Plasma Waves and Instabilities

Lecture notes (under development)

Last modified: September 23, 2025

Contents

Preface

Plasmas support a wide variety of waves. These waves significantly determine plasma dynamics and can be used for plasma manipulation and diagnostics.

Studying plasma waves is an essential part of studying plasma physics and its applications. It is also useful for understanding waves in general, because the complexity of waves in plasmas demands a particularly systematic approach to wave theory.

This course is intended as an introduction into physics of (mostly linear) plasma waves and briefly covers the following general topics:

- concept of linear dispersion;
- dispersion operators and their symbols;
- geometrical-optics approximation;
- envelope equation, ray tracing, mode conversion;
- transport of the wave action, energy, and momentum;
- dispersion properties of nonmagnetized and magnetized plasma within fluid and kinetic models;
- basic types of plasma waves and their applications to plasma manipulation and diagnostics;
- basic instabilities and mechanisms of collisionless dissipation;
- nonlinear saturation of kinetic instabilities, quasilinear theory.

For in-depth discussions of these topics, see additional literature, for example, Refs. [1–8].

Part I

Introduction

The purpose of this first, intentionally haphazard, part of the course is to familiarize readers with some basic vocabulary of plasma-wave theory. The concepts introduced in this part will be used later for developing a more systematic theory.

Lecture 1

Electromagnetic dispersion

In this introductory lecture, we introduce the concept of electromagnetic dispersion in application to a general linear medium, without focusing on effects specific to plasmas.

1.1 Basic equations

1.1.1 Maxwell's equations

Any wave propagating in a medium that contains electric charges involves oscillations of electric currents, which cause oscillations of electromagnetic fields. Because of this, studying waves in such media usually starts with Maxwell's equations:

$$\partial_t \boldsymbol{E} = c\nabla \times \boldsymbol{B} - 4\pi \boldsymbol{j} \qquad \text{(Ampere's law)}, \tag{1.1a}$$

$$\partial_t \boldsymbol{B} = -c\nabla \times \boldsymbol{E} \qquad \text{(Faraday's law)}, \tag{1.1b}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad \text{(electric Gauss's law)}, \tag{1.1c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{(magnetic Gauss's law)}, \tag{1.1d}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field, respectively, $\boldsymbol{j}$ is the current density, ρ is the charge density, and c is the speed of light. (Gaussian units will be used throughout the course.)

Magnetic Gauss's law can be considered as an initial condition for Faraday's law, because

$$\partial_t(\nabla \cdot \boldsymbol{B}) = \nabla \cdot \partial_t \boldsymbol{B} = -c\nabla \cdot (\nabla \times \boldsymbol{E}) = 0, \tag{1.2}$$

where we used the fact that the divergence of a curl is identically zero. Similarly, electric Gauss's law can be considered as an initial condition for Ampere's law, because

$$\partial_t(\nabla \cdot \boldsymbol{E} - 4\pi\rho) = \partial_t(\nabla \cdot \boldsymbol{E}) + 4\pi\nabla \cdot \boldsymbol{j} = c\nabla \cdot (\nabla \times \boldsymbol{B}) = 0, \tag{1.3}$$

where we used charge conservation:

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0. \tag{1.4}$$

In this sense, not all of Eqs. (1.1) are entirely independent from each other, and it will be sufficient for us to use only a subset of them for studying waves. We will return to this subject later.

1.1.2 Electrodynamics in a linear medium

Effects caused by a medium enter Eqs. (1.1) only through ρ and $\boldsymbol{j}$, and ρ can be expressed through $\boldsymbol{j}$ via Eq. (1.4); thus, a model for $\boldsymbol{j}$ is needed. In this course, we will mostly assume that the medium (in our case, plasma) is linear. This means that $\boldsymbol{j}$ will be assumed to depend on $\boldsymbol{E}$ linearly,[1] i.e.,

$$\boldsymbol{j} = \boldsymbol{j}^{(\mathrm{i})} + \boldsymbol{j}^{(\mathrm{f})}, \qquad \boldsymbol{j}^{(\mathrm{i})} = \hat{\boldsymbol{\sigma}}\boldsymbol{E}. \tag{1.5}$$

The "induced" current density $\boldsymbol{j}^{(\mathrm{i})}$ is determined by medium's response to the electric field, and the linear operator $\hat{\boldsymbol{\sigma}}$ that determines this response is called *conductivity*.[2] The remaining, "free" current density, $\boldsymbol{j}^{(\mathrm{f})}$, is independent of $\boldsymbol{E}$; it includes the current density that is prescribed externally, and it can also include a (generally time-dependent) current density that is determined by the initial conditions. For this reason, specifying the representation (1.5) requires specifying the initial moment of time since which the dynamics is considered. We will denote this moment as t_0; then, by definition,

$$\boldsymbol{j}(t = t_0) = \boldsymbol{j}^{(\mathrm{f})}(t = t_0), \qquad \boldsymbol{j}^{(\mathrm{i})}(t = t_0) = 0. \tag{1.6}$$

Below, we consider $\boldsymbol{j}^{(\mathrm{f})}$ as prescribed and discuss how to model the induced current density $\boldsymbol{j}^{(\mathrm{i})}$. In some cases, the latter can be as simple as

$$\boldsymbol{j}^{(\mathrm{i})}(t, \boldsymbol{x}) = \boldsymbol{\sigma}(t, \boldsymbol{x})\boldsymbol{E}(t, \boldsymbol{x}), \tag{1.7}$$

where $\boldsymbol{\sigma}$ is some matrix function or even a scalar. This model is commonly known as Ohm's law. It is also called the local-response model, because it assumes that the current at a given location $(t, \boldsymbol{x})$ is determined by the field only at the same location $(t, \boldsymbol{x})$. Such an approximation can be reasonable, for example, for modeling strongly collisional plasmas. However, in general, $\boldsymbol{j}^{(\mathrm{i})}(t, \boldsymbol{x})$ can also depend on $\boldsymbol{E}(t', \boldsymbol{x}')$ at $(t', \boldsymbol{x}')$ other than $(t, \boldsymbol{x})$. Such media are called dispersive. Nonlocality in time is called *temporal dispersion*, and nonlocality in space is called *spatial dispersion*.

The general induced current in a dispersive medium can be expressed as a functional

$$\boldsymbol{j}^{(\mathrm{i})}(t, \boldsymbol{x}) = \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \boldsymbol{\Sigma}(t, \boldsymbol{x}, t', \boldsymbol{x}')\boldsymbol{E}(t', \boldsymbol{x}'). \tag{1.8}$$

The tensor $\boldsymbol{\Sigma}(t, \boldsymbol{x}, t', \boldsymbol{x}')$, which is the "coordinate" (time-space) representation of $\hat{\boldsymbol{\sigma}}$ (see also Box 1.1), can be formally defined as a functional derivative

$$\boldsymbol{\Sigma}(t, \boldsymbol{x}, t', \boldsymbol{x}') = \frac{\delta \boldsymbol{j}(t, \boldsymbol{x})}{\delta \boldsymbol{E}(t', \boldsymbol{x}')}. \tag{1.9}$$

In other words, it serves as the weight function that determines how much the field $\boldsymbol{E}(t', \boldsymbol{x}')$ contributes to the current $\boldsymbol{j}^{(\mathrm{i})}(t, \boldsymbol{x})$. The fact that the integration domain in Eq. (1.10) is limited to $t' < t$ reflects *causality*; that is, the current at a given time t can be affected by past fields ($t' < t$) but not by future fields ($t' > t$). One can also replace the upper integration limit t in Eq. (1.8) with ∞,

$$\boldsymbol{j}^{(\mathrm{i})}(t, \boldsymbol{x}) = \int_{t_0}^{\infty} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \boldsymbol{\Sigma}(t, \boldsymbol{x}, t', \boldsymbol{x}')\boldsymbol{E}(t', \boldsymbol{x}'), \tag{1.10}$$

assuming the convention that $\boldsymbol{\Sigma}(t, \boldsymbol{x}, t', \boldsymbol{x}') = 0$ for all $t' > t$.

Below, we will often express $\hat{\boldsymbol{\sigma}}$ and related quantities through the frequency operator $\hat{\omega}$ and the wavevector operator $\hat{\boldsymbol{k}}$:

$$\hat{\omega} \doteq i\partial_t, \qquad \hat{\boldsymbol{k}} \doteq -i\nabla, \tag{1.11}$$

[1] We do not involve $\boldsymbol{B}$ here because for oscillatory fields that we are interested in, $\boldsymbol{B}$ can always be expressed through $\boldsymbol{E}$ using Faraday's law. For stationary fields, the general linear model for the induced current density is $\boldsymbol{j}^{(\mathrm{i})} = \hat{\boldsymbol{\sigma}}\boldsymbol{E} + \hat{\boldsymbol{\kappa}}\boldsymbol{B}$.

[2] We use caret ˆ to denote integral operators, including differential operators as a special case.

Box 1.1: An alternative coordinate representation of $\hat{\boldsymbol{\sigma}}$

The current density $\boldsymbol{j}^{(\mathrm{i})}$ is also often expressed in the following alternative form:

$$\boldsymbol{j}^{(\mathrm{i})}(t,\boldsymbol{x}) = \int_{t_0}^{\infty} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \bar{\boldsymbol{\Sigma}}\left(\frac{t+t'}{2}, \frac{\boldsymbol{x}+\boldsymbol{x}'}{2}, t-t', \boldsymbol{x}-\boldsymbol{x}'\right) \boldsymbol{E}(t',\boldsymbol{x}'),$$

where the "symmetrized" kernel $\bar{\boldsymbol{\Sigma}}$ is connected with the previously introduced $\boldsymbol{\Sigma}$ as follows:

$$\bar{\boldsymbol{\Sigma}}(\bar{t},\bar{\boldsymbol{x}},\tau,\boldsymbol{s}) = \boldsymbol{\Sigma}(\bar{t}+\tau/2, \bar{\boldsymbol{x}}+\boldsymbol{s}/2, \bar{t}-\tau/2, \bar{\boldsymbol{x}}-\boldsymbol{s}/2)$$

(and $\bar{\boldsymbol{\mathcal{X}}}$ is defined similarly through $\boldsymbol{\mathcal{X}}$). As to be discussed in Lecture 3, the Fourier transform of $\bar{\boldsymbol{\Sigma}}$ with respect to τ and $\boldsymbol{s}$ is the *Weyl symbol* of $\hat{\boldsymbol{\sigma}}$, which is a more natural quantity than the Fourier transform of $\boldsymbol{\Sigma}$.

where $\doteq$ denotes definitions. In particular, using Eq. (1.4), one can express ρ through $\boldsymbol{E}$ via $\hat{\omega}$:

$$-i\hat{\omega}\rho + \nabla\cdot\boldsymbol{j}^{(\mathrm{f})} + \nabla\cdot(\hat{\boldsymbol{\sigma}}\boldsymbol{E}) = 0. \tag{1.12}$$

Although $\hat{\omega}$ is not invertible, one can define its *right* inverse $\hat{\omega}^{-1}$ as

$$(\hat{\omega}^{-1}f)(t) \doteq -i\int_{t_0}^{t} dt'\, f(t'), \qquad \hat{\omega}^{-1}\hat{\omega} \neq \hat{\omega}\hat{\omega}^{-1} = \hat{1}. \tag{1.13}$$

Using this notation, one can write the solution of Eq. (1.12) as

$$\rho = \rho^{(\mathrm{f})} - \nabla\cdot(i\hat{\omega}^{-1}\hat{\boldsymbol{\sigma}}\boldsymbol{E}), \tag{1.14}$$

where $\rho^{(\mathrm{f})}$ may include contributions from initial conditions and $\boldsymbol{j}^{(\mathrm{f})}$ but is independent of $\boldsymbol{E}$. It is also convenient to introduce the *susceptibility operator*

$$\hat{\boldsymbol{\chi}} = \frac{4\pi i}{\hat{\omega}}\,\hat{\boldsymbol{\sigma}} \tag{1.15}$$

(we assume the notation $1/\hat{\omega} \equiv \hat{\omega}^{-1}$), which we also represent as

$$(\hat{\boldsymbol{\chi}}\boldsymbol{E})(t,\boldsymbol{x}) = \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \boldsymbol{\mathcal{X}}(t,\boldsymbol{x},t',\boldsymbol{x}')\boldsymbol{E}(t',\boldsymbol{x}'), \tag{1.16}$$

and the *dielectric operator*

$$\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{1}} + \hat{\boldsymbol{\chi}}, \tag{1.17}$$

where $\hat{\boldsymbol{1}}$ is a unit matrix operator. Then, Eq. (1.14) can be written as $\rho = \rho^{(\mathrm{f})} + \rho^{(\mathrm{i})}$. Here, the "induced" charge density is given by

$$\rho^{(\mathrm{i})} = -\nabla\cdot\boldsymbol{P} \tag{1.18}$$

and $\boldsymbol{P}$ is the so-called electric polarization, which is understood as the electric dipole moment per unit volume [9] and is defined as[3]

$$\boldsymbol{P} \doteq \frac{\hat{\boldsymbol{\chi}}}{4\pi}\,\boldsymbol{E}. \tag{1.19}$$

[3] In this course, we define $\hat{\boldsymbol{\chi}}$ using Stix's notation [1]. Sometimes, though, the factor 4π is absorbed in the definition of $\hat{\boldsymbol{\chi}}$; then, $\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{1}} + 4\pi\hat{\boldsymbol{\chi}}$ and $\boldsymbol{P} \doteq \hat{\boldsymbol{\chi}}\boldsymbol{E}$.

Also, the field $\hat{\boldsymbol{\epsilon}}\boldsymbol{E} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ is called the electric displacement field. With these, Maxwell's equations (1.1) can be re-written as follows:

$$\partial_t(\hat{\boldsymbol{\epsilon}}\boldsymbol{E}) = c\nabla\times\boldsymbol{B} - 4\pi\boldsymbol{j}^{(\mathrm{f})}, \tag{1.20a}$$

$$\partial_t\boldsymbol{B} = -c\nabla\times\boldsymbol{E}, \tag{1.20b}$$

$$\nabla\cdot(\hat{\boldsymbol{\epsilon}}\boldsymbol{E}) = 4\pi\rho^{(\mathrm{f})}, \tag{1.20c}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{1.20d}$$

1.1.3 Dispersion operators

Waves are usually defined as fields that are periodic or quasiperiodic in time and (or) space.[4] For the purpose of this course, we adopt a different definition; namely, the term "(linear) wave" will be applied to *any* solution of Eqs. (1.20). In particular, we will discuss superpositions of quasiperiodic fields, as well as rapidly growing and dissipating fields, which are generally not quasiperiodic. In this sense, Eqs. (1.20) will be called *wave equations*.

We now seek to represent these equations in a more compact form that contains fewer independent variables. Because the electric and magnetic Gauss's laws can be considered as initial conditions (Sec. 1.1.1), it will be enough for us to limit our consideration to Ampere's and Faraday's laws. Let us consider the curl of Eq. (1.20b) and substitute Eq. (1.20a) for $\nabla\times\boldsymbol{B}$. Then, one obtains

$$\nabla\times\nabla\times\boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2(\hat{\boldsymbol{\epsilon}}\boldsymbol{E})}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial\boldsymbol{j}^{(\mathrm{f})}}{\partial t}. \tag{1.21}$$

We will limit our consideration to regions where the external currents are zero. This does not automatically mean that $\boldsymbol{j}^{(\mathrm{f})}$ is zero, because the initial conditions can give rise to time-dependent currents (Sec. 1.1.2), including currents resonant to waves of interest. Such currents can significantly affect the wave propagation. However, we will postpone a detailed discussion of this subject until Part IV. For the time being, let us simply assume that: (i) the integral (1.8) that defines $\boldsymbol{j}^{(\mathrm{i})}$ converges at $t_0\to-\infty$ and (ii) the initial conditions at $t_0\to-\infty$ do not affect waves of interest at finite t. One can expect this to be a reasonable assumption, for example, in collisional media, because collisions eventually destroy information about the initial conditions. Also, if the wave amplitude grows, then $\boldsymbol{j}^{(\mathrm{i})}$ is determined by $\boldsymbol{E}$ mainly from the recent past and, again, the initial conditions at $t_0\to-\infty$ should not matter. Hence, we adopt $t_0\to-\infty$, so $\hat{\boldsymbol{\sigma}}\boldsymbol{E}$ is given by the following integral that converges by our assumption:

$$(\hat{\boldsymbol{\sigma}}\boldsymbol{E})(t,\boldsymbol{x}) = \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \boldsymbol{\Sigma}(t,\boldsymbol{x},t',\boldsymbol{x}')\boldsymbol{E}(t',\boldsymbol{x}'), \tag{1.22}$$

and we also adopt $\boldsymbol{j}^{(\mathrm{f})} = 0$. Then, Eq. (1.21) can be written as

$$\hat{\boldsymbol{D}}_E\boldsymbol{E} = 0, \qquad \hat{\boldsymbol{D}}_E \doteq \frac{c^2}{\hat{\omega}^2}\,(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger - \boldsymbol{1}\,\hat{k}^2) + \hat{\boldsymbol{\epsilon}}. \tag{1.23}$$

The operator $\hat{\boldsymbol{D}}_E$ will be called the (electromagnetic) dispersion operator.

Note that other representations of the wave equation are also possible. For example, it can be convenient to use the electrostatic potential (Problem PI.1) or the magnetic field (Sec. 2.3.3) instead of $\boldsymbol{E}$, or even give up the very notion of the dielectric operator (Problem PI.2).

[4] Global modes, which are described by discrete variables $\boldsymbol{\psi}(t)$, are also waves in the sense that $\boldsymbol{\psi}(t)$ can be considered as time-dependent fields $\boldsymbol{\psi}(t,\boldsymbol{x})$ on zero-dimensional coordinate space ($\dim\boldsymbol{x} = 0$).

1.2 Waves in homogeneous linear media

Even without source terms, Eq. (1.23) remains a complicated integro-differential equation that, in the general case, can be solved only numerically. However, it can be made tractable if the underlying medium is *homogeneous* (both in time and in space) or *weakly inhomogeneous.* The case of weakly inhomogeneous media is more relevant for practical applications, but the corresponding theory is too complicated for an introductory lecture, so it is left to Part II. Here, we consider waves in strictly homogeneous media in order to introduce some basic concepts that we will need to use later.

1.2.1 Basic concepts

A *homogeneous linear medium* is defined as a medium where $\boldsymbol{\Sigma}(t,\boldsymbol{x},t',\boldsymbol{x}')$ may depend on $(t,\boldsymbol{x})$ only relatively to $(t',\boldsymbol{x}')$. Specifically, this means that $\boldsymbol{\Sigma}$ is representable as[5]

$$\boldsymbol{\Sigma}(t,\boldsymbol{x},t',\boldsymbol{x}') = \bar{\boldsymbol{\Sigma}}(t-t',\boldsymbol{x}-\boldsymbol{x}'), \tag{1.24}$$

which automatically leads to a similar expression for $\boldsymbol{\mathcal{X}}$:

$$\boldsymbol{\mathcal{X}}(t,\boldsymbol{x},t',\boldsymbol{x}') = \bar{\boldsymbol{\mathcal{X}}}(t-t',\boldsymbol{x}-\boldsymbol{x}'). \tag{1.25}$$

Then, from Eq. (1.22),

$$\begin{aligned}(\hat{\boldsymbol{\sigma}}\boldsymbol{E})(t,\boldsymbol{x}) &= \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \bar{\boldsymbol{\Sigma}}(t-t',\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{E}(t',\boldsymbol{x}')\\ &= \int_{0}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{s}\, \bar{\boldsymbol{\Sigma}}(\tau,\boldsymbol{s})\boldsymbol{E}(t-\tau,\boldsymbol{x}-\boldsymbol{s}).\end{aligned} \tag{1.26}$$

Accordingly, $\bar{\Sigma}_{ab}(t,\boldsymbol{x})$ can be understood as the ath component of the current, $j_a(t,\boldsymbol{x})$, produced by the bth component of the electric field of the form[6] $E_b(t',\boldsymbol{x}') = \delta(t'-0)\delta(\boldsymbol{x}')$.

It is easy to see from here that homogeneous linear media support monochromatic waves, that is, waves of the form (Box 1.2)

$$\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{\mathcal{E}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.27}$$

where the amplitude $\boldsymbol{\mathcal{E}}$, the frequency ω, and the wavevector $\boldsymbol{k}$ are complex constants. Indeed, in this case, Eq. (1.26) leads to

$$(\hat{\boldsymbol{\sigma}}\boldsymbol{E})(t,\boldsymbol{x}) = \int_{0}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{s}\, \bar{\boldsymbol{\Sigma}}(\tau,\boldsymbol{s})\boldsymbol{\mathcal{E}}\, e^{-i\omega(t-\tau)+i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{s})} \tag{1.28}$$

$$= \left[\int_{0}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{s}\, \bar{\boldsymbol{\Sigma}}(\tau,\boldsymbol{s})\, e^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}\right] \boldsymbol{E}(t,\boldsymbol{x}). \tag{1.29}$$

Let us denote the expression in the square brackets as $\boldsymbol{\sigma}(\omega,\boldsymbol{k})$,

$$\boldsymbol{\sigma}(\omega,\boldsymbol{k}) \doteq \int_{0}^{\infty} dt \int_{-\infty}^{\infty} d\boldsymbol{x}\, \bar{\boldsymbol{\Sigma}}(t,\boldsymbol{x})\, e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.30}$$

This function is called the *spectral representation* of $\hat{\boldsymbol{\sigma}}$, or simply the spectral conductivity, because $\boldsymbol{\sigma}(\omega,\boldsymbol{k})$ is obtained from $\bar{\boldsymbol{\Sigma}}$ by applying the Fourier transform in space and the Laplace transform in time. The physical meaning of this quantity can be understood by noticing that according to

[5] To put it differently, the function $\bar{\boldsymbol{\Sigma}}$ introduced in Box 1.1 is independent of $\bar{t}$ and $\bar{\boldsymbol{x}}$.

[6] The added 0 in the argument of the delta function denotes an infinitesimal positive value. This shift ensures that the time integral over the domain (t_0,t) is well defined, $\int_{t_0}^{t} dt'\,\delta(t-t'-0) = \int_0^{t-t_0} d\tau\,\delta(\tau-0) = \int_{-\infty}^{\infty} d\tau\,\delta(\tau) = 1$, while $\int_{t_0}^{t} dt'\,\delta(t-t') = \int_0^{\infty} d\tau\,\delta(\tau)$ is undefined. In our special case, $t_0 \to -\infty$, but this is not essential.

Box 1.2: Complex representation of real fields

Although the actual electric field is real, the sourceless wave equations that we work with allow solutions in a complex form. These equations have real coefficients, so for any complex field $\boldsymbol{E}_{\mathrm{c}}$ that is a solution (the index $_{\mathrm{c}}$ stands for "complex"), the complex-conjugate field $\boldsymbol{E}_{\mathrm{c}}^*$ is a solution too, and so is any linear combination of the two. Hence, one can search for real $\boldsymbol{E}$ in the form

$$\boldsymbol{E} = \mathrm{Re}\,\boldsymbol{E}_{\mathrm{c}} = 1/2\,(\boldsymbol{E}_{\mathrm{c}} + \boldsymbol{E}_{\mathrm{c}}^*).$$

To simplify the notation, we will not distinguish $\boldsymbol{E}$ and $\boldsymbol{E}_{\mathrm{c}}$ where this is not essential.

Eq. (1.26), $\boldsymbol{j}$ is a *convolution* of $\bar{\boldsymbol{\Sigma}}$ and $\boldsymbol{E}$; thus, its spectral representation $\boldsymbol{j}(\omega, \boldsymbol{k})$ is simply proportional to the spectral representation $\boldsymbol{E}(\omega, \boldsymbol{k})$ of the electric field with the proportionality coefficient $\boldsymbol{\sigma}(\omega, \boldsymbol{k})$:[7]

$$\boldsymbol{j}(\omega, \boldsymbol{k}) = \boldsymbol{\sigma}(\omega, \boldsymbol{k})\boldsymbol{E}(\omega, \boldsymbol{k}). \tag{1.31}$$

Since $\bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x}) = 0$ at $t < 0$, the integral over t in Eq. (1.30) can be formally extended to $-\infty$,

$$\boldsymbol{\sigma}(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\boldsymbol{x}\, \bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x})\, e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.32}$$

Still, it should not be confused with the Fourier transform, because ω is generally complex. Since $|\bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x})e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}| = |\bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x})|e^{-\omega_i t}$, either $\omega_i \doteq \mathrm{Im}\,\omega$ should be sufficiently large or the response function $\bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x})$ should diminish rapidly enough at $t \to \infty$ for the integral (1.30) to converge. In particular, if $\bar{\boldsymbol{\Sigma}}(t, \boldsymbol{x}) \propto e^{-\nu t}$ with constant ν, then the integrals (1.30) and (1.32) exist provided that

$$\omega_i + \nu > 0. \tag{1.33}$$

These are the same requirements as those adopted in Sec. 1.1.3; that is, applicability of our theory is facilitated by collisions and by exponential growth of the wave amplitude. See also Appendix AI.1 for basic properties of response functions like $\boldsymbol{\sigma}$.

From the spectral representation of $\hat{\boldsymbol{\sigma}}$, one readily finds the spectral representation of $\hat{\boldsymbol{\epsilon}}$, which is known as the *dielectric tensor*:

$$\boldsymbol{\epsilon}(\omega, \boldsymbol{k}) = \mathbf{1} + \boldsymbol{\chi}(\omega, \boldsymbol{k}), \tag{1.34}$$

with $\mathbf{1}$ being the unit matrix. Here, $\boldsymbol{\chi}$ is the spectral representation of $\hat{\boldsymbol{\chi}}$; it is defined by analogy with Eq. (1.30) and given by

$$\boldsymbol{\chi}(\omega, \boldsymbol{k}) = \frac{4\pi i}{\omega}\,\boldsymbol{\sigma}(\omega, \boldsymbol{k}), \tag{1.35}$$

as shown in Box 1.3. Then, the wave equation (1.23) can be written as follows:[8]

$$\boldsymbol{D}_E(\omega, \boldsymbol{k})\boldsymbol{E} = 0, \qquad \boldsymbol{D}_E(\omega, \boldsymbol{k}) = \boldsymbol{N}\boldsymbol{N}^\dagger - \mathbf{1}\,N^2 + \boldsymbol{\epsilon}(\omega, \boldsymbol{k}). \tag{1.36}$$

The matrix $\boldsymbol{D}_E$ is called a dispersion matrix or dispersion tensor, and $\boldsymbol{N}$ is the refractive-index vector:

$$\boldsymbol{N} \doteq c\boldsymbol{k}/\omega. \tag{1.37}$$

[7] See theory of the Fourier and Laplace transforms. We will revisit this topic in Part IV, where we will also discuss properties of the Laplace transform in more detail.

[8] Keep in mind that all functions of $(\omega, \boldsymbol{k})$ are introduced here strictly for homogeneous media. The only fundamental objects in inhomogeneous media are operators, because their representations can be defined in more than one way and can differ significantly. For example, "the dielectric tensor" of inhomogeneous plasma is undefined until a convention is specified for the mapping $\hat{\boldsymbol{\epsilon}} \to \boldsymbol{\epsilon}(t, \boldsymbol{x}, \omega, \boldsymbol{k})$. Except in special settings, such as cold stationary plasmas, one should not expect $\boldsymbol{\epsilon}(t, \boldsymbol{x}, \omega, \boldsymbol{k})$ to be a meaningful quantity if it is obtained simply by taking $\boldsymbol{\epsilon}(\omega, \boldsymbol{k})$ of homogeneous plasma and replacing the constant plasma parameters with functions of $(t, \boldsymbol{x})$. We will revisit this issue in Part II (Box 3.2).

Box 1.3: Derivation of Eq. (1.35)

From Eqs. (1.5), (1.15), (1.16), and (1.25), one finds

$$4\pi \boldsymbol{j}^{(\mathrm{i})} = \partial_t(\hat{\boldsymbol{\chi}}\boldsymbol{E}) = \boldsymbol{F}(t,\boldsymbol{x}) + \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}' \, [\partial_t \bar{\boldsymbol{\mathcal{X}}}(t-t', \boldsymbol{x}-\boldsymbol{x}')]\boldsymbol{E}(t', \boldsymbol{x}'),$$
$$\boldsymbol{F}(t,\boldsymbol{x}) \doteq \int_{-\infty}^{\infty} d\boldsymbol{x}' \, \bar{\boldsymbol{\mathcal{X}}}(0, \boldsymbol{x}-\boldsymbol{x}')\boldsymbol{E}(t, \boldsymbol{x}').$$

For $t = t_0$, this gives $\boldsymbol{F}(t_0, \boldsymbol{x}) = 4\pi \boldsymbol{j}^{(\mathrm{i})}(t_0, \boldsymbol{x}) = 0$ for any $\boldsymbol{E}$; thus, one also has $\bar{\boldsymbol{\mathcal{X}}}(0, \boldsymbol{x}) = 0$ at all $\boldsymbol{x}$. Then, the above equation can be written as

$$4\pi(\hat{\boldsymbol{\sigma}}\boldsymbol{E})(t,\boldsymbol{x}) = \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}' \, [\partial_t \bar{\boldsymbol{\mathcal{X}}}(t-t', \boldsymbol{x}-\boldsymbol{x}')]\boldsymbol{E}(t', \boldsymbol{x}'),$$

which means that $\partial_t \bar{\boldsymbol{\mathcal{X}}} = 4\pi\bar{\boldsymbol{\Sigma}}$. Then, from Eq. (1.30), one finds

$$4\pi\boldsymbol{\sigma}(\omega, \boldsymbol{k}) = \int_0^{\infty} dt \int_{-\infty}^{\infty} d\boldsymbol{x} \, [\partial_t \bar{\boldsymbol{\mathcal{X}}}(t, \boldsymbol{x})] \, e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}} = i\omega\boldsymbol{\chi}(\omega, \boldsymbol{k}),$$

where we used integration by parts and $\bar{\boldsymbol{\mathcal{X}}}(0, \boldsymbol{x}) = 0$. This leads to Eq. (1.35).

1.2.2 Dispersion relations

As seen from the previous section, equations describing electromagnetic waves in linear media away from sources can be expressed as

$$D_{ab}(\omega, \boldsymbol{k})\psi_b = 0. \tag{1.38}$$

The function $\boldsymbol{\psi}$ is an electric field in Eq. (1.36), but in general it can also be a different field (for example, see Problems PI.1 and PI.2), so we leave it unspecified in this section. Equation (1.38) indicates that $\boldsymbol{\psi}$ can be nonzero only if

$$\det \boldsymbol{D}(\omega, \boldsymbol{k}) = 0, \tag{1.39}$$

or simply $D = 0$ when $\boldsymbol{D} = D$ is scalar. This can be considered as an equation for $\omega(\boldsymbol{k})$, which is called a *dispersion relation*. The solutions for ω at given $\boldsymbol{k}$, $\omega = \omega_q(\boldsymbol{k})$, are called *dispersion branches*. Note that they are generally complex, and we will assume the following notation throughout the course:

$$\omega = \omega_r + i\omega_i, \qquad \omega_r \doteq \operatorname{Re}\omega, \qquad \omega_i \doteq \operatorname{Im}\omega. \tag{1.40}$$

Depending on the number of these branches b,[9] or the *order* of the wave equation, the waves are attributed as scalar waves ($\mathsf{b} = 1$) or as vector waves ($\mathsf{b} > 1$); see also Box 1.4.

A spatially monochromatic field with a given wavevector $\boldsymbol{k}$ can contain contributions from multiple branches, called *(eigen)modes*, and can be expressed as follows:[10]

$$\boldsymbol{\psi}(t, \boldsymbol{x}) = \sum_q \bar{\boldsymbol{\psi}}_q e^{-i\omega_q(\boldsymbol{k})t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.41}$$

where the coefficients $\bar{\boldsymbol{\psi}}_q$ are determined by the initial conditions. Each of these coefficients must satisfy $\boldsymbol{D}[\omega_q(\boldsymbol{k}), \boldsymbol{k}]\bar{\boldsymbol{\psi}}_q = 0$. Thus, it can be expressed as $\bar{\boldsymbol{\psi}}_q = A_q \boldsymbol{h}_q(\boldsymbol{k})$, where A_q is a scalar amplitude

[9] Integral wave equations generally yield infinitely many branches and will be discussed in Part IV.

[10] When solutions for ω are degenerate, it is in principle possible to have solutions $\propto t^a e^{-i\omega_q(\boldsymbol{k})t + i\boldsymbol{k}\cdot\boldsymbol{x}}$ with natural a. However, this is not typical for waves of our interest, so we will not consider such cases.

Box 1.4: Scalar waves vs. vector waves

Any differential equation $\partial_t^m \boldsymbol{\psi} = \mathsf{F}[\boldsymbol{\psi}, \partial_t \boldsymbol{\psi}, \ldots \partial_t^{m-1} \boldsymbol{\psi}]$ (where F may be an integral transform in $\boldsymbol{x}$) can be represented as $\partial_t \tilde{\boldsymbol{\psi}} = \tilde{\mathsf{F}}(\tilde{\boldsymbol{\psi}})$, where $\tilde{\boldsymbol{\psi}} = (\tilde{\boldsymbol{\psi}}_1, \ldots \tilde{\boldsymbol{\psi}}_{m-1})^\intercal$ is a vector of dimension $m \times \dim \boldsymbol{\psi}$, with $\tilde{\boldsymbol{\psi}}_1 \doteq \boldsymbol{\psi}$, $\tilde{\boldsymbol{\psi}}_2 \doteq \partial_t \boldsymbol{\psi}$, $\ldots$, $\tilde{\boldsymbol{\psi}}_m \doteq \partial_t^{m-1} \tilde{\boldsymbol{\psi}}$. Assuming F is linear, such an equation generally has $\mathsf{b} = m \times \dim \boldsymbol{\psi}$ roots, so it describes vector waves unless $\boldsymbol{\psi}$ is scalar *and* $m = 1$.

For example, consider the Schrödinger equation and the Klein–Gordon equation from quantum mechanics. (As will be discussed later, they also emerge in plasma-wave theory.) The Schrödinger equation has $\dim \boldsymbol{\psi} = m = 1$, so $\mathsf{b} = 1$; i.e., it has only one dispersion branch and thus the corresponding waves are true scalar waves. The Klein–Gordon equation also has $\dim \boldsymbol{\psi} = 1$; but in this case, $m = 2$, so $\mathsf{b} = 2$, and the corresponding waves are vector waves. The representation of the Klein–Gordon equation as a first-order equation for a two-dimensional vector is known as the Feshbach–Villars representation. A similar reduction of plasma-wave equations to first-order vector equations is discussed, for example, in Problem PI.2.

and $\boldsymbol{h}_q(\boldsymbol{k})$ is a unit *polarization vector*[11] defined via

$$\boldsymbol{D}[\omega_q(\boldsymbol{k}), \boldsymbol{k}]\, \boldsymbol{h}_q(\boldsymbol{k}) = 0, \qquad |\boldsymbol{h}_q| = 1. \tag{1.42}$$

In general, $\dim \boldsymbol{h}_q = \dim \boldsymbol{\psi}$ can be different from $\dim \boldsymbol{x}$ (Problem PI.2), in which case $\boldsymbol{h}_q$ may not have a clear physical meaning. But let us mention the important special case when $\boldsymbol{\psi}$ is a complexified (Box 1.2) single-mode electric field on a three-dimensional space, with the real field being

$$\boldsymbol{E}(t, \boldsymbol{x}) = \operatorname{Re}\left[\boldsymbol{\mathcal{E}}\, e^{-i\omega_q(\boldsymbol{k})t + i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \qquad \boldsymbol{\mathcal{E}} = \boldsymbol{h}_q \mathcal{E}. \tag{1.43}$$

If $\boldsymbol{h}_q$ is parallel to some fixed coordinate axis a (say, $\boldsymbol{h}_q = \bar{\boldsymbol{e}}_a$, where $\bar{\boldsymbol{e}}_a$ is the unit vector along the a axis), then $\boldsymbol{E}$ remains parallel to the a axis at all times:

$$\boldsymbol{E}(t, \boldsymbol{x}) = \bar{\boldsymbol{e}}_a \mathcal{E} \cos\theta, \tag{1.44}$$

where $\theta = -\omega_q t + \boldsymbol{k}\cdot\boldsymbol{x} + \text{const}$. Such a field is called linearly polarized. Similarly, if $\boldsymbol{h}_q = (\bar{\boldsymbol{e}}_a \pm i\bar{\boldsymbol{e}}_b)/\sqrt{2}$ (here $\sqrt{2}$ is added only to ensure the normalization $|\boldsymbol{h}_q| = 1$), then

$$\boldsymbol{E}(t, \boldsymbol{x}) = (\bar{\boldsymbol{e}}_a \cos\theta \mp \bar{\boldsymbol{e}}_b \sin\theta)\mathcal{E}/\sqrt{2}, \tag{1.45}$$

so $E_a^2 + E_b^2$ is conserved, while E_a^2 and E_b^2 are not. This indicates that $\boldsymbol{E}$ at given $\boldsymbol{x}$ rotates in the (a, b) plane with a constant amplitude. Such a field is called circularly polarized in the (a, b) plane. Also, if $\boldsymbol{h}_q = (\bar{\boldsymbol{e}}_a + i\varsigma\bar{\boldsymbol{e}}_b)/\sqrt{2}$ with constant ς that is not purely imaginary, the trajectory of $\boldsymbol{E}$ in the (a, b) plane at fixed $\boldsymbol{x}$ is an ellipse, so the field is called elliptically polarized.

