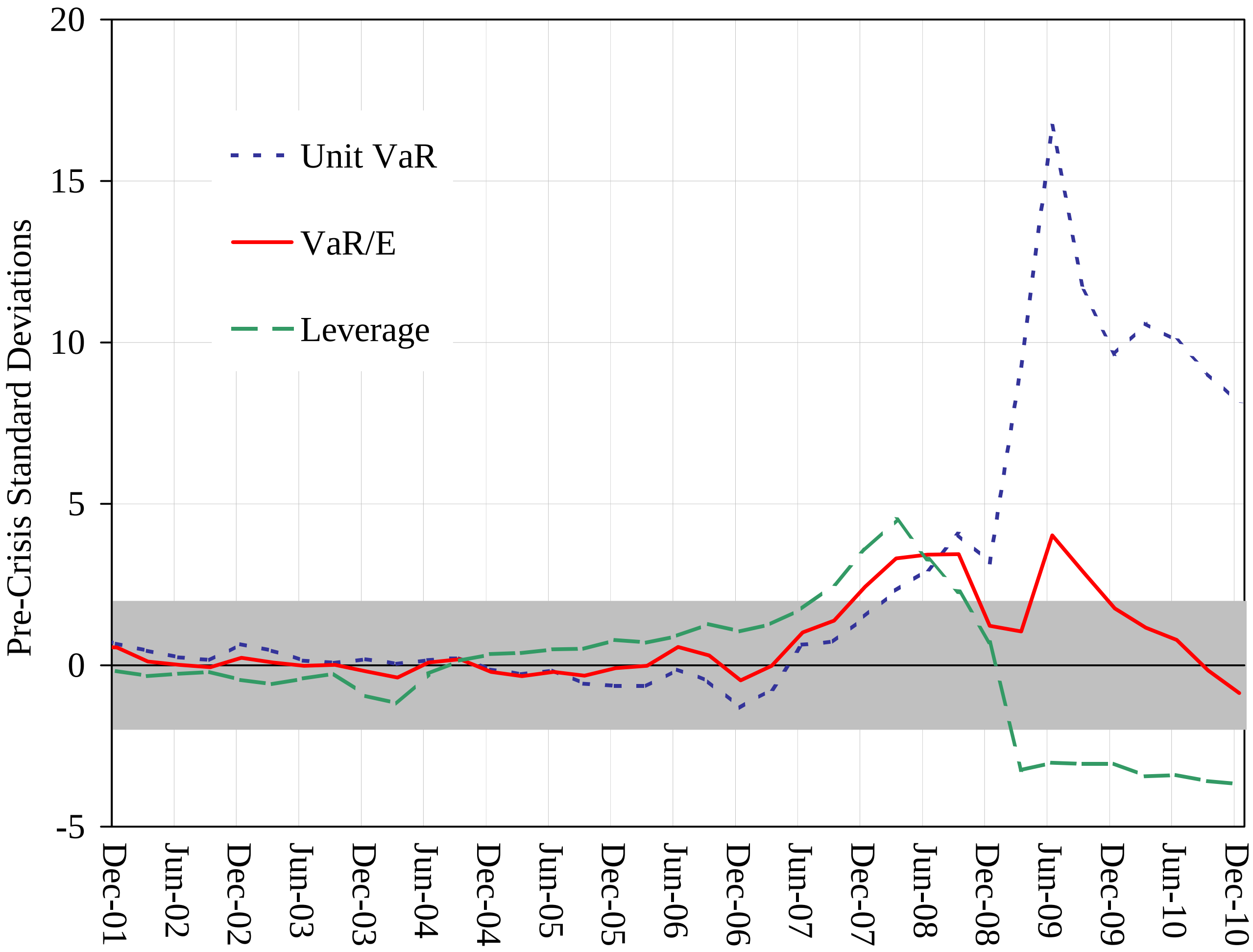
Scatter chart of leverage changes against unit VaR changes should have slope -1.

