Balance Sheet Capacity and Endogenous Risk

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December 2010
Balance Sheet Capacity

• Balance sheet capacity is capacity of banking (intermediary) sector to channel credit

• Balance sheet capacity is impaired during crises
  — Erosion of bank capital due to credit losses
  — Deleveraging

• One rationale for central bank purchase of risky securities is to restore lost balance sheet capacity
Procyclical Leverage

Adrian and Shin (2010)
Balance Sheet Capacity Decomposed

Asset growth \[= \log A(t+1) - \log A(t)\]

Leverage Growth \[= \log A(t+1) - \log A(t) - (\log E(t+1) - \log E(t))\]

45 degree line represents constant equity line:

\[\log E(t+1) - \log E(t) = 0\]

Above 45 degree line, equity increasing.
Below 45 degree line, equity decreasing.

Straight line with slope 1 is constant equity growth line, with intercept being growth rate
Balance Sheet Capacity Decomposed

• Equity seems to play role of exogenous variable
  — Contrast to corporate finance textbooks extending MM

• Balance sheet capacity is

\[
\text{Equity} \times \text{Permitted Leverage}
\]

• Our approach: intermediaries are subject to Value-at-Risk constraint

\[
\text{Equity} = \text{Total Assets} \times \text{VaR per dollar of assets}
\]

\[
\text{Leverage} = \frac{\text{Assets}}{\text{Equity}} = \frac{1}{\text{Unit VaR}}
\]
Endogeneity of Risk

• True risks impacting financial markets are attributable (at least in part) to actions of economic agents.

• Endogenous risk is fixed point of mapping:

\[ \text{perceived risk} \Rightarrow \text{actual risk} \]

• Solve for volatility \( \sigma \) and risk premium \( \mu - r \) in closed form, where
  
  —Banking sector capital (equity) is state variable
  —Fundamental uncertainty is constant
Some Themes

• Volatility depends on balance sheet capacity, even though fundamental uncertainty is constant ("Endogenous Risk" Danielsson and Shin (2003))

• Risk premium depends on balance sheet capacity
  —Lagrange multiplier of VaR constraint enters like a risk aversion parameter
  —"As if" preferences + endogenous risk generate risk premiums that depend on balance sheet capacity

• Implication: intermediary balance sheet size forecasts asset returns
  —Adrian, Moench and Shin (2009) for asset returns in US
  —Etula (2009) for commodities
  —Adrian, Etula and Shin (2009) for FX returns for USD cross rates
Simplified Financial System

- Banks
  - (Active Investors)
  - Debt Claims
- Households
  - (Passive Investors)
- Intermediated Credit
- Directly granted credit
- end-user borrowers

Intermediated and Directly Granted Credit
Model

• Time indexed by $t \in [0, \infty)$. 

• $N > 0$ non-dividend paying risky securities (date $t$ price of $i$th risky security is $P_t^i$)

• One risk-free bond ($B_0 = 1$, $dB_t = rB_tdt$, with $r$ constant)

• Two types of traders
  — Active (risk-neutral, with VaR constraints) - banks
  — Passive (residual demand/supply curves) - households, value investors
Posit equilibrium of form:

\[
\begin{bmatrix}
\frac{dP^1_t}{P^1_t} \\
\vdots \\
\frac{dP^N_t}{P^N_t}
\end{bmatrix} =
\begin{bmatrix}
\mu^1_t \\
\vdots \\
\mu^N_t
\end{bmatrix}
\, dt +
\begin{bmatrix}
-\sigma^1_t & - \\
-\sigma^N_t & -
\end{bmatrix}
\begin{bmatrix}
\frac{dW^1_t}{dW^1_t} \\
\vdots \\
\frac{dW^N_t}{dW^N_t}
\end{bmatrix}
\]

\(\{W^i_t\}\) independent Brownian motions (fundamental shocks enter via passive traders’ demands)

Scalars \(\{\mu^i_t\}\) and \(1 \times N\) vectors \(\{\sigma^i_t\}\) are as yet undetermined coefficients to be solved in equilibrium

Solve for rational expectations equilibrium (REE) with respect to active traders’ beliefs
Portfolio Choice of Active Traders

- Risk-neutral traders
- (Ultra) short horizons
- Value-at-Risk (VaR) constraint: capital (i.e. equity) should be large enough to meet VaR

$D^i_t$ is dollar holding of $i$th security at $t$

$V_t$ is trader’s capital (notice no superscript for trader, due to aggregation result, to follow)

Balance sheet identity

$$b_t B_t = V_t - \sum_i D^i_t$$
Evolution of capital

\[ dV_t = [rV_t + D^\top_t (\mu_t - r)] \, dt + D^\top_t \sigma_t dW_t \]

\( D^\top \) is transpose of \( D \), \( \sigma_t \) is the \( N \times N \) diffusion matrix, \( r = (r, \ldots, r)^\top \).