1.2.3 Quasimonochromatic waves

Let us consider localized wave packets, which can be viewed as superpositions of monochromatic waves with various real $\boldsymbol{k}$:

$$\boldsymbol{\psi}(t, \boldsymbol{x}) = \sum_q \boldsymbol{\psi}_q(t, \boldsymbol{x}), \qquad \boldsymbol{\psi}_q = \int_{-\infty}^{\infty} \frac{d\boldsymbol{k}}{(2\pi)^n}\, \bar{\boldsymbol{\psi}}_q(\boldsymbol{k})\, e^{-i\omega_q(\boldsymbol{k})t + i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.46}$$

Here, $n \doteq \dim \boldsymbol{x}$ is the dimension of the underlying coordinate space ($\dim \boldsymbol{x}$ does not have to be equal to $\dim \boldsymbol{\psi}$) and the factor $(2\pi)^n$ has been added for convenience; see below. Equation (1.46) shows

[11] A polarization vector should not be confused with the electric polarization introduced in Sec. 1.1.2.

that individual branches of the dispersion relation contribute to the total field *independently*.[12] Hence, from now on, we will consider only one branch and omit the branch index q for brevity:

$$\boldsymbol{\psi}(t,\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{d\boldsymbol{k}}{(2\pi)^n}\,\bar{\boldsymbol{\psi}}(\boldsymbol{k})\,e^{-i\omega(\boldsymbol{k})t+i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{1.47}$$

In particular, for real fields ($\psi = \psi^*$), taking the complex conjugate of this expression and making a variable transformation $\boldsymbol{k} \mapsto -\boldsymbol{k}$ readily yields

$$\omega_r(-\boldsymbol{k}) = -\omega_r(\boldsymbol{k}), \qquad \omega_i(-\boldsymbol{k}) = \omega_i(\boldsymbol{k}), \qquad \bar{\boldsymbol{\psi}}(-\boldsymbol{k}) = \bar{\boldsymbol{\psi}}^*(\boldsymbol{k}). \tag{1.48}$$

Suppose a *quasimonochromatic* complex wave, specifically, a wave such that $\bar{\boldsymbol{\psi}}(\boldsymbol{k})$ is localized around some $\boldsymbol{k} = \boldsymbol{k}_0$, so only small $\boldsymbol{\kappa} \doteq \boldsymbol{k} - \boldsymbol{k}_0$ contribute to the integral (1.47).[13] For simplicity, let us assume that ω_i is small enough, so we can expand $\omega(\boldsymbol{k})$ as follows:

$$\omega = \omega_0 + i\gamma + \boldsymbol{v}_g \cdot \boldsymbol{\kappa}, \tag{1.49}$$

where $\omega_0 \doteq \omega_r(\boldsymbol{k}_0)$, $\gamma \doteq \omega_i(\boldsymbol{k}_0)$, and $\boldsymbol{v}_g$ is the *group velocity* given by

$$\boldsymbol{v}_g \doteq \left[\frac{\partial\omega_r(\boldsymbol{k})}{\partial\boldsymbol{k}}\right]_{\boldsymbol{k}=\boldsymbol{k}_0}. \tag{1.50}$$

This leads to

$$\boldsymbol{\psi}(t,\boldsymbol{x}) = e^{i\boldsymbol{k}_0\cdot(\boldsymbol{x}-\boldsymbol{v}_p t)+\gamma t} \int_{-\infty}^{\infty} \frac{d\boldsymbol{\kappa}}{(2\pi)^n}\,\bar{\boldsymbol{\psi}}(\boldsymbol{k}_0+\boldsymbol{\kappa})\,e^{i\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{v}_g t)}, \tag{1.51}$$

where we introduced the "phase velocity"

$$\boldsymbol{v}_p \doteq \frac{\boldsymbol{k}_0}{k_0}\frac{\omega_0}{k_0}. \tag{1.52}$$

(The corresponding phase *speed* is $v_p = \omega_0/k_0$.) This can also be written equivalently as

$$\boldsymbol{\psi}(t,\boldsymbol{x}) = \boldsymbol{\Psi}(t,\boldsymbol{x})\,e^{i\boldsymbol{k}_0\cdot(\boldsymbol{x}-\boldsymbol{v}_p t)}, \tag{1.53}$$

$$\boldsymbol{\Psi}(t,\boldsymbol{x}) \doteq e^{\gamma t} \int_{-\infty}^{\infty} \frac{d\boldsymbol{\kappa}}{(2\pi)^n}\,\bar{\boldsymbol{\psi}}(\boldsymbol{k}_0+\boldsymbol{\kappa})\,e^{i\boldsymbol{\kappa}\cdot(\boldsymbol{x}-\boldsymbol{v}_g t)}, \tag{1.54}$$

and, as can be checked by direct substitution, $\boldsymbol{\Psi}$ satisfies

$$(\partial_t + \boldsymbol{v}_g\cdot\nabla - \gamma)\boldsymbol{\Psi} = 0. \tag{1.55}$$

Aside from the uniform amplification factor $e^{\gamma t}$, the wave envelope $\boldsymbol{\Psi}(t,\boldsymbol{x})$ depends on $(t,\boldsymbol{x})$ only through the combination $\boldsymbol{x} - \boldsymbol{v}_g t$, so the envelope is stationary in the frame moving with velocity $\boldsymbol{v}_g$. In Lecture 5, we will show that, for linear waves, $\boldsymbol{v}_g$ is also the velocity of the wave action ("photons"), energy, and momentum. Likewise, the phase factor in Eq. (1.53) depends on $(t,\boldsymbol{x})$ only through the combination $\boldsymbol{x} - \boldsymbol{v}_p t$, so $\boldsymbol{v}_p$ serves as the velocity of phase fronts; hence the name "phase velocity". The phase velocity can be very different from the group velocity and can even be pointed in the opposite direction. Also note that we have neglected $\partial^2_{\boldsymbol{k}\boldsymbol{k}}\omega_r$ here. This term can be important at large t, when the pulse starts to experience distortion due to the difference in the group velocities that correspond to different $\boldsymbol{k}$ (Exercise 1.1). Confusingly enough, this distortion is called *dispersion* too (cf. Sec. 1.1.2), and waves with zero $\partial^2_{\boldsymbol{k}\boldsymbol{k}}\omega_r$ are called dispersionless. Such waves include light in vacuum and sound waves in neutral gases.

The above discussion shows that dispersion relations $\omega(\boldsymbol{k})$ contain information not only about monochromatic oscillations but also about the evolution of wave packets. This makes dispersion relations particularly important, which is why we will focus on dispersion relations a lot in this course.

[12] This is true only in a homogeneous medium, as will be discussed later.

[13] A real quasimonochromatic field has $\bar{\boldsymbol{\psi}}(\boldsymbol{k})$ localized near $\boldsymbol{k} = \pm\boldsymbol{k}_0$, see Eq. (1.48) and Box 1.2.

Exercise 1.1: Show that if the first-order expansion of ω_r in $\boldsymbol{\kappa}$ [Eq. (1.49)] is replaced with the second-order expansion of ω_r in $\boldsymbol{\kappa}$, then Eq. (1.55) acquires a form similar to the Schrödinger equation of a free quantum particle,

$$(\partial_t + \boldsymbol{v}_g \cdot \nabla - \gamma)\boldsymbol{\Psi} = {}^{i}\!/\!{}_{2}\, \boldsymbol{\Theta} : \nabla\nabla\boldsymbol{\Psi}, \tag{1.56}$$

where $\boldsymbol{\Theta} \doteq \partial^2_{\boldsymbol{k}\boldsymbol{k}}\omega_r$ is a symmetric matrix and : denotes double contraction; i.e., $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) \equiv A_{ab}B_{ba}$. What new effect does the term $\boldsymbol{\Theta}$ bring in? Estimate the time scale on which this effect becomes noticeable for a pulse with a given initial width.

Lecture 2

A sneak preview: waves in cold nonmagnetized plasma

In this lecture, we illustrate the key concepts introduced in Lecture 1 and also give a sneak preview of the generic transformations of waves in inhomogeneous plasmas, which will be discussed more systematically in Part II. To do this, we describe waves in cold nonmagnetized plasmas as a simple example. We will also discuss these waves later in a broader context.

2.1 Basic equations

2.1.1 Plasma model

As discussed in Lecture 1, describing electromagnetic waves in a dispersive medium starts with developing a model for the electric current density $\boldsymbol{j}$. In classical plasma, $\boldsymbol{j}$ can be written as

$$\boldsymbol{j} = \sum_s e_s n_s \boldsymbol{v}_s = \sum_s e_s (n_{0s} + \tilde{n}_s)(\boldsymbol{v}_{0s} + \tilde{\boldsymbol{v}}_s). \tag{2.1}$$

Here, the summation is taken over all species s, e_s are charges of these species (e.g., for electrons, $e_e = -e < 0$), n_s are the corresponding densities, and $\boldsymbol{v}_s$ are the corresponding flow velocities. Tildes denote that the corresponding quantities are small linear perturbations, while background currents and fields are denoted with index 0. For simplicity, we assume that there are no background flows, $\boldsymbol{v}_{0s} = 0$, so n_{s0} may not depend on t but may depend on $\boldsymbol{x}$. In this case,

$$\boldsymbol{j} = \tilde{\boldsymbol{j}} = \sum_s e_s n_{0s} \tilde{\boldsymbol{v}}_s + \sum_s e_s \tilde{n} \tilde{\boldsymbol{v}}_s \approx \sum_s e_s n_{s0} \tilde{\boldsymbol{v}}_s, \tag{2.2}$$

where $\tilde{n}_s \tilde{\boldsymbol{v}}_s$ is omitted because it is of the second order in $\tilde{\boldsymbol{E}}$.

To calculate $\tilde{\boldsymbol{v}}_s$, let us consider the momentum equation

$$\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} + (\tilde{\boldsymbol{v}}_s \cdot \nabla)\, \tilde{\boldsymbol{v}}_s = \frac{e_s}{m_s} \left[\tilde{\boldsymbol{E}} + \frac{1}{c}\, \tilde{\boldsymbol{v}}_s \times (\boldsymbol{B}_0 + \tilde{\boldsymbol{B}}) \right] - \frac{\nabla P_s}{m_s n_s} + \boldsymbol{C}_s. \tag{2.3}$$

Let us assume for now that there is no background magnetic field ($\boldsymbol{B}_0 = 0$); then, the magnetic part of the Lorentz force is quadratic in $\tilde{\boldsymbol{E}}$ and thus can be neglected. The term $(\tilde{\boldsymbol{v}}_s \cdot \nabla)\, \tilde{\boldsymbol{v}}_s$ is negligible for the same reason. Let us also assume that the pressure term ∇P_s can be neglected too. (The validity conditions of this approximation will become clear when we study kinetic theory.) For the collision term $\boldsymbol{C}_s$, we adopt a simple model $\boldsymbol{C}_s = -\nu_s \tilde{\boldsymbol{v}}_s$, where $\nu_s = \nu_s(\boldsymbol{x})$ serves as the collision rate. Then, Eq. (2.3) becomes

$$\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} = -\nu_s \tilde{\boldsymbol{v}}_s + \frac{e_s}{m_s}\,\tilde{\boldsymbol{E}}. \tag{2.4}$$

This leads to the following equation for the current density:

$$\frac{\partial \tilde{\boldsymbol{j}}_s}{\partial t} = -\nu_s \tilde{\boldsymbol{j}}_s + \frac{\omega_{ps}^2}{4\pi}\,\tilde{\boldsymbol{E}}, \tag{2.5}$$

where we introduced the so-called plasma frequencies

$$\omega_{ps} \doteq \sqrt{\frac{4\pi n_{s0} e_s^2}{m_s}}. \tag{2.6}$$

2.1.2 Dispersion properties

Equation (2.5) is an inhomogeneous partial differential equation (PDE), so its general solution is a general solution $\tilde{\boldsymbol{j}}_s^{(\mathrm{f})}(t, \boldsymbol{x})$ of the corresponding homogeneous equation plus a particular solution $\tilde{\boldsymbol{j}}_s^{(\mathrm{i})}(t, \boldsymbol{x})$ of the inhomogeneous equation:

$$\tilde{\boldsymbol{j}}_s = \tilde{\boldsymbol{j}}_s^{(\mathrm{f})} + \frac{i\omega_{ps}^2}{4\pi(\hat{\omega} + i\nu_s)}\,\tilde{\boldsymbol{E}}, \qquad \frac{\partial \tilde{\boldsymbol{j}}_s^{(\mathrm{f})}}{\partial t} = -\nu_s \tilde{\boldsymbol{j}}_s^{(\mathrm{f})}. \tag{2.7}$$

Since the total current density is a sum of $\tilde{\boldsymbol{j}}_s$, one can introduce the conductivity $\hat{\boldsymbol{\sigma}}_s$ for each species independently, with the total conductivity $\hat{\boldsymbol{\sigma}}$ being the sum over those of the individual species,

$$\hat{\boldsymbol{\sigma}} = \sum_s \hat{\boldsymbol{\sigma}}_s. \tag{2.8}$$

Also note that in our problem, all these operators are scalar operators, $\hat{\boldsymbol{\sigma}}_s = \hat{\sigma}_s$, and the same applies to $\hat{\boldsymbol{\chi}}$ and $\hat{\boldsymbol{\epsilon}}$. In summary then, an inhomogeneous stationary cold nonmagnetized plasma without average flows is characterized by the following operators:

$$\hat{\sigma}_s = \frac{i\omega_{ps}^2}{4\pi(\hat{\omega} + i\nu_s)}, \qquad \hat{\chi}_s = -\frac{\omega_{ps}^2}{\hat{\omega}(\hat{\omega} + i\nu_s)}, \qquad \hat{\epsilon} = 1 - \sum_s \frac{\omega_{ps}^2}{\hat{\omega}(\hat{\omega} + i\nu_s)}. \tag{2.9}$$

In order to find the operators $(\hat{\omega} + i\nu_s)^{-1}$ explicitly, let us directly integrate Eq. (2.5). This leads to (Exercise 2.1)

$$\tilde{\boldsymbol{j}}_s^{(\mathrm{f})}(t, \boldsymbol{x}) = \tilde{\boldsymbol{j}}_s^{(\mathrm{f})}(t_0, \boldsymbol{x})\, e^{-\nu_s(\boldsymbol{x})(t - t_0)}, \qquad \tilde{\boldsymbol{j}}_s^{(\mathrm{i})}(t, \boldsymbol{x}) = \frac{\omega_{ps}^2}{4\pi} \int_{t_0}^{t} dt'\, e^{\nu_s(\boldsymbol{x})(t' - t)}\, \tilde{\boldsymbol{E}}(t', \boldsymbol{x}). \tag{2.10}$$

Exercise 2.1: Derive Eq. (2.10).

From here, we can readily infer the coordinate representation (1.24) of the conductivity operator:

$$\boldsymbol{\Sigma}_s = \Sigma_s, \qquad \Sigma_s(t, \boldsymbol{x}, t', \boldsymbol{x}') = \frac{\omega_{ps}^2(\boldsymbol{x})}{4\pi}\, e^{-\nu_s(\boldsymbol{x})(t - t')} \delta(\boldsymbol{x} - \boldsymbol{x}') H(t - t'), \tag{2.11}$$

where we have added the Heaviside step function $H(t-t')$ to emphasize that $\boldsymbol{\Sigma}(t,\boldsymbol{x},t',\boldsymbol{x}') \equiv 0$ for $t' > t$. The presence of the delta function in Eq. (2.11) signifies that the plasma response is local in space, so there is no *spatial* dispersion.[1]

The effect of the conductivity operator on the electric field can now be written as

$$\begin{aligned}
(\hat{\boldsymbol{\sigma}}_s \boldsymbol{E})(t,\boldsymbol{x}) &= \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}' \, \frac{\omega_{ps}^2(\boldsymbol{x})}{4\pi} \, e^{-\nu_s(\boldsymbol{x})(t-t')} \delta(\boldsymbol{x}-\boldsymbol{x}') \boldsymbol{E}(t',\boldsymbol{x}') \\
&= \frac{\omega_{ps}^2(\boldsymbol{x})}{4\pi} \int_{t_0}^{t} dt' e^{-\nu_s(\boldsymbol{x})(t-t')} \boldsymbol{E}(t',\boldsymbol{x}).
\end{aligned} \tag{2.12}$$

For fields monochromatic in time with frequency $\omega_i > -\nu_s$, this yields

$$(\hat{\boldsymbol{\sigma}}_s \boldsymbol{E})(t,\boldsymbol{x}) = \frac{i\omega_{ps}^2(\boldsymbol{x})}{4\pi(\omega + i\nu_s(\boldsymbol{x}))} \, \boldsymbol{E}(t,\boldsymbol{x}). \tag{2.13}$$

Because $\boldsymbol{x}$ enters here only as a parameter, one can introduce the local conductivities, the local susceptibilities, and the local dielectric tensor as functions of $(\omega;\boldsymbol{x})$:

$$\sigma_s(\omega;\boldsymbol{x}) = \frac{i\omega_{ps}^2}{4\pi(\omega+i\nu_s)}, \qquad \chi_s(\omega;\boldsymbol{x}) = -\frac{\omega_{ps}^2}{\omega(\omega+i\nu_s)}, \qquad \epsilon(\omega;\boldsymbol{x}) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega+i\nu_s)}, \tag{2.14}$$

where the dependence on $\boldsymbol{x}$ enters through ω_{ps}^2 and ν_s.

2.2 Homogeneous plasma

2.2.1 General considerations

In plasma that is spatially homogeneous, the functions (2.14) are independent of $\boldsymbol{x}$, so there exist waves that are monochromatic in $\boldsymbol{x}$, i.e., have a well-defined wavevector $\boldsymbol{k}$. Let us search for such waves assuming the coordinates $\boldsymbol{x} = \{x, y, z\}$ such that $\boldsymbol{k}$ is directed along the x axis. The corresponding dispersion tensor [Eq. (1.36)] can be written as

$$\boldsymbol{D}_E(\omega,\boldsymbol{k}) = \begin{pmatrix} \epsilon(\omega) & 0 & 0 \\ 0 & \epsilon(\omega) - N^2 & 0 \\ 0 & 0 & \epsilon(\omega) - N^2 \end{pmatrix}, \tag{2.15}$$

and, as a reminder, $N \doteq ck/\omega$. The corresponding dispersion relation $\det \boldsymbol{D}_E(\omega,\boldsymbol{k}) = 0$ can be written as

$$\epsilon(\omega)\,[\epsilon(\omega) - N^2]^2 = 0, \tag{2.16}$$

so it has solutions of two types:

$$\epsilon(\omega) = 0 \qquad \text{and} \qquad \epsilon(\omega) = N^2. \tag{2.17}$$

Since $\omega_{ps}^2 \propto m_s^{-1}$ and $m_i \gg m_e$, the ion contribution to the dispersion of cold nonmagnetized plasma is typically negligible. Assuming that this is the case, we obtain

$$\epsilon(\omega) \approx 1 - \frac{\omega_{pe}^2}{\omega(\omega + i\nu_e)}. \tag{2.18}$$

[1] The absence of spatial dispersion is due to the fact that we have neglected the plasma temperature and average flows. These effects will be discussed later.

Although exact solutions of Eq. (2.17) for $\omega(k)$ are possible, it is more instructive to consider the regime when ν_e is small. By assuming $\omega \gg \nu_e$ and justifying this assumption *a posteriori* one obtains

$$\omega \approx \pm\omega_{pe} - \frac{i\nu_e}{2}, \tag{2.19a}$$

$$\omega \approx \pm\sqrt{\omega_{pe}^2 + c^2k^2} - \frac{i\nu_e}{2}\,\frac{\omega_{pe}^2}{c^2k^2 + \omega_{pe}^2}. \tag{2.19b}$$

If, instead, one starts with the assumption that ω is of order ν_e, one similarly obtains

$$\omega \approx -i\nu_e\,\frac{c^2k^2}{c^2k^2 + \omega_{pe}^2}. \tag{2.19c}$$

Because $\epsilon(\omega) = 0$ gives quadratic equation for ω and thus has two solutions, and $\epsilon(\omega) = N^2$ gives a cubic equation for ω and thus has three solutions, Eqs. (2.19) exhaust all possible solutions of Eqs. (2.17) and thus of Eq. (2.16) as well.

Note that all five modes (2.19) satisfy $\omega_i > -\nu_e$ (or $|\omega_i| < \nu_e$). Thus, the applicability condition of our theory, $\nu_e + \omega_i > 0$, is satisfied and the result (2.19) can be trusted. One might wonder if this is by accident, and the answer is no. As seen from Eq. (1.8), $\bar{\boldsymbol{\Sigma}}$ is just the current density produced by a delta-shaped field. In contrast, the solutions (2.19) describe collective oscillations, in which particles remain coupled with nonzero self-consistent field at all t. As collisions dissipate the electron energy, this loss is partially compensated from the energy of the field, so the current dissipates slower than electrons would slow down without the wave. This argument also extends to general collisional plasmas, so in such plasmas one can always obtain $\boldsymbol{\sigma}(\omega, \boldsymbol{k})$ from $\hat{\boldsymbol{\sigma}}$ simply by replacing $\hat{\omega}$ with ω (assuming that the plasma parameters are time-independent). However, this is not the case in collisionless plasmas,[2] where the total macroscopic field can dissipate while the response of individual particles does not. For the time being, we will always assume some amount of collisional dissipation to ensure the applicability of our general approach; i.e., collisionless plasmas will be considered as collisional plasmas with infinitesimally small but positive ν_s:

$$\nu_s \to 0+\,. \tag{2.20}$$

In what follows, Eq. (2.20) will always be assumed for simplicity, in which case ϵ is simplified as

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \qquad \omega_p \doteq \sqrt{\sum_s \omega_{ps}^2}. \tag{2.21}$$

"The" plasma frequency ω_p contains information about both electrons and ions, so we do not have to neglect the ion response to treat dispersion relations analytically. Then the resulting three wave modes are as follows.

2.2.2 Static magnetic-field mode

Let us discuss Eq. (2.19c) first. In the collisionless limit, this mode has zero frequency, $\omega = 0$, so, often, it is not even considered as a wave *per se*. This mode consists of static currents, which create a nonuniform magnetic field but no electric field. Such static magnetic-field "waves" are of interest in some cases [10] but will not be discussed further.

[2] Since no plasma is purely collisionless, the distinction between "collisionless" and "collisional" can be tricky. In a nutshell, (relevant) wave modes are defined differently when the collision rates are small enough, and then, in a way, small collision rates cease to matter. We will return to this subject in the second half of the course.

2.2.3 Electrostatic Langmuir oscillations

Equation (2.19a), which corresponds to $\epsilon(\omega) = 0$, in the collisionless limit becomes

$$\omega^2 = \omega_p^2. \tag{2.22}$$

This shows that ω_p is the frequency of natural oscillations of cold plasma. Consequently, it is also common to refer to the mode (2.22) as "the" plasma oscillations, as opposed to other waves and oscillations that can also exist in plasma.

The waves governed by Eq. (2.22) are nondissipative and can have *any* phase velocity $\boldsymbol{v}_p = (\boldsymbol{k}/k)(\omega_p/k)$, because while ω is fixed by Eq. (2.22), the vector $\boldsymbol{k}$ can be anything. In contrast, the group velocity $\boldsymbol{v}_g \doteq \partial_{\boldsymbol{k}}\omega(\boldsymbol{k})$ is identically zero for these waves. Also, the wave polarization $\boldsymbol{h}$ is found from the field equation $\boldsymbol{D}_E(\omega, \boldsymbol{k})\boldsymbol{h} = 0$, where $\boldsymbol{\mathcal{E}}$ is the complex amplitude of the electric field (Sec. 1.2.2). At $\omega^2 = \omega_p^2$, this equation becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & N^2 & 0 \\ 0 & 0 & N^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0, \tag{2.23}$$

so $h_y = h_z = 0$. In other words, the field is linearly polarized (Sec. 1.2.2) parallel to $\boldsymbol{k}$, so the wave is *longitudinal* and therefore electrostatic (Exercise 2.2).

> **Exercise 2.2:** Derive a PDE for the electron-density perturbations in this wave using the linearized continuity equation for the electron density, the linearized momentum equation for the electron velocity, and the electrostatic dispersion relation obtained in Lecture 1.

In electron–ion plasma, one has $\omega_p \approx \omega_{pe}$ (due to $m_i \gg m_e$), in which case the plasma oscillations can be interpreted as oscillations of electrons relative to the stationary neutralizing ion background. These oscillations were discovered by Langmuir and Tonks in the 1920s [11], so they are commonly called electron Langmuir oscillations. The terms is also often shortened to just "Langmuir oscillations", but note that ion Langmuir oscillations are possible too. They occur at $\omega_p \approx \omega_{pi}$ when electrons are hot enough to have negligible susceptibility. We will discuss this subject in Lecture 7.

2.2.4 Transverse electromagnetic waves

Equation (2.19b), which corresponds to $\epsilon(\omega) = N^2$, in the collisionless limit becomes

$$\omega^2 = \omega_p^2 + k^2c^2. \tag{2.24}$$

Unlike Langmuir oscillations, these are electromagnetic waves with transverse polarization. This means that their electric field is perpendicular to the wavevector, so there are two independent transverse modes overall. They can be chosen as the modes with the polarization vector $\boldsymbol{h}$ along, say, the y axis and the z axis, respectively. Alternatively, one can define them as the two circularly polarized (Sec. 1.2.2) modes with opposite, left and right, polarization. (This flexibility is removed when plasma is magnetized; see Lecture 6.)

The transverse electromagnetic waves have a superluminal phase velocity,

$$v_p = \frac{\omega}{k} > \frac{kc}{k} = c, \tag{2.25}$$

and their group velocity is given by

$$v_g = \frac{c^2}{v_p} < c. \tag{2.26}$$

At $\omega \gg \omega_p$, these waves become the usual vacuum light waves.

2.3 Wave transformations in inhomogeneous plasma

Now let us discuss how waves propagate in *inhomogeneous* cold nonmagnetized stationary plasma. Because there is no spatial dispersion in cold stationary plasma, the dispersion operator in this case is the same as the matrix (2.15) except $\boldsymbol{k}$ must be replaced with the operator $\hat{\boldsymbol{k}} = -i\nabla$. We will focus on transverse electromagnetic waves described by Eq. (2.24) and consider how they transform in simple geometries. The transformations that we will discuss are of interest in that they are characteristic of generic waves in inhomogeneous media. Hence, the material below is intended as a motivation for a more general theory, a subject that will be discussed in Part II.

2.3.1 WKB approximation

Let us start with considering a stationary ($\partial_t = -i\omega$) transverse wave propagating along the x axis in collisionless stationary plasma. Finding the wave field in this case is a boundary-value problem. We will assume that the field at the boundary has no dependence on y or z and that $\omega_p = \omega_p(x)$, so we can adopt $\partial_y = \partial_z = 0$ everywhere. The direction of the field polarization $\boldsymbol{h} \doteq \tilde{\boldsymbol{E}}/\tilde{E}$ is perpendicular to the x axis but otherwise unimportant. In any case, $\nabla \cdot \tilde{\boldsymbol{E}} = 0$, so Eq. (1.23) becomes

$$\frac{d^2\mathcal{E}}{dx^2} + q(x)\mathcal{E} = 0, \tag{2.27}$$

where we have adopted the notation $\tilde{\boldsymbol{E}} = \operatorname{Re}[\boldsymbol{h}\mathcal{E}(x)e^{-i\omega t}]$ and introduced

$$q(x) \doteq \frac{\omega^2}{c^2}\,\epsilon(\omega;x) = \frac{\omega^2 - \omega_p^2(x)}{c^2}. \tag{2.28}$$

Now that the coefficients of the wave equation explicitly depend on x, a spatially monochromatic wave is no longer a solution. But let us assume (until Sec. 2.3.2) that the plasma parameters change along x slowly. Then locally q can be considered constant, so $\sqrt{q}$ serves as the local wavenumber k. [This can be seen by comparing Eq. (2.28) with the homogeneous-plasma dispersion relation (2.24) or simply by solving Eq. (2.27) locally under the approximation of constant q.] This interpretation is justified provided that k is reasonably well defined, i.e., if the corresponding wavelength $\lambda \doteq 2\pi/k$ has a small gradient (which is a dimensionless quantity):

$$\frac{d\lambda}{dx} \ll 1, \quad \lambda = \frac{2\pi}{\sqrt{q}}. \tag{2.29}$$

Then, it is possible to construct an asymptotic solution of Eq. (2.27) using the so-called Wentzel–Kramers–Brillouin (WKB) approximation. We will not review the general WKB method here; rather, we will present an *ad hoc* solution of Eq. (2.27) sufficient for our purposes. (In Part II, we will adopt an alternative approach, which is easier to apply in the general case.)

Let $\varepsilon \ll 1$ be the corresponding characteristic value of $d\lambda/dx \sim (L_c\sqrt{q})^{-1}$, where L_c is some constant characteristic scale of the plasma density. Then, Eq. (2.27) can be written as

$$\varepsilon^2\mathcal{E}'' + Q(\xi)\mathcal{E} = 0, \tag{2.30}$$

where the prime denotes $d/d\xi$, $\xi \doteq x/L_c$, and $Q(\xi) \doteq (\varepsilon L_c)^2 q(x)$ is an order-one function with $Q' \sim Q$. Let us search for a solution in the form $\mathcal{E} = e^{iS(\xi)/\varepsilon}$, where S is a complex function of the form $S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 \ldots$ with order-one S_n. Then, Eq. (2.30) yields[3]

$$Q - S_0'^2 + \varepsilon(iS_0'' - 2S_0'S_1') + \mathcal{O}(\varepsilon^2) = 0. \tag{2.31}$$

[3] $f = \mathcal{O}(\varepsilon^\alpha)$ means "f is of order ε^α", or more precisely, there is constant M such that $|f| \leq M\varepsilon^\alpha$ for small enough ε.

Since the equality is supposed to hold for all small ε, this leads to

$$S_0'^2 = Q, \qquad S_1' = {}^i\!/_2\, S_0''/S_0', \tag{2.32}$$

and the terms $S_{n>1}$ will be ignored, because including them changes $\mathcal{E}$ only by $\mathcal{O}(\varepsilon)$. Then,

$$S_0 = \pm \int d\xi\, \sqrt{Q(\xi)}, \qquad S_1 = \frac{i}{4}\, \ln Q(\xi) + \text{const}, \tag{2.33}$$

so the field can be expressed as follows:

$$\mathcal{E}(x) = \frac{A}{\sqrt[4]{q(x)}}\, \exp\left[\pm i \int dx\, \sqrt{q(x)}\right], \tag{2.34}$$

where A is an arbitrary complex constant and $\pm$ correspond to the waves propagating in the $\pm x$ directions, respectively. Since Eq. (2.27) is linear, any linear combination of the solutions with plus and minus sign is also a solution.

The WKB solution (2.34) corroborates our expectation that the local wavevector $k(x)$, defined as the gradient of the phase, is the same[4] as that predicted by the homogeneous-plasma dispersion relation (2.24); i.e., $k(x) = \pm\sqrt{q(x)}$. Equation (2.34) also shows the field amplitude changes as $|\mathcal{E}| \propto [q(x)]^{-1/4}$, so

$$|\mathcal{E}(x)|^2 \sqrt{q(x)} = \text{const}. \tag{2.35}$$

This coincides with the well-known adiabatic invariant of a harmonic oscillator [our original Eq. (2.27) *is* the equation of a harmonic oscillator], which is the ratio of oscillator's energy and frequency [12]. A more general form of this conservation law will be discussed in Lecture 5.

2.3.2 Field structure near a cutoff

Equations (2.34) and (2.35) predict $|\mathcal{E}| \to \infty$ at $q \to 0$, or at the "critical density", which corresponds to $\omega^2 = \omega_p^2$. This is an artifact of the WKB approximation, which is inapplicable at small q, as seen from Eq. (2.29). The critical point $q = 0$ corresponds to a *cutoff* (a type of caustic), where k^2 changes sign according to the homogeneous-plasma dispersion relation (2.24) (Box 2.1). Beyond the cutoff (i.e., at $\omega_p^2 > \omega^2$), the wavenumber is imaginary. Such waves are called *evanescent* and do not transport energy (Exercise 2.3). This means that a wave has to experience reflection near a cutoff.

Exercise 2.3: Show that the time-average Poynting vector of an evanescent wave with a single imaginary wavevector is zero. Explain how waves can carry electromagnetic energy across a *finite-width* region where they are evanescent.

Let us assume a linear approximation for $\omega_p^2(x)$ and choose coordinates such that the cutoff corresponds to $x = 0$, with waves propagating at $x < 0$ and evanescent at $x > 0$. Then locally,

$$\omega_p^2(x) = \omega^2 \left(1 + \frac{x}{L_c}\right), \qquad L_c > 0. \tag{2.36}$$

Equation (2.27) becomes

$$\frac{d^2\mathcal{E}}{d\eta^2} - \eta\mathcal{E} = 0, \qquad \eta \doteq \frac{x}{\ell}, \qquad \ell \doteq \left(\frac{L_c c^2}{\omega^2}\right)^{1/3}. \tag{2.37}$$

[4] Although this is always true approximately at small ε, we will show in Lecture 4 that a small deviation from the homogeneous-plasma dispersion relation is possible in some cases.

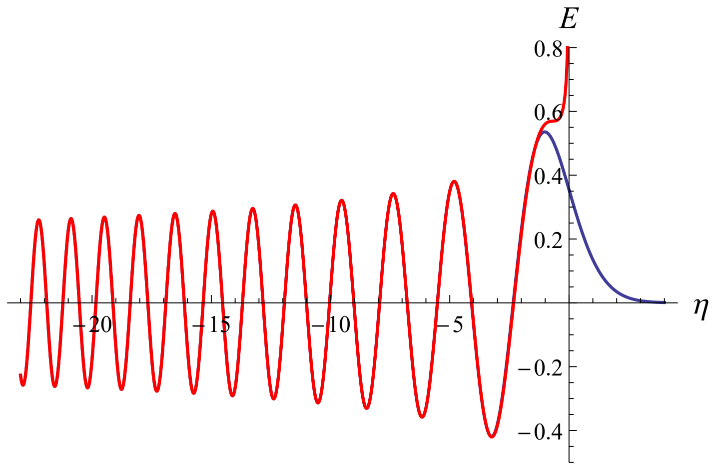


Figure 2.1: The Airy function Ai(η) (blue) vs. the WKB approximation (2.40) (red), with $x = \eta\ell$, $q = -\eta\ell^2$, and $C = 1$. The field is a standing wave at $\eta < 0$ and an evanescent wave at $\eta > 0$.

This is the so-called Airy equation, and its general solution is a linear superposition of the Airy functions of the first and second kind, Ai(η) and Bi(η), respectively. Assuming that a wave is incident from the left, we are interested in a solution that corresponds to $\mathcal{E}(x \to \infty) = 0$. This leaves us with $\mathcal{E} = C\mathrm{Ai}(\eta)$, where C is an arbitrary constant that determines the maximum near-cutoff magnitude, $\mathcal{E}_{\max} \sim C$ (Fig. 2.1). One can connect C with the constants A_+ and A_- of the general WKB solution,

$$\mathcal{E}(x) = \frac{A_+ e^{i\theta(x)} + A_- e^{-i\theta(x)}}{[q(x)]^{1/4}}, \qquad \theta(x) = \frac{\pi}{4} + \int_0^x dx' \sqrt{q(x')}, \tag{2.38}$$

by comparing its asymptotic behavior with the known asymptotic behavior of Ai(η) at $\eta \to -\infty$:

$$\mathrm{Ai}(\eta) \approx \frac{\cos[\theta_A(\eta)]}{\sqrt{\pi}\,(-\eta)^{1/4}}, \qquad \theta_A(\eta) = \frac{\pi}{4} - \frac{2}{3}(-\eta)^{3/2}. \tag{2.39}$$

Near the cutoff, $q \approx -\eta/\ell^2$, which leads to $\theta(x) \approx \theta_A(\eta)$. Then, for the solution (2.38) to match $\mathcal{E} = C\mathrm{Ai}(\eta)$ at not-too-large negative η, one should adopt $A_+ = A_- = 2\sqrt{\pi\ell}C$, which leads to

$$\mathcal{E}(x) = \frac{C\cos[\theta(x)]}{\sqrt{\pi}[q(x)\ell^2]^{1/4}}. \tag{2.40}$$

In other words, the WKB approximation predicts that at large negative η, the field is a standing wave with the local wavenumber $k(x) = \sqrt{q(x)}$ [same as predicted by the homogeneous-plasma dispersion relation (2.24)] and the amplitude $|\mathcal{E}|$ that satisfies

$$\frac{|\mathcal{E}|}{\mathcal{E}_{\max}} \sim \frac{1}{(q\ell^2)^{1/4}} = \frac{1}{\sqrt[4]{\epsilon}}\left(\frac{c}{\omega L_c}\right)^{1/6}. \tag{2.41}$$

Since the right-hand side depends on the plasma parameters slowly, the field amplification near the cutoff is not as dramatic as one might imagine based on the WKB approximation. Nevertheless, the WKB approximation describes the field well all the way up to $\eta \sim -1$ (Fig. 2.1).