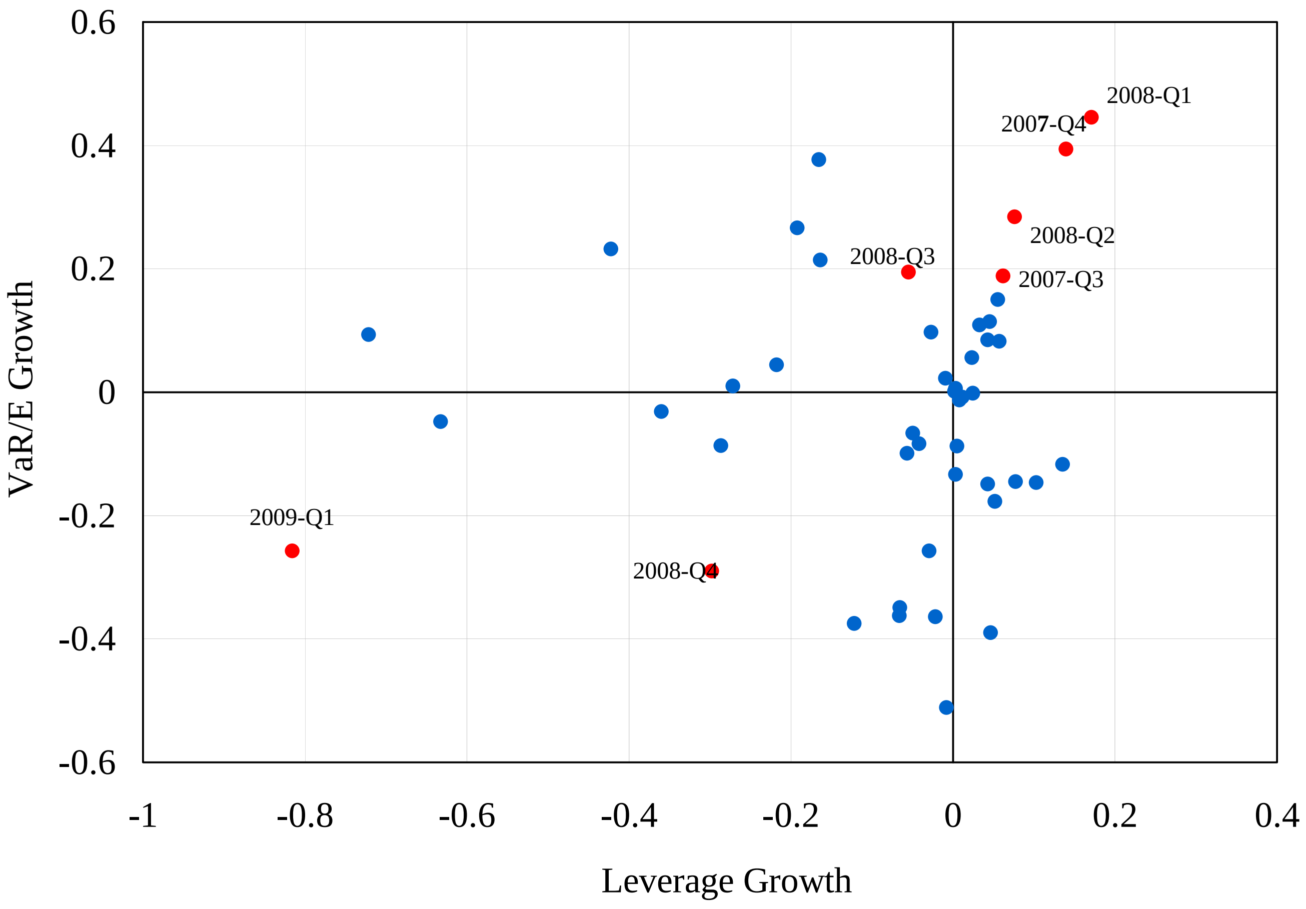
Evidence?