Expected capital gain:

\[ E_t[dV_t] = [rV_t + D^\top_t (\mu_t - r)] dt \] (1)

Variance of capital:

\[ \text{Var}_t(dV_t) = D^\top_t \sigma_t \sigma_t^\top D_t dt \] (2)

Trader maximizes (1) subject to VaR constraint, where VaR is \( \alpha \) times forward-looking standard deviation of return on equity.
Assuming trader is solvent ($V_t > 0$) maximization problem is

$$\max_{D_t} \ rV_t + D_t^\top (\mu_t - r) \quad \text{subject to} \quad \alpha \sqrt{D_t^\top \Sigma_t \Sigma_t^\top D_t} \leq V_t$$

First-order condition

$$\mu_t - r = \alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t \Sigma_t D_t$$

where $\gamma_t$ is Lagrange multiplier for VaR constraint, and $\Sigma_t := \sigma_t \sigma_t^\top$. 

$$D_t = \frac{1}{\alpha (D_t^\top \Sigma_t D_t)^{-1/2} \gamma_t \Sigma_t^{-1}} (\mu_t - r)$$
Constraint binds due to risk-neutrality

\[ V_t = \alpha \sqrt{D_t^\top \Sigma_t D_t} \]  \hspace{1cm} (3)

Therefore

\[ D_t = \frac{V_t}{\alpha^2 \gamma_t} \Sigma_t^{-1}(\mu_t - r) \]

"As if" preferences. Optimal portfolio is similar to mean-variance optimal portfolio where the Lagrange multiplier \( \gamma_t \) appears like a risk-aversion coefficient.

Substitute into (3) to solve for Lagrange multiplier

\[ \gamma_t = \frac{\sqrt{\xi_t}}{\alpha} \]
where

$$\xi_t := (\mu_t - r)^\top \Sigma_t^{-1} (\mu_t - r)$$

Lagrange multiplier $\gamma_t$ is

- proportional to generalized Sharpe ratio $\sqrt{\xi}$
- does not depend directly on equity $V_t$

**Interpretation.** Additional unit of capital relaxes VaR constraint by multiple $\alpha$ of standard deviation, raising expected return by risk-premium on the portfolio per unit of standard deviation
Finally, solve for optimal portfolio:

\[ D_t = \frac{V_t}{\alpha \sqrt{\xi_t}} \Sigma_t^{-1}(\mu_t - r) \]

Optimal portfolio is homogeneous of degree one in equity \( V_t \)

**Aggregation result.** Portfolio depends on \( V_t \), total capital of active trading (banking?) sector, not on profile of individual equity capital.

⇒ Take aggregate capital, \( V_t \), as state variable
Closing the Model

Passive traders in aggregate have vector-valued exogenous demand schedule for the risky assets, \( y_t = (y_t^1, \ldots, y_t^N) \) where

\[
y_t = \Sigma_t^{-1} \begin{bmatrix}
\delta^1 (z_t^1 - \ln P_t^1) \\
\vdots \\
\delta^N (z_t^N - \ln P_t^N)
\end{bmatrix}
\]

\( \delta^i \) is scaling parameter for slope of the demand curve

\( z_t^i \) is positive demand shock for asset \( i \)

\[
dz_t^i = r^* dt + \eta \sigma_z^i dW_t
\]
The market-clearing condition $D_t + y_t = 0$ gives

$$\frac{V_t}{\alpha \sqrt{\xi_t}}(\mu_t - r) + \left[ \begin{array}{c} \delta^1 (z^1_t - \ln P^1_t) \\ \vdots \\ \delta^N (z^N_t - \ln P^N_t) \end{array} \right] = 0$$

Equilibrium prices are

$$P^i_t = \exp \left( \frac{V_t}{\alpha \delta^i \sqrt{\xi_t}}(\mu^i_t - r) + z^i_t \right); \quad i = 1, \ldots, N$$
Single Risky Asset

Look for equilibrium of form:

\[
\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t \tag{4}
\]

\(\mu_t\) and \(\sigma_t\) are undetermined coefficients to be solved in equilibrium, \(W_t\) is standard (scalar) Brownian motion. Equivalently,

\[
d\ln P_t = \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t \tag{5}
\]