2.3.3 Oblique incidence on a cutoff region

Let us also consider oblique incidence of a transverse wave on a cutoff. As in the previous section, we assume a plane incident wave and $\omega_p = \omega_p(x)$. However, now we allow for a nonzero constant k_y, so

Box 2.1: Reinstating the WKB approximation near cutoffs

In principle, the WKB approximation can be reinstated near a cutoff by changing the representation of the field equation. For example, consider the Airy equation (2.37) in the operator form, $\hat{k}^2\mathcal{E} + \hat{x}\mathcal{E} = 0$. In the usual, coordinate representation, one has $\hat{x} = x$ and $\hat{k} = -i\partial_x$. (Here, we blur the distinction between the dimensionless η and $-i\partial_\eta$ on one side and the dimensional x and $-i\partial_x$ on the other side for simplicity.) However, we can adopt the wavevector representation (cf. momentum representation in quantum mechanics), i.e., take the Fourier transform of the Airy equation. Then, $\hat{x} = i\partial_k$ and $\hat{k} = k$, so the equation becomes

$$k^2\bar{\mathcal{E}} + i\partial_k\bar{\mathcal{E}} = 0,$$

where $\bar{\mathcal{E}}$ is the Fourier spectrum of $\mathcal{E}$. In this form, the equation *can* be solved using the WKB method (in the k space). In fact, the corresponding WKB approximation, $\bar{\mathcal{E}} = \text{const} \times \exp(ik^3/3)$, is an exact solution. Mapping it back to the coordinate space is only a matter of calculating the integral $\int dk\, \exp(ikx + ik^3/3)$, which can be done at various levels of accuracy.

One can also use a more general class of transforms called metaplectic transforms. The latter can rotate (more generally, perform symplectic transformations of) the "phase space" (x, k) such that, in the new phase space (x', k'), cutoffs are eliminated. (The Fourier transform can be considered as a particular case of the metaplectic transform that corresponds to 90° rotation.) This is illustrated in Fig. 2.2. The natural GO parameter in the rotated frame is the squared curvature of the ray trajectory in the phase space. However, note that, because the phase-space is a symplectic space rather than a metric space, the phase-space coordinates must be properly rescaled for this symplectic curvature to be well defined.

the electric and magnetic fields can be written as follows:

$$\tilde{\boldsymbol{E}} = \operatorname{Re}[\boldsymbol{\mathcal{E}}(x)e^{-i\omega t + ik_y y}], \qquad \tilde{\boldsymbol{B}} = \operatorname{Re}[\boldsymbol{\mathcal{B}}(x)e^{-i\omega t + ik_y y}]. \tag{2.42}$$

Two distinct wave patterns can be formed in this case, depending on the wave polarization. Below, we briefly discuss both cases.

TE wave

Let us start with the transverse-electric (TE) polarization (Fig. 2.3), which corresponds to $\mathcal{E}_x = \mathcal{E}_y = 0$. Since $\partial_z\mathcal{E}_z = 0$, one has $\nabla \cdot \tilde{\boldsymbol{E}} = 0$, which leads to the following equation for $\mathcal{E}_z$:

$$0 = \nabla^2\mathcal{E}_z + \frac{\omega^2}{c^2}\,\epsilon(\omega; x)\mathcal{E}_z = \frac{d^2\mathcal{E}_z}{dx^2} + \frac{1}{c^2}\,[\omega^2 - c^2k_y^2 - \omega_p^2(x)]\mathcal{E}_z. \tag{2.43}$$

This equation is identical to Eq. (2.27) up to replacing $\omega^2 \to \omega^2 - c^2k_y^2$. Thus, the field forms the same WKB/Airy profile (Fig. 2.1), except the cutoff is shifted to the location where $\omega_p^2(x) = \omega^2 - k_y^2c^2$, in agreement with the homogeneous-plasma dispersion relation (2.24).

TM wave

Let us also consider the transverse-magnetic (TM) polarization (Fig. 2.3). In this case, $\nabla \cdot \tilde{\boldsymbol{E}}$ is nonzero, so a transverse wave is coupled to the oscillations of the charge density, $\tilde{\rho}$, through Gauss's law (1.1c). Similarly, $\tilde{\rho}$ is coupled to $\tilde{\boldsymbol{E}}$ via (Exercise 2.4)

$$\partial_t^2\tilde{\rho} + \omega_p^2\tilde{\rho} = \tilde{f}, \qquad \tilde{f} = -\frac{e^2}{m_e}\,\nabla n_{0e} \cdot \tilde{\boldsymbol{E}}. \tag{2.44}$$

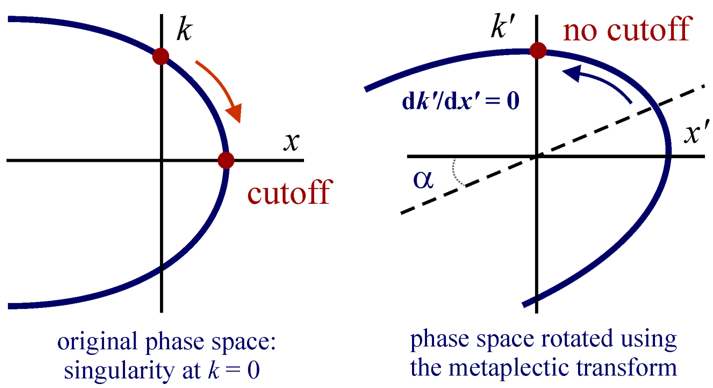


Figure 2.2: An illustration of cutoff elimination using the metaplectic transform (Box 2.1): the phase space is continuously transformed such that the wavenumber remains constant and thus a wave never sees a cutoff in this transforming frame. In the simplest case, the transformation is a rotation by some nonzero angle α, with $\alpha = 90°$ corresponding to the Fourier transform.

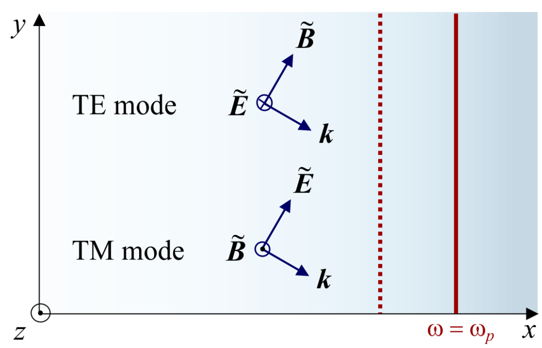


Figure 2.3: A schematic of transverse-wave oblique incidence on plasma with background electron density $n_{0e} = n_{0e}(x)$ illustrating the difference between a TE wave and a TM wave. The vertical red line indicates the critical-density region, where $\omega_p^2(x) = \omega^2$. The dashed red line indicates the cutoff region, where $\omega_p^2(x) = \omega^2 - k_y^2 c^2$. Waves are propagating on the left and evanescent on the right. The background color intensity illustrates the background density, with darker colors corresponding to higher densities.

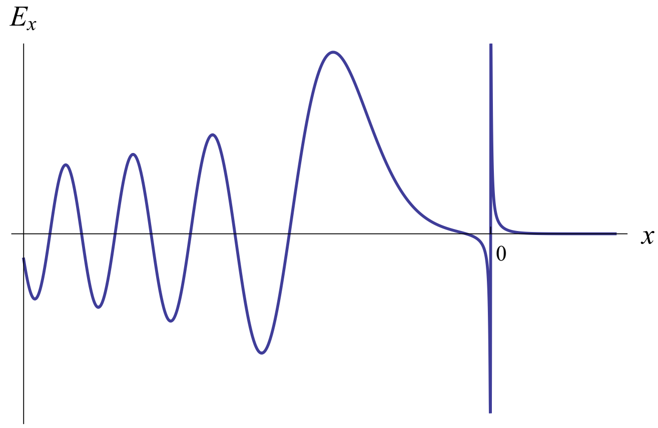


Figure 2.4: A typical $\mathcal{E}_x$ [Eq. (2.46), numerical solution] for intermediate β and weak dissipation, with the boundary conditions corresponding to vanishing $\mathcal{B}_z$ at $x \to \infty$. Here, $\epsilon(\omega; x)$ is given by Eq. (2.18), with $\omega_p^2(x)$ given by Eq. (2.36).

Equation (2.44) can be understood as the electrostatic wave equation for free Langmuir oscillations driven by an external force $\tilde{f} \propto \tilde{\boldsymbol{E}}$. If this force has a resonant frequency $\omega = \pm\omega_p$, the magnitude of $\tilde{\rho}$ can become large. Then, several scenarios are possible.

When the angle β between $\boldsymbol{k}$ and ∇n_{0e} approaches zero, one recovers the usual WKB/Airy profile discussed above. Likewise, if β is large, one can expect from Eq. (2.24) that the wave is reflected much earlier than it reaches the critical-density region. Then, $\tilde{f}$ is extremely small in the resonance region, so Langmuir oscillations are not excited (unless the collision rate is also extremely small) and a TM wave behaves just like a TE wave. However, at intermediate β, the coupling of a TM wave with Langmuir oscillations can be substantial and results in a significant peak of the wave field near the critical region. This can be described by the following equations (Exercise 2.4)

$$\frac{d^2\mathcal{B}_z}{dx^2} - \frac{\partial \ln \epsilon(\omega; x)}{\partial x}\frac{d\mathcal{B}_z}{dx} + \left[\frac{\omega^2}{c^2}\,\epsilon(\omega; x) - k_y^2\right]\mathcal{B}_z = 0, \tag{2.45}$$

$$\mathcal{E}_x = -\frac{ck_y\mathcal{B}_z}{\omega\epsilon(\omega; x)}, \qquad \mathcal{E}_y = \frac{ic}{\omega\epsilon(\omega; x)}\,\frac{d\mathcal{B}_z}{dx}. \tag{2.46}$$

A typical $\mathcal{E}_x$ [Eq. (2.46)] for intermediate β is shown in Fig. 2.4, where an Airy pattern is seen on the left and a large near-singular field is seen near the Langmuir resonance. (For details and asymptotic analytic solutions, see Ref. [13].) Note that a small collision rate has to be added to keep the wave field finite. This is due to the fact that energy is continuously deposited by the incoming wave into Langmuir oscillations and cannot escape the critical-density region, because Langmuir waves are tied to this region and because they cannot dissipate collisionlessly in cold plasma. One can show that adding thermal effects removes both these limitations.

The coupling of TM waves with Langmuir oscillations is an example of *mode conversion*, which is mathematically similar to quantum tunneling. This effect is completely missed within the homogeneous-plasma analysis (Sec. 2.2), where the two transverse modes have identical properties and are completely uncoupled from Langmuir oscillations. To capture polarization effects robustly and in a general geometry, a more systematic theory is needed. Later, we will present such theory, which will be readily applicable to almost any dispersion operators.

Exercise 2.4: Derive Eqs. (2.44)–(2.46).

Appendices for Part I

AI.1 Analytic properties of response functions

Here, we briefly summarize some important properties of response functions that commonly emerge in theory of dispersion. We will be concerned only with temporal dynamics, so possible dependence of these functions on spatial coordinates and wavevectors will be ignored.

Let us consider a general system whose response j (which may or may not be the current density) to an external force E (which may or may not be the electric field) is linear, i.e., can be expressed in terms of some Green's function $\bar{\Sigma}$:

$$j(t) = \int_0^t dt' \, \bar{\Sigma}(t - t') E(t'). \tag{2.47}$$

Let us assume that our system is stable in the sense that a response to a delta-shaped field eventually fades away, i.e., $\bar{\Sigma}(t \to +\infty) = 0$.[5] Then, the integral that determines its Laplace image,

$$\sigma(\omega) = \int_0^\infty dt \, e^{i\omega t} \bar{\Sigma}(t), \tag{2.48}$$

converges for all ω that satisfy $\operatorname{Im} \omega \geq 0$. Similarly, if $\bar{\Sigma}(t)$ fades away faster than any power of t at $t \to \infty$ (which is typically the case, as discussed in Lecture 8), then all integrals of the form

$$J_n[\bar{\Sigma}](\omega) \doteq \int_0^\infty dt \, e^{i\omega t} (it)^n \bar{\Sigma}(t) \tag{2.49}$$

converge too. [The square brackets denote that J_n is a functional of $\bar{\Sigma}$, and (ω) denotes that J_n depends on ω, as usual.] Since $J_n[\bar{\Sigma}](\omega) = \sigma^{(n)}(\omega)$, where ${}^{(n)}$ is the nth derivative, this means that all derivatives of $\sigma(\omega)$ are well defined at $\operatorname{Im} \omega \geq 0$. This proves the following theorem:

> **Theorem:** $\sigma(\omega)$ is analytic in the upper half of the complex-ω plane.

Next, note that the function $\sigma(\omega) \equiv J_0[\bar{\Sigma}](\omega) = \int_0^\infty dt \, e^{i\omega t} \bar{\Sigma}(t)$ can be expressed as follows:

$$\begin{aligned} J_0[\bar{\Sigma}](\omega) &= \int_0^\infty dt \left\{ \frac{d}{dt} \left[\frac{e^{i\omega t}}{i\omega} \bar{\Sigma}(t) \right] - \frac{e^{i\omega t}}{i\omega} \bar{\Sigma}'(t) \right\} \\ &= \frac{1}{i\omega} \left[e^{i\omega t} \bar{\Sigma}(t) \right] \Big|_0^\infty - \frac{1}{i\omega} \int_0^\infty dt \, e^{i\omega t} \bar{\Sigma}'(t) \\ &= -\frac{\bar{\Sigma}(0)}{i\omega} - \frac{J_0[\bar{\Sigma}'](\omega)}{i\omega}. \end{aligned} \tag{2.50}$$

[5] This does not rule out collective instabilities, where the field itself grows in time. In the sense assumed here, the response of a current to an external field, which is described by the conductivity operator $\hat{\boldsymbol{\sigma}}$, is typically stable. (Systems exhibiting avalanche ionization are exceptions.) In contrast, the response of a self-consistent field to an external current, which is described by $\hat{\boldsymbol{D}}_E^{-1}$, is often not.

[Here, we have assumed that the function $\bar{\Sigma}$ is also sufficiently well behaved so that $\bar{\Sigma}(0)$ and $J[\bar{\Sigma}']$ are finite.] The function $\bar{\Sigma}(t)$ is determined by microscopic processes that have some nonzero minimum time scale $\mathcal{T}$. Then, roughly, $|J[\bar{\Sigma}']| \lesssim |J[\bar{\Sigma}]|/\mathcal{T}$. Hence, at $\omega \gg \mathcal{T}^{-1}$, the second term on the right-hand side of Eq. (2.50) can be neglected compared to the term on the left-hand side. Thus, at large enough ω, one has $\sigma(\omega) \approx i\bar{\Sigma}(0)/\omega$, and in particular,

$$\sigma(\omega \to \infty) = 0. \tag{2.51}$$

In combination with Eq. (2.51), the analyticity of $\sigma(\omega)$ in the upper half of the complex-ω plane (proven above) leads to some interesting properties of the functions

$$\sigma_r(\omega) \doteq \operatorname{Re}\sigma(\omega), \qquad \sigma_i(\omega) \doteq \operatorname{Im}\sigma(\omega). \tag{2.52}$$

These properties are derived as follows. Let us consider real ω_0 and the integral

$$\mathcal{I}(\omega_0) \doteq \int_C d\omega\, \frac{\sigma(\omega)}{\omega - \omega_0}, \tag{2.53}$$

where C is a contour that goes along the real axis and encircles the pole at $\omega = \omega_0$ from above. On one hand, one can rewrite $\mathcal{I}(\omega_0)$ as

$$\mathcal{I}(\omega_0) = \fint_{-\infty}^{\infty} d\omega\, \frac{\sigma(\omega)}{\omega - \omega_0} - i\pi\sigma(\omega_0), \tag{2.54}$$

where $\fint$ denotes the Cauchy principal value of the corresponding integral. On the other hand, one can close the contour C through a semicircle C_+ with radius $R \to \infty$ at $\operatorname{Im}\omega > 0$, because the integral over C_+ is zero. [This is because at $\omega \to \infty$, the integrand decreases faster than $1/\omega$ due to $\sigma(\omega) \to 0$.] But $\sigma(\omega)/(\omega - \omega_0)$ is analytic everywhere within the closed contour, so the integral over this closed contour must be zero. Therefore, $\mathcal{I}(\omega_0) = 0$, which means

$$\sigma(\omega_0) = -\frac{i}{\pi} \fint_{-\infty}^{\infty} d\omega\, \frac{\sigma(\omega)}{\omega - \omega_0}. \tag{2.55}$$

By taking the real and imaginary parts of Eq. (2.55), one obtains that σ_r and σ_i are connected via the Hilbert transform; namely,

$$\sigma_r(\omega_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} d\omega\, \frac{\sigma_i(\omega)}{\omega - \omega_0}, \tag{2.56a}$$

$$\sigma_i(\omega_0) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} d\omega\, \frac{\sigma_r(\omega)}{\omega - \omega_0}. \tag{2.56b}$$

These are known as the *Kramers–Kronig relations*. They show that having nonzero σ_i at some frequencies implies having nonzero σ_r at some (possibly, different) frequencies, and vice versa. Notably, the collisionless dissipation discussed in Part IV can be anticipated from the Kramers–Kronig relations.

Problems for Part I

PI.1 Electrostatic approximation

In some cases, a wave can be considered electrostatic, meaning that its electric field is approximately representable as (minus) the gradient of an electrostatic potential, $\boldsymbol{E} \approx -\nabla\varphi$. Such waves can be easier to describe, because instead of the multiple components of $\boldsymbol{E}$, one can work with a single function φ. Here, you are asked to explore the electrostatic-wave approximation for a homogeneous stationary medium.

(a) Consider the components of $\boldsymbol{E}$ parallel and perpendicular to the wavevector, $\boldsymbol{E}_\parallel$ and $\boldsymbol{E}_\perp$. Using Eq. (1.36), estimate the ratio of $|\boldsymbol{E}_\parallel|$ and $|\boldsymbol{E}_\perp|$ at sufficiently large N and argue that

$$N^2 \gg \epsilon_{ab} \quad \text{for all } a \text{ and } b \tag{2.57}$$

is a sufficient condition for a wave to be electrostatic.

(b) Assuming that the field is electrostatic and that the medium is described by an unspecified dielectric tensor $\boldsymbol{\epsilon}$, derive the corresponding dispersion relation from Gauss's law. Show that the same result is obtained from Eq. (1.36) if one takes for granted that the field is electrostatic.

(c) In a gyrotropic medium with no spatial dispersion, the dielectric tensor has the form

$$\boldsymbol{\epsilon}(\omega, \boldsymbol{k}) = \begin{pmatrix} \epsilon_\perp(\omega) & -ig(\omega) & 0 \\ ig(\omega) & \epsilon_\perp(\omega) & 0 \\ 0 & 0 & \epsilon_\parallel(\omega) \end{pmatrix}. \tag{2.58}$$

Consider an electrostatic wave propagating with a such medium with $\boldsymbol{k} = \{k_\perp, 0, k_\parallel\}$. (One can always choose axes such that $k_y = 0$.) Substitute this $\boldsymbol{k}$ and Eq. (2.58) for $\boldsymbol{\epsilon}$ into the dispersion relation derived in problem (b). Using your result, show that the group velocity of such a wave is orthogonal to the wave phase velocity.

PI.2 Photon wave function in cold magnetized plasma

Consider a cold plasma formed by $\mathcal{N}$ species with charges e_s, masses m_s, and unperturbed densities $n_{0s} = n_{0s}(\boldsymbol{x})$. Assume that this plasma is immersed in a stationary magnetic field $\boldsymbol{B}_0(\boldsymbol{x})$ and has no average flows.

(a) Show that the combination of the linearized momentum equation for the fluid velocities $\tilde{\boldsymbol{v}}_s$, Ampere's law for the wave electric field $\tilde{\boldsymbol{E}}$, and Faraday's law for the wave magnetic field $\tilde{\boldsymbol{B}}$ can be represented together in the form

$$i\partial_t \boldsymbol{\psi} = \hat{\boldsymbol{H}}\boldsymbol{\psi}, \tag{2.59}$$

or equivalently, $\hat{\boldsymbol{D}}_\psi \boldsymbol{\psi} = 0$ with $\hat{\boldsymbol{D}}_\psi = \hat{\omega} - \hat{\boldsymbol{H}}$, where $\boldsymbol{\psi}$ is a $3(\mathcal{N}+2)$-dimensional real vector field and the Hamiltonian $\hat{\boldsymbol{H}}$ being the following operator:

$$\hat{\boldsymbol{H}} = \begin{pmatrix} -\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_1(\boldsymbol{x}) & 0 & \dots & 0 & i\omega_{p1}(\boldsymbol{x}) & 0 \\ 0 & -\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_2(\boldsymbol{x}) & \dots & 0 & i\omega_{p2}(\boldsymbol{x}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_{\mathcal{N}}(\boldsymbol{x}) & i\omega_{p\mathcal{N}}(\boldsymbol{x}) & 0 \\ -i\omega_{p1}(\boldsymbol{x}) & -i\omega_{p2}(\boldsymbol{x}) & \dots & -i\omega_{p\mathcal{N}}(\boldsymbol{x}) & 0 & ic\boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}} \\ 0 & 0 & \dots & 0 & -ic\boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}} & 0 \end{pmatrix}. \tag{2.60}$$

The reason for introducing the imaginary unit into Eq. (2.59) is that in this particular form $\hat{\boldsymbol{H}}$ is Hermitian. Specifically, $\boldsymbol{\Omega}_s \doteq e_s \boldsymbol{B}_0(\boldsymbol{x})/(m_s c)$, $\omega_{ps} \doteq e_s\sqrt{4\pi n_{0s}(\boldsymbol{x})/m_s}$ (note that this definition is slightly different from the one that we used earlier), and $\boldsymbol{\alpha} \doteq (\alpha_x, \alpha_y, \alpha_z)^\intercal$ is the column vector comprised of the following Hermitian matrices:[6]

$$\alpha_x \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \alpha_y \doteq \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \alpha_z \doteq \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.61}$$

Note that Eq. (2.59) is similar to the Schrödinger equation for vector particles. (This becomes even more evident if one multiplies it by $\hbar$ and expresses the right-hand side through the momentum operator $\hat{\boldsymbol{p}} \doteq \hbar\hat{\boldsymbol{k}}$ instead of the wavevector operator $\hat{\boldsymbol{k}}$.) Thus, $\boldsymbol{\psi}$ can be understood as the photon wave function in cold plasma (cf. the photon wave function in vacuum [14]).

> **Hint:** Use $\boldsymbol{\psi} = (\tilde{\boldsymbol{\zeta}}_1, \tilde{\boldsymbol{\zeta}}_2, \dots, \tilde{\boldsymbol{\zeta}}_{\mathcal{N}}, \tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}})^\intercal/\sqrt{8\pi}$, where $\tilde{\boldsymbol{\zeta}}_s$ is a rescaled $\boldsymbol{v}_s$, with the rescaling factor that you are asked to find. Also use that for any three-component column vectors $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$, one has $\boldsymbol{\mathcal{A}}\times\boldsymbol{\mathcal{B}} = -i(\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{A}})\boldsymbol{\mathcal{B}}$, as can be verified by direct calculation.

(b) Calculate $|\boldsymbol{\psi}|^2$, which is the same as $\boldsymbol{\psi}^2$ here. Show that $\int d\boldsymbol{x}\,|\boldsymbol{\psi}(t,\boldsymbol{x})|^2 \equiv \langle\boldsymbol{\psi}|\boldsymbol{\psi}\rangle$ is conserved.

> **Hint:** In the last question, use Eq. (2.59) and the fact that $\hat{\boldsymbol{H}}$ is Hermitian. Do *not* use the explicit formula (2.60), or your calculations will be much longer than necessary.

(c) Argue that cold-plasma modes governed by Eq. (2.59) cannot be unstable (irrespective of how inhomogeneous the plasma and $\boldsymbol{B}_0$ are[7].)

> **Hint:** This can be argued in one sentence.

PI.3 Beam–plasma instability (cold electrostatic limit)

Consider one-dimensional nonmagnetized collisionless cold homogeneous stationary electron plasma with motionless ions. Suppose that bulk electrons have the average density n_0 and zero average velocity. Consider also a cold electron beam with some average density n_b and some average velocity v_b.

(a) Consider bulk electrons and beam electrons as different species. Show that the beam-electron susceptibility in the spectral representation is given by $\chi_b(\omega, k) = -\omega_b^2/(\omega - kv_b)^2$, where $\omega_b \doteq \sqrt{4\pi n_b e^2/m_e}$. (You may adopt $\hat{\omega} = \omega$ and $\hat{k} = k$.)

[6] Notably, α_a belong to the family of Gell–Mann matrices, which serve as infinitesimal generators of SU(3).
[7] This is true only for cold plasma without flows. Otherwise, $\hat{\boldsymbol{H}}$ is different and generally has different properties.

(b) Calculate the plasma dielectric function $\epsilon(\omega, k)$ and write down the general dispersion relation of electrostatic oscillations.

(c) How many branches does this dispersion relation have?

(d) Show graphically, by qualitatively analyzing the function $\epsilon(\omega, k)$ derived in (b) and using your answer from (c), that such plasma is unstable at small enough k.

(e) For the unstable regime, plot $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ as functions of k. (You may do it qualitatively or by solving the dispersion relation numerically, but make sure to plot all branches.)

(f) Assuming $\eta \doteq n_b/n_0$ is a small parameter, show that the maximum of the growth rate γ scales as $\gamma_{\max} \propto \eta^{1/3}$. (If a beam is warm, which case will be studied later in this course, the scaling for γ can be different.)

> **Hint:** Use the fact that, to the zeroth order in η, the unstable branch has $\omega = kv_b$ [as you should be able to see from (e)]. Show that to the leading order, $\gamma \to \infty$ at $k = \pm\omega_{p0}/v_b$, where $\omega_{p0}^2 \doteq \sqrt{4\pi n_0 e^2/m_e}$. Hence, adopt $k = \pm\omega_{p0}/v_b$ as the optimum wavenumber for the instability and then find γ more accurately.

PI.4 Surface waves

Here, you are asked to study azimuthally symmetric electromagnetic waves propagating along a cold-plasma column of finite radius a. Assume the cylindrical coordinates (r, θ, z), where z is the axis of symmetry, and $\tilde{\boldsymbol{E}} = \mathrm{Re}\,(\boldsymbol{\mathcal{E}}\, e^{-i\omega t + ik_z z})$, with constant ω and k_z. For simplicity, consider only TM waves, i.e., waves with $\tilde{B}_z = 0$. Also assume that the plasma density at $r < a$ is constant.

> **Hint:** Remember that the tangential component of the electric field is always continuous at the plasma boundary. Make sure you understand why.

(a) Using Maxwell's equations for the three components of the electric field $\tilde{\boldsymbol{E}}$ and the three components of the magnetic field $\tilde{\boldsymbol{B}}$, show that the above assumptions lead to $\tilde{E}_\theta = \tilde{B}_r = \tilde{B}_z = 0$ at $r \neq a$. (Assume that $\epsilon \neq 0$.) Argue that *in this problem*, $\tilde{B}_\theta$ is continuous at the plasma boundary.

(b) Show that $\nabla \cdot \tilde{\boldsymbol{E}} = 0$ at $r \neq a$. Using this, show that $\mathcal{E}_z$ satisfies the modified Bessel equation

$$\frac{d^2\mathcal{E}_z}{d\zeta^2} + \frac{1}{\zeta}\frac{d\mathcal{E}_z}{d\zeta} - \mathcal{E}_z = 0. \tag{2.62}$$

Here $\zeta \doteq \kappa r$, $\kappa^2 \doteq k_z^2 + \omega_p^2/c^2 - \omega^2/c^2$, $\omega_p(r < a) = \omega_{p0}$ is a nonzero constant, and $\omega_p(r > a) = 0$. Using the continuity of E_z at $r = a$, find $\mathcal{E}_z(r)$ as a piecewise-analytic function. (Remember that your solution is supposed to be finite everywhere, including $r \to 0$ and $r \to \infty$.)

(c) Using the continuity of $\tilde{B}_\theta$ at $r = a$ (which can be expressed through $d\mathcal{E}_z/d\zeta$), show that the dispersion relation is

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{1}{\kappa_{\text{in}} a}\frac{I_1(\kappa_{\text{in}} a)}{I_0(\kappa_{\text{in}} a)} + \frac{1}{\kappa_{\text{out}} a}\frac{K_1(\kappa_{\text{out}} a)}{K_0(\kappa_{\text{out}} a)} = 0. \tag{2.63}$$

Here, I_n and K_n are modified Bessel functions of the first and second kind, respectively; also, $\kappa_{\text{in}} \doteq \kappa(r < a)$ and $\kappa_{\text{out}} \doteq \kappa(r > a)$.

(d) Solve Eq. (2.63) numerically. Plot $\omega(k_z)$ and $\mathcal{E}_z(r)$ for several values of a/δ_c, where $\delta_p \doteq c/\omega_{p0}$. Explain the results qualitatively.

Part II

Basic theory of quasimonochromatic waves

In this part of the course, we outline a general asymptotic theory of quasimonochromatic waves in linear dispersive media. In the context of fusion research, our discussion is particularly relevant to waves in the electron-cyclotron and lower-hybrid ranges of frequencies, which correspond to mm and cm wavelengths, respectively (Part III). For waves with longer wavelengths, such as those in the ion-cyclotron range, our discussion is relevant qualitatively.

Lecture 3

Asymptotic expansion of dispersion operators

In this lecture, we derive an approximate envelope equation for a general quasimonochromatic wave by asymptotically expanding the wave dispersion operator. In doing so, we also introduce the Wigner–Weyl transform, which is a central element of modern wave theory and will be used also in other parts of the course.

3.1 Problem setup

Let us consider a general wave $\boldsymbol{\psi}$ governed by

$$\hat{\boldsymbol{D}}\boldsymbol{\psi} = 0, \tag{3.1}$$

where $\hat{\boldsymbol{D}}$ is a linear dispersion operator. We will assume that the wave remains quasimonochromatic, so $\boldsymbol{\psi}$ can be split into a rapid *real* phase θ ("eikonal", to be specified in Lecture 4) and a slow complex envelope $\boldsymbol{\Psi}$:

$$\boldsymbol{\psi} = \mathsf{e}^{i\theta}\boldsymbol{\Psi}. \tag{3.2}$$

Then, the quantities

$$\bar{w} \doteq -\partial_t\theta, \qquad \bar{\boldsymbol{k}} \doteq \nabla\theta \tag{3.3}$$

represent the local frequency and the local wavevector (note that they are real by definition), and

$$\tau \doteq \frac{2\pi}{\bar{w}}, \qquad \lambda \doteq \frac{2\pi}{\bar{k}} \tag{3.4}$$

are the local temporal period and the local spatial period (wavelength). Suppose the characteristic temporal and spatial scales of $\boldsymbol{\Psi}$, $\bar{w}$, $\bar{\boldsymbol{k}}$, and of the medium are limited from below by some values T_c and L_c, respectively. Then, we introduce a "geometrical-optics parameter"

$$\varepsilon \doteq \max\left\{\frac{\tau}{T_c}, \frac{\lambda}{L_c}\right\} \ll 1. \tag{3.5}$$

Our goal is to develop an approximate *geometrical-optics* (GO)[1] theory that captures the leading-order effects caused by nonzero ε. This will be done by approximating the dispersion operator $\hat{\boldsymbol{D}}$ to the first order in ε, which involves three main steps:

[1] The name "GO" is determined by the fact that the regime $\varepsilon \ll 1$ was initially studied for optical waves. Higher-order theories include the quasioptical approximation and the paraxial approximation as a special case (Box 3.1).

(i) map the operator $\hat{\boldsymbol{D}}$ to some function $\boldsymbol{D}$,

(ii) approximate the function $\boldsymbol{D}$ using ε as a small parameter,

(iii) and map the approximated function back to the operator space.

To ensure that $\boldsymbol{D}$ be efficiently expandable in ε, the mapping will be done using the *Wigner–Weyl transform* (WWT).[2] But let us introduce some basic notation first.

3.2 Notation

To shorten the calculations, we will describe the wave propagation in terms of spacetime coordinates x^α, or in the invariant notation, $\mathsf{x} = \{x^0, \boldsymbol{x}\}$, where $x^0 \doteq ct$ and x^a with $a > 0$ are spatial coordinates. The dimension of the coordinate space $n \doteq \dim \boldsymbol{x}$ can be any $n \geq 0$. The spacetime dimension is $\mathsf{n} \doteq \dim \mathsf{x} = n + 1$ and restricted to $\mathsf{n} \geq 1$. For simplicity, we restrict our consideration to spacetimes M_n with the Minkowski metric $g_{\alpha\beta} = g^{\alpha\beta} = \operatorname{diag}\{-1, 1, 1, \ldots\}$.[3] Then, $\bar{w}$ and $\bar{\boldsymbol{k}}$ can be expressed through the components of the row vector

$$\bar{k}_\alpha \doteq \partial_\alpha \theta = (-\bar{w}/c, \bar{\boldsymbol{k}})_\alpha, \tag{3.6}$$

where $\partial_\alpha \doteq \partial/\partial x^\alpha$; in particular, $\bar{k}_0 = -\bar{w}/c$. The corresponding column vector has components $k^\alpha = g^{\alpha\beta} k_\beta$, where $g^{\alpha\beta}$ is the inverse metric. In case of the Minkowski metric, $g^{\alpha\beta} = g_{\alpha\beta}$, so $k^0 = -\bar{k}_0 = \bar{w}/c$ and $\bar{k}^a = \bar{k}_a$, or in the invariant form, $\bar{\mathsf{k}} = (\bar{w}/c, \bar{\boldsymbol{k}})^\intercal$. This implies

$$\bar{\mathsf{k}} \cdot \mathsf{x} = g_{\alpha\beta} \bar{k}^\alpha x^\beta = \bar{k}_\alpha x^\alpha = -\bar{k}^0 x^0 + \bar{k}^a x^a = -\bar{w}t + \bar{\boldsymbol{k}} \cdot \boldsymbol{x}. \tag{3.7}$$

The operators $\hat{\omega} \doteq i\partial_t$ and $\hat{\boldsymbol{k}} \doteq -i\nabla$ that we introduced earlier can be similarly expressed through

$$\hat{k}_\alpha \doteq -i\partial_\alpha. \tag{3.8}$$

By analogy with quantum mechanics, let us consider scalar fields on M_n as vectors $|\psi\rangle$ in the corresponding Hilbert space. Then, any scalar function $\psi(\mathsf{x})$ can be understood as the x-representation of the corresponding vector $|\psi\rangle$; namely, $\psi(\mathsf{x}') = \langle \mathsf{x}'|\psi\rangle$. Here, $|\mathsf{x}'\rangle$ is the normalized eigenvector of the coordinate operator $\hat{\mathsf{x}}$ that corresponds to the eigenvalue x'; i.e.,

$$\hat{\mathsf{x}}\,|\mathsf{x}'\rangle = \mathsf{x}'\,|\mathsf{x}'\rangle\,, \qquad \langle \mathsf{x}'|\mathsf{x}''\rangle = \delta(\mathsf{x}' - \mathsf{x}''), \tag{3.9}$$

so $\int d\mathsf{x}'\ |\mathsf{x}'\rangle\,\langle \mathsf{x}'|$ is a unit operator. (All integrals are taken on $(-\infty, +\infty)$ unless specified otherwise.) Accordingly, any $(\hat{M}\psi)(\mathsf{x})$ can be expressed as

$$(\hat{M}\psi)(\mathsf{x}) = \langle \mathsf{x}|\hat{M}\psi\rangle \equiv \langle \mathsf{x}|\hat{M}|\psi\rangle = \int d\mathsf{x}'\,\langle \mathsf{x}|\hat{M}|\mathsf{x}'\rangle\,\langle \mathsf{x}'|\psi\rangle = \int d\mathsf{x}'\,\mathfrak{M}(\mathsf{x}, \mathsf{x}')\psi(\mathsf{x}'), \tag{3.10}$$

where $\mathfrak{M}(\mathsf{x}, \mathsf{x}') \doteq \langle \mathsf{x}|\hat{M}|\mathsf{x}'\rangle$ is the x-representation of $\hat{M}$. Similarly, one can introduce the normalized eigenvectors of the wavevector operator $\hat{\mathsf{k}}$,

$$\hat{\mathsf{k}}\,|\mathsf{k}'\rangle = \mathsf{k}'\,|\mathsf{k}'\rangle\,, \qquad \langle \mathsf{k}'|\mathsf{k}''\rangle = \delta(\mathsf{k}' - \mathsf{k}''), \tag{3.11}$$

and define the k-representation of vectors as $\langle \mathsf{k}|\psi\rangle$ and of operators as $\langle \mathsf{k}|\hat{M}|\mathsf{k}'\rangle$ (Exercise 3.1).