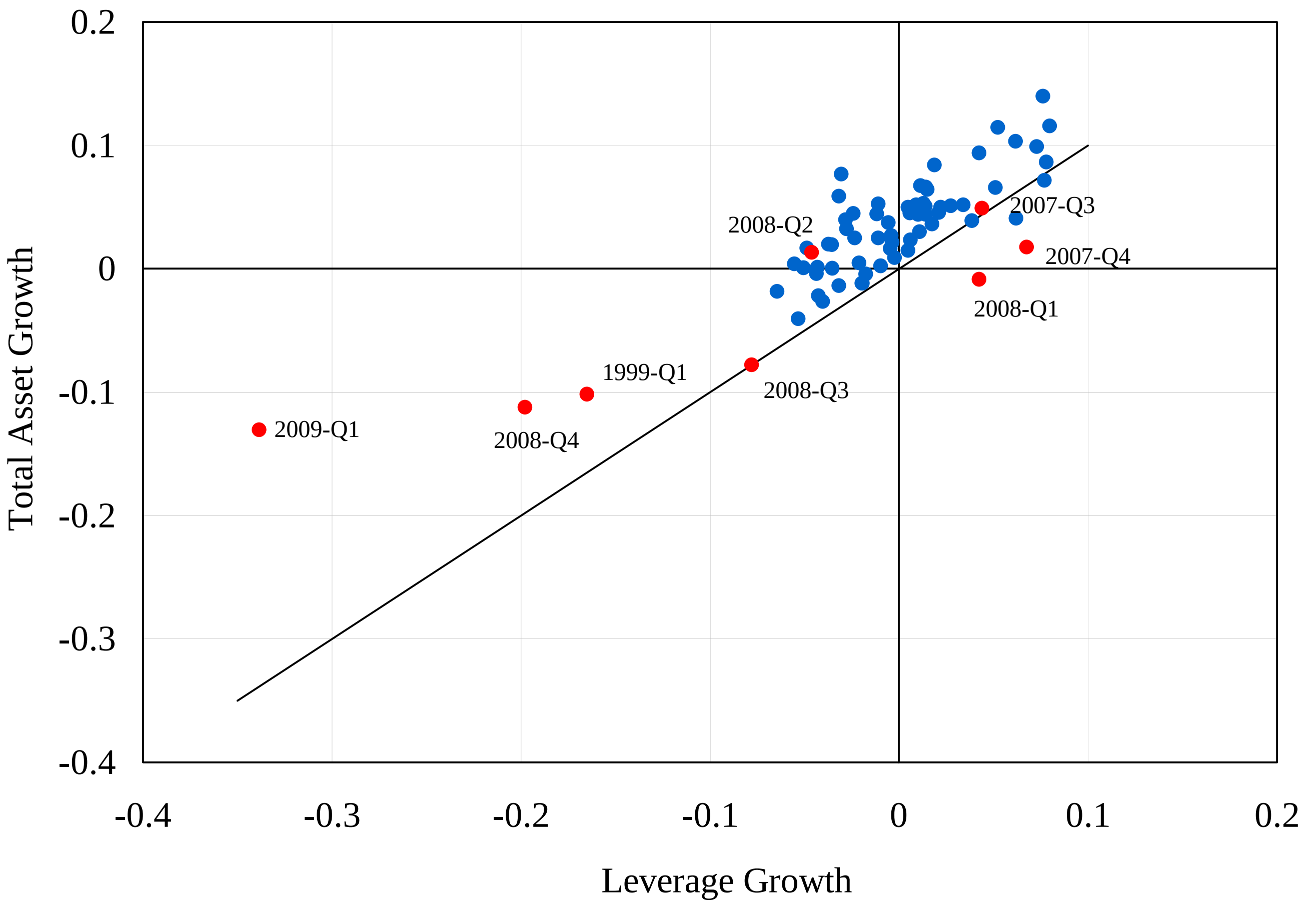

1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
Unit VaR Growth
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Leverage Growth
2009-Q1
2008-Q3
2007-Q4
2008-Q1
2008-Q2
2007-Q3
2008-Q4



Pre-Crisis Standard Deviations
20
15
10
5
0
-5
Unit VaR
Implied Vol
CDS Spread
Dec-01
Jun-02
Dec-02
Jun-03
Dec-03
Jun-04
Dec-04
Jun-05
Dec-05
Jun-06
Dec-06
Jun-07
Dec-07
Jun-08
Dec-08
Jun-09
Dec-09
Jun-10
Dec-10


Pre-Crisis Standard Deviations
20
15
10
5
0
-5
Unit VaR
VaR/E
Leverage
Dec-01
Jun-02
Dec-02
Jun-03
Dec-03
Jun-04
Dec-04
Jun-05
Dec-05
Jun-06
Dec-06
Jun-07
Dec-07
Jun-08
Dec-08
Jun-09
Dec-09
Jun-10
Dec-10


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
VaR/E Growth
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
Leverage Growth
2008-Q1
2007-Q4
2008-Q2
2008-Q3
2007-Q3
2009-Q1
2008-Q4


0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
Total Asset Growth
Leverage Growth
2008-Q2
2007-Q3
2007-Q4
2008-Q1
1999-Q1
2008-Q3
2009-Q1
2008-Q4

Model

Agency model in the spirit of Holmstrom and Tirole (1997)

- Moral hazard comes from choosing sub-optimally risky assets
- Temptation payoff is the additional option value of bad action
- Agent is the borrower (e.g. investment bank).
- Principal is creditor to the bank (e.g. money market fund)

Two dates: date 0 and date 1. Balance sheet at date 0 in market values:

Assets	Liabilities
Assets A	Debt D Equity E

Balance sheet in notional (face) values

Assets	Liabilities
Assets $A(1+\bar{r})$	Debt $\bar{D}$ Equity $\bar{E}$

Interpretation as repo transaction: Sell A worth of securities for D at date 0, repurchase for $\bar{D}$ at date 1. Then, $A-D$ is "haircut" and $\bar{D}/D-1$ is repo interest rate

Solving for Leverage and Balance Sheet Size

Fix E.

Optimal contract maximizes agent's payoff by choice of A, D and $\bar{D}$ subject to (IC) and (IR).