“Seeds” of fundamental shocks given by exogenous shocks to passive traders’ demands:

\[
dz_t = r^* dt + \eta \sigma_z dW_t, \quad \text{for known constants } \eta > 0, \sigma_z > 0
\]
Start with market-clearing price with VaR-constrained traders

\[ P_t = \exp \left( z_t + \frac{\sigma_t V_t}{\alpha \delta} \right) \]

Take logs and apply Ito’s Lemma

\[ d \ln P_t = d \left( z_t + \frac{\sigma_t V_t}{\alpha \delta} \right) \]

\[ = r^* dt + \eta \sigma_z dW_t + \frac{1}{\alpha \delta} \left( \sigma_t dV_t + V_t d\sigma_t + dV_t d\sigma_t \right) \quad (6) \]

- Unpack \( dV_t \) and \( d\sigma_t \), and substitute back into (6)

- Compare coefficients with (5)
Step 1. Unpack $dV_t$ as an Itô process

$$
\begin{align*}
    dV_t &= [rV_t + D_t(\mu_t - r)]dt + D_t\sigma_t dW_t \\
    &= \left[ rV_t + \frac{V_t(\mu_t - r)}{\alpha\sigma_t} \right] dt + \frac{V_t}{\alpha} dW_t
\end{align*}
$$

Key step is the simplification allowed by binding VaR constraint:

$$
\alpha\sigma_tD_t = V_t
$$
Step 2. Unpack $d\sigma_t$ as an Ito process

From Itô’s Lemma on $\sigma(V_t)$,

$$
d\sigma_t = \frac{\partial \sigma}{\partial V_t} dV_t + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} (dV_t)^2
= \left\{ \frac{\partial \sigma}{\partial V_t} \left[ rV_t + \frac{V_t(\mu_t - r)}{\alpha \sigma_t} \right] + \frac{1}{2} \frac{\partial^2 \sigma}{\partial (V_t)^2} \left( \frac{V_t}{\alpha} \right)^2 \right\} dt + \frac{\partial \sigma}{\partial V_t} \frac{V_t}{\alpha} dW_t \tag{7}
$$

where we substitute in $dV_t$ and where $(dV_t)^2 = \left( \frac{V_t}{\alpha} \right)^2 dt$
Substitute everything back into (6) and re-arrange:

\[
d\ln P_t = \text{(drift term)}\ d\tau + \left[ \eta \sigma_{\tau} + \frac{1}{\alpha \delta} \left( \sigma_{t} \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right) \right] \ dW_t
\]  

(\*)

By hypothesis,

\[
d\ln P_t = \left( \mu_{t} - \frac{1}{2} \sigma_{t}^2 \right) \ d\tau + \sigma_{t} \ dW_t
\]  

(\**

Comparing coefficients between (*) and (**),

\[
\sigma(V_t) = \eta \sigma_{\tau} + \frac{1}{\alpha \delta} \left( \sigma_{t} \frac{V_t}{\alpha} + V_t \frac{\partial \sigma_t}{\partial V_t} \frac{V_t}{\alpha} \right)
\]
giving ordinary differential equation:

$$V_t^2 \frac{\partial \sigma}{\partial V_t} = \alpha^2 \delta (\sigma_t - \eta \sigma_z) - V_t \sigma_t$$

Generic solution:

$$\sigma(V_t) = \frac{1}{V_t} e^{-\frac{\alpha^2 \delta}{V_t}} \left[ c - \alpha^2 \delta \eta \sigma_z \int_{-\frac{\alpha^2 \delta}{V_t}}^{\infty} \frac{e^{-u}}{u} du \right]$$

where $c$ is a constant of integration, and $\text{Ei}(z)$ function ("exponential integral" function) $- \int_{-z}^{\infty} \frac{e^{-u}}{u} du$ is defined provided $z \neq 0$. 

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Constant of integration \( c \) can be tied down as follows.

Consider limit case \( \delta \to 0 \) where “fundamental shocks” are shut down

As \( \delta \to 0 \)

\[
\sigma (V_t) \to \frac{c}{V_t}
\]

Restriction. In the absence of fundamental shocks, volatility is zero

This implies \( c = 0 \) so that

\[
\sigma (V_t) = \eta \sigma_z \frac{\alpha^2 \delta}{V_t} \exp \left\{ -\frac{\alpha^2 \delta}{V_t} \right\} \times \text{Ei} \left( \frac{\alpha^2 \delta}{V_t} \right)
\]
Equilibrium drift $\mu_t$

$$
\mu_t = r + \frac{\sigma_t}{2\alpha \eta \sigma_z} \left\{ 2\alpha (r^* - r) + \alpha \sigma_t^2 - \eta \sigma_z + (\sigma_t - \eta \sigma_z) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\}
$$