[2]Reported here is a modern approach to GO [15, 16]. Earlier formulations of GO [17] do not use the term WWT explicitly but still follow similar steps.

[3]Readers who are not familiar with tensor analysis may consider upper-index and lower-index quantities as independent functions formally defined through the formulas below.

Exercise 3.1: Show that $\langle \mathsf{x}|\mathsf{k}\rangle = \langle \mathsf{k}|\mathsf{x}\rangle^* = (2\pi)^{-\mathsf{n}/2} e^{i\mathsf{k}\cdot\mathsf{x}}$.

For N-component *vector* fields

$$\boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_{\mathsf{N}} \end{pmatrix}, \tag{3.12}$$

where ψ_u are scalar fields on M_n and N may or may not coincide with n, we similarly introduce

$$|\boldsymbol{\psi}\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ \dots \\ |\psi_{\mathsf{N}}\rangle \end{pmatrix} \tag{3.13}$$

via $\psi_u(\mathsf{x}') = \langle \mathsf{x}'|\psi_u\rangle$. Then, operators on vector fields are considered as matrices of operators that act on the field components; i.e.,

$$(\hat{M}_{uv}\psi_v)(\mathsf{x}) = \int d\mathsf{x}'\, \mathfrak{M}_{uv}(\mathsf{x},\mathsf{x}')\psi_v(\mathsf{x}'), \qquad \mathfrak{M}_{uv}(\mathsf{x},\mathsf{x}') = \langle \mathsf{x}|\hat{M}_{uv}|\mathsf{x}'\rangle\,, \tag{3.14}$$

the k-representations of matrix operators are introduced similarly.

3.3 Wigner–Weyl transform

The WWT[4] can be understood as a "mixed", or "phase-space", representation of an operator. Unlike the x- and k-representations, which involve projections *either* on $|\mathsf{x}\rangle$ *or* on $|\mathsf{k}\rangle$, the WWT projects an operator on both $|\mathsf{x}\rangle$ and $|\mathsf{k}\rangle$ simultaneously such that the resulting projection has "natural" properties. This is more easily seen from the definition of the WWT that is formulated in terms of the "phase-space" coordinates $\mathsf{z} \doteq \{\mathsf{x},\mathsf{k}\}$; for example, see Refs. [2,18] or Ref. [16, Supplemental Material]. Here, though, we will use a less elegant definition that is perhaps easier to absorb. This definition is as follows.

For any given scalar operator $\hat{M}$, its Weyl image $M(\mathsf{x},\mathsf{k}) \equiv \mathscr{W}[\hat{M}]$, also called the *(Weyl) symbol* of $\hat{M}$, is a function defined on the $2n$-dimensional real space (x,k) as follows:

$$M(\mathsf{x},\mathsf{k}) \doteq \int d\mathsf{s}\, \langle \mathsf{x}+\mathsf{s}/2|\hat{M}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}. \tag{3.15}$$

One can also understand $M(\mathsf{x},\mathsf{k})$ as the Fourier image of $\bar{\mathfrak{M}}(\bar{\mathsf{x}},\mathsf{s}) \doteq \mathfrak{M}(\bar{\mathsf{x}}+\mathsf{s}/2, \bar{\mathsf{x}}-\mathsf{s}/2)$ with respect to s at fixed $\bar{\mathsf{x}} = \mathsf{x}$ (cf. Box 1.1):

$$M(\mathsf{x},\mathsf{k}) = \int d\mathsf{s}\, \bar{\mathfrak{M}}(\mathsf{x},\mathsf{s})\, e^{-i\mathsf{k}\cdot\mathsf{s}}. \tag{3.16}$$

The *inverse* WWT[4] $\mathscr{W}^{-1}$ maps any given function $M(\mathsf{x},\mathsf{k})$ to an operator $\hat{M}$ via [16, Supplemental Material]

$$\hat{M} = \frac{1}{(2\pi)^{\mathsf{n}}} \int d\mathsf{x}\, d\mathsf{k}\, d\mathsf{s}\; |\mathsf{x}+\mathsf{s}/2\rangle\, M(\mathsf{x},\mathsf{k})\, e^{i\mathsf{k}\cdot\mathsf{s}}\, \langle \mathsf{x}-\mathsf{s}/2|\,. \tag{3.17}$$

[4] The direct WWT is also called the Wigner transform. The inverse WWT is also called the Weyl transform.

For $\hat{\boldsymbol{M}}$ that is a matrix of operators $\hat{M}_{uv}$, the corresponding symbol $\boldsymbol{M}(\mathsf{x},\mathsf{k})$ is defined as the matrix whose elements are the symbols of $\hat{M}_{uv}$. Also note that the WWT commutes with the Hermitian conjugation:

$$\mathscr{W}[\hat{M}^\dagger] = M^\dagger(\mathsf{x},\mathsf{k}). \tag{3.18}$$

As a corollary (Problem 3.2), the Hermitian and anti-Hermitian parts [Eq. (C.3)] of $\hat{M}$ and M satisfy

$$M_{\rm H}(\mathsf{x},\mathsf{k}) = \mathscr{W}[\hat{M}_{\rm H}], \qquad M_{\rm A}(\mathsf{x},\mathsf{k}) = \mathscr{W}[\hat{M}_{\rm A}]. \tag{3.19}$$

Exercise 3.2: Prove Eqs. (3.18) and (3.19). Show that the symbol of a Hermitian operator is a Hermitian matrix or, in case of a scalar operator, a real function.

The WWT is a natural mapping between operators and functions on phase space in the following sense. The symbol of a unit operator is unity:

$$\begin{aligned}
\mathscr{W}[\hat{1}] &= \int d\mathsf{s}\,\langle \mathsf{x}+\mathsf{s}/2|\hat{1}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= \int d\mathsf{s}\,\langle \mathsf{x}+\mathsf{s}/2|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= \int d\mathsf{s}\,\delta(\mathsf{s})\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= 1.
\end{aligned} \tag{3.20}$$

The symbol of an operator that can be represented as a function of the coordinate operator $\hat{\mathsf{x}}$ is the same function of x:

$$\begin{aligned}
\mathscr{W}[F(\hat{\mathsf{x}})] &= \int d\mathsf{s}\,\langle \mathsf{x}+\mathsf{s}/2|F(\hat{\mathsf{x}})|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= \int d\mathsf{s}\,F(\mathsf{x}-\mathsf{s}/2)\,\langle \mathsf{x}+\mathsf{s}/2|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= \int d\mathsf{s}\,F(\mathsf{x}-\mathsf{s}/2)\delta(\mathsf{s})\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\
&= F(\mathsf{x}).
\end{aligned} \tag{3.21}$$

Similarly, the symbol of an operator that can be represented as a function of the wavevector operator $\hat{\mathsf{k}}$ is the same function of k (Exercise 3.3):

$$\mathscr{W}[G(\hat{\mathsf{k}})] = \int d\mathsf{s}\,\langle \mathsf{x}+\mathsf{s}/2|G(\hat{\mathsf{k}})|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}} = G(\mathsf{k}). \tag{3.22}$$

Also, the symbol of an operator representable as $F(\hat{\mathsf{x}}) + G(\hat{\mathsf{k}})$ is $F(\mathsf{x}) + G(\mathsf{k})$. In summary then,

$$\hat{1} \Leftrightarrow 1, \qquad \hat{\mathsf{x}} \Leftrightarrow \mathsf{x}, \qquad \hat{\mathsf{k}} \Leftrightarrow \mathsf{k}, \qquad F(\hat{\mathsf{x}}) + G(\hat{\mathsf{k}}) \Leftrightarrow F(\mathsf{x}) + G(\mathsf{k}), \tag{3.23}$$

where $\Leftrightarrow$ denotes the correspondence between operators and their symbols.

Exercise 3.3: Prove Eq. (3.22) using the result from Exercise 3.1.

In the general case, though, the correspondence $\Leftrightarrow$ is more complicated than swapping $(\hat{\mathsf{x}}, \hat{\mathsf{k}})$ and (x, k). For example,

$$\begin{aligned}
\mathscr{W}[\hat{\mathsf{k}}F(\hat{\mathsf{x}})] &= \int d\mathsf{s}\, e^{-i\mathsf{k}\cdot\mathsf{s}}\, \langle \mathsf{x}+\mathsf{s}/2|\hat{\mathsf{k}}F(\hat{\mathsf{x}})|\mathsf{x}-\mathsf{s}/2\rangle \\
&= \int d\mathsf{k}'\, d\mathsf{s}\, e^{-i\mathsf{k}\cdot\mathsf{s}}\, \langle \mathsf{x}+\mathsf{s}/2|\mathsf{k}'\rangle\, \hat{\mathsf{k}}F(\mathsf{x}-\mathsf{s}/2)\, \langle \mathsf{k}'|\mathsf{x}-\mathsf{s}/2\rangle \\
&= \int d\mathsf{k}'\, d\mathsf{s}\, e^{-i\mathsf{k}\cdot\mathsf{s}}\, \langle \mathsf{x}+\mathsf{s}/2|\mathsf{k}'\rangle\, \mathsf{k}'F(\mathsf{x}-\mathsf{s}/2)\, \langle \mathsf{k}'|\mathsf{x}-\mathsf{s}/2\rangle \\
&= \int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, \mathsf{k}'F(\mathsf{x}-\mathsf{s}/2)\, e^{i(\mathsf{k}'-\mathsf{k})\cdot\mathsf{s}} \\
&= \int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, (\mathsf{k}'+\mathsf{k})F(\mathsf{x}-\mathsf{s}/2)\, e^{i\mathsf{k}'\cdot\mathsf{s}} \\
&= \mathsf{k}\int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, F(\mathsf{x}-\mathsf{s}/2)\, e^{i\mathsf{k}'\cdot\mathsf{s}} - i\int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, F(\mathsf{x}-\mathsf{s}/2)\, \partial_{\mathsf{s}} e^{i\mathsf{k}'\cdot\mathsf{s}} \\
&= \mathsf{k}\int d\mathsf{s}\, F(\mathsf{x}-\mathsf{s}/2)\delta(\mathsf{s}) + i\int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, \partial_{\mathsf{s}}[F(\mathsf{x}-\mathsf{s}/2)]\, e^{i\mathsf{k}'\cdot\mathsf{s}} \\
&= \mathsf{k}F(\mathsf{x}) - \frac{i}{2}\,\partial_{\mathsf{x}}\int \frac{d\mathsf{k}'}{(2\pi)^{\mathsf{n}}}\, d\mathsf{s}\, F(\mathsf{x}-\mathsf{s}/2)\, e^{i\mathsf{k}'\cdot\mathsf{s}} \\
&= \mathsf{k}F(\mathsf{x}) - \frac{i}{2}\,\partial_{\mathsf{x}}F(\mathsf{x}).
\end{aligned} \tag{3.24}$$

Using Eq. (3.18), one also obtains

$$\begin{aligned}
\mathscr{W}[F(\hat{\mathsf{x}})\hat{\mathsf{k}}] &= \big(\mathscr{W}[(F(\hat{\mathsf{x}})\hat{\mathsf{k}})^\dagger]\big)^\dagger \\
&= \big(\mathscr{W}[\hat{\mathsf{k}}F^\dagger(\hat{\mathsf{x}})]\big)^\dagger \\
&= \big[\mathsf{k}F^\dagger(\mathsf{x}) - {}^{i}\!/\!_{2}\,\partial_{\mathsf{x}}F^\dagger(\mathsf{x})\big]^\dagger \\
&= F(\mathsf{x})\mathsf{k} + {}^{i}\!/\!_{2}\,\partial_{\mathsf{x}}F(\mathsf{x}),
\end{aligned} \tag{3.25}$$

and as a corollary,

$$\mathsf{k}F(\mathsf{x}), F(\mathsf{x})\mathsf{k} \;\;\Leftrightarrow\;\; {}^{1}\!/\!_{2}\,[F(\hat{\mathsf{x}})\hat{\mathsf{k}} + \hat{\mathsf{k}}F(\hat{\mathsf{x}})]\,. \tag{3.26}$$

The terms $\partial_\alpha F$ in Eqs. (3.24) and (3.25) emerge because $\hat{\mathsf{x}}$ and $\hat{\mathsf{k}}$ do not commute,

$$[\hat{x}^\alpha, \hat{k}_\beta] = i\delta^\alpha_\beta. \tag{3.27}$$

These terms are of order ε and vanish in the GO limit ($\varepsilon \to 0$). Similarly, for a general $\hat{M} = M^{(0)}(\hat{\mathsf{x}}, \hat{\mathsf{k}})$, where $M^{(0)}$ is any combination of $\hat{\mathsf{x}}$ and $\hat{\mathsf{k}}$, one has

$$\mathscr{W}[\hat{M}] = M^{(0)}(\mathsf{x}, \mathsf{k}) + \mathcal{O}\big(\varepsilon M^{(0)}\big). \tag{3.28}$$

3.4 Envelope equation

Now, let us return to Eq. (3.1), multiply it by $e^{-i\theta(\mathsf{x})}$, and rewrite the result as follows:

$$\hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} = 0, \qquad \hat{\boldsymbol{\mathcal{D}}} \doteq e^{-i\theta(\mathsf{x})}\,\hat{\boldsymbol{D}}\,e^{i\theta(\mathsf{x})}. \tag{3.29}$$

The operator $\hat{\boldsymbol{\mathcal{D}}}$ serves as the *envelope* dispersion operator and can also be expressed in the following invariant form:

$$\hat{\boldsymbol{\mathcal{D}}} = e^{-i\theta(\hat{\mathsf{x}})}\,\hat{\boldsymbol{D}}\,e^{i\theta(\hat{\mathsf{x}})}. \tag{3.30}$$

By definition [Eq. (3.15)], the symbol of $\hat{\boldsymbol{\mathcal{D}}}$ is

$$\begin{aligned}\boldsymbol{\mathcal{D}}(\mathsf{x},\mathsf{k}) &= \int d\mathsf{s}\ \langle \mathsf{x}+\mathsf{s}/2|e^{-i\theta(\hat{\mathsf{x}})}\hat{\boldsymbol{D}}\,e^{i\theta(\hat{\mathsf{x}})}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\ &= \int d\mathsf{s}\ \langle \mathsf{x}+\mathsf{s}/2|e^{-i\theta(\mathsf{x}+\mathsf{s}/2)}\hat{\boldsymbol{D}}\,e^{i\theta(\mathsf{x}-\mathsf{s}/2)}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i\mathsf{k}\cdot\mathsf{s}}\\ &= \int d\mathsf{s}\ \langle \mathsf{x}+\mathsf{s}/2|\hat{\boldsymbol{D}}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i[\theta(\mathsf{x}+\mathsf{s}/2)-\theta(\mathsf{x}-\mathsf{s}/2)+\mathsf{k}\cdot\mathsf{s}]}. \end{aligned}\tag{3.31}$$

Since the gradient of θ is assumed to be a slow function, we can use

$$\theta(\mathsf{x}\pm\mathsf{s}/2)=\theta(\mathsf{x})\pm\frac{s^\alpha}{2}\frac{\partial\theta}{\partial x^\alpha}+\frac{s^\alpha s^\beta}{8}\frac{\partial^2\theta}{\partial x^\alpha\partial x^\beta}\pm\frac{s^\alpha s^\beta s^\gamma}{48}\frac{\partial^3\theta}{\partial x^\alpha\partial x^\beta\partial x^\gamma}+\dots,\tag{3.32}$$

where summation over repeated indexes is assumed. Then,

$$\begin{aligned} e^{-i[\theta(\mathsf{x}+\mathsf{s}/2)-\theta(\mathsf{x}-\mathsf{s}/2)+\mathsf{k}\cdot\mathsf{s}]} &= \left(1-\frac{i}{24}\,s^\alpha s^\beta s^\gamma\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}+\dots\right)e^{-i[\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}]\cdot\mathsf{s}}\\ &= \left(1-\frac{1}{24}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3}{\partial k_\alpha\partial k_\beta\partial k_\gamma}+\dots\right)e^{-i[\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}]\cdot\mathsf{s}}. \end{aligned}\tag{3.33}$$

This leads to

$$\begin{aligned}\boldsymbol{\mathcal{D}}(\mathsf{x},\mathsf{k}) &= \left(1-\frac{1}{24}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3}{\partial k_\alpha\partial k_\beta\partial k_\gamma}+\dots\right)\int d\mathsf{s}\ \langle \mathsf{x}+\mathsf{s}/2|\hat{\boldsymbol{D}}|\mathsf{x}-\mathsf{s}/2\rangle\, e^{-i[\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}]\cdot\mathsf{s}}\\ &= \left(1-\frac{1}{24}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3}{\partial k_\alpha\partial k_\beta\partial k_\gamma}+\dots\right)\boldsymbol{D}\left(\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}\right)\\ &= \boldsymbol{D}'\left(\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}\right), \end{aligned}\tag{3.34}$$

where we have introduced

$$\boldsymbol{D}'(\mathsf{x},\mathsf{k})\doteq\boldsymbol{D}(\mathsf{x},\mathsf{k})-\frac{1}{24}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3\boldsymbol{D}(\mathsf{x},\mathsf{k})}{\partial k_\alpha\partial k_\beta\partial k_\gamma}+\dots\tag{3.35}$$

Note that the inverse WWT will, loosely speaking, turn the variable k into the operator $\hat{\mathsf{k}}$. The latter will act on the wave envelope, which is considered slow in the coordinate representation, so $\hat{\mathsf{k}}\,|\boldsymbol{\Psi}\rangle=\mathcal{O}(\varepsilon)$. In this sense, $\mathsf{k}=\mathcal{O}(\varepsilon)$ in Eq. (3.34). Then, $\boldsymbol{D}'(\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k})$ can be Taylor-expanded in k, which corresponds to expanding $\hat{\boldsymbol{\mathcal{D}}}$ in ε. Next, notice that

$$\begin{aligned}\varepsilon' &\doteq D[\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}]^{-1}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3 D[\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}]}{\partial k_\alpha\partial k_\beta\partial k_\gamma}\\ &\approx D(\mathsf{x},\bar{\mathsf{k}})^{-1}\frac{\partial^2\bar{k}_\gamma}{\partial x^\alpha\partial x^\beta}\frac{\partial^3 D(\mathsf{x},\bar{\mathsf{k}})}{\partial\bar{k}_\alpha\partial\bar{k}_\beta\partial\bar{k}_\gamma}\\ &\sim\left(\frac{\bar{k}}{L_c^2 D}\right)\left(\frac{D}{\bar{k}^3}\right)=\mathcal{O}(\varepsilon^2), \end{aligned}\tag{3.36}$$

where we have assumed $\partial/\partial x^\alpha\sim 1/L_c$ and $\partial/\partial\bar{\mathsf{k}}\sim 1/\bar{\mathsf{k}}$.[5] Because we are interested only in $\mathcal{O}(\varepsilon)$ corrections, this means that on the right-hand side of Eq. (3.35), the second term can be neglected compared to the first term. In other words, $\boldsymbol{D}'(\mathsf{x},\mathsf{k})\approx\boldsymbol{D}(\mathsf{x},\mathsf{k})$ and thus

$$\boldsymbol{\mathcal{D}}(\mathsf{x},\mathsf{k})\approx\boldsymbol{D}\left(\mathsf{x},\bar{\mathsf{k}}(\mathsf{x})+\mathsf{k}\right),\tag{3.37}$$

[5] A more accurate estimate is as follows. Let us consider $\boldsymbol{D}(\mathsf{x},\mathsf{k})$ in the form (3.16), i.e., $\boldsymbol{D}(\mathsf{x},\mathsf{k})=\int d\mathsf{s}\,\bar{\boldsymbol{\mathfrak{D}}}(\mathsf{x},\mathsf{s})\,e^{-i\mathsf{k}\cdot\mathsf{s}}$. Then, $\partial\boldsymbol{D}/\partial k_\alpha=-i\int d\mathsf{s}\,\bar{\boldsymbol{\mathfrak{D}}}(\mathsf{x},\mathsf{s})s^\alpha e^{-i\mathsf{k}\cdot\mathsf{s}}\sim\mathfrak{s}^\alpha\boldsymbol{D}$, where $\mathfrak{s}^\alpha$ is the characteristic scale on which $\bar{\boldsymbol{\mathfrak{D}}}(\mathsf{x},\mathsf{s})$ fades away along the s^α axis. Then, $\varepsilon'\sim(\bar{k}L_c)^{-2}(\bar{k}\mathfrak{s})^3\sim\varepsilon^2(\bar{k}\mathfrak{s})^3$. To ensure that $\varepsilon'\ll\varepsilon$, one must require $(\bar{k}\mathfrak{s})^3\varepsilon\ll 1$. Strictly speaking, this is an additional requirement of GO, independent of Eq. (3.5). The scale $\mathfrak{s}$ can be estimated in the homogeneous-plasma limit, when $\boldsymbol{D}(\mathsf{x},\mathsf{k})$ is just the dispersion tensor, $\boldsymbol{D}(\mathsf{x},\mathsf{k})\to\boldsymbol{D}(\mathsf{k})$.

Box 3.1: Quasioptical and paraxial approximation

In practice, waves often propagate as narrow beams whose transverse scale $L_\perp$ is much less than the longitudinal scale $L_\parallel$, which is comparable to that of the medium. For diffraction-limited beams, one has $\varepsilon_\parallel \sim \varepsilon_\perp^2$, where $\varepsilon_\parallel \doteq (\bar{k}L_\parallel)^{-1}$ and $\varepsilon_\perp \doteq (\bar{k}L_\perp)^{-1}$, so when retaining terms of the first order in $\varepsilon_\parallel$, one must retain effects of the second order in $\varepsilon_\perp$. Because ε' [as in Eq. (3.36)] is of order $\varepsilon_\parallel^2$, Eq. (3.37) is still applicable, but Eq. (3.38) must be replaced with the second order expansion in $k_\perp$. The corresponding envelope equation is

$$\boldsymbol{D}\left[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})\right] \boldsymbol{\Psi} - \frac{i}{2}\,(\partial_\alpha \boldsymbol{\mathcal{V}}^\alpha)\boldsymbol{\Psi} - i\boldsymbol{\mathcal{V}}^\alpha \partial_\alpha \boldsymbol{\Psi} - \frac{1}{2}\,(\boldsymbol{\Phi}_{\alpha\beta} : \nabla_{\perp\alpha}\nabla_{\perp\beta})\boldsymbol{\Psi} = 0,$$

where $\boldsymbol{\Phi}_{\alpha\beta} \doteq \partial^2 \boldsymbol{D}/\partial k_\alpha\, \partial k_\beta$ evaluated at $[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]$. This model is called the paraxial approximation if the beam axis is a straight line, or in the general case, the quasioptical approximation [16]. It is widely used, for example, in optics and also for modeling mm- and cm-wave beams in magnetically confined plasmas.

or using the first-order Taylor expansion,

$$\boldsymbol{\mathcal{D}}(\mathsf{x}, \mathsf{k}) \approx \boldsymbol{D}\left(\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})\right) + \mathsf{k}_\alpha \boldsymbol{\mathcal{V}}^\alpha(\mathsf{x}), \qquad \boldsymbol{\mathcal{V}}^\alpha(\mathsf{x}) \doteq \left(\frac{\partial \boldsymbol{D}(\mathsf{x}, \mathsf{k})}{\partial k_\alpha}\right)_{\mathsf{k}=\bar{\mathsf{k}}(\mathsf{x})}. \tag{3.38}$$

By Eq. (3.26), the inverse WWT of this expression is

$$\hat{\boldsymbol{\mathcal{D}}} \approx \boldsymbol{D}\left(\hat{\mathsf{x}}, \bar{\mathsf{k}}(\hat{\mathsf{x}})\right) + \frac{1}{2}\,[\hat{\mathsf{k}}_\alpha \boldsymbol{\mathcal{V}}^\alpha(\hat{\mathsf{x}}) + \boldsymbol{\mathcal{V}}^\alpha(\hat{\mathsf{x}})\hat{\mathsf{k}}_\alpha], \tag{3.39}$$

or in the coordinate representation,

$$\hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} \approx \boldsymbol{D}(\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x}))\boldsymbol{\Psi}(\mathsf{x}) - \frac{i}{2}\,\partial_\alpha(\boldsymbol{\mathcal{V}}^\alpha(\mathsf{x})\boldsymbol{\Psi}(\mathsf{x})) - \frac{i}{2}\,\boldsymbol{\mathcal{V}}^\alpha(\mathsf{x})\partial_\alpha \boldsymbol{\Psi}(\mathsf{x}). \tag{3.40}$$

Then, the envelope equation $\hat{\boldsymbol{\mathcal{D}}}\boldsymbol{\Psi} = 0$ can be approximately expressed as follows:

$$\boldsymbol{D}\left(\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})\right) \boldsymbol{\Psi} - \frac{i}{2}\,(\partial_\alpha \boldsymbol{\mathcal{V}}^\alpha)\boldsymbol{\Psi} - i\boldsymbol{\mathcal{V}}^\alpha \partial_\alpha \boldsymbol{\Psi} = 0, \tag{3.41}$$

which can be considered as an extension of Eq. (1.55) to a general inhomogeneous medium (Box 3.1). We will further simplify this equation in Lecture 4.

3.5 How to use the envelope equation

In order to use Eq. (3.41), one must know $\boldsymbol{D}$ to the first order in ε, which can be challenging to calculate in practice. Fortunately, this problem is often alleviated by the following. As seen from Lecture 2 and Problem PI.2, collisionless cold static (CCS) plasmas are naturally described by Hermitian dispersion operators that satisfy (Exercise 3.4)

$$\boldsymbol{D}_{\text{CCS}} = \boldsymbol{D}_{\text{CCS}}^{(0)}, \tag{3.42}$$

where the notation is the same as in Sec. 3.3. Deviations from CCS models due to thermal effects and collisions often result in only small corrections $\hat{\boldsymbol{\Delta}}$ to the dispersion operator $\hat{\boldsymbol{D}}$. Then, the symbol of this operator, $\boldsymbol{D}$, satisfies

$$\boldsymbol{D} - \boldsymbol{D}^{(0)} = \left(\boldsymbol{D}_{\text{CCS}} + \boldsymbol{\Delta}\right) - \left(\boldsymbol{D}_{\text{CCS}}^{(0)} + \boldsymbol{\Delta}^{(0)}\right) = \boldsymbol{\Delta} - \boldsymbol{\Delta}^{(0)} = \mathcal{O}(\varepsilon\Delta), \tag{3.43}$$

Box 3.2: Subtleties with approximating $\boldsymbol{D}$ in other formulations of GO

Although GO can be constructed based on a transform other than the WWT, the approximation (3.45) relies on the WWT property (3.19) that the other transform may not have. Then, one might not be able to approximate the corresponding symbol of $\hat{\boldsymbol{D}}$ with the (well-known) dispersion tensor of the wave in a homogeneous medium.

For example, sometimes GO is constructed based on the transform

$$\bar{\mathscr{W}}[\hat{M}] \doteq \int d\mathsf{s}\, \mathfrak{M}(\mathsf{x}, \mathsf{x}-\mathsf{s})\, e^{-i\mathsf{k}\cdot\mathsf{s}} = \frac{\langle \mathsf{x}|\hat{M}|\mathsf{k}\rangle}{\langle \mathsf{x}|\mathsf{k}\rangle}$$

(e.g., see Ref. [19]). To the extent that the commutator $[\hat{\mathsf{x}}, \hat{\mathsf{k}}]$ is negligible, one has $\bar{\mathscr{W}}[\hat{M}] \approx M^{(0)}(\mathsf{x}, \mathsf{k})$, so $\bar{\mathscr{W}}$ satisfies Eq. (3.28) just like the WWT. However, symbols produced by the transform $\bar{\mathscr{W}}$ do not have the property (3.18). As a result,

$$(\bar{\mathscr{W}}[\hat{\boldsymbol{D}}])_{\mathrm{A}} = \boldsymbol{D}^{(0)}_{\mathrm{A}} + [\mathcal{O}(\varepsilon \boldsymbol{D}^{(0)})]_{\mathrm{A}} = \boldsymbol{D}^{(0)}_{\mathrm{A}} + \mathcal{O}(\varepsilon).$$

Both terms on the right-hand side are of the same order, so the anti-Hermitian part of the symbol differs from the corresponding matrix in homogeneous plasma by $\sim \partial^2 \boldsymbol{D}^{(0)}_{\mathrm{H}}/\partial \mathsf{x}^\alpha \partial \mathsf{k}_\alpha$. This leads to alternative GO equations, which often results in confusion and stirred a controversy [20, 21].

where we have used Eq. (3.28) to obtain the last equality. Because both ε and Δ are *both* small, the term $\mathcal{O}(\varepsilon\Delta)$ is often negligible [as opposed to $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\Delta)$], i.e., one can adopt

$$\boldsymbol{D} \approx \boldsymbol{D}^{(0)}. \tag{3.44}$$

This means that the symbol of the dispersion operator can be approximated with the dispersion tensor of homogeneous plasma. In fact, the two are often not even distinguished explicitly in literature.

Exercise 3.4: Calculate the symbols of: (a) $\hat{\boldsymbol{D}}_E$ for waves in CCS nonmagnetized plasma (Lecture 2) and (b) $\hat{\boldsymbol{D}}_\psi$ for waves in CCS magnetized plasma (Problem PI.2).

Because $\boldsymbol{D}^{(0)}_{\mathrm{CCS}}$ is Hermitian, Eq. (3.44) is often further reduced to (Box 3.2)

$$\boldsymbol{D}_{\mathrm{H}} \approx \boldsymbol{D}^{(0)}_{\mathrm{CCS}} + \boldsymbol{\Delta}^{(0)}_{\mathrm{H}} \approx \boldsymbol{D}^{(0)}_{\mathrm{CCS}}, \qquad \boldsymbol{D}_{\mathrm{A}} \approx 0 + \boldsymbol{\Delta}^{(0)}_{\mathrm{A}} = \boldsymbol{\Delta}^{(0)}_{\mathrm{A}}. \tag{3.45}$$

This is justified in the following sense. In $\boldsymbol{D}_{\mathrm{H}}$, which determines wave propagation (Lecture 5), the main contribution is provided by $\boldsymbol{D}_{\mathrm{CCS}}$, so $\boldsymbol{\Delta}_{\mathrm{H}}$ is only a small correction, which is often negligible. In contrast, $\boldsymbol{D}_{\mathrm{A}}$, which determines wave dissipation (Lecture 5), is *entirely* determined by $\boldsymbol{\Delta}_{\mathrm{A}}$, so unlike $\boldsymbol{\Delta}_{\mathrm{H}}$, the function $\boldsymbol{\Delta}_{\mathrm{A}}$ must be kept unless dissipation is entirely negligible. The advantage of the model (3.45) is that $\boldsymbol{D}^{(0)}_{\mathrm{CCS}}$ (Part III) is simpler than $\boldsymbol{D}^{(0)}_{\mathrm{H}}$ (Part IV) and thus easier to implement numerically. (However, note that this model can be insufficient when dissipation is strongly inhomogeneous. Sometimes this happens for resonant absorption, which is naturally localized in space.)

The arguments presented in this section do not apply to corrections $\hat{\boldsymbol{\Delta}}_m$ caused by plasma motion. In this case, $\hat{\boldsymbol{\Delta}}^{(0)}_m = 0$, so Eq. (3.43) becomes $\boldsymbol{D} - \boldsymbol{D}^{(0)} = \boldsymbol{\Delta}_m = \mathcal{O}(\epsilon)$. Ignoring this correction can lead to significant violation of energy conservation on times $t = \mathcal{O}(\epsilon^{-1})$ and is typically not acceptable. One must actually recalculate the dispersion operator for moving plasma to fix this issue. (But if one needs an equation just for the energy, the problem can be bypassed as discussed in the next lectures.) This asymmetry in how GO tolerates various corrections is due to the fact that plasma temporal dispersion is typically much stronger than plasma spatial dispersion.

Lecture 4

Equations of geometrical optics

In this lecture, we derive a complete set of equations that describe scalar waves: the local dispersion relation, the polarization equation, the consistency relations, the ray equations, and scalar-amplitude equations, including an equation for the wave action. These are known as the equations of GO.