Solve for

- Leverage $\lambda = A/E$
- Balance sheet size A

Overview of Solution

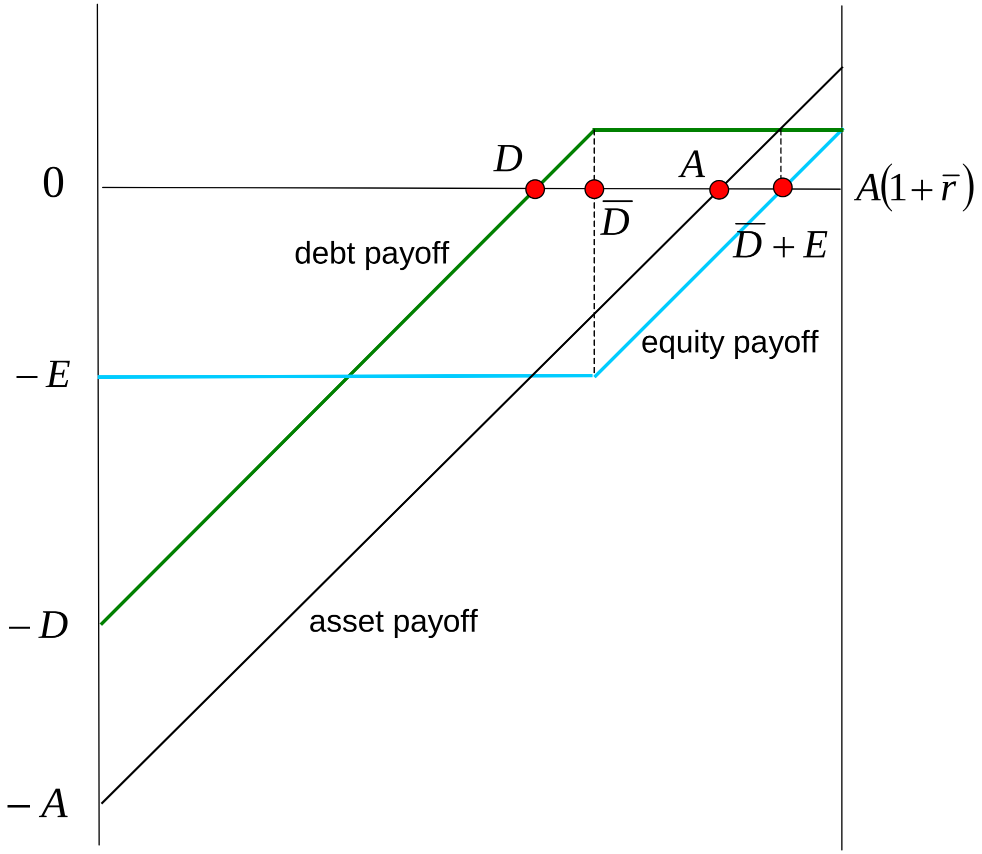
Creditor's payoff has embedded short put option

Broker's payoff has embedded long put option

- IC constraint ties down
  - strike price of put option
  - face value of debt

- IC and IR constraints together tie down
  - market value of debt
  - leverage

Net Payoffs


0
$-E$
$-D$
$-A$
D
$\overline{D}$
A
$\overline{D}+E$
$A(1+\overline{r})$
debt payoff
equity payoff
asset payoff

Moral Hazard

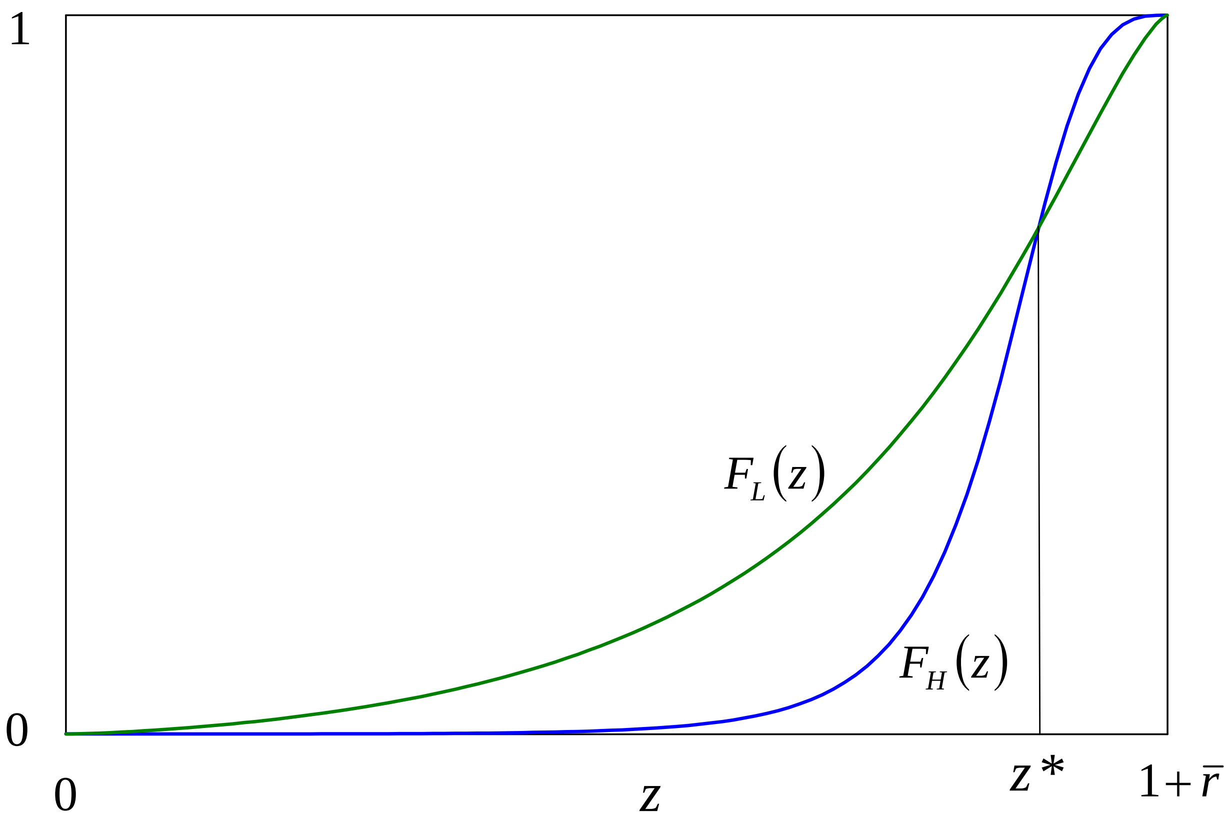
Choice between two types of securities. One dollar of good security has expected payoff

$$1 + r_H$$

outcome density $f_H(.)$. Bad security has expected payoff $1 + r_L$ with density $f_L(.)$.

$$r_L < 0 < r_H$$

But bad security has higher upside potential. F_H cuts F_L precisely once from below.