Sharpe ratio:

$$
\frac{\mu_t - r}{\sigma_t} = \frac{1}{2\alpha \eta \sigma_z} \left\{ 2\alpha (r^* - r) + \alpha \sigma_t^2 - \eta \sigma_z + (\sigma_t - \eta \sigma_z) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\}
$$
Example

\[ r = 0.01, \ r^* = 0.047, \ \delta = 6, \ \alpha = 2.7, \ \sigma_z = 0.3, \ \eta = 1.5 \]
Risk and Return

![Graph showing risk and return](image)

Higher risk premium
higher risk
greater "as if" risk aversion

Equity
Risk and Return

Loss of balance sheet capacity
equity down
Var up
Risk and Return

Equity

σ  μ

Loss of balance sheet capacity

equity down

VaR up

Procyclical leverage
Leverage under P

Asset Growth vs. Leverage growth
Shadow value of bank capital turns up as leverage increases.
Many Risky Assets

Special case of \( N \) risky securities case can be solved using ODE solution from the single risky asset case.

Assumption (Symmetry)

1. Diffusion matrix for \( z \) is \( \tilde{\sigma}_z I_N \) where \( \tilde{\sigma}_z > 0 \) is a scalar and \( I_N \) is the \( n \times n \) identity matrix.

2. \( \delta^i = \delta \) for all \( i \).

Let \( \sigma_t^{ij} \) be coefficient giving effect of change in demand shock of \( j \)th security on price of \( i \)th security.
From assumption of symmetry, we only need to solve for one diffusion variable, $\sigma_{t}^{ii} = \sigma_{t}^{11}$, since for $i \neq j$ the cross effects are tied down by $\sigma_{t}^{ij} = \sigma_{t}^{12} = \sigma_{t}^{11} - \eta \tilde{\sigma}_z$.

Define $x_t \equiv x(V_t)$ the solution to the ODE for single risky asset with $\eta$ replaced by $\frac{\eta}{N}$.

**Proposition 1.** Assume $(S)$. The following is a REE.

The REE diffusion coefficients are $\sigma_{t}^{ii} = x_t + \frac{N-1}{N} \eta \tilde{\sigma}_z$, and for $i \neq j$, $\sigma_{t}^{ij} = x_t - \frac{1}{N} \eta \tilde{\sigma}_z$. Also, $\Sigma_{t}^{ii} = \text{Var}_t(\text{return on security } i) = \eta^2 \tilde{\sigma}_z^2 + \frac{1}{N} \left( N^2 x_t^2 - \eta^2 \tilde{\sigma}_z^2 \right)$, and for $i \neq j$, $\Sigma_{t}^{ij} = \text{Cov}_t(\text{return on security } i, \text{return on security } j) = \frac{1}{N} \left( N^2 x_t^2 - \eta^2 \tilde{\sigma}_z^2 \right)$ and $\text{Corr}_t(\text{return on security } i, \text{return on security } j) = \frac{N x_t^2 - \frac{1}{N} \eta^2 \tilde{\sigma}_z^2}{N x_t^2 + \frac{1}{N} \eta^2 \tilde{\sigma}_z^2}$.

Risky holdings are $D_t^i = \frac{V_t}{\alpha N^{3/2} x_t}$.
The risk-reward relationship is given by

\[
\frac{\mu_t^i - r}{x_t} = \frac{1}{2\alpha} \eta \sigma_z \left\{ \alpha \left( x_t + \frac{N - 1}{N} \eta \sigma_z \right)^2 - \frac{\eta}{\sqrt{N}} \sigma_z + \sqrt{N} \left( x_t - \frac{\eta}{N} \sigma_z \right) \left[ 2\alpha^2 r + \frac{\alpha^2 \delta}{V_t} - 2 \right] \right\} 
\]

(8)

The intuition and form of the drift term is very similar to the \( N = 1 \) case and reduces to it if \( N \) is set equal to 1.
Related Literature

• Two strands coming together


• Asset pricing taking account of balance sheet constraints: Adrian, Etula and Shin (2009) [for exchange rates], Etula (2009) [commodities], Adrian, Moench and Shin (2009)
Avenues for Further Research

• Having $V_t$ as sole state variable is limiting
  — Single state variable cannot take account of time dependence
  — “Long period of tranquility builds up vulnerabilities…”

• Backward-looking learning (e.g. RiskMetrics)
  — In practice, banks use historical data to forecast volatility
  — … also encouraged by regulators (Basel II)
  — This is one way to tie down beliefs, and overcome multiplicity of equilibrium
  — Encompasses history-dependence