4.1 Scalar-wave model

Based on the reasoning presented in Sec. 3.5, let us assume the following ordering:

$$\boldsymbol{D}_{\mathrm{H}}(\mathsf{x}, \mathsf{k}) = \mathcal{O}(1), \qquad \boldsymbol{D}_{\mathrm{A}}(\mathsf{x}, \mathsf{k}) = \mathcal{O}(\varepsilon). \tag{4.1}$$

Then, since $\partial_\alpha = \mathcal{O}(\varepsilon)$ and $\boldsymbol{D}_{\mathrm{A}} = \mathcal{O}(\varepsilon)$, the envelope equation (3.41) can be simplified as follows:

$$\boldsymbol{D}_{\mathrm{H}}\left[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})\right] \boldsymbol{\Psi} + i\boldsymbol{D}_{\mathrm{A}}\left[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})\right] \boldsymbol{\Psi} - \frac{i}{2}\,(\partial_\alpha \boldsymbol{\mathcal{V}}^\alpha_{\mathrm{H}})\boldsymbol{\Psi} - i\boldsymbol{\mathcal{V}}^\alpha_{\mathrm{H}}\partial_\alpha \boldsymbol{\Psi} = 0, \tag{4.2}$$

where $\boldsymbol{\mathcal{V}}^\alpha_{\mathrm{H}}$ is the Hermitian part of $\boldsymbol{\mathcal{V}}^\alpha$. It is readily seen from here that

$$\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{\Psi} = \mathcal{O}(\varepsilon). \tag{4.3}$$

This means that the field polarization must approximately satisfy the same equation as in the corresponding homogeneous medium, $\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{\Psi} \approx 0$. Hence, it is convenient to represent $\boldsymbol{\Psi}$ in the basis of the orthonormal eigenvectors $\boldsymbol{h}_v(\mathsf{x})$ of $\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]$,

$$\boldsymbol{\Psi}(\mathsf{x}) = \sum_{v=1}^{\mathsf{N}} \boldsymbol{h}_v(\mathsf{x})\Psi_v(\mathsf{x}), \qquad \boldsymbol{h}_u \cdot \boldsymbol{h}_v \equiv \boldsymbol{h}^\dagger_u(\mathsf{x})\boldsymbol{h}_v(\mathsf{x}) = \delta_{uv} \tag{4.4}$$

(as earlier, $\mathsf{N} \doteq \dim \boldsymbol{\Psi}$), and the corresponding eigenvalues $\Lambda_v(\mathsf{x})$, which satisfy

$$\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{h}_v(\mathsf{x}) = \Lambda_v(\mathsf{x})\boldsymbol{h}_v(\mathsf{x}). \tag{4.5}$$

Let us consider the scalar product of Eq. (4.3) with $\boldsymbol{h}_u(\mathsf{x})$:

$$\begin{aligned}
\mathcal{O}(\varepsilon) &= \sum_{v=1}^{\mathsf{N}} \boldsymbol{h}^\dagger_u(\mathsf{x})\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{h}_v(\mathsf{x})\Psi_v(\mathsf{x}) \\
&= \sum_{v=1}^{\mathsf{N}} \Lambda_v(\mathsf{x})\boldsymbol{h}^\dagger_u(\mathsf{x})\boldsymbol{h}_v(\mathsf{x})\Psi_v(\mathsf{x}) \\
&= \Lambda_u(\mathsf{x})\Psi_u(\mathsf{x}).
\end{aligned} \tag{4.6}$$

Equation (4.6) shows that Ψ_u can be order-one only if the corresponding eigenvalue is small, $\Lambda_u = \mathcal{O}(\varepsilon)$. This can be satisfied for more than one mode at a time, in which case "mode conversion" is possible (Problem PII.2). Here, we will consider a simpler case, when only one, say uth, mode has a noticeable amplitude. Specifically, suppose $\Lambda_u = \mathcal{O}(\varepsilon)$, and $\Lambda_{v\neq u} = \mathcal{O}(1)$; then, $\Psi_u = \mathcal{O}(1)$ is allowed, and $\Psi_{v\neq u} = \mathcal{O}(\varepsilon)$. Let us re-examine the wave equation under these conditions more carefully. Consider the product of Eq. (4.2) with $\boldsymbol{h}_u^\dagger$:

$$\begin{aligned} 0 &= \sum_{v=1}^{\mathsf{N}} \left[\Lambda_v \boldsymbol{h}_u^\dagger \boldsymbol{h}_v \Psi_v + i\boldsymbol{h}_u^\dagger \boldsymbol{D}_\mathrm{A} \boldsymbol{h}_v \Psi_v - \frac{i}{2}\,\boldsymbol{h}_u^\dagger(\partial_\alpha \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha)\boldsymbol{h}_v\Psi_v - i\boldsymbol{h}_u^\dagger \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha \partial_\alpha(\boldsymbol{h}_v\Psi_v)\right] \\ &\approx \sum_{v=1}^{\mathsf{N}} (\Lambda_v \delta_{uv}\Psi_v) + i(\boldsymbol{h}_u^\dagger \boldsymbol{D}_\mathrm{A}\boldsymbol{h}_u)\Psi_u - \frac{i}{2}\,\boldsymbol{h}_u^\dagger(\partial_\alpha \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha)\boldsymbol{h}_u\Psi_u - i\boldsymbol{h}_u^\dagger \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha \partial_\alpha(\boldsymbol{h}_u\Psi_u) \\ &= \Lambda_u\Psi_u + i\mathit{\Gamma}_u\Psi_u - \frac{i}{2}\,\boldsymbol{h}_u^\dagger(\partial_\alpha \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha)\boldsymbol{h}_u\Psi_u - i(\boldsymbol{h}_u^\dagger \boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\partial_\alpha\boldsymbol{h}_u)\Psi_u - iV_u^\alpha\partial_\alpha\Psi_u. \end{aligned} \tag{4.7}$$

(Although u is a repeating index, no summation over u is assumed.) Here,

$$\Lambda_u = \boldsymbol{h}_u^\dagger(\mathsf{x})\boldsymbol{D}_\mathrm{H}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{h}_u(\mathsf{x}), \tag{4.8a}$$

$$\mathit{\Gamma}_u \doteq \boldsymbol{h}_u^\dagger(\mathsf{x})\boldsymbol{D}_\mathrm{A}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]\boldsymbol{h}_u(\mathsf{x}), \tag{4.8b}$$

$$V_u^\alpha \doteq \boldsymbol{h}_u^\dagger(\mathsf{x})\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha(\mathsf{x})\boldsymbol{h}_u(\mathsf{x}) \tag{4.8c}$$

are real scalar functions. Also note that

$$\begin{aligned} &\frac{i}{2}\,\boldsymbol{h}_u^\dagger(\partial_\alpha\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha)\boldsymbol{h}_u + i\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\partial_\alpha\boldsymbol{h}_u \\ &\quad= \frac{i}{2}\partial_\alpha(\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\boldsymbol{h}_u) - \frac{i}{2}(\partial_\alpha\boldsymbol{h}_u^\dagger)\boldsymbol{\mathcal{V}}^\alpha\boldsymbol{h}_u - \frac{i}{2}\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha(\partial_\alpha\boldsymbol{h}_u) + i\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha(\partial_\alpha\boldsymbol{h}_u) \\ &\quad= \frac{i}{2}\,(\partial_\alpha V_u^\alpha) + \frac{i}{2}\left[\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha(\partial_\alpha\boldsymbol{h}_u) - (\partial_\alpha\boldsymbol{h}_u^\dagger)\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\boldsymbol{h}_u\right] \\ &\quad= \frac{i}{2}\,(\partial_\alpha V_u^\alpha) + Q_u, \end{aligned} \tag{4.9}$$

where Q_u is a real function given by[1]

$$Q_u \doteq \frac{i}{2}\left[\boldsymbol{h}_u^\dagger\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha(\partial_\alpha\boldsymbol{h}_u) - (\partial_\alpha\boldsymbol{h}_u^\dagger)\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\boldsymbol{h}_u\right] = \operatorname{Im}\left[(\partial_\alpha\boldsymbol{h}_u^\dagger)\boldsymbol{\mathcal{V}}_\mathrm{H}^\alpha\boldsymbol{h}_u\right]. \tag{4.10}$$

Then, we obtain the following scalar equation:

$$\left[\Lambda_u - Q_u + i\mathit{\Gamma}_u - \frac{i}{2}\,(\partial_\alpha V_u^\alpha) - iV_u^\alpha\partial_\alpha\right]\Psi_u = 0. \tag{4.11}$$

Recall now that $\bar{\mathsf{k}}$ was introduced as the gradient of θ [Eq. (3.3)] and θ has not been defined yet. Let us define it now by requiring that[2]

$$\Lambda_u(\mathsf{x}, \partial\theta) = 0, \quad \text{i.e.,} \quad \Lambda_u(\mathsf{x}, \bar{\mathsf{k}}) = 0, \tag{4.12}$$

which can be understood as a Hamilton–Jacobi equation for θ [12]. This leads to the following:

[1] If $\mathsf{N} = 1$, then there is only one normalized polarization vector, $\boldsymbol{h}_u \equiv 1$. Accordingly, $\partial_\alpha\boldsymbol{h}_u^\dagger = 0$, so $Q_u = 0$.

[2] This is the most common approach to formulating GO. Another natural way to define θ is to require that Ψ be real; then Eq. (4.12) is replaced with $\Lambda_u - Q_u = 0$. This is further discussed in Sec. 4.4.

- Equation (4.12) connects the local frequency $\bar{w}(t,\boldsymbol{x})$ with the local wavevector $\bar{\boldsymbol{k}}(t,\boldsymbol{x})$. This means that $\bar{w}$ can be expressed through $\bar{\boldsymbol{k}}$:

$$\bar{w}(t,\boldsymbol{x}) = \omega[t,\boldsymbol{x},\bar{\boldsymbol{k}}(t,\boldsymbol{x})], \tag{4.13}$$

  where $\omega(t,\boldsymbol{x},\bar{\boldsymbol{k}})$ is a solution of the local dispersion relation

$$\Lambda_u[t,\boldsymbol{x},\omega(t,\boldsymbol{x},\bar{\boldsymbol{k}}),\bar{\boldsymbol{k}}] = 0. \tag{4.14}$$

  Also note that because $\det \boldsymbol{D}_{\mathrm{H}} = \prod_v \Lambda_v$, Eq. (4.14) can be considered as a particular solution of the more general dispersion relation

$$\det \boldsymbol{D}_{\mathrm{H}}[t,\boldsymbol{x},\omega(t,\boldsymbol{x},\bar{\boldsymbol{k}}),\bar{\boldsymbol{k}}] = 0, \tag{4.15}$$

  which is not specific to $v = u$ but describes all modes simultaneously.

- By Eq. (4.5) with $v = u$, Eq. (4.12) also yields

$$\boldsymbol{D}_{\mathrm{H}}\boldsymbol{h}_u = 0, \tag{4.16}$$

  which determines the mode polarization. This equation is not in violation of the assumed ordering (4.1), because the latter characterizes $\boldsymbol{D}_{\mathrm{H}}(\mathsf{x},\mathsf{k})$ at generic (x,k) while Eq. (4.16) characterizes $\boldsymbol{h}$ specifically on solutions of Eq. (4.12).

- By combining Eqs. (4.11) and (4.12), one obtains the following equation for Ψ_u (Box 4.1):

$$V_u^\alpha \partial_\alpha \Psi_u + \frac{1}{2}(\partial_\alpha V_u^\alpha)\Psi_u = (\Gamma_u + iQ_u)\Psi_u. \tag{4.17}$$

  Since the phase of Ψ_u is often not of interest, let us transform this into an equation for the real field $|\Psi|^2$. To do this, let us multiply Eq. (4.21c) by Ψ^* and sum up the resulting equation with its complex conjugate,

$$\Psi_u^*(V_u^\alpha \partial_\alpha)\Psi_u + \frac{1}{2}(\partial_\alpha V_u^\alpha)|\Psi_u|^2 = (\Gamma_u + iQ_u)|\Psi_u|^2, \tag{4.18}$$

$$\Psi_u(V_u^\alpha \partial_\alpha)\Psi_u^* + \frac{1}{2}(\partial_\alpha V_u^\alpha)|\Psi_u|^2 = (\Gamma_u - iQ_u)|\Psi_u|^2. \tag{4.19}$$

  This leads to

$$\partial_\alpha(V_u^\alpha |\Psi_u|^2) = 2\Gamma_u|\Psi_u|^2. \tag{4.20}$$

Let us summarize these results and simplify the notation. Specifically, we now omit the bar in $\bar{\boldsymbol{k}}$ and drop the mode index $_u$. Then, our main equations can be written as follows (Exercise 4.1):

$$\Lambda[t,\boldsymbol{x},\omega(t,\boldsymbol{x},\boldsymbol{k}),\boldsymbol{k}] = 0, \tag{4.21a}$$

$$\boldsymbol{D}_{\mathrm{H}}\boldsymbol{h} = 0, \tag{4.21b}$$

$$\partial_\alpha(V^\alpha |\Psi|^2) = 2\Gamma|\Psi|^2. \tag{4.21c}$$

This is a complete set of GO equations that allows one to find the whole vector field $\boldsymbol{\psi} = \boldsymbol{h}\Psi e^{i\theta} + \mathcal{O}(\varepsilon)$ up to the $\mathcal{O}(\varepsilon)$ term, which is often considered negligible and otherwise can be found perturbatively. The effect of other modes is completely ignored in this approximation, so effectively, any wave is modeled as a scalar wave. (A more general model, which allows for linear resonant coupling of multiple GO modes, is addressed in Problem PII.2.) Below, we discuss how to solve these equations in practice.

Box 4.1: Amplitude equation for narrow beams

If a wave beam is narrow enough such that variations of V_u^α across the beam can be ignored, then $V_u^\alpha \approx V_u^\alpha(l)$, $V_u^\alpha \partial_\alpha \Psi_u \approx V_u^l \partial_l \Psi_u$, and $\partial_\alpha V_u^\alpha \approx d_l V_u^l$, where l is the coordinate along the beam axis. In this case, Eq. (4.17) yields the following equation for $\bar{\Psi}_u \doteq \Psi_u (V_u^l)^{1/2}$:

$$V_u^l \partial_l \bar{\Psi}_u = (\mathit{\Gamma}_u + iQ_u)\bar{\Psi}_u.$$

This equation does not contain $\partial_\alpha V_u^\alpha$, so may be easier to implement numerically than Eq. (4.17). Also notably, when V_u^l turns to zero, the field is singular. Beyond the GO approximation, this corresponds to a caustic like those discussed in Sec. 2.3.

Exercise 4.1: Show that at $\mathit{\Gamma} = 0$, Eq. (4.21c) can be derived, along with Eq. (4.21a), from the least-action principle with the action integral $\mathsf{S} = \int d\mathsf{x}\, \Lambda(\mathsf{x}, \partial\theta) A$. Here, θ and $A \doteq |\Psi|^2$ are considered as independent functions of x and Λ is a given function.

4.2 Ray equations

4.2.1 Consistency relations

As the first step, one needs to solve Eq. (4.21a) for ω and Eq. (4.21b) for $\boldsymbol{h}$, so one knows

$$\omega = \omega(t, \boldsymbol{x}, \boldsymbol{k}), \qquad \boldsymbol{h} = \boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{k}). \tag{4.22}$$

The next step is to calculate $\boldsymbol{k}(t, \boldsymbol{x})$. This can be done by revisiting the definitions of w and $\boldsymbol{k}$ [see Eq. (3.3), where the bars have been dropped to simplify the notation]:

$$w(t, \boldsymbol{x}) \doteq -\partial_t \theta(t, \boldsymbol{x}), \qquad \boldsymbol{k}(t, \boldsymbol{x}) \doteq \nabla\theta(t, \boldsymbol{x}). \tag{4.23}$$

Since $\nabla \times \nabla = 0$ and $\partial_t \nabla\theta = \nabla\partial_t\theta$, one obtains the following "consistency relations":

$$\nabla \times \boldsymbol{k} = 0, \qquad \partial_t \boldsymbol{k}(t, \boldsymbol{x}) = -\nabla w(t, \boldsymbol{x}). \tag{4.24}$$

The first consistency relation yields $\partial k_b/\partial x^a = \partial k_a/\partial x^b$. With this and Eq. (4.13), the second consistency relation yields the following nonlinear PDE for $\boldsymbol{k}(t, \boldsymbol{x})$:

$$\begin{aligned}
\frac{\partial k_a}{\partial t} &= -\frac{\partial w}{\partial x^a} \\
&= -\frac{\partial \omega}{\partial x^a} - \frac{\partial \omega}{\partial k_b}\frac{\partial k_b}{\partial x^a} \\
&= -\frac{\partial \omega}{\partial x^a} - \frac{\partial \omega}{\partial k_b}\frac{\partial k_a}{\partial x^b}
\end{aligned}$$

Box 4.2: Conservation of wave crests

In a one-dimensional system, the first consistency relation can be written as a continuity equation, $\partial_t k + \partial_x(v_p k) = 0$, where $v_p = \omega/k$ is the phase velocity. This can be understood as a conservation of wave crests (or zeros) in GO [22].

Box 4.3: Connection between the two consistency relations

By taking the curl of $\partial_t \boldsymbol{k} = -\nabla w$, one obtains $\partial_t(\nabla \times \boldsymbol{k}) = 0$. Therefore, the second consistency relation can be considered as the initial condition for the first consistency relation, much like magnetic Gauss's law serves as the initial condition for Faraday's law (Lecture 1.1.1).

$$= -\frac{\partial \omega}{\partial x^a} - \boldsymbol{v}_g \cdot \frac{\partial k_a}{\partial \boldsymbol{x}}, \tag{4.25}$$

where the spatial derivative of ω on the right-hand side is taken at fixed $\boldsymbol{k}$. The same can be written in the following vector form:

$$[\partial_t + (\boldsymbol{v}_g \cdot \nabla)]\boldsymbol{k} = -\partial_{\boldsymbol{x}}\omega. \tag{4.26}$$

In other words, the evolution of the field $\boldsymbol{k}(t, \boldsymbol{x})$ in the frame moving at the group velocity $\boldsymbol{v}_g$ is determined by the explicit dependence of $\omega(t, \boldsymbol{x}, \boldsymbol{k})$ on $\boldsymbol{x}$.

4.2.2 Hamilton's equations for rays

Consider trajectories $\boldsymbol{x}(t)$, called "characteristics" or "GO rays", that are governed by

$$d_t\boldsymbol{x}(t) = \boldsymbol{v}_g[t, \boldsymbol{x}(t), \boldsymbol{k}(t)] \equiv \partial_{\boldsymbol{k}}\omega[t, \boldsymbol{x}(t), \boldsymbol{k}(t)], \tag{4.27}$$

with $\boldsymbol{k}(t) \doteq \boldsymbol{k}[t, \boldsymbol{x}(t)]$. Then, Eq. (4.26) yields

$$d_t\boldsymbol{k}(t) = \left\{[\partial_t + (\boldsymbol{v}_g \cdot \nabla)]\boldsymbol{k}(t, \boldsymbol{x})\right\}_{\boldsymbol{x}=\boldsymbol{x}(t)} = -\partial_{\boldsymbol{x}}\omega[t, \boldsymbol{x}(t), \boldsymbol{k}(t)], \tag{4.28}$$

which is an *ordinary* differential equation (ODE) for $\boldsymbol{k}(t)$. Also notice that the instantaneous frequency $\omega(t) \doteq \omega[t, \boldsymbol{x}(t), \boldsymbol{k}(t)]$ is governed by

$$d_t\omega = \partial_t\omega + \partial_{\boldsymbol{x}}\omega \cdot d_t\boldsymbol{x} + \partial_{\boldsymbol{k}}\omega \cdot d_t\boldsymbol{k} = \partial_t\omega + \partial_{\boldsymbol{x}}\omega \cdot \partial_{\boldsymbol{k}}\omega - \partial_{\boldsymbol{k}}\omega \cdot \partial_{\boldsymbol{x}}\omega = \partial_t\omega. \tag{4.29}$$

In summary then,

$$d_t\boldsymbol{x} = \partial_{\boldsymbol{k}}\omega(t, \boldsymbol{x}, \boldsymbol{k}), \qquad d_t\boldsymbol{k} = -\partial_{\boldsymbol{x}}\omega(t, \boldsymbol{x}, \boldsymbol{k}), \qquad d_t\omega = \partial_t\omega(t, \boldsymbol{x}, \boldsymbol{k}). \tag{4.30}$$

Notably, Eqs. (4.30) remain applicable near reflection points as well, even though the amplitude equation breaks down there (Box 4.1) and the GO parameter is not small. The reason for this is that the GO approximation can be reinstated near cutoffs using phase-space rotation (Box 2.1) that does not affect the ray equations.

To better understand Eqs. (4.30), notice the following. The quantities $\boldsymbol{p} \doteq \hbar\boldsymbol{k}$ and $H \doteq \hbar\omega$ are commonly interpreted as the photon canonical momentum and energy. Assuming this interpretation, the ray equations can be viewed as Hamilton's equations of the photon motion in phase space $(\boldsymbol{x}, \boldsymbol{p})$, with $H(t, \boldsymbol{x}, \boldsymbol{p})$ serving as the Hamiltonian:[3]

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{\partial(\hbar\omega)}{\partial(\hbar\boldsymbol{k})} = \frac{\partial H}{\partial\boldsymbol{p}}, \tag{4.31a}$$

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(\hbar\boldsymbol{k})}{dt} = -\frac{\partial(\hbar\omega)}{\partial\boldsymbol{x}} = -\frac{\partial H}{\partial\boldsymbol{x}}, \tag{4.31b}$$

$$\frac{dH}{dt} = \hbar\frac{d\omega}{dt} = \hbar\frac{\partial\omega}{\partial t} = \frac{\partial H}{\partial t}. \tag{4.31c}$$

[3]Strictly speaking, photons are defined through *global* modes and thus cannot be assigned specific coordinates. For example, in a stationary GO wave, each photon is spread out along the whole propagation distance. However, this does not limit the applicability of the ray equations, because they are derived from classical considerations and independently from the photon analogy.

Box 4.4: Covariant form of the ray equations

Equations (4.34) can also be derived as follows. Let us introduce *some* auxiliary time τ and consider solutions of the original equation $\hat{\boldsymbol{D}}\boldsymbol{\psi} = 0$ as "stationary" ($\partial_\tau \boldsymbol{\psi} = 0$) solutions of $(\hat{\boldsymbol{D}} - i\partial_\tau)\boldsymbol{\psi} = 0$. The latter can be considered as $\hat{\boldsymbol{D}}_{\text{ext}}\boldsymbol{\psi} = 0$, where $\hat{\boldsymbol{D}}_{\text{ext}} \doteq \hat{\boldsymbol{D}} - \hat{\varpi}$ is an "extended" dispersion operator and $\hat{\varpi} \doteq i\partial_\tau$ is the frequency operator associated with the new time τ. Then, one can build GO for $\hat{\boldsymbol{D}}_{\text{ext}}$, in which case $x^0 \equiv ct$ and $k_0 \equiv -\omega/c$ are treated on the same footing as $\boldsymbol{x}$ and $\boldsymbol{k}$. The corresponding dispersion relation is readily found to be $\varpi = \Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k})$, and Eqs. (4.34) emerge as the covariant ray equations in spacetime:

$$\frac{dx^\alpha}{d\tau} = \frac{\partial \Lambda(\mathsf{x}, \mathsf{k})}{\partial k_\alpha}, \qquad \frac{dk_\alpha}{d\tau} = -\frac{\partial \Lambda(\mathsf{x}, \mathsf{k})}{\partial x^\alpha}.$$

4.2.3 Alternative forms of the ray equations

Another useful form of the ray equations is derived as follows. Because Eq. (4.21a) holds for all t, $\boldsymbol{x}$, and $\boldsymbol{k}$, one can differentiate it with respect to all these variables to obtain

$$0 = d_t \Lambda[t, \boldsymbol{x}, \omega(t, \boldsymbol{x}, \boldsymbol{k}), \boldsymbol{k}] = (\partial_\omega \Lambda)(\partial_t \omega) + \partial_t \Lambda, \tag{4.32a}$$
$$0 = d_{\boldsymbol{x}} \Lambda[t, \boldsymbol{x}, \omega(t, \boldsymbol{x}, \boldsymbol{k}), \boldsymbol{k}] = (\partial_\omega \Lambda)(\partial_{\boldsymbol{x}} \omega) + \partial_{\boldsymbol{x}} \Lambda, \tag{4.32b}$$
$$0 = d_{\boldsymbol{k}} \Lambda[t, \boldsymbol{x}, \omega(t, \boldsymbol{x}, \boldsymbol{k}), \boldsymbol{k}] = (\partial_\omega \Lambda)(\partial_{\boldsymbol{k}} \omega) + \partial_{\boldsymbol{k}} \Lambda. \tag{4.32c}$$

These equations lead to

$$\frac{\partial_t \Lambda}{\partial_\omega \Lambda} = -\frac{\partial \omega}{\partial t}, \qquad \frac{\partial_{\boldsymbol{x}} \Lambda}{\partial_\omega \Lambda} = -\frac{\partial \omega}{\partial \boldsymbol{x}}, \qquad \frac{\partial_{\boldsymbol{k}} \Lambda}{\partial_\omega \Lambda} = -\frac{\partial \omega}{\partial \boldsymbol{k}} = -\boldsymbol{v}_g. \tag{4.33}$$

Let us define an auxiliary time variable τ via $d_t \tau = -1/\partial_\omega \Lambda$.[4] Using $d_t = (d_t \tau) d_\tau$ and Eqs. (4.33), one can rewrite Eqs. (4.30) as follows:

$$\frac{dt}{d\tau} = -\frac{\partial \Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k})}{\partial \omega}, \qquad \frac{d\omega}{d\tau} = +\frac{\partial \Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k})}{\partial t}, \tag{4.34a}$$
$$\frac{d\boldsymbol{x}}{d\tau} = +\frac{\partial \Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k})}{\partial \boldsymbol{k}}, \qquad \frac{d\boldsymbol{k}}{d\tau} = -\frac{\partial \Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k})}{\partial \boldsymbol{x}}, \tag{4.34b}$$

where the former equation is the definition of τ included for completeness. (An alternative path to these equations is presented in Box 4.4.) As a corollary, $\boldsymbol{k}$ is conserved if the medium is spatially homogeneous ($\partial_{\boldsymbol{x}} \Lambda = 0$), and ω is conserved if the medium is stationary ($\partial_t \Lambda = 0$).

4.3 Amplitude equation

Now, let us figure out how to marry the ray equations with the amplitude equation (4.21c). Note that

$$\begin{aligned}
V^\alpha &= \boldsymbol{h}^\dagger \frac{\partial \boldsymbol{D}_{\text{H}}}{\partial k_\alpha} \boldsymbol{h} \\
&= \frac{\partial}{\partial k_\alpha}(\boldsymbol{h}^\dagger \boldsymbol{D}_{\text{H}} \boldsymbol{h}) - \left(\frac{\partial \boldsymbol{h}^\dagger}{\partial k_\alpha}\right) \boldsymbol{D}_{\text{H}} \boldsymbol{h} - \boldsymbol{h}^\dagger \boldsymbol{D}_{\text{H}} \left(\frac{\partial \boldsymbol{h}}{\partial k_\alpha}\right) \\
&= \frac{\partial \Lambda}{\partial k_\alpha} - \left(\frac{\partial \boldsymbol{h}^\dagger}{\partial k_\alpha}\right) \times 0 - 0 \times \left(\frac{\partial \boldsymbol{h}}{\partial k_\alpha}\right)
\end{aligned}$$

[4] Other definitions of τ can be used as well and lead to different but equivalent ray equations.

$$= \frac{\partial \Lambda}{\partial k_\alpha}, \tag{4.35}$$

where we have used Eq. (4.21b) and its conjugate. Then, we obtain

$$\frac{\partial}{\partial x^0}\left(\frac{\partial \Lambda}{\partial k_0}|\Psi|^2\right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left(\frac{\partial \Lambda}{\partial \boldsymbol{k}}|\Psi|^2\right) = 2\mathit{\Gamma}|\Psi|^2, \tag{4.36}$$

where the first term can as well be expressed as follows:

$$\frac{\partial}{\partial x^0}\left(\frac{\partial \Lambda}{\partial k_0}|\Psi|^2\right) = \frac{\partial}{\partial (ct)}\left(\frac{\partial \Lambda}{\partial (-\omega/c)}|\Psi|^2\right) = -\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \omega}|\Psi|^2\right). \tag{4.37}$$

Let us introduce

$$\mathcal{I} \doteq \frac{\partial \Lambda}{\partial \omega}|\Psi|^2 = \boldsymbol{h}^\dagger \frac{\partial \boldsymbol{D}_{\mathrm{H}}}{\partial \omega}\boldsymbol{h}|\Psi|^2 = \boldsymbol{\Psi}^\dagger \frac{\partial \boldsymbol{D}_{\mathrm{H}}}{\partial \omega}\boldsymbol{\Psi}, \tag{4.38}$$

$$\mathcal{J}^a \doteq -\frac{\partial \Lambda}{\partial k_a}|\Psi|^2 = -\boldsymbol{h}^\dagger \frac{\partial \boldsymbol{D}_{\mathrm{H}}}{\partial k_a}\boldsymbol{h}|\Psi|^2 = -\boldsymbol{\Psi}^\dagger \frac{\partial \boldsymbol{D}_{\mathrm{H}}}{\partial k_a}\boldsymbol{\Psi}. \tag{4.39}$$

Importantly, one can also express $\boldsymbol{\mathcal{J}}$ through the eigenvalue Λ as

$$\boldsymbol{\mathcal{J}} = -\frac{\partial_{\boldsymbol{k}}\Lambda}{\partial_\omega \Lambda}\mathcal{I} = \boldsymbol{v}_g \mathcal{I}, \tag{4.40}$$

where we have used Eq. (4.33). Then, Eq. (4.36) can be expressed as a *continuity equation for* $\mathcal{I}$,

$$\partial_t \mathcal{I} + \nabla \cdot (\boldsymbol{v}_g \mathcal{I}) = 2\gamma \mathcal{I}, \tag{4.41}$$

where γ is given by

$$\gamma \doteq -\frac{\mathit{\Gamma}}{\partial_\omega \Lambda} = -\frac{\boldsymbol{h}^\dagger \boldsymbol{D}_{\mathrm{A}} \boldsymbol{h}}{\boldsymbol{h}^\dagger (\partial_\omega \boldsymbol{D}_{\mathrm{H}})\, \boldsymbol{h}} = -\frac{\boldsymbol{h}^\dagger \boldsymbol{D}_{\mathrm{A}} \boldsymbol{h}}{\partial_\omega \Lambda}. \tag{4.42}$$

The quantity $\mathcal{I}$ that enters this equation [and is given by Eq. (4.38)] is called the *action density*, and accordingly, $\boldsymbol{\mathcal{J}}$ can be understood as the action flux density. The physical meaning of these quantities will be discussed in detail in Lecture 5. Meanwhile, note that Eq. (4.41) can be rewritten as

$$[\partial_t + (\boldsymbol{v}_g \cdot \nabla)]\, \mathcal{I} = -\mathcal{I}\nabla \cdot \boldsymbol{v}_g + 2\gamma \mathcal{I}. \tag{4.43}$$

This readily leads to an ODE for the action density on a ray, $\mathcal{I}(t) \doteq \mathcal{I}[t, \boldsymbol{x}(t)]$, namely,[5]

$$d_t \mathcal{I} = 2\gamma_{\mathrm{eff}}(t)\mathcal{I}, \qquad \gamma_{\mathrm{eff}}(t) \doteq (\gamma - \nabla \cdot \boldsymbol{v}_g/2)_{[t, \boldsymbol{x}(t), \boldsymbol{k}(t)]}. \tag{4.44}$$

By integrating Eq. (4.44), one obtains

$$\mathcal{I}(t) = \exp\left[\int_0^t dt'\, 2\gamma_{\mathrm{eff}}(t')\right]\mathcal{I}_0, \tag{4.45}$$

where $\mathcal{I}_0$ is determined by initial conditions.

In summary then, GO equations are applied as follows. First, one finds $\omega(t, \boldsymbol{x}, \boldsymbol{k})$ from

$$\Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = 0. \tag{4.46}$$

Then, one calculates $\boldsymbol{k}(t, \boldsymbol{x})$ by solving either the consistency relation (4.26) as a PDE or the ray equations (4.30) as ODEs. Next, one finds the wave polarization $\boldsymbol{h}$ from $\boldsymbol{D}_{\mathrm{H}}\boldsymbol{h} = 0$. After that, one solves the equation for the action density, which also can be viewed either as a PDE [Eq. (4.41)] or as an ODE on the rays [Eq. (4.44)]. Finally, one calculates the wave amplitude $|\Psi|$ by definition of $\mathcal{I}$ [Eq. (4.38)]. In Lecture 5, we will show how to do this for electromagnetic waves in particular.

[5]For narrow wave beams, $\nabla \cdot \boldsymbol{v}_g$ can be removed from Eqs. (4.43) and (4.44) using the same idea as in Box 4.1.

4.4 *Spin Hall effect of light

The dispersion relation (4.46) is postulated in a somewhat arbitrary manner and is not entirely consistent. The term iQ in the amplitude equation (4.17) can, with enough time, produce an arbitrarily large gradient of $\arg\Psi$. Because Q does not enter Eq. (4.41), such a gradient has no effect on $|\Psi|$ within the GO approximation, However, it can eventually undermine the validity of the GO approximation, because the latter requires that Ψ be a slow function.

To prevent this, one can define θ such that Ψ be real. Then, by taking the real part of Eq. (4.11), one arrives at a dispersion relation

$$\Lambda(t, \boldsymbol{x}, \omega, \boldsymbol{k}) - Q(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = 0 \tag{4.47}$$

instead of Eq. (4.12). This complicates the polarization equation, which now becomes

$$\boldsymbol{D}_{\mathrm{H}}\boldsymbol{h} = \Lambda\boldsymbol{h} = Q\boldsymbol{h} \tag{4.48}$$

[cf. Eq. (4.21b)]. However, the amplitude equation is simplified:

$$V^\alpha \partial_\alpha \Psi + \frac{1}{2}\,(\partial_\alpha V^\alpha)\Psi = \mathit{\Gamma}\Psi. \tag{4.49}$$

The expression for V^α is not affected:

$$\begin{aligned}
V^\alpha &= \boldsymbol{h}^\dagger \frac{\partial \boldsymbol{D}_{\mathrm{H}}}{\partial k_\alpha}\,\boldsymbol{h} \\
&= \frac{\partial}{\partial k_\alpha}(\boldsymbol{h}^\dagger \boldsymbol{D}_{\mathrm{H}} \boldsymbol{h}) - \left(\frac{\partial \boldsymbol{h}^\dagger}{\partial k_\alpha}\right)\boldsymbol{D}_{\mathrm{H}}\boldsymbol{h} - \boldsymbol{h}^\dagger \boldsymbol{D}_{\mathrm{H}}\left(\frac{\partial \boldsymbol{h}}{\partial k_\alpha}\right) \\
&= \frac{\partial \Lambda}{\partial k_\alpha} - \left(\frac{\partial \boldsymbol{h}^\dagger}{\partial k_\alpha}\right)Q\boldsymbol{h} - \boldsymbol{h}^\dagger Q\left(\frac{\partial \boldsymbol{h}}{\partial k_\alpha}\right) \\
&= \frac{\partial \Lambda}{\partial k_\alpha} - Q\,\frac{\partial(\boldsymbol{h}^\dagger\boldsymbol{h})}{\partial k_\alpha} \\
&= \frac{\partial \Lambda}{\partial k_\alpha}
\end{aligned} \tag{4.50}$$

[cf. Eq. (4.35); here we have substituted Eq. (4.48) and $\boldsymbol{h}^\dagger\boldsymbol{h} = 1$]. However, remember that redefining θ entails a modification of $\boldsymbol{k}$, so V^α is now evaluated on a different $\boldsymbol{k}$ and thus *is* affected. Thus in reality, a wave propagates somewhat differently than as predicted by the conventional ray equations (4.30). This effect is known as the spin Hall effect of light and has been observed experimentally [23].

The spin Hall effect is typically negligible in practical applications, so it is usually ignored or not even recognized. Still, this effect is interesting because of its analogy with the spin–orbital interaction known for quantum particles. (The GO limit in quantum mechanics is known as the semiclassical approximation, and ray equations for quantum waves are known as classical mechanics.) When placed in an external magnetic field, quantum particles exhibit different trajectories depending on their spin state, for example, as seen in the famous Stern–Gerlach experiment. The analog of the spin for electromagnetic waves is their polarization state, and parameters of the medium serve as the vector potential, whose derivatives create an effective "magnetic field" for the "spin" to interact with. This analogy can be made quantitative and in fact the spin Hall effect of light and the Stern–Gerlach effect are mathematically identical [18, 24, 25].

Lecture 5

Wave action, energy, and momentum

In this lecture, we discuss the physical meaning of the wave action and also introduce the wave energy, the wave momentum, and the corresponding transport equations.

5.1 Wave action

As shown in Lecture 4, the amplitude equation for GO waves can be represented in the form

$$\partial_t \mathcal{I} + \nabla \cdot (\boldsymbol{v}_g \mathcal{I}) = 2\gamma \mathcal{I}, \tag{5.1}$$

where $\mathcal{I} = \boldsymbol{\Psi}^\dagger (\partial_\omega \boldsymbol{D}_\mathrm{H}) \boldsymbol{\Psi}$ is the wave action density and $\gamma = -\boldsymbol{h}^\dagger \boldsymbol{D}_\mathrm{A} \boldsymbol{h} / \partial_\omega \Lambda$. By integrating Eq. (5.1) over the whole space, one obtains

$$\frac{d}{dt} \int d\boldsymbol{x}\, \mathcal{I} + \int d\boldsymbol{x}\, \nabla \cdot (\boldsymbol{v}_g \mathcal{I}) = \int d\boldsymbol{x}\, 2\gamma \mathcal{I}. \tag{5.2}$$

By Gauss's theorem, the second term equals the flux through the infinite surface and thus is zero, assuming that the field is localized. Then, if $\gamma = 0$, the total action $I \doteq \int d\boldsymbol{x}\, \mathcal{I}$ is conserved:

$$\frac{dI}{dt} = 0. \tag{5.3}$$

The reason for this conservation law is that at zero γ, the wave is a Lagrangian system whose Lagrangian density does not depend on θ explicitly (Box 5.1). This makes I a Noether invariant of an asymptotic theory, an *adiabatic invariant*. The well-known adiabatic invariant of a harmonic oscillator can be understood as a special case of I (Box 5.2). In a broader context, Eqs. (5.1)–(5.3) generalize the WKB conservation law (2.35) that was derived in Lecture 2 for a specific wave in a simple geometry.

Because $\mathcal{I}$ satisfies a continuity equation (modulo the local dissipation term γ), it can be interpreted as the density of quasiparticles that travel with velocity $\boldsymbol{v}_g$. With the same reservations as in Lecture 4, these particles can be identified as photons. We will return to the photon analogy later in this lecture.

Box 5.1: Variational approach

At $\gamma = 0$, the envelope equation (4.11) becomes

$$0 = [\Lambda - Q - {}^{i}\!/_{2}\,(\partial_\alpha V^\alpha) - iV^\alpha \partial_\alpha]\,\Psi \equiv \hat{\mathscr{L}}\Psi, \tag{5.4}$$

where the mode index is omitted and $\Gamma \propto \gamma$ has vanished. The operator $\hat{\mathscr{L}}$ is Hermitian, so this equation satisfies the least-action principle $\delta\mathsf{S}[\Psi^*, \Psi] = 0$ with $\mathsf{S} = \int dt\, d\boldsymbol{x}\, \Psi^* \hat{\mathscr{L}}\Psi$, where Ψ and Ψ^* are independent variables.[a] At $\varepsilon \to 0$, when Q and ∂_α vanish, one obtains

$$\mathsf{S} \approx \int dt\, d\boldsymbol{x}\, \Lambda(t, \boldsymbol{x}, -\partial_t\theta, \nabla\theta)|\Psi|^2, \tag{5.5}$$

where we used $w = -\partial_t\theta$ and $\boldsymbol{k} = \nabla\theta$. (See also Exercise 5.2.) If θ and $A \doteq |\Psi|^2$ are treated as independent variables, this S readily leads to the following Euler–Lagrange equations:

$$0 = \delta_A \mathsf{S} = \Lambda(t, \boldsymbol{x}, w, \boldsymbol{k}), \tag{5.6}$$

$$0 = \delta_\theta \mathsf{S} = \partial_t \mathcal{I} + \nabla \cdot (\boldsymbol{v}_g \mathcal{I}), \tag{5.7}$$

which are the anticipated GO dispersion relation and the action-conservation theorem. This will be discussed in more detail in Lecture 15.