Cumulative distribution functions

Creditor's Payoff

Creditor's (gross) payoff is portfolio consisting of

- cash of $\bar{D}$
- short put position on assets with strike $\bar{D}$.

$$\begin{aligned} & \bar{D} - A\pi_H\left(\bar{d}\right) \\ = \; & A\left(\bar{d} - \pi_H\left(\bar{d}\right)\right) \end{aligned}$$

where $\bar{d} \equiv \bar{D}/A$

$\pi_H\left(\bar{d}\right)$ is price of put on 1 dollar's worth of assets, with strike $\bar{d}$.

(Merton (JF 1974))

Creditor's *net* expected payoff is

$$V = A\left(\bar{d} - d - \pi_H\left(\bar{d}\right)\right)$$

where $d \equiv D/A$ is debt/asset ratio.

Participation constraint is

$$\bar{d} - d - \pi_H\left(\bar{d}\right) \geq 0 \tag{IR}$$

Equity Holder's Payoff

The bank's equity holder is residual claimant.

Net expected payoff is

$$\begin{aligned} U(A) &= Ar - A\left(\bar{d} - d - \pi\left(\bar{d}\right)\right) \\ &= A\left(r - \bar{d} + d + \pi\left(\bar{d}\right)\right) \end{aligned}$$

The equity holder's stake is a portfolio consisting of

- put option on the assets of the bank with strike price $\bar{D}$
- risky asset with expected payoff $A\left(r - \bar{d} + d\right)$

When the asset portfolio consists of the good asset

$$A\left(r_H - \bar{d} + d + \pi_H\left(\bar{d}\right)\right)$$

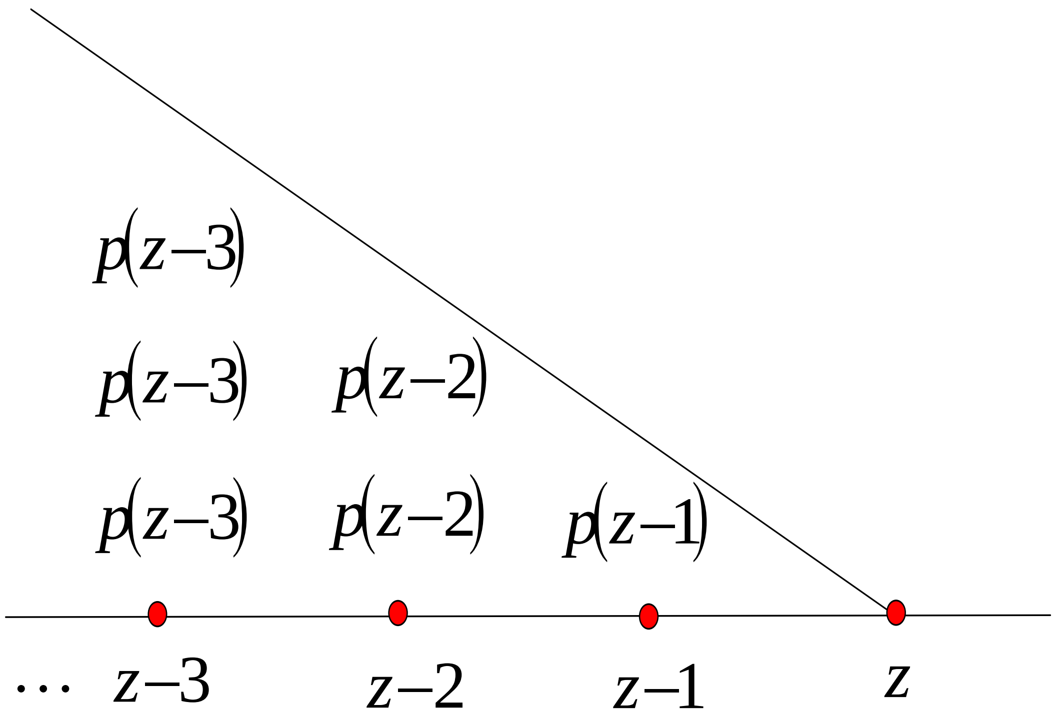
with bad assets, it is $A\left(r_L - \bar{d} + d + \pi_L\left(\bar{d}\right)\right)$. Incentive compatibility constraint

$$\begin{aligned} r_H - r_L &\geq \pi_L\left(\bar{d}\right) - \pi_H\left(\bar{d}\right) \\ &= \Delta\pi\left(\bar{d}\right) \end{aligned} \tag{IC}$$

where $\Delta\pi\left(\bar{d}\right) \equiv \pi_L\left(\bar{d}\right) - \pi_H\left(\bar{d}\right)$

(analogous to private benefit B in Holmstrom-Tirole model).

Lemma 1. *$\Delta\pi(z)$ is a single-peaked function of z, and is maximized at the value of z where F_H cuts F_L from below.*



Outcome space is $\{0, 1, 2, \cdots, Z\}$.

$p(s)$ is price Arrow-Debreu contingent claim at outcome s

Put option with strike price z has price

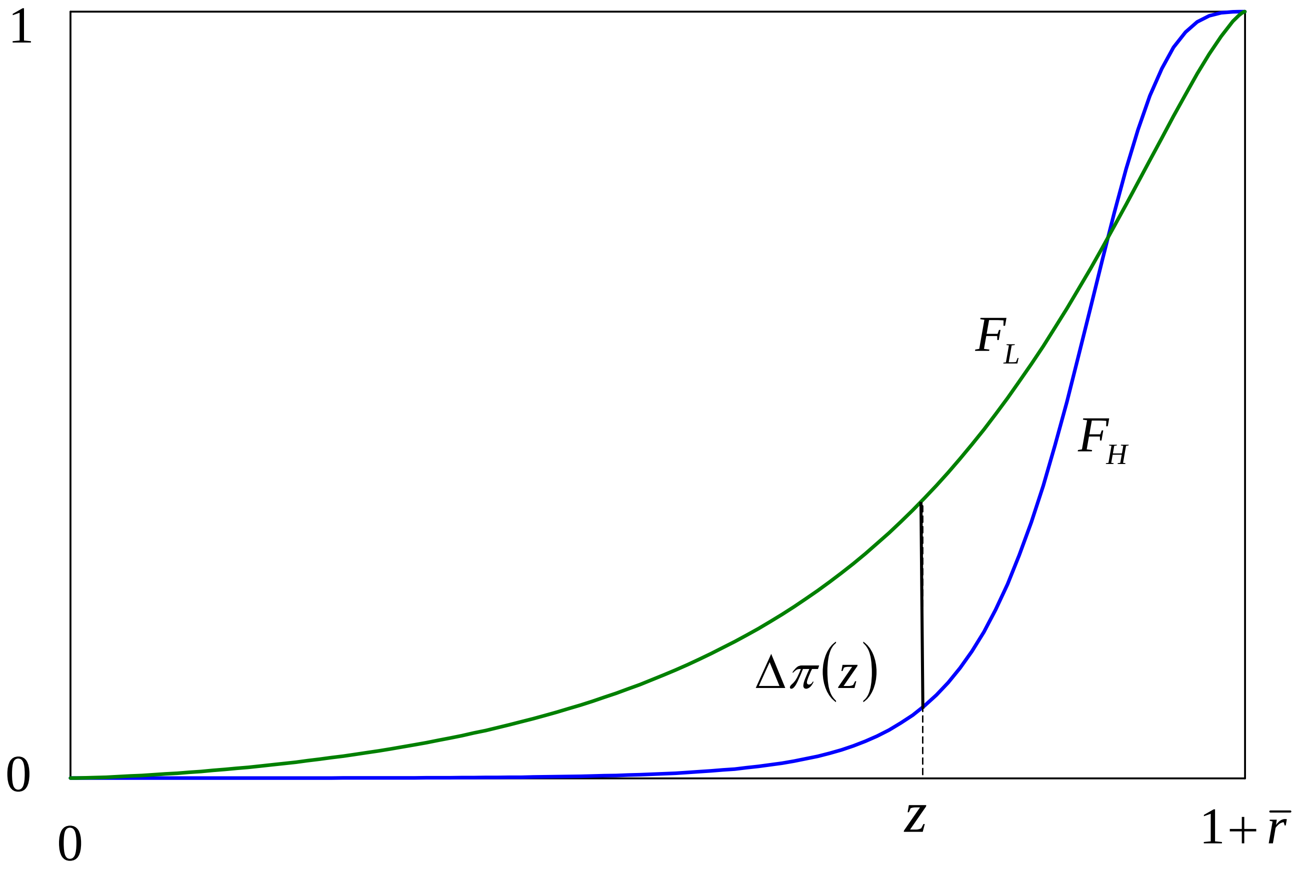
$$p\left(z-1\right)+2p\left(z-2\right)+3p\left(z-3\right)+\cdots+zp\left(0\right)$$

First difference in option prices is

$$\begin{aligned}\pi\left(z\right)-\pi\left(z-1\right) &= p\left(z-1\right)+2p\left(z-2\right)+3p\left(z-3\right)+\cdots+zp\left(0\right)\\ &\qquad -p\left(z-2\right)-2p\left(z-3\right)-\cdots-\left(z-1\right)p\left(0\right)\\ &= p\left(z-1\right)+p\left(z-2\right)+p\left(z-3\right)+\cdots+p\left(0\right)\end{aligned}$$

State price density is *second difference* in option price with respect to the strike price.

(Breeden and Litzenberger (1978))



Difference in Option Values $\Delta\pi\,(z)$

Leverage Constraint

Assume (IC) binds. Then optimal $\bar{d}^*$ is smallest solution to the equation:

$$\Delta\pi\left(\bar{d}\right) = r_H - r_L$$

$\bar{d}^*$ is mixes notional and market values. This is natural, since option specifies strike price in terms of notional value.