Note that there is no need to retain $\mathcal{O}(\varepsilon)$ corrections when deriving the action-conservation theorem from a variational principle. Also note that one obtains a coninuity equation for $\mathcal{I} \doteq \partial_\omega \mathfrak{L}$ for any $\mathsf{S} \approx \int dt\, d\boldsymbol{x}\, \mathfrak{L}$ such that the wave Lagrangian density has the form $\mathfrak{L} = \mathfrak{L}(\mathsf{x}, \partial\theta)$. However, if $\mathfrak{L}$ explicitly depends on θ rather than $\partial\theta$, then the wave action is not conserved.

[a] Because Ψ and Ψ^* are linearly independent combinations of $\operatorname{Re}\Psi$ and $\operatorname{Im}\Psi$, and because $\operatorname{Re}\Psi$ and $\operatorname{Im}\Psi$ can be treated as independent, the complex functions Ψ and Ψ^* can be treated as independent too.

Application to electromagnetic waves: action density

Let us now apply the above machinery to electromagnetic waves as a special case. In this case, $\boldsymbol{\Psi} = \boldsymbol{\mathcal{E}}$ is the electric-field complex amplitude and[1]

$$\hat{\boldsymbol{D}}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = \frac{1}{16\pi}\left[\frac{c^2}{\hat{\omega}^2}(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger - \mathbf{1}\,\hat{k}^2) + \hat{\boldsymbol{\epsilon}}\right]. \tag{5.8}$$

The corresponding symbol is

$$\boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = \frac{1}{16\pi}\left[\frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2) + \boldsymbol{\epsilon}(t, \boldsymbol{x}, \omega, \boldsymbol{k})\right], \tag{5.9}$$

where $\boldsymbol{\epsilon}$ is the symbol of $\hat{\boldsymbol{\epsilon}}$. Also, from Faraday's law the magnetic-field complex envelope is, to the leading order, $\boldsymbol{\mathcal{B}} = (c/\omega)(\boldsymbol{k} \times \boldsymbol{\mathcal{E}})$.

The corresponding action density can be calculated as follows:

$$\mathcal{I} = \frac{1}{16\pi}\,\boldsymbol{\mathcal{E}}^\dagger \partial_\omega\left[\frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2) + \boldsymbol{\epsilon}_{\mathrm{H}}\right]\boldsymbol{\mathcal{E}} = \frac{1}{16\pi}\,\boldsymbol{\mathcal{E}}^\dagger\left[-\frac{2c^2}{\omega^3}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2) + \partial_\omega \boldsymbol{\epsilon}_{\mathrm{H}}\right]\boldsymbol{\mathcal{E}}. \tag{5.10}$$

Using $\boldsymbol{D}_{\mathrm{H}}\boldsymbol{\mathcal{E}} = 0$, one can rewrite this as follows:

$$\mathcal{I} = \frac{1}{16\pi\omega}\,\boldsymbol{\mathcal{E}}^\dagger[-2(16\pi \boldsymbol{D}_{\mathrm{H}} - \boldsymbol{\epsilon}_{\mathrm{H}}) + \omega\partial_\omega \boldsymbol{\epsilon}_{\mathrm{H}}]\boldsymbol{\mathcal{E}}$$

[1] A dispersion operator of a free wave is defined only up to a constant factor. Here, we have introduced an additional factor $(16\pi)^{-1}$ (compared to Lecture 1) in order to simplify the interpretation of $\mathcal{I}$ in later sections.

Box 5.2: Example: linear oscillator

Our formulation of GO applies not just to electromagnetic waves but also to any linear oscillating systems that satisfy the assumed orderings. For example, consider a dissipative harmonic oscillator with time-dependent frequency $\omega_0(t)$:

$$\ddot{q} + 2\nu\dot{q} + \omega_0^2(t)q = 0, \qquad q = \mathrm{Re}\,(a\,\mathrm{e}^{i\theta}),$$

assuming the damping coefficient ν is a small constant. We can rewrite this as $\hat{D}q = 0$, where

$$\hat{D} \doteq \hat{\omega}^2 + 2i\nu\hat{\omega} - \omega_0^2(\hat{t}), \qquad \hat{\omega} = i\partial_t.$$

The symbol of this $\hat{D}$ is

$$D(t,\omega) = \omega^2 + 2i\nu\omega - \omega_0^2(t).$$

Since D is a scalar, one has $\Lambda = D_{\mathrm{H}} = \omega^2 - \omega_0^2(t)$, so the local dispersion relation is $\omega^2 = \omega_0^2(t)$. The action density $\mathcal{I}$, which is the same as the total action I here, is given by

$$\mathcal{I} \doteq (\partial_\omega\Lambda)\,|a|^2 = 2\omega_0|a|^2$$

and satisfies $\dot{\mathcal{I}} = 2\gamma\mathcal{I}$, where

$$\gamma = -\frac{D_{\mathrm{A}}}{\partial_\omega\Lambda} = -\nu.$$

Note that this $\mathcal{I} = I$ is proportional to U/ω, where U is the time-averaged energy:

$$U = \frac{1}{2}\,\langle\dot{q}^2\rangle + \frac{1}{2}\,\omega_0^2\langle q^2\rangle = \omega_0^2\langle q^2\rangle = \frac{1}{2}\,\omega_0^2|a|^2.$$

Thus, the action I of a harmonic oscillator is the same as the well-known adiabatic invariant [12] that is conserved at $\nu = 0$.

$$\begin{aligned} &= \frac{1}{16\pi\omega}\,\boldsymbol{\mathcal{E}}^\dagger(\omega\partial_\omega\boldsymbol{\epsilon}_{\mathrm{H}} + 2\boldsymbol{\epsilon}_{\mathrm{H}})\boldsymbol{\mathcal{E}} \\ &= \frac{1}{16\pi\omega^2}\,\boldsymbol{\mathcal{E}}^\dagger\,\partial_\omega(\omega^2\boldsymbol{\epsilon}_{\mathrm{H}})\boldsymbol{\mathcal{E}}. \end{aligned} \tag{5.11}$$

Using that

$$\boldsymbol{\mathcal{E}}^\dagger(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2)\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^\dagger[\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\mathcal{E}}) - \boldsymbol{\mathcal{E}}k^2] = \boldsymbol{\mathcal{E}}\cdot[\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{\mathcal{E}})] = (\boldsymbol{\mathcal{E}}\times\boldsymbol{k})\cdot(\boldsymbol{k}\times\boldsymbol{\mathcal{E}}) = -|\boldsymbol{k}\times\boldsymbol{\mathcal{E}}|^2$$

(assuming the notation as described in Appendix B), one can also express $\mathcal{I}$ as follows:

$$\begin{aligned} \mathcal{I} &= \frac{1}{16\pi\omega}\,\boldsymbol{\mathcal{E}}^\dagger\Big[-\frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2) - (16\pi\boldsymbol{D}_{\mathrm{H}} - \boldsymbol{\epsilon}_{\mathrm{H}}) + \omega\partial_\omega\boldsymbol{\epsilon}_{\mathrm{H}}\Big]\boldsymbol{\mathcal{E}} \\ &= \frac{1}{16\pi\omega}\,\boldsymbol{\mathcal{E}}^\dagger\Big[\partial_\omega(\omega\boldsymbol{\epsilon}_{\mathrm{H}}) - \frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\,k^2)\Big]\boldsymbol{\mathcal{E}} \\ &= \frac{1}{16\pi\omega}\Big[\boldsymbol{\mathcal{E}}^\dagger\partial_\omega(\omega\boldsymbol{\epsilon}_{\mathrm{H}})\boldsymbol{\mathcal{E}} + \frac{c^2}{\omega^2}|\boldsymbol{k}\times\boldsymbol{\mathcal{E}}|^2\Big] \\ &= \frac{1}{16\pi\omega}\Big[\boldsymbol{\mathcal{E}}^\dagger\partial_\omega(\omega\boldsymbol{\epsilon}_{\mathrm{H}})\boldsymbol{\mathcal{E}} + |\boldsymbol{\mathcal{B}}|^2\Big]. \end{aligned} \tag{5.12}$$

Then in summary,

$$\mathcal{I} = \frac{1}{16\pi\omega^2}\, \boldsymbol{\mathcal{E}}^\dagger\, \partial_\omega(\omega^2 \boldsymbol{\epsilon}_{\rm H})\boldsymbol{\mathcal{E}} = \frac{1}{16\pi\omega}[\boldsymbol{\mathcal{E}}^\dagger \partial_\omega(\omega \boldsymbol{\epsilon}_{\rm H})\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{B}}^\dagger \boldsymbol{\mathcal{B}}], \tag{5.13}$$

or in terms of real fields $\tilde{\boldsymbol{E}} = \operatorname{Re}(\boldsymbol{\mathcal{E}} e^{i\theta})$ and $\tilde{\boldsymbol{B}} = \operatorname{Re}(\boldsymbol{\mathcal{B}} e^{i\theta})$ (Exercise 5.1),

$$\mathcal{I} = \frac{1}{8\pi\omega^2}\, \langle \tilde{\boldsymbol{E}}^\dagger\, \partial_\omega(\omega^2 \boldsymbol{\epsilon}_{\rm H})\tilde{\boldsymbol{E}}\rangle = \frac{1}{8\pi\omega}\, \big[\langle \tilde{\boldsymbol{E}}^\dagger \partial_\omega(\omega \boldsymbol{\epsilon}_{\rm H})\tilde{\boldsymbol{E}}\rangle + \langle \tilde{\boldsymbol{B}}^2\rangle\big], \tag{5.14}$$

where the angular brackets denote average over the fast oscillations.

Exercise 5.1: Consider any $a = \operatorname{Re}(a_{\rm c} e^{i\theta})$ and $b = \operatorname{Re}(b_{\rm c} e^{i\theta})$ with slow complex envelopes $a_{\rm c}$ and $b_{\rm c}$ and fast real phase θ. Show that their θ-averaged product can be expressed as follows:

$$\langle ab \rangle = {}^1\!/_2 \operatorname{Re}(a_{\rm c} b_{\rm c}^*) = {}^1\!/_2 \operatorname{Re}(a_{\rm c}^* b_{\rm c}).$$

Application to electromagnetic waves: action flux density

The action flux density can be calculated either via $\boldsymbol{\mathcal{J}} = \boldsymbol{v}_g \mathcal{I}$ (Sec. 5.1) or directly as follows:

$$\boldsymbol{\mathcal{J}} = -\frac{1}{16\pi}\, \boldsymbol{\mathcal{E}}^\dagger \partial_{\boldsymbol{k}} \left[\frac{c^2}{\omega^2}(\boldsymbol{k}\boldsymbol{k}^\dagger - \mathbf{1}\, k^2) + \boldsymbol{\epsilon}_{\rm H}(t, \boldsymbol{x}, \omega, \boldsymbol{k})\right] \boldsymbol{\mathcal{E}} = \frac{\boldsymbol{S} + \boldsymbol{K}}{\omega}. \tag{5.15}$$

Then, the gth component of the vector $\boldsymbol{S}$ is defined as

$$\begin{aligned}
S_g &\doteq -\frac{c^2}{16\pi\omega}\frac{\partial}{\partial k_g}(k_a k_b - \delta_{ab} k_c k_c)\mathcal{E}_a^* \mathcal{E}_b \\
&= -\frac{c^2}{16\pi\omega}\,(\delta_{ag} k_b + k_a \delta_{bg} - 2\delta_{ab}\delta_{cg} k_c)\mathcal{E}_a^* \mathcal{E}_b \\
&= \frac{c^2}{16\pi\omega}\,(2k_g \mathcal{E}_a^* \mathcal{E}_a - \mathcal{E}_g^* k_b \mathcal{E}_b - \mathcal{E}_g k_a \mathcal{E}_a^*) \\
&= \frac{c^2}{16\pi\omega}\,[\boldsymbol{\mathcal{E}} \times (\boldsymbol{k} \times \boldsymbol{\mathcal{E}}^*) + \boldsymbol{\mathcal{E}}^* \times (\boldsymbol{k} \times \boldsymbol{\mathcal{E}})]_g \\
&= \frac{c^2}{8\pi\omega}\operatorname{Re}[\boldsymbol{\mathcal{E}} \times (\boldsymbol{k} \times \boldsymbol{\mathcal{E}}^*)]_g \\
&= \frac{c}{8\pi}\operatorname{Re}(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}^*)_g,
\end{aligned} \tag{5.16}$$

so $\boldsymbol{S}$ is the average Poynting vector,

$$\boldsymbol{S} = \frac{c}{8\pi}\operatorname{Re}(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}^*) = \frac{c}{4\pi}\,\langle \tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{B}}\rangle. \tag{5.17}$$

Then $\boldsymbol{K}$, whose components are

$$K_a \doteq -\frac{\omega}{16\pi}\,\boldsymbol{\mathcal{E}}^\dagger \frac{\partial \boldsymbol{\epsilon}_{\rm H}}{\partial k_a}\boldsymbol{\mathcal{E}}, \tag{5.18}$$

must be the kinetic action flux density. We will symbolically express it in the vector form as follows:

$$\boldsymbol{K} = -\frac{\omega}{16\pi}\,\boldsymbol{\mathcal{E}}^\dagger \partial_{\boldsymbol{k}} \boldsymbol{\epsilon}_{\rm H} \boldsymbol{\mathcal{E}} = -\frac{\omega}{8\pi}\,\langle \tilde{\boldsymbol{E}}^\dagger \partial_{\boldsymbol{k}} \boldsymbol{\epsilon}_{\rm H} \tilde{\boldsymbol{E}}\rangle. \tag{5.19}$$

5.2 Wave energy

Equation (4.41) for the wave action density can be used to generate various corollaries. For example, for any function $\mathsf{X}(t, \boldsymbol{x})$, one obtains

$$\begin{aligned}\partial_t(\mathsf{X}\mathcal{I}) + \nabla\cdot(\mathsf{X}\mathcal{I}\boldsymbol{v}_g) &= (\partial_t\mathsf{X})\mathcal{I} + \mathsf{X}(\partial_t\mathcal{I}) + [\nabla\cdot(\mathcal{I}\boldsymbol{v}_g)]\mathsf{X} + \mathcal{I}(\boldsymbol{v}_g\cdot\nabla)\mathsf{X}\\ &= \mathcal{I}[\partial_t + (\boldsymbol{v}_g\cdot\nabla)]\mathsf{X} + \mathsf{X}[\partial_t\mathcal{I} + \nabla\cdot(\mathcal{I}\boldsymbol{v}_g)]\\ &= \mathcal{I}\, d_t\mathsf{X} + 2\gamma\mathsf{X}\mathcal{I}.\end{aligned} \tag{5.20}$$

Below, we consider the choices of X that are particularly important.

First, consider $\mathsf{X} = \omega$. In this case, Eq. (5.20) becomes

$$\partial_t\,\mathcal{U} + \nabla\cdot(\mathcal{U}\boldsymbol{v}_g) = \mathcal{I}d_t\omega + 2\gamma\omega\mathcal{I}, \tag{5.21}$$

where we used the ray equation $d_t\omega = \partial_t\omega$ and introduced

$$\mathcal{U} \doteq \omega\mathcal{I}. \tag{5.22}$$

Then, Eq. (5.21) can be written as follows:

$$\partial_t\,\mathcal{U} + \nabla\cdot(\mathcal{U}\boldsymbol{v}_g) = \mathcal{I}\partial_t\omega + 2\gamma\,\mathcal{U}. \tag{5.23}$$

Adiabatic waves in stationary media ($\gamma = 0,\ \partial_t\omega = 0$)

First, suppose that there is no dissipation ($\gamma = 0$), in which case the system becomes Lagrangian. Also suppose that the medium is stationary ($\partial_t\omega = 0$). Then, this equation becomes conservative [cf. Eq. (5.3)]:

$$\partial_t\,\mathcal{U} + \nabla\cdot(\mathcal{U}\boldsymbol{v}_g) = 0, \qquad \int d\boldsymbol{x}\,\mathcal{U} = \text{const.} \tag{5.24}$$

In other words, the invariance of a Lagrangian system with respect to translations in time leads to the conservation of $\int d\boldsymbol{x}\,\mathcal{U}$. This means that *by definition* [12], $\mathcal{U}$ is the density of the wave canonical energy, at least up to a constant factor α. This also means that the energy of a linear wave propagates at the group velocity.[2]

In order to determine α for electromagnetic waves, consider the following. From Eq. (5.13),

$$\mathcal{U} = \frac{1}{16\pi\omega}\,\boldsymbol{\mathcal{E}}^\dagger\,\partial_\omega(\omega^2\boldsymbol{\epsilon}_{\rm H})\boldsymbol{\mathcal{E}} = \frac{1}{16\pi}[\boldsymbol{\mathcal{E}}^\dagger\partial_\omega(\omega\boldsymbol{\epsilon}_{\rm H})\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{B}}^\dagger\boldsymbol{\mathcal{B}}], \tag{5.25}$$

or in terms of the reals fields,

$$\mathcal{U} = \frac{1}{8\pi\omega}\,\langle\tilde{\boldsymbol{E}}^\dagger\,\partial_\omega(\omega^2\boldsymbol{\epsilon}_{\rm H})\tilde{\boldsymbol{E}}\rangle = \frac{1}{8\pi}\left[\langle\tilde{\boldsymbol{E}}^\dagger\partial_\omega(\omega\boldsymbol{\epsilon}_{\rm H})\tilde{\boldsymbol{E}}\rangle + \langle\tilde{\boldsymbol{B}}^2\rangle\right]. \tag{5.26}$$

Consider a wave in vacuum, in which case $\boldsymbol{\epsilon} = 1$. Then, $\mathcal{U}$ exactly coincides with the known formula for the vacuum-wave energy density [27], and thus $\alpha = 1$. Accordingly, $\mathcal{U}\boldsymbol{v}_g$ has the meaning of the energy flux density. Due to Eq. (5.15), it can be expressed as

$$\mathcal{U}\boldsymbol{v}_g = \boldsymbol{S} + \boldsymbol{K}, \tag{5.27}$$

where $\boldsymbol{S}$ [Eq. (5.17)] is the Poynting vector and $\boldsymbol{K}$ [Eq. (5.18)] has the meaning of the kinetic-energy flux density. Notably, Eq. (5.27) also leads to an alternative formula for the group velocity:

$$\boldsymbol{v}_g = \frac{\boldsymbol{S} + \boldsymbol{K}}{\mathcal{U}}. \tag{5.28}$$

[2]This conclusion is invalid for nonlinear waves, which have different propagation velocities for the action, energy, and momentum. Some authors *define* the group velocity $\boldsymbol{v}_g$ for nonlinear waves as the energy velocity, but this is an arbitrary definition that can lead to confusion. In particular, such $\boldsymbol{v}_g$ will not be the velocity on the characteristics. Moreover, there are two families of characteristics in this case, which have *different* velocities. For details on these issues, see Ref. [22] or Ref. [26, Appendix B].

Adiabatic waves in nonstationary media ($\gamma = 0,\ \partial_t\omega \neq 0$)

Let us compare Eq. (5.22) with the expression for the wave energy density expected from quantum mechanics,

$$\mathcal{U} = \hbar\omega n_{\rm ph}, \tag{5.29}$$

where $n_{\rm ph}$ is the photon density. It is seen from here that $\mathcal{I} = \hbar n_{\rm ph}$; i.e., $\mathcal{I}$ is just the photon density in units $\hbar^{-1}$ (cf. Exercise 5.2). Accordingly, Eq. (5.21) can be understood as follows. Let us assume for now that $\gamma = 0$. Then, the action conservation (5.1) says that the number of photons (action/$\hbar$) is conserved, and thus the wave energy can evolve only through the evolution of the energies of individual photons, $H = \hbar\omega$. This regime is called adiabatic (cf. Box 5.2). The corresponding power density is

$$n_{\rm ph}\, \mathsf{d}_t H = (\mathcal{I}/\hbar)\, \mathsf{d}_t(\hbar\omega) = \mathcal{I}\, \mathsf{d}_t\omega = \mathcal{I}\partial_t\omega, \tag{5.30}$$

which is in agreement with Eq. (5.25) at zero γ.

Exercise 5.2: One can use the Taylor expansion of Eq. (5.6) around the solution of the dispersion relation $w = \omega(t, \boldsymbol{x}, \boldsymbol{k})$ to obtain $\Lambda(t, \boldsymbol{x}, w, \boldsymbol{k}) \approx (w - \omega(t, \boldsymbol{x}, \boldsymbol{k}))\partial_\omega\Lambda$. Then, using $\partial_\omega\Lambda|\Psi|^2 = \mathcal{I}$, one arrives at the "canonical" form of the wave Lagrangian density [22, 28]

$$\mathfrak{L} = -(\partial_t\theta + \omega(t, \boldsymbol{x}, \boldsymbol{k}))\mathcal{I}, \tag{5.31}$$

where the independent functions are θ and $\mathcal{I}$. Show that in the "point-particle limit", when $\mathcal{I}(t, \boldsymbol{x}) = \hbar\delta[\boldsymbol{x} - \boldsymbol{X}(t)]$,[a] this action becomes $\mathsf{S}[\boldsymbol{X}, \boldsymbol{P}] = \int \mathsf{d}t\, [\boldsymbol{P}\cdot\dot{\boldsymbol{X}} - H(t, \boldsymbol{X}, \boldsymbol{P})]$ and leads to Hamilton's equations (4.31), with $H = \hbar\omega(t, \boldsymbol{X}, \boldsymbol{P})$ as the Hamiltonian, $\boldsymbol{X}(t)$ as the coordinate, and $\boldsymbol{P}(t) \doteq \nabla\theta[t, \boldsymbol{X}(t)]$ as the canonical momentum [29].

[a]Here, "$\delta[\boldsymbol{x} - \boldsymbol{X}(t)]$" denotes a profile that is narrow compared to the scale of the background medium but still wide compared to the wavelength, so GO is still applicable.

Dissipative waves ($\gamma \neq 0$)

Now, let us allow for non-conservation of photons, i.e., nonzero γ. To the extent that $\mathcal{U}$ still can be understood as the wave energy density,[3] the function $\mathscr{P}_{\rm abs} \doteq -2\gamma\mathcal{U}$ can be interpreted as the wave power density absorbed irreversibly. (The sign is chosen such that $\mathscr{P}_{\rm abs} > 0$ when a wave is *losing* action.) This is understood because dissipation is due to the loss of photons; the loss of a single photon results in the energy loss equal to $\hbar\omega$, so the loss of $2\gamma\mathcal{I}/\hbar$ photons per unit time per unit volume results in the loss of energy $2\gamma\mathcal{U}$.

Using Eq. (4.42), one finds that

$$\gamma\mathcal{U} = \frac{1}{16\pi}\left(-\frac{\boldsymbol{h}^\dagger\boldsymbol{\epsilon}_{\rm A}\boldsymbol{h}}{\partial_\omega\Lambda}\right)\omega\mathcal{I} = \frac{\omega}{16\pi}\left(-\frac{\boldsymbol{h}^\dagger\boldsymbol{\epsilon}_{\rm A}\boldsymbol{h}}{\partial_\omega\Lambda}\right)\partial_\omega\Lambda\,|\mathcal{E}|^2 = -\frac{\omega}{16\pi}\,\boldsymbol{\mathcal{E}}^\dagger\boldsymbol{\epsilon}_{\rm A}\boldsymbol{\mathcal{E}}. \tag{5.32}$$

Hence,

$$\mathscr{P}_{\rm abs} = \frac{\omega}{8\pi}\,\boldsymbol{\mathcal{E}}^\dagger\boldsymbol{\epsilon}_{\rm A}\boldsymbol{\mathcal{E}}. \tag{5.33}$$

[3]Strictly speaking, the concept of energy is undefined for dissipative systems. However, if such a system transitions from a conservative state into another conservative one, then the *change* of its energy is well defined. The function $\mathcal{U}$ can be used as a means to calculate this change, because it represents the true energy density before and after the transition and its governing equation (5.21) holds at all times. The physical meaning of $\mathcal{U}$ during the transition is irrelevant, but one might as well call $\mathcal{U}$ energy density for the lack of a better definition of the energy density.

Also note that

$$\begin{aligned}
\mathscr{P}_{\text{abs}} &= \frac{\omega}{8\pi}\boldsymbol{\mathcal{E}}^\dagger \left(1+\frac{4\pi i\boldsymbol{\sigma}}{\omega}\right)_{\text{A}} \boldsymbol{\mathcal{E}} \\
&= \frac{1}{2}\boldsymbol{\mathcal{E}}^\dagger \boldsymbol{\sigma}_{\text{H}} \boldsymbol{\mathcal{E}} \\
&= \frac{1}{2}\,\text{Re}\,(\boldsymbol{\mathcal{E}}^\dagger \boldsymbol{\sigma}_{\text{H}} \boldsymbol{\mathcal{E}}) \\
&= \frac{1}{2}\,\text{Re}\,(\boldsymbol{\mathcal{E}}^\dagger \boldsymbol{\sigma} \boldsymbol{\mathcal{E}}) \\
&= \langle \tilde{\boldsymbol{j}}_{\text{ind}} \cdot \tilde{\boldsymbol{E}} \rangle,
\end{aligned} \tag{5.34}$$

where $\tilde{\boldsymbol{j}}_{\text{ind}}$ and $\tilde{\boldsymbol{E}} = \text{Re}\,(\boldsymbol{\mathcal{E}} e^{i\theta})$ are the real current density and the real electric field (Exercise 5.1). Therefore, $\mathscr{P}_{\text{abs}}$ represents the Joule-heating power.

5.3 Wave momentum

Let us return to Eq. (5.20) and substitute $\mathsf{X} = k_a$, where k_a is any of the components of the wave vector. This leads to

$$\partial_t(k_a \mathcal{I}) + \nabla \cdot (k_a \mathcal{I} \boldsymbol{v}_g) = \mathcal{I}\, \boldsymbol{d}_t k_a + 2\gamma k_a \mathcal{I}. \tag{5.35}$$

Let us use the ray equation $\boldsymbol{d}_t \boldsymbol{k} = -\partial_{\boldsymbol{x}}\omega$ and introduce

$$\boldsymbol{\mathcal{P}} \doteq \boldsymbol{k}\mathcal{I}. \tag{5.36}$$

Then, Eq. (5.35) can be written as follows:

$$\partial_t \mathcal{P}_a + \nabla \cdot (\mathcal{P}_a \boldsymbol{v}_g) = -\mathcal{I}\partial_a \omega + 2\gamma \mathcal{P}_a. \tag{5.37}$$

Suppose there is no dissipation ($\gamma = 0$), in which case the system becomes Lagrangian. Also suppose that the medium is homogeneous ($\partial_a \omega = 0$). Then, this equation becomes conservative:

$$\partial_t \mathcal{P}_a + \nabla \cdot (\mathcal{P}_a \boldsymbol{v}_g) = 0, \qquad \int \boldsymbol{dx}\, \mathcal{P}_a = \text{const.} \tag{5.38}$$

In other words, the invariance of a Lagrangian system with respect to translations in space leads to the conservation of $\int \boldsymbol{dx}\, \boldsymbol{\mathcal{P}}$. This means that *by definition* [12], $\boldsymbol{\mathcal{P}}$ is the density of the wave canonical momentum, at least up to a constant factor. This also means that the energy of a linear wave propagates at the group velocity. One can also extend this discussion by analogy with Sec. 5.2.

Finally, notice that together, Eqs. (5.36) with Eq. (5.22) lead to the following fundamental relation between the wave momentum density and the wave energy density:

$$\boldsymbol{\mathcal{P}} = \frac{\boldsymbol{k}}{\omega}\,\mathcal{U}. \tag{5.39}$$

Equation (5.39) is similar to the relation between photon momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ and the photon energy $H = \hbar\omega$. Because of this, some authors derive Eq. (5.39) from the quantum analogy. However, as shown above, Eq. (5.39) can be derived using purely classical arguments as well.

5.4 Example: α channeling

Equation (5.39) can be useful for understanding basic physics of wave absorption. For example, consider the interaction between a wave and a particle in a static magnetic field $\boldsymbol{B}_0 = \nabla \times \boldsymbol{A}$. We will

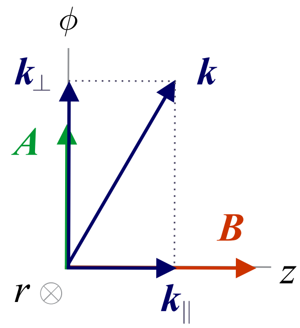


Figure 5.1: Schematic of the geometry assumed in Sec. 5.4.

assume the planar geometry but adopt the notation for the coordinates as in the cylindrical geometry such as that in a tokamak with a high aspect ratio. Assume that the vector potential $\boldsymbol{A}$ has the form $\boldsymbol{A} = \boldsymbol{e}_\phi \Psi(r)$, so $\boldsymbol{B}_0 = \boldsymbol{e}_z \Psi'(r)$ (Fig. 5.1). A radial static electric field is also allowed but will not be important. Suppose that a wave has deposited energy ΔU through the interaction with a particle with charge q. Since the overall system is symmetric in ϕ, the total canonical momentum in the ϕ direction must be conserved. Hence, the particle must change its canonical momentum in the ϕ direction by $k_\perp \Delta U/\omega$. This leads to

$$\frac{k_\perp}{\omega} \Delta U = \Delta \left[p_\phi + \frac{q}{c} \Psi(r) \right], \tag{5.40}$$

where $\boldsymbol{p}$ is the particle *kinetic* momentum. After the interaction, the left-hand side does not change, so it is equal to its own time average. But $\langle p_\phi \rangle = 0$ (the average perpendicular velocity of a particle that experiences stationary Larmor rotation in a dc magnetic field is zero), so one obtains

$$\frac{k_\perp}{\omega} \Delta U \approx \frac{q\Psi'(r)}{c} \langle \Delta r \rangle = \frac{qB_0}{c} \Delta r_{\text{gc}}, \tag{5.41}$$

where $\Delta r_{\text{gc}} = \langle \Delta r \rangle$ is the displacement of the particle guiding center. This shows that whenever a particle extracts energy ΔU from the wave, it also gets shifted radially by

$$\Delta r_{\text{gc}} = \frac{ck_\perp}{qB_0\omega} \Delta U. \tag{5.42}$$

This equation plays an important role in tokamak plasmas. In particular, it underlies the α channeling effect, and it is also relevant in the context of wave-induced plasma rotation.

Problems for Part II

PII.1 Single-wave dynamics within geometrical optics

Consider one-dimensional propagation of a stationary transverse electromagnetic wave in inhomogeneous nonmagnetized CCS plasma with unperturbed density $n_e(x) = (1 + x/L_c)n_0$, where n_0 and L_c are positive constants. Assume that the wave is launched at $x = 0$ with a wavevector $k_0 > 0$ and that the GO approximation holds at all x.

(a) Write down the ray equations explicitly. Find the wave frequency $\omega(x)$, the wavevector $k(x)$, and the group velocity $v_g(x)$. At what x is the wave reflected?

(b) Find and plot the ray trajectory $x(t)$. Find the time at which the envelope arrives at the reflection point.

(c) Calculate the wave energy density. How much of it is stored in: (i) the electric field, (ii) the magnetic field, and (iii) plasma oscillations?

(d) Write down the conservation law for the wave energy in a differential form. Using the above results, find the electric field amplitude as a function of x for a stationary wave. Compare the result with Eq. (2.35) obtained from the WKB approximation.

(e) Suppose now that the plasma is homogeneous but undergoes mechanical compression transversely to the wavevector. What is conserved in this case? Assuming for simplicity that $\partial_x n_e = 0$, find the wave *total* energy as a function of the instantaneous density $n_e(t)$.

PII.2 Coupling of resonant waves, mode conversion

Suppose a wave field $\boldsymbol{\psi}$ that is quasimonochromatic in the region of interest, i.e., can be expressed as $\boldsymbol{\psi} = \boldsymbol{\Psi} e^{i\theta}$, where $\boldsymbol{\Psi}$ and $\mathsf{k}_\alpha \doteq \partial_\alpha \theta$ may slowly depend on spacetime coordinates x. Unlike in Lecture 4, here we consider the case when the field consists of *two* modes,

$$\boldsymbol{\Psi} = \boldsymbol{h}_1 \Psi_1 + \boldsymbol{h}_2 \Psi_2 + \mathcal{O}(\varepsilon) \equiv \boldsymbol{\Xi} \bar{\boldsymbol{\Psi}} + \mathcal{O}(\varepsilon), \tag{5.43}$$

where $\boldsymbol{h}_{1,2}$ are normalized eigenvectors of $\boldsymbol{D}_{\mathrm{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})]$, $\boldsymbol{\Xi} \doteq (\boldsymbol{h}_1, \boldsymbol{h}_2)$ is a 3×2 matrix that has $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ as its columns, and $\bar{\boldsymbol{\Psi}}$ is a two-dimensional column vector with components Ψ_1 and Ψ_2. The derivation of the approximate equation for $\bar{\boldsymbol{\Psi}}$ is identical to that presented in Lecture 4 up to replacing Ψ with $\bar{\boldsymbol{\Psi}}$, $\boldsymbol{h}$ with $\boldsymbol{\Xi}$, and $\boldsymbol{h}^\dagger$ with $\boldsymbol{\Xi}^\dagger$, and one obtains

$$\left[\boldsymbol{\Lambda} - \boldsymbol{Q} + i\boldsymbol{\Gamma} - \frac{i}{2}(\partial_\alpha \boldsymbol{V}^\alpha) - i\boldsymbol{V}^\alpha \partial_\alpha\right] \bar{\boldsymbol{\Psi}} \approx 0, \tag{5.44}$$

which you may consider as given. Here, $\boldsymbol{\Lambda}$ and $\boldsymbol{V}^\alpha$ are 2×2 real diagonal matrices given by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad \boldsymbol{V}^\alpha \approx \begin{pmatrix} \partial\Lambda_1/\partial \mathsf{k}_\alpha & 0 \\ 0 & \partial\Lambda_2/\partial \mathsf{k}_\alpha \end{pmatrix}, \tag{5.45}$$

$\boldsymbol{Q}$ is a nondiagonal Hermitian matrix that determines mode coupling, and $\boldsymbol{\Gamma}$ is a (generally, also nondiagonal) Hermitian matrix that determines dissipation. For simplicity, let us suppose that dissipation is negligible ($\boldsymbol{\Gamma} \approx 0$) and that the problem is one-dimensional, i.e., all functions depend on only one spacetime coordinate l (which can be the time, a spatial coordinate, or a linear combination thereof). Suppose also that $\partial\Lambda_1/\partial\mathsf{k}_l$ and $\partial\Lambda_2/\partial\mathsf{k}_l$ have the same sign, for example, both are positive.

(a) Consider $\boldsymbol{\zeta} \doteq \sqrt{\boldsymbol{V}^l}\bar{\boldsymbol{\Psi}}$, where $\sqrt{\boldsymbol{V}^l}$ is a (diagonal) matrix whose square equals $\boldsymbol{V}^l$. Show that $\boldsymbol{\zeta}$ satisfies the following equation, where $\boldsymbol{\mathcal{H}}$ is a Hermitian 2×2 matrix:[4]

$$i\,\frac{d\boldsymbol{\zeta}}{dl} = \boldsymbol{\mathcal{H}}\boldsymbol{\zeta}. \tag{5.46}$$

(b) Because $\boldsymbol{\mathcal{H}}$ is Hermitian, Eq. (5.46) has a conservation law. Write down this law in terms of $\boldsymbol{\zeta}$, then express it in terms of $\bar{\boldsymbol{\Psi}}$. What is the physical meaning of the conserved quantity?

(c) Like in Eq. (4.11), $\bar{\mathsf{k}}(\mathsf{x})$ must be chosen such that the right side of Eq. (5.46) be small, namely, $\mathcal{O}(\varepsilon)$. We cannot choose $\bar{\mathsf{k}}(\mathsf{x})$ to make the *whole* $\boldsymbol{\Lambda}$ zero, because $\boldsymbol{\Lambda}$ is now a matrix that is determined by two independent scalar functions, Λ_1 and Λ_2. But by our assumption that $\boldsymbol{\psi}$ is quasimonochromatic, the two modes must be approximately in resonance; thus, if we choose $\bar{\mathsf{k}}(\mathsf{x})$, say, such that Λ_1 be small, then Λ_2 will be small automatically, and vice versa. Like in Lecture 4, the specific choice of $\bar{\mathsf{k}}(\mathsf{x})$ is a matter of convenience; for example, one can choose $\bar{\mathsf{k}}(\mathsf{x})$ such that $c_1\Lambda_1 + c_2\Lambda_2 = 0$, where $c_{1,2}$ can have any values of order one or less. It is often convenient to adopt $\operatorname{tr}\boldsymbol{\mathcal{H}}[\mathsf{x}, \bar{\mathsf{k}}(\mathsf{x})] = 0$; then, $\boldsymbol{\mathcal{H}}$ can be parametrized as follows:[5]

$$\boldsymbol{\mathcal{H}} = \begin{pmatrix} -\alpha & -i\beta \\ i\beta^* & \alpha \end{pmatrix}, \tag{5.47}$$

where α is real and $\alpha, \beta = \mathcal{O}(\varepsilon)$. Show that the variable transformation $a_1 \doteq \zeta_1 e^{-i\gamma/2}$ and $a_2 \doteq \zeta_2 e^{i\gamma/2}$, where $\gamma \doteq \arg\beta$, leads to the following equation for $a_{1,2}$:

$$i\,\frac{d}{d\tau}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -u(\tau) & -i \\ i & u(\tau) \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{5.48}$$

where $\tau \doteq \int dl\,|\beta|$ and $u \doteq \alpha/|\beta| - \dot{\gamma}/2$, and the dot denotes $d/d\tau$. Equation (5.48) is the canonical form of the envelope equation describing one-dimensional coupling of two (nondissipative) resonant modes. Examples of such "mode conversion" in plasma will be discussed in Part III.

(d) For constant u, search for the eigenmodes of Eq. (5.48) in the form $a_{1,2} = A_{1,2}e^{-i\Omega\tau}$. The values of A_1 and A_2 characterize how close these eigenmodes are to Mode I and Mode II of the corresponding homogeneous medium ($\boldsymbol{Q} = 0$). [For example, if $\boldsymbol{A} = (1, 0)^\intercal$, then the value of a_2 in such eigenmode is zero, i.e., the eigenmode is purely Mode I.] Plot Ω and $A_{1,2}$ as functions of u for both eigenmodes and explain where each eigenmode is close to Mode I and where it is close to Mode II.

[4] Notably, Eq. (5.46) is similar to the equation governing a two-level quantum system.

[5] Choosing an alternative convention leads to equations that are equivalent to Eq. (5.48) but have a different form. This is due to the fact that redefining $\bar{\mathsf{k}}(\mathsf{x})$ implies that the total phase of $\boldsymbol{\psi}$ is split differently between $\boldsymbol{\Psi}$ and θ, resulting in a different definition of the envelope functions $a_{1,2}$. In other words, the alternative equations are different because they describe the same total field $\boldsymbol{\psi}$ in different variables.

(e) Suppose $u(\tau) = \mu\tau$, where μ is constant, and assume $\mu > 0$ for simplicity. Here, $\tau = 0$ is understood as the moment when Mode I and Mode II are exactly in resonance. In quantum mechanics, this is known as the Landau–Zener problem [30, 31]. One way to solve Eq. (5.48) in this case is to rewrite it as the Weber equation, $\ddot{a}_1 + (1 + \mu^2\tau^2 - i\mu)a_1 = 0$, and then explore the asymptotic behavior of its solutions, which are given by parabolic cylinder functions. But let us adopt a more intuitive approach (which is an abridgment of the approach used in Ref. [32]).

First, show that Eq. (5.48) can be expressed as follows:

$$\hat{\xi}\hat{\eta}\,a_1 = -\frac{a_1}{2\mu}, \qquad \hat{\xi} \doteq \frac{s - i\partial_s}{\sqrt{2}}, \qquad \hat{\eta} \doteq \frac{s + i\partial_s}{\sqrt{2}}, \qquad s \doteq \tau\sqrt{\mu}. \tag{5.49}$$

For clarity, let us call s a "time" variable, so $i\partial_s$ can be viewed as the s-representation of the corresponding "frequency" operator $\hat{\omega}_s$. (Alternatively, one can view s as a spatial variable and $-i\partial_s$ as the corresponding wavevector operator, or momentum operator.) Then, one can write $\hat{\xi}$ and $\hat{\eta}$ in the following invariant form: $\hat{\xi} = (\hat{s} - \hat{\omega}_s)/\sqrt{2}$ and $\hat{\eta} = (\hat{s} + \hat{\omega}_s)/\sqrt{2}$. Show that $[\hat{\xi}, \hat{\eta}] = -i$. This is identical to $[\hat{s}, \hat{\omega}_s]$, so one can view $\hat{\xi}$ and $\hat{\eta}$ as the new time and frequency operators in the phase space rotated by 45° with respect to (s, ω_s).[6] Accordingly, in the ξ-representation, Eq. (5.49) has the form

$$i\xi\,\frac{da_1}{d\xi} = -\frac{a_1}{2\mu}. \tag{5.50}$$

(You are not asked to prove this formally but you can try to do it if you are interested; otherwise, see the solutions later.) Integrate Eq. (5.50) and show that the "transmission coefficient" $\mathsf{T} \doteq a_1(+\infty)/a_1(-\infty)$ is given by[7]

$$\mathsf{T} = e^{-\pi/(2\mu)}. \tag{5.51}$$

Hint: In order to encircle the pole at $\xi = 0$ correctly, you will need to introduce infinitesimal positive dissipation, i.e., replace $\mp u$ with $\mp u - i0$ [cf. the discussion preceding Eq. (2.20)]. If you did not study complex analysis, know that for any g,

$$\lim_{\nu\to 0+} \int_a^b d\xi\,\frac{g(\xi)}{\xi + i\nu} = -i\pi g(0) + ⨍_a^b d\xi\,\frac{g(\xi)}{\xi}, \tag{5.52}$$

where $⨍ d\xi\,(...)$ is the Cauchy principal value of the corresponding integral. This is known as the Sokhotski–Plemelj theorem. It will be discussed in detail in Part IV.

(f) Using Eq. (5.51) and the result of part (b), qualitatively explain the result of mode conversion at $|\mu| \ll 1$ ("adiabatic" regime) and at $|\mu| \gg 1$ ("diabatic" regime).

(g) Show that μ is of the order of the largest relative rate at which the beat period $\mathcal{T}_\text{b} \doteq 2\pi/|\Omega_\text{I} - \Omega_\text{II}|$ evolves with τ; i.e., show that $|\mu| \sim \max|d\mathcal{T}_\text{b}/d\tau|$.[8] Using this, formulate the condition under which the interaction between two modes can be neglected, i.e., the single-mode model studied in Lecture 4 can be used instead of the more complicated two-mode theory discussed above.

[6] This is an example of the metaplectic transform mentioned in Box 2.1.
[7] Remember that $\mu > 0$ is assumed here. For general μ, Eq. (5.51) becomes $\mathsf{T} = e^{-\pi/(2|\mu|)}$.
[8] Because $d\mathcal{T}_\text{b}/d\tau$ is dimensionless, this derivative can just as well be calculated in the original dimensional units.

Part III

Waves in plasmas: fluid theory

In this part of the course, we overview basic waves in plasmas within fluid models. Our overview is intended as introductory rather than exhaustive. For more information, see, for example, Refs. [1, 3, 33].

Lecture 6

Waves in cold magnetized plasma

In this lecture, we study waves in cold magnetized plasma in the same manner as we studied waves in cold nonmagnetized plasma in Lecture 2.

6.1 Basic equations

Let us explore dispersion properties of cold collisionless plasma with nonzero background magnetic field $\boldsymbol{B}_0$. This topic has been partly discussed in Problem PI.2, but here we will formulate it in a more conventional form, namely, in terms of the dielectric operator.

Let us assume the same plasma model as in Sec. 2.1.1, except now the background magnetic field is nonzero and we assume coordinates such that $\boldsymbol{B}_0 = \bar{\boldsymbol{e}}_z B_0$, where $\bar{\boldsymbol{e}}_z$ is a unit vector along the z axis. Then, the fluid velocity of any given species s is governed by

$$\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} = \frac{e_s}{m_s}\,\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{v}}_s \times \boldsymbol{\Omega}_s - \nu_s \tilde{\boldsymbol{v}}_s, \tag{6.1}$$

where $\boldsymbol{\Omega}_s \doteq \boldsymbol{e}_z \Omega_s$ and $\Omega_s \doteq e_s B_0/(m_s c)$ is the gyrofrequency. This leads to the following equations for the velocity components:

$$\frac{\partial \tilde{v}_{s,x}}{\partial t} = \frac{e_s}{m_s}\,\tilde{E}_x + \Omega_s \tilde{v}_{s,y} - \nu_s \tilde{v}_{s,x}, \tag{6.2a}$$

$$\frac{\partial \tilde{v}_{s,y}}{\partial t} = \frac{e_s}{m_s}\,\tilde{E}_y - \Omega_s \tilde{v}_{s,x} - \nu_s \tilde{v}_{s,y}, \tag{6.2b}$$

$$\frac{\partial \tilde{v}_{s,z}}{\partial t} = \frac{e_s}{m_s}\,\tilde{E}_z - \nu_s \tilde{v}_{s,z}. \tag{6.2c}$$

The equation for $\tilde{v}_{s,z}$ can be approached just like in Lecture 2, so let us focus on the equations for $\tilde{v}_{s,x}$ and $\tilde{v}_{s,y}$. Let us multiply Eq. (6.2b) by the imaginary unit and add it to (subtract it from) Eq. (6.2a). Assuming the notation

$$\tilde{j}_{s,\pm} \doteq \tilde{j}_{s,x} \pm i\tilde{j}_{s,y}, \qquad \tilde{E}_\pm \doteq \tilde{E}_x \pm i\tilde{E}_y, \tag{6.3}$$

one can write the result as follows:

$$\frac{\partial \tilde{j}_{s,\pm}}{\partial t} = -(\nu_s \pm i\Omega_s)\tilde{j}_{s,\pm} + \frac{\omega_{ps}^2}{4\pi}\,\tilde{E}_\pm, \tag{6.4}$$

and the real current densities are found from

$$\tilde{j}^{(\mathrm{i})}_{s,x} = \mathrm{Re}\,\tilde{j}^{(\mathrm{i})}_{s,\pm}, \qquad \tilde{j}^{(\mathrm{i})}_{s,y} = \pm\mathrm{Im}\,\tilde{j}^{(\mathrm{i})}_{s,\pm}. \tag{6.5}$$

In case of a homogeneous plasma, one can assume all the "tilded" quantities to be $\propto e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$ and infer the spectral conductivity as described in Lecture 1. More generally, though, we are interested in the Weyl symbol of $\hat{\boldsymbol{\sigma}}$, which is derived as follows. Equation (6.4) has the same form as Eq. (2.5), and similarly, its general solution of Eq. (6.4) can be written as $\tilde{j}_{s,\pm} = \tilde{j}^{(\mathrm{f})}_{s,\pm} + \tilde{j}^{(\mathrm{i})}_{s,\pm}$,

$$\tilde{j}^{(\mathrm{f})}_{s,\pm}(t,\boldsymbol{x}) = \tilde{j}^{(\mathrm{f})}_{s,\pm}(t_0,\boldsymbol{x})\, e^{-(\nu_s \pm i\Omega)(t-t_0)}, \tag{6.6a}$$

$$\tilde{j}^{(\mathrm{i})}_{s,\pm}(t,\boldsymbol{x}) = \frac{\omega_{ps}^2}{4\pi}\int_{t_0}^{t} dt'\, e^{(\nu_s \pm i\Omega)(t'-t)}\, \tilde{E}_\pm(t',\boldsymbol{x}). \tag{6.6b}$$

(Here and further, ω_{ps}, Ω_s, and ν_s are evaluated at location $\boldsymbol{x}$.) Then, from Eq. (6.5), one finds

$$\tilde{j}^{(\mathrm{i})}_{s,x}(t,\boldsymbol{x}) = \frac{\omega_{ps}^2}{4\pi}\int_{t_0}^{t} dt' \left\{+\cos[\Omega_s(t-t')]\,\tilde{E}_x(t',\boldsymbol{x}) + \sin[\Omega_s(t-t')]\,\tilde{E}_y(t',\boldsymbol{x})\right\} e^{-\nu_s(t-t')}, \tag{6.7a}$$

$$\tilde{j}^{(\mathrm{i})}_{s,y}(t,\boldsymbol{x}) = \frac{\omega_{ps}^2}{4\pi}\int_{t_0}^{t} dt' \left\{-\sin[\Omega_s(t-t')]\,\tilde{E}_x(t',\boldsymbol{x}) + \cos[\Omega_s(t-t')]\,\tilde{E}_y(t',\boldsymbol{x})\right\} e^{-\nu_s(t-t')}, \tag{6.7b}$$

where Ω_s and ν_s are evaluated at location $\boldsymbol{x}$. From here,

$$\Sigma_{s,xx}(t,\boldsymbol{x},t',\boldsymbol{x}') = +\Sigma_{s,yy}(t,\boldsymbol{x},t',\boldsymbol{x}') = \frac{\omega_{ps}^2}{4\pi}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\cos[\Omega_s(t-t')]\,e^{-\nu_s(t-t')}, \tag{6.8a}$$

$$\Sigma_{s,xy}(t,\boldsymbol{x},t',\boldsymbol{x}') = -\Sigma_{s,yx}(t,\boldsymbol{x},t',\boldsymbol{x}') = \frac{\omega_{ps}^2}{4\pi}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\sin[\Omega_s(t-t')]\,e^{-\nu_s(t-t')}, \tag{6.8b}$$

$$\Sigma_{s,zz}(t,\boldsymbol{x},t',\boldsymbol{x}') = \frac{\omega_{ps}^2}{4\pi}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,e^{-\nu_s(t-t')}, \tag{6.8c}$$

and the remaining elements of $\boldsymbol{\Sigma}_s$ are zero. Like in nonmagnetized plasma, the presence of $\delta(\boldsymbol{x}-\boldsymbol{x}')$ signifies that the plasma response is local in space, so there is no spatial dispersion.

Remember that $\boldsymbol{\Sigma}(t,\boldsymbol{x},t',\boldsymbol{x}') \equiv 0$ for $t' > t$. This means that for general t', the Heaviside step function $H(t-t')$ should be added as a factor on the right-hand side in all Eqs. (6.8). Then, each Σ_{ab} has a form

$$\Sigma_{ab}(t,\boldsymbol{x},t',\boldsymbol{x}') = f_{ab}(t-t',\boldsymbol{x})\,H(t-t')\,\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{6.9}$$

The corresponding symbol is, by definition [Eq. (3.15)], as follows:

$$\begin{aligned}
\sigma_{ab}(t,\boldsymbol{x},\omega,\boldsymbol{k}) &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{s}\,\Sigma_{ab}(t+\tau/2,\boldsymbol{x}+\boldsymbol{s}/2,t-\tau/2,\boldsymbol{x}-\boldsymbol{s}/2)\,e^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}\\
&= \int_{0}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{s}\, f_{ab}(\tau,\boldsymbol{x}+\boldsymbol{s}/2)\,\delta(\boldsymbol{s})\,e^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}\\
&= \int_{0}^{\infty} d\tau\, f_{ab}(\tau,\boldsymbol{x})\,e^{i\omega\tau}.
\end{aligned} \tag{6.10}$$

The right-hand side is independent of t, which is due to the fact that the plasma is assumed stationary; however, it generally depends on $\boldsymbol{x}$ through ω_{ps}, Ω_s, and ν_s. After substituting the corresponding f_{ab} and performing the integration, one obtains

$$\sigma_{s,xx} = +\sigma_{s,yy} = +\frac{\omega_{ps}^2}{4\pi}\,\frac{i(\omega+i\nu_s)}{(\omega+i\nu_s)^2-\Omega_s^2}, \tag{6.11a}$$

$$\sigma_{s,xy} = -\sigma_{s,yx} = -\frac{\omega_{ps}^2}{4\pi}\,\frac{\Omega_s}{(\omega+i\nu_s)^2-\Omega_s^2}, \tag{6.11b}$$

$$\sigma_{s,zz} = -\frac{\omega_{ps}^2}{4\pi i(\omega + i\nu_s)}, \tag{6.11c}$$

where the time integrals (6.10) converge because $\operatorname{Im}\omega = 0$ and $\nu_s > 0$. Because this result extends to homogeneous plasma as is, we will ignore the distinction between the symbol of $\hat{\boldsymbol{\sigma}}$ from the spectral conductivity in this lecture. For the same reason, we will not distinguish the symbol of $\hat{\boldsymbol{\chi}}$ from the spectral susceptibility and the symbol of $\hat{\boldsymbol{\epsilon}}$ from the dielectric tensor. We will also assume for simplicity that the plasma is collisionless, i.e., $\nu_s \to 0+$ [cf. Eq. (2.20)].

6.2 Susceptibility and dielectric tensor

In the collisionless limit, the susceptibility $\boldsymbol{\chi} = 4\pi i\boldsymbol{\sigma}/\omega$ corresponding to the conductivity (6.11) is

$$\chi_{s,xx} = +\chi_{s,yy} = -\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}, \tag{6.12a}$$

$$\chi_{s,xy} = -\chi_{s,yx} = -\frac{i\Omega_s}{\omega}\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2}, \tag{6.12b}$$

$$\chi_{s,zz} = -\frac{\omega_{ps}^2}{\omega^2}. \tag{6.12c}$$

Using Eqs. (6.12), one readily obtains the corresponding dielectric tensor (Box 6.1):

$$\boldsymbol{\epsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}, \tag{6.13}$$

which is Hermitian.[1] The notation assumed here is the traditional notation from Ref. [1]:

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} = \frac{1}{2}(R + L), \tag{6.14a}$$

$$D = \sum_s \frac{\Omega_s}{\omega}\frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} = \frac{1}{2}(R - L), \tag{6.14b}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}, \tag{6.14c}$$

$$R, L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \pm \Omega_s)}. \tag{6.14d}$$

If ω is much larger than all Ω_s and ω_{ps}, then $\boldsymbol{\epsilon} \approx \mathbf{1}$. In the somewhat more general case when $\omega \gg \Omega_e$ yet $\omega \sim \omega_{pe}$, the tensor $\boldsymbol{\epsilon}$ is the same as in nonmagnetized plasma. Let us also consider the low-frequency limit, $\omega \ll \Omega_i$, which is known as the magnetohydrodynamic (MHD) limit. There, one has $S \approx 1 + \gamma_A$, where

$$\gamma_A \doteq \sum_s \frac{\omega_{ps}^2}{\Omega_s^2} = \sum_s \frac{4\pi n_{s0} e_s^2}{m_s}\frac{m_s^2 c^2}{e_s^2 B_0^2} = c^2 \frac{4\pi}{B_0^2}\sum_s n_{s0} m_s = \frac{c^2}{V_A^2}. \tag{6.16}$$

Here, we have also introduced the so-called Alfvén speed,

$$V_A \doteq \frac{B_0}{\sqrt{4\pi\rho_m}}, \tag{6.17}$$

[1] The regions $\omega \to 0$ and $\omega \to \pm\Omega_s$ are exceptions. There, one should account for nonzero ν_s, or other dissipation, even in the limit when ν_s is infinitesimally small (Problem PIV.8).

Box 6.1: Alternative form of $\boldsymbol{\epsilon}$

Using the Gell–Mann matrices $\boldsymbol{\alpha}$ introduced in Problem PI.2, one can also express $\boldsymbol{\epsilon}$ in the following invariant form that allows for an arbitrary orientation of $\boldsymbol{B}_0$:

$$\boldsymbol{\epsilon} = \mathbf{1} + \sum_s \left[-\mathbf{1}\,\frac{\omega_{ps}^2}{\omega^2} + \frac{\omega_{ps}^2\,(\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_s)}{\omega(\omega^2-\Omega_s^2)} - \frac{\omega_{ps}^2}{\omega^2}\,\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}_s)^2}{\omega^2-\Omega_s^2} \right]. \tag{6.15}$$

and the mass density $\rho_m \doteq \sum_s n_{s0} m_s$. Also,

$$\begin{aligned} \mathsf{D} &= -\sum_s \frac{\Omega_s}{\omega}\,\frac{\omega_{ps}^2}{\Omega_s^2\,(1-\omega^2/\Omega_s^2)} \\ &\approx -\sum_s \frac{\Omega_s}{\omega}\,\frac{\omega_{ps}^2}{\Omega_s^2}\left(1+\frac{\omega^2}{\Omega_s^2}\right) \\ &= -\frac{1}{\omega}\sum_s \frac{\omega_{ps}^2}{\Omega_s} - \sum_s \frac{\omega}{\Omega_s}\,\frac{\omega_{ps}^2}{\Omega_s^2}. \end{aligned} \tag{6.18}$$

Here,

$$\sum_s \frac{\omega_{ps}^2}{\Omega_s} = \sum_s \frac{4\pi n_{0,s} e_s^2}{m_s}\,\frac{m_s c}{e_s B_0} = \frac{4\pi c}{B_0}\sum_s n_{0,s} e_s = 0 \tag{6.19}$$

due to plasma neutrality, so

$$\mathsf{D} = -\sum_s \frac{\omega}{\Omega_s}\,\frac{\omega_{ps}^2}{\Omega_s^2} \sim \frac{\omega}{\Omega_i}\gamma_A \ll \mathsf{S}. \tag{6.20}$$

Hence, we obtain the following low-frequency limit of $\boldsymbol{\epsilon}$, which we will use later (Exercise 6.1):

$$\boldsymbol{\epsilon} \approx \begin{pmatrix} 1+\gamma_A & 0 & 0 \\ 0 & 1+\gamma_A & 0 \\ 0 & 0 & \mathsf{P} \end{pmatrix}. \tag{6.21}$$

Exercise 6.1: At $\omega \to 0$, the wave field becomes stationary, and one might expect that a stationary field cannot create a current perpendicular to $\boldsymbol{B}_0$. Nevertheless, the above calculation shows that $\epsilon_{xx} = \epsilon_{yy} \to 1+\gamma_A$, which is not unity. (In fact, γ_A is often large.) This means that plasma *does* respond to such field. What is the physical nature of this response?

6.3 General dispersion relation

Now let us explore the general dispersion relation (4.15) and the equation for the field polarization (4.16), which we copy here in the following form (since $\boldsymbol{D}_E$ is Hermitian):

$$\det \boldsymbol{D}_E(\boldsymbol{x},\omega,\boldsymbol{k}) = 0, \qquad \boldsymbol{D}_E(\boldsymbol{x},\omega,\boldsymbol{k})\boldsymbol{h} = 0. \tag{6.22}$$

Without loss of generality, one can choose the coordinate axes such that $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$, where $k_\perp = k\sin\theta$, $k_\parallel = k\cos\theta$, and θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$. A similar notation will be assumed for the refractive index $\boldsymbol{N}$. Then, the dispersion tensor (or the symbol of $\hat{\boldsymbol{D}}$) is given by

$$\boldsymbol{D}_E = \begin{pmatrix} S - N_\parallel^2 & -iD & N_\perp N_\parallel \\ iD & S - N^2 & 0 \\ N_\perp N_\parallel & 0 & P - N_\perp^2 \end{pmatrix}, \tag{6.23}$$

and thus,

$$\begin{aligned}
&\det \boldsymbol{D}_E \\
&= (S - N^2\cos^2\theta)(S - N^2)(P - N^2\sin^2\theta) - (-iD)(iD)(P - N^2\sin^2\theta) - (N^2\sin\theta\cos\theta)^2(S - N^2) \\
&= (S - N^2)(SP - SN^2\sin^2\theta - PN^2\cos^2\theta + N^4\sin^2\theta\cos^2\theta - N^4\sin^2\theta\cos^2\theta) - D^2(P - N^2\sin^2\theta) \\
&= (S - N^2)(SP - SN^2\sin^2\theta - PN^2\cos^2\theta) - D^2(P - N^2\sin^2\theta) \\
&= S^2P - S^2N^2\sin^2\theta - SPN^2\cos^2\theta - SPN^2 + SN^4\sin^2\theta + PN^4\cos^2\theta - D^2P + D^2N^2\sin^2\theta \\
&= N^4(S\sin^2\theta + P\cos^2\theta) + N^2(-S^2\sin^2\theta - SP\cos^2\theta - SP + D^2\sin^2\theta) + S^2P - D^2P \\
&= N^4(S\sin^2\theta + P\cos^2\theta) - N^2[(S^2 - D^2)\sin^2\theta + PS(1 + \cos^2\theta)] + P(S^2 - D^2).
\end{aligned}$$

Let us notice that

$$S^2 - D^2 = \frac{1}{4}(R + L)^2 - \frac{1}{4}(R - L)^2 = RL \tag{6.24}$$

and introduce

$$A \doteq S\sin^2\theta + P\cos^2\theta, \qquad B \doteq RL\sin^2\theta + PS\left(1 + \cos^2\theta\right), \qquad C \doteq PRL. \tag{6.25}$$

Then, we can also rewrite our equation as follows:

$$AN^4 - BN^2 + C = 0. \tag{6.26}$$

This biquadratic equation for N has the following solutions:

$$N^2 = \frac{B \pm F}{2A}, \tag{6.27}$$

where

$$\begin{aligned}
F^2 &\doteq B^2 - 4AC \\
&= [RL\sin^2\theta + PS(1 + \cos^2\theta)]^2 - 4(S\sin^2\theta + P\cos^2\theta)PRL \\
&= [(RL - PS)\sin^2\theta + 2PS]^2 - 4PRL(S\sin^2\theta + P\cos^2\theta) \\
&= (RL - PS)^2\sin^4\theta + 4PS(RL - PS)\sin^2\theta + 4P^2S^2 - 4PRLS\sin^2\theta - 4P^2RL\cos^2\theta \\
&= (RL - PS)^2\sin^4\theta + 4P^2S^2\cos^2\theta - 4P^2RL\cos^2\theta \\
&= (RL - PS)^2\sin^4\theta + P^2(R + L)^2\cos^2\theta - 4P^2RL\cos^2\theta \\
&= (RL - PS)^2\sin^4\theta + P^2(R - L)^2\cos^2\theta \\
&= (RL - PS)^2\sin^4\theta + 4P^2D^2\cos^2\theta.
\end{aligned}$$

Note that $F^2 > 0$ at all real frequencies, so N^2 is real too. Thus, N is either real or imaginary.

Consider also $\tau \doteq \tan^2\theta$ and note that

$$\cos^2\theta = \frac{1}{1 + \tau}, \qquad \sin^2\theta = \frac{\tau}{1 + \tau}. \tag{6.28}$$

Then, we obtain

$$\begin{aligned}
0 &= (S\tau + P)N^4 - [RL\tau + PS(2+\tau)]N^2 + PRL(1+\tau) \\
&= \tau(SN^4 - RLN^2 - PSN^2 + PRL) + PN^4 - 2PSN^2 + PRL \\
&= \tau[N^2(N^2 - P)S - RL(N^2 - P)] + P(N^4 - 2SN^2 + RL) \\
&= \tau(N^2 S - RL)(N^2 - P) + P[N^4 - (R+L)N^2 + RL] \\
&= \tau(N^2 S - RL)(N^2 - P) + P(N^2 - R)(N^2 - L). \qquad (6.29)
\end{aligned}$$

This leads to the following handy formula, which will be used below:

$$\tau = -\frac{P(N^2 - R)(N^2 - L)}{(N^2 S - RL)(N^2 - P)}. \qquad (6.30)$$

6.4 Eigenmodes

Solutions of the dispersion relation (6.27) can be qualitatively understood by analyzing the characteristic frequencies and limits, which are as follows.

6.4.1 Cutoffs and resonances

Let us start with cutoffs, where $N = 0$.[2] According to Eq. (6.26), cutoffs correspond to $C = 0$, or $PRL = 0$, which requires

$$P = 0 \quad \text{or} \quad R = 0 \quad \text{or} \quad L = 0. \qquad (6.31)$$

The former is satisfied when $\omega^2 = \omega_p^2$. The other two equations generally have multiple solutions depending on the number of species; but there are only two solutions in case of electron–ion plasma with single type of ions, $\omega = \omega_{R,L}$, and $\omega_R \gg \omega_L$.

Other notable frequencies are resonances, which correspond to $N \to \infty$. Resonances can be found by taking the corresponding limit in Eq. (6.30) or, equivalently, by noticing that our solution for N^2 predicts infinite refractive index at $A = 0$. In either case, one finds that such points are located where

$$S \sin^2\theta + P \cos^2\theta = 0. \qquad (6.32)$$

This equation coincides with the electrostatic dispersion relation discussed in Problem PI.1. However, note that the resonance condition is necessary but insufficient for a field to be electrostatic (Exercise 6.2).

> **Exercise 6.2:** Show that the electrostatic dispersion relation discussed in Problem PI.1 leads to Eq. (6.32). Using the condition (2.57), explain which resonances in cold magnetized plasma (discussed below) are electrostatic and which are not.

At $\theta = 0$, the resonance condition can be satisfied at $P = 0$, or at $\omega^2 = \omega_p^2$. Alternatively, as seen from $\tau(N^2 \to \infty) \to -P/S$, the resonance condition can be satisfied at $S \to \infty$, which corresponds to cyclotron resonances, $\omega = \Omega_s$. At $\theta = \pi/2$, the resonance condition becomes $S = 0$. Solutions of this equation are called *hybrid resonances*. For a plasma that consists on M species, $S = 0$ can be represented as an Mth-order polynomial equation for ω^2 (excluding special cases like electron-positron plasma), so such a plasma has M hybrid resonances. (We count only positive frequencies

[2] Cutoffs can also be defined depending on the problem. For example, consider a problem where N_z is conserved and N_x is not, as in a tokamak with a large aspect ratio or as in Problem PIII.2. Then, N_z can be considered as a constant parameter of the problem and cutoffs are defined as locations where $N_x = 0$. Examples will be discussed later.

here.) For electron–ion plasma with just one type of ions ($M = 2$), these resonances can be found exactly, but it is more instructive to obtain them approximately, by looking for a dominant balance in the corresponding equation,

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} = 0. \tag{6.33}$$

Clearly, one of the solutions corresponds to the case when the first two terms are dominant. The corresponding solution is known as the upper-hybrid frequency, which is determined just by electrons:

$$\omega_{\text{uh}} = \sqrt{\omega_{pe}^2 + \Omega_e^2}. \tag{6.34}$$

The remaining solution, $\omega = \omega_{\text{lh}}$, is known as the lower-hybrid frequency. Two asymptotic forms are used for it, depending on the value of ω_{pe}^2/Ω_e^2. They are found as follows. First, suppose that dominant in (6.33) are the second term and the third term. Then $\Omega_i \ll \omega \ll \Omega_e$, so one obtains

$$0 \approx -\frac{\omega_{pe}^2}{\omega_{\text{lh}}^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega_{\text{lh}}^2 - \Omega_i^2} \approx \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega_{\text{lh}}^2}. \tag{6.35}$$

Since $Z_i n_i = n_e$, where Z_i is the ion charge state, this leads to

$$\omega_{\text{lh}} \approx \sqrt{\frac{Z_i m_e}{m_i}}\, |\Omega_e| = |\Omega_i \Omega_e|^{1/2}. \tag{6.36}$$

Since this is based on the assumption that

$$1 \ll \frac{\omega_{pi}^2}{\omega_{\text{lh}}^2 - \Omega_i^2} \approx \frac{\omega_{pi}^2}{\omega_{\text{lh}}^2} \approx \frac{\omega_{pe}^2}{\Omega_e^2}, \tag{6.37}$$

the formula (6.36) is valid only for extremely overdense plasma. In the opposite case, when the plasma is extremely underdense, dominant in Eq. (6.33) are the first term and the third term. That leads to

$$\omega_{\text{lh}} = \sqrt{\omega_{pi}^2 + \Omega_i^2}. \tag{6.38}$$

In fusion plasmas, where ω_{pe} and Ω_e are typically comparable, equations (6.36) and (6.38) give comparable results, so it is common to assume $\omega_{\text{lh}} \sim |\Omega_i \Omega_e|^{1/2}$ as an estimate. Also note the following useful equality:

$$\mathsf{S} = \frac{(\omega^2 - \omega_{\text{uh}}^2)(\omega^2 - \omega_{\text{lh}}^2)}{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2)}. \tag{6.39}$$

At intermediate θ, the corresponding resonances are shown in Fig. 6.1. Plasmas with multiple ion species also have ion–ion hybrid resonances, but we will not consider them in detail here.

6.4.2 Low-frequency limit

At $\omega \ll \Omega_i$, Eq. (6.23) can be simplified using the low-frequency limit of $\boldsymbol{\epsilon}$ given by Eq. (6.21):

$$\begin{pmatrix} 1 + \gamma_A - N_\parallel^2 & 0 & N_\perp N_\parallel \\ 0 & 1 + \gamma_A - N^2 & 0 \\ N_\perp N_\parallel & 0 & \mathsf{P} - N_\perp^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0. \tag{6.40}$$

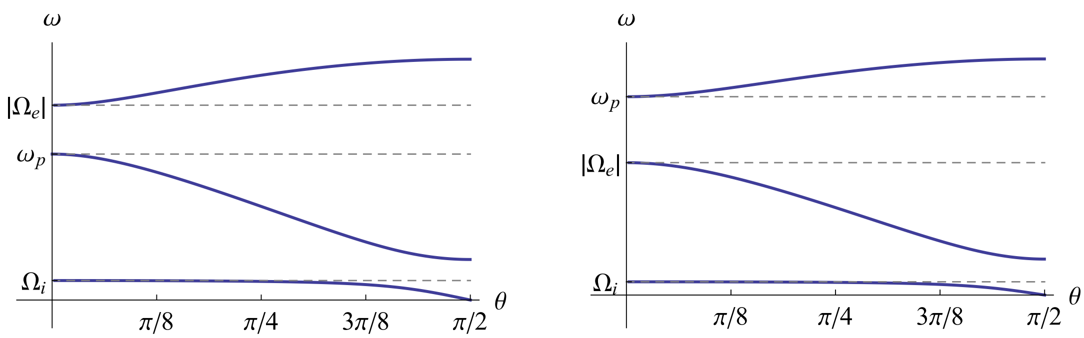


Figure 6.1: Resonance frequencies in electron–ion plasma with one type of ions [numerical solution of Eq. (6.32)]. Left – underdense plasma ($\omega_p < |\Omega_e|$). Right – overdense plasma ($\omega_p > |\Omega_e|$). In both cases, there are three resonances at $\theta = 0$ (at frequencies ω_p, Ω_e, and Ω_i; all dashed), and there are two resonances at $\theta = \pi/2$ (at frequencies $\omega_{\rm lh}$ and $\omega_{\rm uh}$).

At small enough ω, we also have large P[3] while the refraction index remains finite, as we will find shortly. Then, $h_z \approx -(N_\perp N_\parallel / P) h_x \ll h_x$, so the first two field equations can be written as follows:

$$\begin{pmatrix} 1 + \gamma_A - N_\parallel^2 & 0 \\ 0 & 1 + \gamma_A - N^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix} = 0. \tag{6.41}$$

Equation (6.41) indicates that there are two modes in this limit. The first one, known as the shear Alfvén wave, is x-polarized ($h_y = 0$) and satisfies

$$N_\parallel^2 = 1 + \gamma_A, \tag{6.42}$$

or equivalently,

$$\omega^2 = \frac{k_\parallel^2 V_A^2}{1 + \gamma_A^{-1}}. \tag{6.43}$$

The second mode, known as the compressional Alfvén wave, is y-polarized ($h_x = 0$) and satisfies

$$N^2 = 1 + \gamma_A, \tag{6.44}$$

or equivalently,

$$\omega^2 = \frac{k^2 c^2}{1 + \gamma_A} = \frac{k^2 c^2}{\gamma_A (1 + \gamma_A^{-1})} = \frac{k^2 V_A^2}{1 + \gamma_A^{-1}}. \tag{6.45}$$

Many plasmas of practical interest, including magnetically confined fusion plasmas, have $\gamma_A \gg 1$, so the term γ_A^{-1} in Eqs. (6.43) and (6.45) is often neglected. These waves are further studied in Problem PIII.3.