From participation constraint,

$$\begin{aligned} d^* &= \bar{d}^* - \pi_H\left(\bar{d}^*\right) \\ &= \int_0^{1+\bar{r}} \min\left\{\bar{d}^*, s\right\} f_H\left(s\right) ds \end{aligned}$$

gives debt ratio in market values.

Solving for Balance Sheet Size

Bank equity holder's expected payoff under the optimal contract is:

$$U\left(A\right) \equiv A\left(r_H - \bar{d}^* + d^* + \pi_H\left(\bar{d}^*\right)\right)$$

Expression inside the brackets is strictly positive, since the equity holder extracts the full surplus.

Equity holder's payoff is strictly increasing in A. For

$$\lambda^* \equiv \frac{1}{1 - d^*}$$

We have

$$A = \lambda^* E$$

Comparative Statics

Outcome densities parameterized by σ

Higher σ indicating mean preserving spreads.

$\pi_H(z, \sigma)$ is value of put option parameterized by σ.

$$\sigma' > \sigma \Rightarrow \pi_H(z, \sigma') > \pi_H(z, \sigma)$$

Proposition 2. *If* $\Delta\pi\left(z,\sigma\right)$ *is increasing in* σ*, then both* $\bar{d}^*$ *and* d^* *are decreasing in* σ*.*

Step 1. Show $\bar{d}^*$ is decreasing in σ by IC constraint, and use $\Delta\pi\left(z,\sigma\right)$ is increasing in σ.

From (IC) and scalability,

$$\Delta\pi\left(\bar{d}^*\left(\sigma\right),\sigma\right)=r_H-r_L$$

LHS is increasing in both arguments.

Hence, $\bar{d}^*\left(\sigma\right)$ is decreasing in σ.

Step 2.

From IR

$$\begin{aligned} d^* &= \bar{d}^* - \pi_H\left(\bar{d}^*\right) \\ &= \int_0^{1+\bar{r}} \min\left\{\bar{d}^*, s\right\} f_H(s)\, ds \end{aligned} \tag{1}$$

Since $\bar{d}^*$ is decreasing in σ, d^* is decreasing in σ.

Value at Risk

Value at risk (VaR) is smallest non-negative number V such that

$$\text{Prob}\,(A_1 < A - V) \leq 1 - c$$

Generalized extreme value distribution

$$G\,(z) = \exp\left\{-\left(1 + \xi\left(\frac{z-\theta}{\sigma}\right)\right)^{-1/\xi}\right\}$$

Parameter ξ is any real number, support is $(-\infty, \theta - \sigma/\xi)$ when $\xi < 0$.

Take $\xi = -1$, $\sigma = 1$. Family $\{G_L, G_H\}_\theta$ parametrized by θ

$$G_L\left(z;\theta\right) = \exp\left\{z - \theta\right\} \quad \text{and} \quad G_H\left(z;\theta\right) = \exp\left\{z - k - \theta\right\}$$

where $k > 0$.

Condition. There is $\hat{z}$ such that for all $z \in (0, \hat{z})$

$$F_L\left(z;\theta\right) = G_L\left(z;\theta\right) \text{ and } F_H\left(z;\theta\right) = G_H\left(z;\theta\right)$$

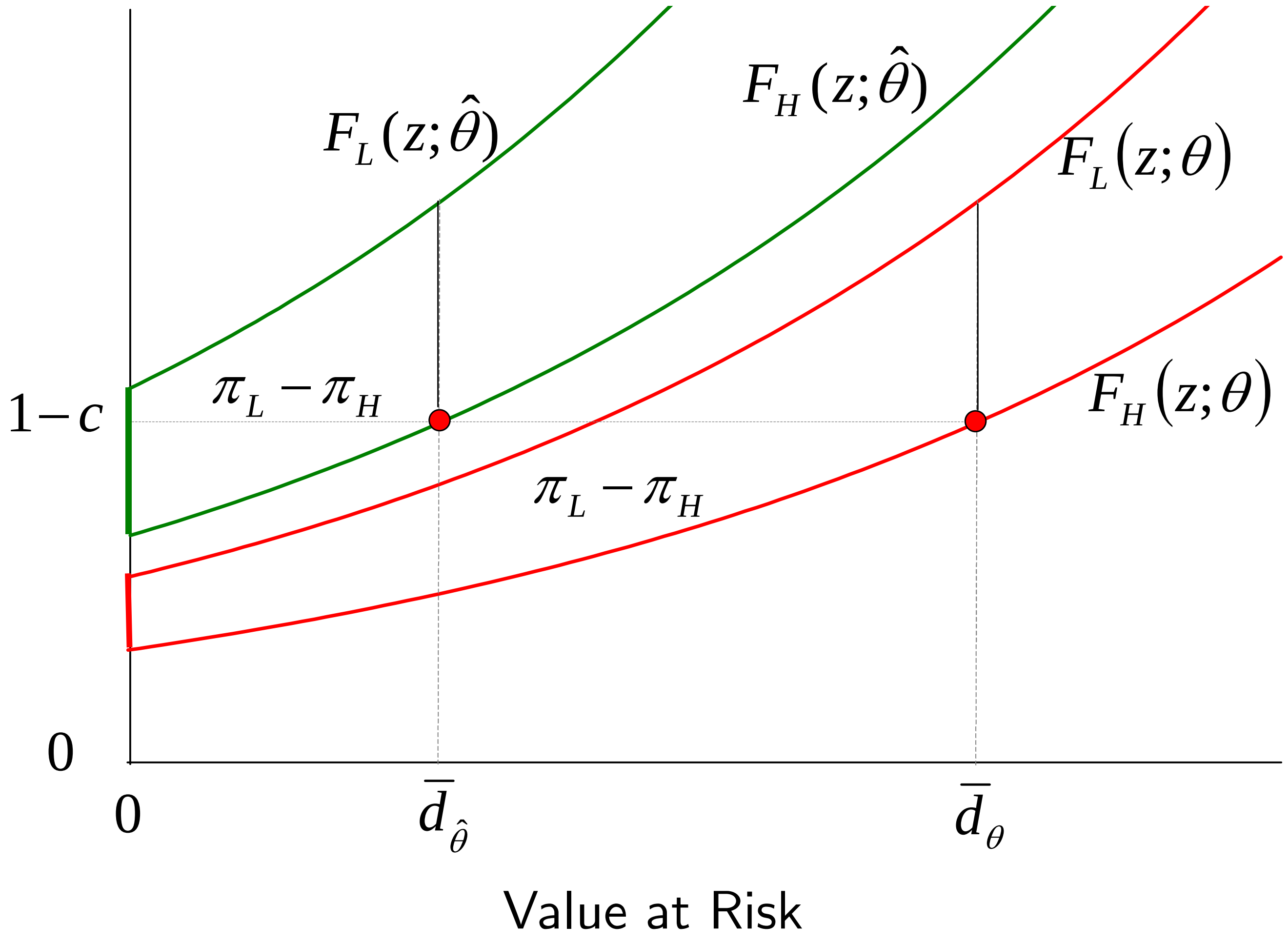
When $z = 0$

$$F_L\left(0;\theta\right) = \int_{-\infty}^{0} G_L\left(s;\theta\right) ds \text{ and } F_H\left(0;\theta\right) = \int_{-\infty}^{0} G_H\left(s;\theta\right) ds$$

Let $\bar{d}^*\left(\theta\right)$ be value of $\bar{d}^*$ in contracting problem parameterized by θ.

Proposition 3. *For all* $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$ *suppose that* $\bar{d}^{*}(\theta) < \hat{z}$*. Suppose also that* $r_H - r_L$ *stays constant to shifts in* θ*. Finally, suppose that above condition holds. Then the probability that the bank defaults is constant over all optimal contracts parameterized by* $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$*.*

Corollary. Under the conditions of the proposition, assets of the bank are adjusted so that total Value-at-Risk is kept constant and set equal to E in spite of shifts in parameter θ.


$F_L(z;\hat{\theta})$
$F_H(z;\hat{\theta})$
$F_L(z;\theta)$
$F_H(z;\theta)$
$\pi_L - \pi_H$
$\pi_L - \pi_H$
$1-c$
0
0
$\bar{d}_{\hat{\theta}}$
$\bar{d}_{\theta}$
Value at Risk

Proof. For $z \in (0, \hat{z})$

$$\frac{F_L(z;\theta)}{F_H(z;\theta)} = \frac{G_L(z;\theta)}{G_H(z;\theta)} = e^k > 1$$

Then

$$\begin{aligned}
\Delta\pi(z;\theta) &= \int_0^z \left(F_L(s;\theta) - F_H(s;\theta)\right) ds \\
&= \int_{-\infty}^z \left(G_L(s;\theta) - G_H(s;\theta)\right) ds \\
&= \left(e^k - 1\right) \int_{-\infty}^z G_H(s;\theta)\, ds \\
&= \left(e^k - 1\right) G_H(z;\theta) \qquad (2)
\end{aligned}$$

From the IC constraint,

$$\Delta\pi\left(\bar{d}^{*};\theta\right)=r_{H}-r_{L},$$

so

$$\left(e^{k}-1\right)G_{H}\left(\bar{d}^{*};\theta\right)=r_{H}-r_{L} \tag{3}$$

From (2) and (3)

$$G_{H}\left(\bar{d}^{*};\theta\right)=\frac{r_{H}-r_{L}}{e^{k}-1}=\text{ constant}$$

Two Features of Exponential Tail

The result comes from two features:

- Exponential tail means that the ratios of c.d.f.s are constant:

$$\int_{-\infty}^{z} \left(G_L\left(s;\theta\right) - G_H\left(s;\theta\right)\right) ds = \left(e^k - 1\right) \int_{-\infty}^{z} G_H\left(s;\theta\right) ds$$

- Exponential tail means that *integral* of c.d.f. (to get option value) is the just the c.d.f. itself

$$\left(e^k - 1\right) \int_{-\infty}^{z} G_H\left(s;\theta\right) ds = \left(e^k - 1\right) G_H\left(z;\theta\right)$$

Alternative Modeling Approaches

- VaR constraint is imposed by the creditors (as here).
  - Haircuts in repo contracts

- VaR constraint is self-imposed by equity holders who maximize long-term payoff
  - Future payoffs are attainable only if the bank remains solvent
  - Trade off the increased short term payoff from large balance sheet against greater probability of default