6.4.3 Parallel propagation ($\theta = 0$)

Now, let us allow for general frequencies but focus on the case of parallel propagation, i.e., $\theta = 0$. In this case, Eq. (6.22) becomes

$$\begin{pmatrix} S - N^2 & -iD & 0 \\ iD & S - N^2 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0 \tag{6.46}$$

[3] The low-ω limit of P is determined by thermal corrections discussed in Lecture 7 and later.

and Eq. (6.30) has three solutions (Figs. 6.2 and 6.3). The first one is $P = 0$, which corresponds to the Langmuir oscillations with $\boldsymbol{\mathcal{E}} \parallel \boldsymbol{B}_0$. The other two solutions are

$$N^2 = R, \qquad N^2 = L, \tag{6.47}$$

so they are called the R wave and the L wave, respectively. The R wave satisfies

$$\begin{pmatrix} S - R & -iD & 0 \\ iD & S - R & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0. \tag{6.48}$$

This gives $h_z = 0$ and

$$\frac{h_x}{h_y} = -\frac{S - R}{iD} = -\frac{R + L - 2R}{i(R - L)} = -i, \tag{6.49}$$

so this wave is circularly polarized in the (x, y) plane.[4] For the L wave, one similarly obtains $h_x/h_y = i$, which corresponds to the opposite circular polarization (Exercise 6.3).

> **Exercise 6.3:** Which direction do the R and L waves rotate relative to the particle rotation? Can you answer this without doing calculations?

6.4.4 Perpendicular propagation ($\theta = \pi/2$)

At perpendicular propagation, which corresponds to $\theta = \pi/2$, Eq. (6.23) takes the form

$$\begin{pmatrix} S & -iD & 0 \\ iD & S - N^2 & 0 \\ 0 & 0 & P - N^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0, \tag{6.50}$$

and the equation for τ has two solutions (Figs. 6.2 and 6.3). The first one,

$$N^2 - P = 0, \tag{6.51}$$

corresponds to the same dispersion as in nonmagnetized plasma,

$$\omega^2 = \omega_p^2 + c^2k^2, \tag{6.52}$$

with a cutoff at $P = 0$. The corresponding mode is called the O wave ("ordinary wave"). The fact that the O wave is insensitive to B_0 is explained by the wave polarization. One can see from

$$\begin{pmatrix} S & -iD & 0 \\ iD & S - N^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0 \tag{6.53}$$

that the corresponding $\boldsymbol{\mathcal{E}}$ is parallel to $\boldsymbol{B}_0$. Such field causes oscillations of (cold) particles parallel to $\boldsymbol{B}_0$, so the magnetic Lorentz force on the particles in the O wave is zero.

The other mode, called the X wave ("extraordinary wave"), corresponds to

$$N^2 = RL/S. \tag{6.54}$$

[4] In the regime $\Omega_i \ll \omega \lesssim \Omega_e$, the R branch is known as the whistler wave and will be studied separately.

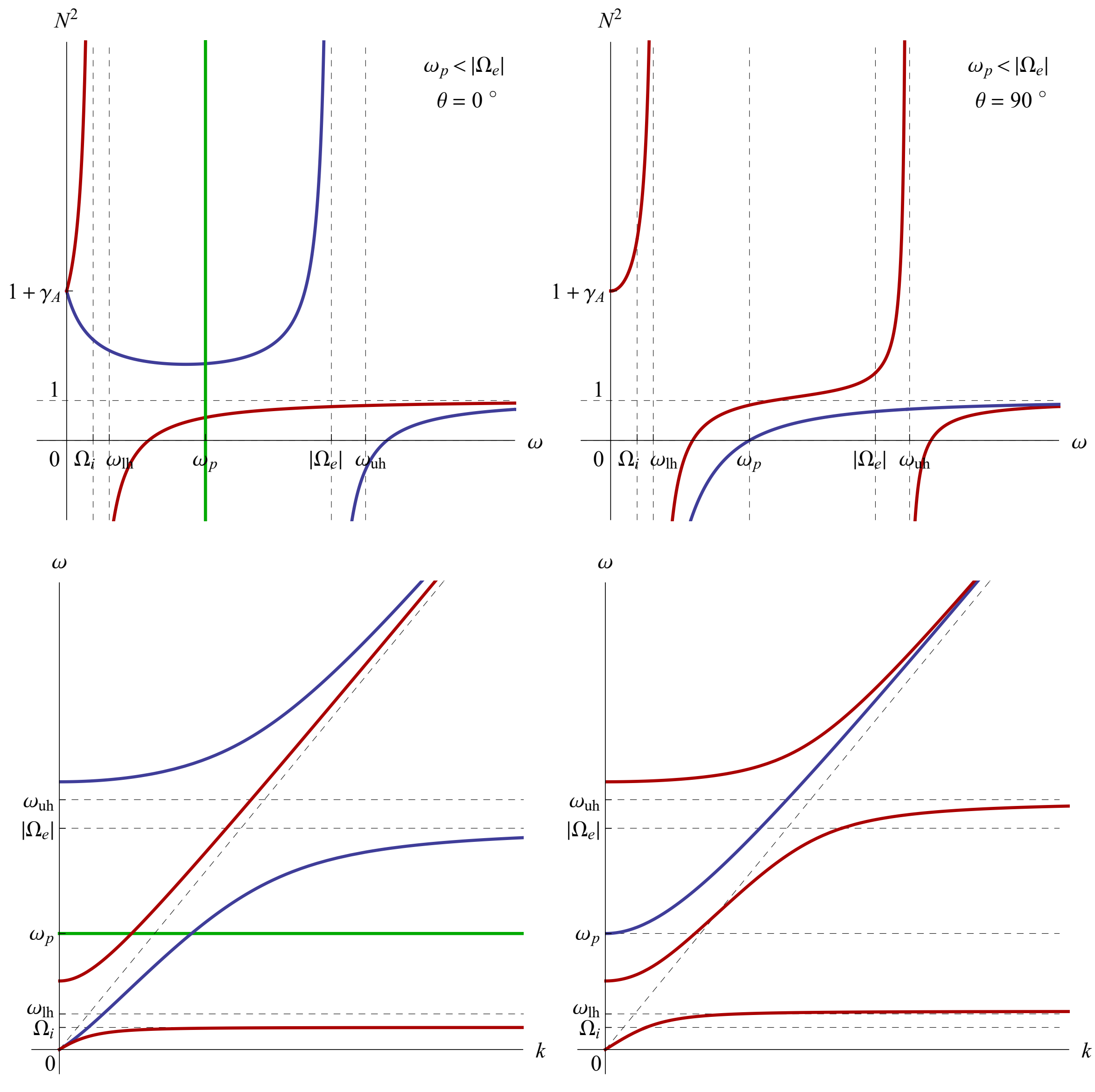


Figure 6.2: The dispersion curves of underdense plasma: $N^2(\omega)$ (upper row) and $\omega(k)$ (lower row). The left column corresponds to $\theta = 0°$: red – L mode, blue – R mode, green – Langmuir oscillations. The right column corresponds to $\theta = 90°$: red – X mode, blue – O mode. The dashed diagonals correspond to $\omega = ck$.

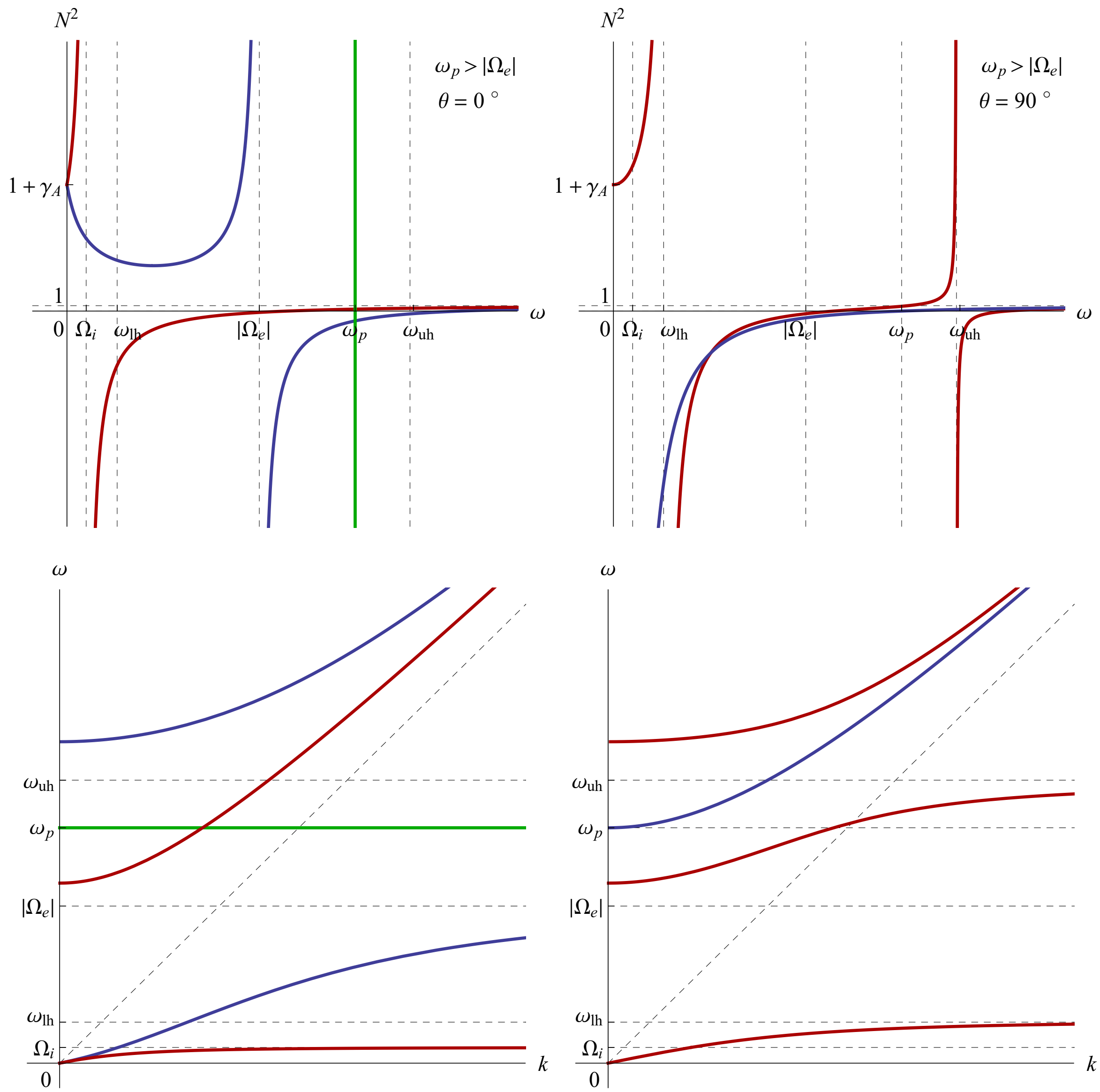


Figure 6.3: The dispersion curves of overdense plasma: $N^2(\omega)$ (upper row) and $\omega(k)$ (lower row). The left column corresponds to $\theta = 0°$: red – L mode, blue – R mode, green – Langmuir oscillations. The right column corresponds to $\theta = 90°$: red – X mode, blue – O mode. The dashed diagonals correspond to $\omega = ck$.

It has cutoffs at $R = 0$ and $L = 0$. The resonances are zeros of S, i.e., hybrid resonances. The polarization is found from

$$\begin{pmatrix} S & -iD & 0 \\ iD & (S^2 - RL)/S & 0 \\ 0 & 0 & P - (S^2 - RL)/S \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0. \tag{6.55}$$

This shows that $h_z = 0$ and $h_x/h_y = iD/S$, so the X-wave polarization is elliptic in the (x, y) plane. At the electron cyclotron resonance in particular, one has $S \approx R/2 \approx D$, so $h_x/h_y \approx i$, which corresponds to a circular polarization. Similarly, at ion cyclotron resonances, one has a circular polarization in the opposite direction, $h_x/h_y = -i$.

6.4.5 Propagation at a general angle

At general θ, the structure of the dispersion curves is qualitatively similar except the branches usually do not cross but rather "repel" each other (Fig. 6.4), forming frequency gaps (Exercise 6.4). This is a generic feature of dispersion curves (Sec. 6.4.6), which fact helps plotting the dispersion curves qualitatively without actually calculating them. One needs to determine only the limits and the asymptotes, then one can connect the corresponding curves by continuity such that they do not intersect.

Exercise 6.4: Qualitatively plot the dispersion curves for plasma that has two types of ions.

6.4.6 *Level repulsion

In this (optional) section, we explain the cause of level repulsion as a generic effect. Consider a general wave system governed by $\boldsymbol{D}\boldsymbol{\Psi} = 0$ with Hermitian $\boldsymbol{D}$. Suppose there are two waves that are in exact resonance (have their dispersion curves crossed) at some $\omega = \omega_0$, $k = k_0$, and $\theta = \theta_0$. Let us perturb k by some small $\delta k \doteq k - k_0$ and θ by some small $\delta\theta \doteq \theta - \theta_0$ and consider the resulting frequency shifts of the eigenmodes, $\delta\omega \doteq \omega - \omega_0$. These shifts are determined by the field equation that for homogeneous waves can be written as follows:

$$(\bar{\boldsymbol{D}} + \delta\boldsymbol{D})\boldsymbol{\Psi} = 0. \tag{6.56}$$

Here, $\bar{\boldsymbol{D}} \doteq \boldsymbol{D}(\omega, k, \theta_0)$ and $\delta\boldsymbol{D} \doteq \boldsymbol{D}(\omega, k, \theta) - \bar{\boldsymbol{D}}$ is a small matrix induced by $\delta\theta$. Let us search for the field in the form $\boldsymbol{\Psi} = \boldsymbol{h}_1\Psi_1 + \boldsymbol{h}_2\Psi_2 + \mathcal{O}(\delta\theta)$, where $\boldsymbol{h}_{1,2}$ are the eigenvectors of $\bar{\boldsymbol{D}}$ that correspond to the two resonant modes of interest. This can also be written symbolically as $\boldsymbol{\Psi} = \boldsymbol{\Xi}\bar{\boldsymbol{\Psi}} + \mathcal{O}(\delta\theta)$, where $\boldsymbol{\Xi}$ is a (non-square) matrix that has $\boldsymbol{h}_{1,2}$ as its columns and $\bar{\boldsymbol{\Psi}}$ is a column vector with elements $\Psi_{1,2}$ (cf. Problem PII.2). Upon multiplying Eq. (6.56) by $\boldsymbol{\Xi}^\dagger$, one obtains

$$\left[\boldsymbol{\Lambda} + \boldsymbol{\Xi}^\dagger(\delta\boldsymbol{D})\,\boldsymbol{\Xi}\right]\bar{\boldsymbol{\Psi}} = 0, \tag{6.57}$$

where $\boldsymbol{\Lambda} = \operatorname{diag}\{\Lambda_1, \Lambda_2\}$ is a diagonal 2×2 matrix. By Taylor-expanding this matrix in $\delta\omega$ and in δk, one obtains $\boldsymbol{\Lambda} \approx \boldsymbol{V}_\omega\delta\omega + \boldsymbol{V}_k\delta k$, where $\boldsymbol{V}_\omega \doteq \partial_\omega\boldsymbol{\Lambda}(\omega_0, k_0, \theta_0)$ and $\boldsymbol{V}_k \doteq \partial_k\boldsymbol{\Lambda}(\omega_0, k_0, \theta_0)$ are also diagonal matrices. Then, assuming the notation $\boldsymbol{\zeta} \doteq \boldsymbol{V}_\omega^{1/2}\bar{\boldsymbol{\Psi}}$, one can rewrite Eq. (6.57) as follows:

$$(\bar{\boldsymbol{\mathcal{H}}} - \mathbf{1}\,\delta\omega)\boldsymbol{\zeta} = 0, \tag{6.58}$$

where $\bar{\boldsymbol{\mathcal{H}}}$ is the following matrix:

$$\bar{\boldsymbol{\mathcal{H}}} \doteq \underbrace{\begin{pmatrix} v_1\delta k & 0 \\ 0 & v_2\delta k \end{pmatrix}}_{-\boldsymbol{V}_\omega^{-1/2}\boldsymbol{V}_k\boldsymbol{V}_\omega^{-1/2}\delta k} + \underbrace{\begin{pmatrix} \Delta_1 & \beta_1 \\ \beta_2 & \Delta_2 \end{pmatrix}}_{-\boldsymbol{V}_\omega^{-1/2}\boldsymbol{\Xi}^\dagger(\delta\boldsymbol{D})\,\boldsymbol{\Xi}\boldsymbol{V}_\omega^{-1/2}} \tag{6.59}$$

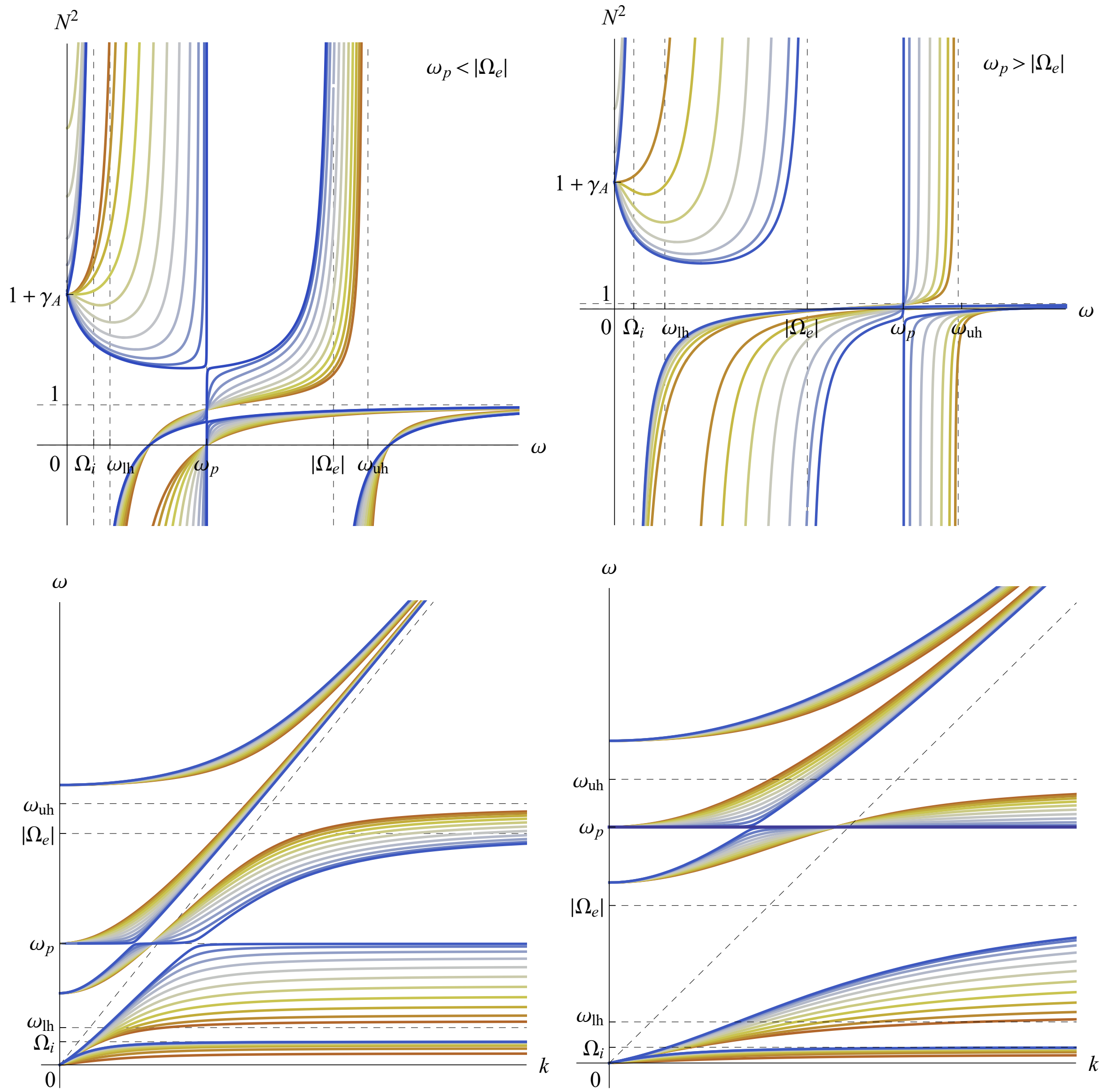


Figure 6.4: The dispersion curves for various θ (colder colors correspond to smaller θ): $N^2(\omega)$ (upper row) and $\omega(k)$ (lower row). The left column corresponds to underdense plasma, the right column corresponds to overdense plasma. The dashed diagonals correspond to $\omega = ck$.

and the second term is small so it can be evaluated at $\omega = \omega_0$. Because $\boldsymbol{V}_\omega$ and $\boldsymbol{V}_k$ are diagonal and real, and because $\boldsymbol{\Xi}^\dagger(\delta \boldsymbol{D})\,\boldsymbol{\Xi}$ is Hermitian, one can readily show that $v_{1,2}$ and $\Delta_{1,2}$ are real and

$$\beta_1\beta_2 = |\beta|^2\,\mathrm{sign}\,(V_{\omega,1}V_{\omega,2}), \tag{6.60}$$

where β is a complex number. The corresponding frequency shifts $\delta\omega$ are found from $\det(\bar{\boldsymbol{\mathcal{H}}} - \mathbf{1}\,\delta\omega) = 0$. The latter is a quadratic equation for $\delta\omega$, so it has two roots that satisfy $\delta\omega_1 = \delta\omega_2^*$:

$$\delta\omega_{1,2} = \frac{1}{2}\,[\Delta_1 + \Delta_2 + (v_1 + v_2)\delta k] \pm \frac{1}{2}\,\sqrt{\beta_1\beta_2 + [\Delta_1 - \Delta_2 + (v_1 - v_2)\delta k]^2}. \tag{6.61}$$

If δk is sufficiently large, such that $\beta_1\beta_2$ and $\Delta_{1,2}$ are negligible, then Eq. (6.61) leads to

$$\delta\omega_{1,2} \approx \frac{1}{2}\,\big\{(v_1 + v_2)\delta k \pm |(v_1 - v_2)\delta k|\big\} \to \left\{ \begin{array}{l} v_1\delta k, \\ v_2\delta k, \end{array} \right. \tag{6.62}$$

which is just the linear approximation to the unperturbed dispersion curves that correspond to $\delta\boldsymbol{D} \approx 0$. These *asymptotics* cross at $\delta k = 0$; but when δk is small, the terms $\beta_1\beta_2$ and $\Delta_{1,2}$ are not negligible. To understand what happens then, consider the following. If the signs of $V_{\omega,1}$ and $V_{\omega,2}$ are opposite, then $\beta_1\beta_2 = -|\beta|^2 < 0$, so the square root in Eq. (6.61) becomes imaginary at δk close enough to $-(\Delta_1 - \Delta_2)/(v_1 - v_2)$. This signifies an instability. If the system has no free energy, though (as is the case in cold plasma), it cannot support instabilities in principle. This guarantees that the signs of $V_{\omega,1}$ and $V_{\omega,2}$ in such a system are the same, so $\beta_1\beta_2 = |\beta|^2 > 0$. Then, $\delta\omega_{1,2}$ are real and

$$\delta\omega_1 - \delta\omega_2 \geq |\beta| \tag{6.63}$$

at all δk. This shows that unless $\beta = 0$ (a degenerate case), $\delta\theta$ induces a nonvanishing "frequency gap" between the curves $\delta\omega_1(\delta k)$ and $\delta\omega_2(\delta k)$. This constitutes "level repulsion".

Lecture 7

Waves in warm fluid plasma

The cold-plasma model that was considered in the previous lecture misses thermal effects that can be important. Here, we explore some of these effects semi-qualitatively within a basic fluid model. A more accurate (kinetic) description will be presented in Part IV.

7.1 Introduction

Like in Lecture 2, let us start with the momentum equation

$$\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} + (\tilde{\boldsymbol{v}}_s \cdot \nabla)\, \tilde{\boldsymbol{v}}_s = \frac{e_s}{m_s} \left[\tilde{\boldsymbol{E}} + \frac{1}{c}\, \tilde{\boldsymbol{v}}_s \times (\boldsymbol{B}_0 + \tilde{\boldsymbol{B}}) \right] - \frac{\nabla P_s}{m_s n_s} + \boldsymbol{C}_s. \tag{7.1}$$

Because $\nabla P_s / n_s$ is now retained, this equation must be complemented with equations for the density n_s and the pressure P_s. The density can be obtained from the continuity equation,

$$\partial_t n_s + \nabla \cdot (n_s \boldsymbol{v}_s) = 0, \tag{7.2}$$

but handling the pressure is more complicated. A rigorous way to calculate P_s is to use the kinetic approach (Part IV); but then the theory becomes more complicated and less transparent. Rigorous fluid theories can be constructed asymptotically for some regimes (e.g., when plasma is strongly collisional or not-so-warm), but they have limited applicability. Here, we adopt an alternative approach, which is less rigorous but adequate qualitatively and has wider applicability.

First, consider a plasma that is in a *global* equilibrium. Then, as commonly done in thermodynamics, one can assume the following simple model:

$$P_s = P_{0s} \left(\frac{n_s}{n_{0s}} \right)^{\gamma_s}. \tag{7.3}$$

Here, the constants P_{0s} and n_{0s} are the pressure and density of some fixed reference state and the constant γ_s is called a polytropic index. This index depends on processes of interest and can be derived from statistical physics. For example, isotropic processes correspond to $\gamma_s = 1$, as seen from the equation of state for the (ideal) gas of species s, $P_s = n_s T_s$, where T_s is the sth-species temperature. Adiabatic dynamics corresponds to $\gamma_s = (D_s + 2)/D_s$ [34], where D_s is the number of relevant degrees of freedom. Which degrees of freedom are relevant can be guessed based on qualitative arguments or by comparing with kinetic theory.

A natural generalization of Eq. (7.3) is the model in which plasma may evolve such that each fluid element still remains in its own *local* equilibrium and thus conserves its own value of $P_s / n_s^{\gamma_s}$. This corresponds to conservation of $P_s / n_s^{\gamma_s}$ in the frame moving with the fluid velocity $\boldsymbol{v}_s$; i.e.,

$$0 = \frac{d}{dt} \left(\frac{P_s}{n_s^{\gamma_s}} \right) \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{v}_s \cdot \nabla \right) \left(\frac{P_s}{n_s^{\gamma_s}} \right). \tag{7.4}$$

Below, we use Eq. (7.4) along with Eqs. (7.1) and (7.2) to study linear plasma waves.[1]

7.2 Nonmagnetized plasma

7.2.1 Basic equations

Let us consider nonmagnetized plasma first. We will assume the same model as in Lecture 2, except for the following: (i) we will adopt the collisionless limit ($\nu_s \to 0+$) from the start; (ii) we will assume that the plasma is homogeneous both in time *and* in space;[2] and (iii) we will allow for nonzero P_s. From Eq. (7.4), the latter satisfies the following linearized equation:

$$0 \approx \frac{\partial}{\partial t}\left(\frac{P_s}{n_s^{\gamma_s}}\right) \approx \frac{1}{n_{s0}^{\gamma_s}}\frac{\partial \tilde{P}_s}{\partial t} - \frac{\gamma_s P_{s0}}{n_{s0}^{\gamma_s+1}}\frac{\partial \tilde{n}_s}{\partial t} = \frac{1}{n_{s0}^{\gamma_s}}\left(\frac{\partial \tilde{P}_s}{\partial t} - \gamma_s T_{s0}\frac{\partial \tilde{n}_s}{\partial t}\right). \tag{7.5}$$

Then, the complete set of linearized material equations is as follows:

$$\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} = \frac{e_s}{m_s}\tilde{\boldsymbol{E}} - \frac{\nabla \tilde{P}_s}{m_s n_{s0}}, \tag{7.6a}$$

$$\frac{\partial \tilde{P}_s}{\partial t} = \gamma_s m_s v_{Ts}^2 \frac{\partial \tilde{n}_s}{\partial t}, \tag{7.6b}$$

$$\frac{\partial \tilde{n}_s}{\partial t} = -n_{0s}\nabla\cdot\tilde{\boldsymbol{v}}_s, \tag{7.6c}$$

where $v_{Ts} \doteq T_{0s}/m_s$ is the unperturbed thermal speed of species s.

By differentiating Eq. (7.6a) with respect to t, one obtains

$$\frac{\partial^2 \tilde{\boldsymbol{v}}_s}{\partial t^2} = \frac{e_s}{m_s}\frac{\partial \tilde{\boldsymbol{E}}}{\partial t} - \frac{\nabla \partial_t \tilde{P}_s}{m_s n_{s0}} = \frac{e_s}{m_s}\frac{\partial \tilde{\boldsymbol{E}}}{\partial t} - \frac{\gamma_s v_{Ts}^2}{n_{s0}}\frac{\partial \nabla \tilde{n}_s}{\partial t}. \tag{7.7}$$

From the continuity equation, we obtain

$$\frac{\partial \nabla \tilde{n}_s}{\partial t} = -n_{0s}\nabla\left(\nabla\cdot\tilde{\boldsymbol{v}}_s\right). \tag{7.8}$$

Then, finally,

$$\frac{\partial^2 \tilde{\boldsymbol{v}}_s}{\partial t^2} - \gamma_s v_{Ts}^2 \nabla\left(\nabla\cdot\tilde{\boldsymbol{v}}_s\right) = \frac{e_s}{m_s}\frac{\partial \tilde{\boldsymbol{E}}}{\partial t}. \tag{7.9}$$

Correspondingly, the linear current $\tilde{\boldsymbol{j}}_s = e_s n_{0s}\tilde{\boldsymbol{v}}_s$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - \gamma_s v_{Ts}^2 \nabla\nabla\right)\tilde{\boldsymbol{j}}_s = \frac{\omega_{ps}^2}{4\pi}\frac{\partial \tilde{\boldsymbol{E}}}{\partial t}. \tag{7.10}$$

7.2.2 Dielectric tensor

It is seen from Eq. (7.10) that the spectral representations of the induced current and the electric field are related by

$$\left(-\omega^2 \mathbf{1} + \gamma_s v_{Ts}^2 \boldsymbol{k}\boldsymbol{k}\right)\tilde{\boldsymbol{j}}_s^{(\mathrm{i})} = -i\omega\,\frac{\omega_p^2}{4\pi}\,\tilde{\boldsymbol{E}}. \tag{7.11}$$

[1]Keep in mind that the results presented below have limited applicability. As to be discussed in Part IV, some of the waves in warm plasma are in fact heavily damped due to kinetic effects, particularly at large k.

[2]Considering the general case requires cumbersome calculations, because warm plasma exhibits spatial dispersion.

Let us consider projections of this equations on the axes perpendicular and parallel to $\boldsymbol{k}$ by applying the corresponding projection matrices:

$$\boldsymbol{\Pi}_\perp \doteq \mathbf{1} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}, \qquad \boldsymbol{\Pi}_\parallel \doteq \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}. \tag{7.12}$$

Since $\boldsymbol{\Pi}_\perp \boldsymbol{\Pi}_\parallel = 0$, one obtains that $\tilde{\boldsymbol{j}}^{(\mathrm{i})}_{s\perp} \doteq \boldsymbol{\Pi}_\perp \tilde{\boldsymbol{j}}^{(\mathrm{i})}_s$ and $\tilde{\boldsymbol{j}}^{(\mathrm{i})}_{s\parallel} \doteq \boldsymbol{\Pi}_\parallel \tilde{\boldsymbol{j}}^{(\mathrm{i})}_s$ satisfy

$$\tilde{\boldsymbol{j}}^{(\mathrm{i})}_{s\perp} = \frac{i\omega}{4\pi}\frac{\omega_p^2}{\omega^2}\,\tilde{\boldsymbol{E}}_\perp, \qquad \tilde{\boldsymbol{j}}^{(\mathrm{i})}_{s\parallel} = \frac{i\omega}{4\pi}\frac{\omega_p^2}{\omega^2 - \gamma_s k^2 v_{Ts}^2}\,\tilde{\boldsymbol{E}}_\parallel. \tag{7.13}$$

Assuming that $\boldsymbol{k}$ is directed along the x axis, the corresponding susceptibility tensor is as follows:

$$\boldsymbol{\chi}_s(\omega, \boldsymbol{k}) = \begin{pmatrix} -\frac{\omega_p^2}{\omega^2 - \gamma_s k^2 v_{Ts}^2} & 0 & 0 \\ 0 & -\frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & -\frac{\omega_p^2}{\omega^2} \end{pmatrix} \tag{7.14}$$

and the dielectric tensor is

$$\boldsymbol{\epsilon}(\omega, \boldsymbol{k}) = \mathbf{1} + \sum_s \boldsymbol{\chi}_s(\omega, \boldsymbol{k}) = \begin{pmatrix} \epsilon_\parallel(\omega, k) & 0 & 0 \\ 0 & \epsilon_\perp(\omega) & 0 \\ 0 & 0 & \epsilon_\perp(\omega) \end{pmatrix}. \tag{7.15}$$

The corresponding dispersion function is

$$\boldsymbol{D}_E(\omega, \boldsymbol{k}) = \begin{pmatrix} \epsilon_\parallel(\omega, k) & 0 & 0 \\ 0 & \epsilon_\perp(\omega) - N^2 & 0 \\ 0 & 0 & \epsilon_\perp(\omega) - N^2 \end{pmatrix}, \tag{7.16}$$

and the dispersion relation is

$$\epsilon_\parallel(\omega, k)\left[\epsilon_\perp(\omega) - N^2\right]^2 = 0. \tag{7.17}$$

Just like in the case of cold nonmagnetized plasma (Lecture 2), there are two types of nonzero-frequency waves in this case. One type corresponds to (Figs. 7.1 and 7.2)

$$N^2 = \epsilon_\perp(\omega) = 0, \quad \text{i.e.,} \quad \omega^2 = \omega_p^2 + k^2 c^2. \tag{7.18}$$

These are the same transverse waves as in cold plasma (Sec. 2.2.4). Due to their transverse polarization, the corresponding oscillation quiver velocities satisfy $\nabla \cdot \tilde{\boldsymbol{v}}_s = 0$; then, $\partial_t \tilde{n}_s = 0$, and thus $\partial_t \tilde{P}_s = 0$, which is why these waves are independent of the plasma temperature.

The other type of waves predicted by Eq. (7.17) are longitudinal waves. Their dispersion relation is

$$0 = \epsilon_\parallel(\omega, \boldsymbol{k}) = 1 + \sum_s \left[\chi_\parallel(\omega, \boldsymbol{k})\right]_s, \tag{7.19}$$

$$\left[\chi_\parallel(\omega, \boldsymbol{k})\right]_s \doteq -\frac{\omega_{ps}^2}{\omega^2 - \gamma_s k^2 v_{Ts}^2}. \tag{7.20}$$

In case of low-frequency oscillations, we expect oscillations to be isothermal; then $\gamma_s = 1$, so

$$\chi_{\parallel,s}(\omega, \boldsymbol{k}) \approx \frac{\omega_{ps}^2}{k^2 v_{Ts}^2} \equiv \frac{1}{k^2 \lambda_{Ds}^2}, \tag{7.21}$$

where $\lambda_{Ds} \doteq v_{Ts}/\omega_{ps}$ is the Debye length. In contrast, high-frequency oscillations are expected to be adiabatic and effectively one-dimensional ($D_s = 1$). Hence, $\gamma_s = (D_s + 2)/D_s = 3$, so we expect

$$\chi_{\parallel,s}(\omega, \boldsymbol{k}) = -\frac{\omega_{ps}^2}{\omega^2 - 3k^2 v_{Ts}^2}. \tag{7.22}$$

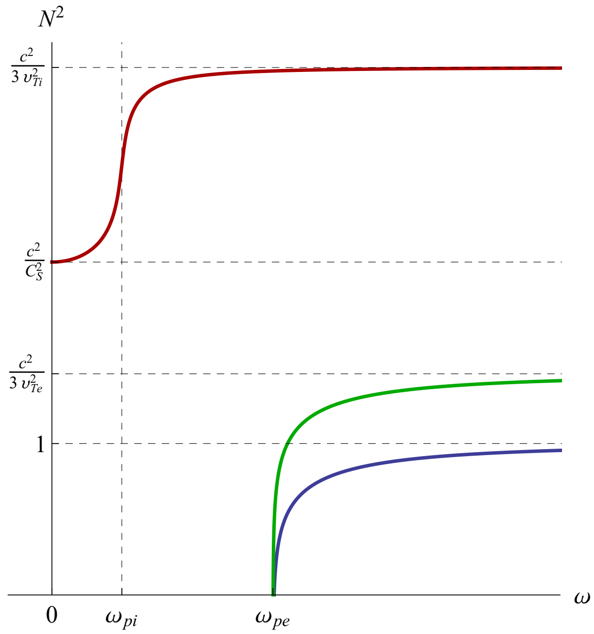


Figure 7.1: The dispersion curves $N^2(k)$ for the three types of waves that can propagate in warm non-magnetized plasma within the fluid approximation: blue – transverse electromagnetic waves, green – electron plasma waves, red – the branch known as the ion acoustic wave (IAW) at $\omega \ll \omega_{pi}$ and as the ion plasma wave (IPW) at $\omega \gtrsim \omega_{pi}$.

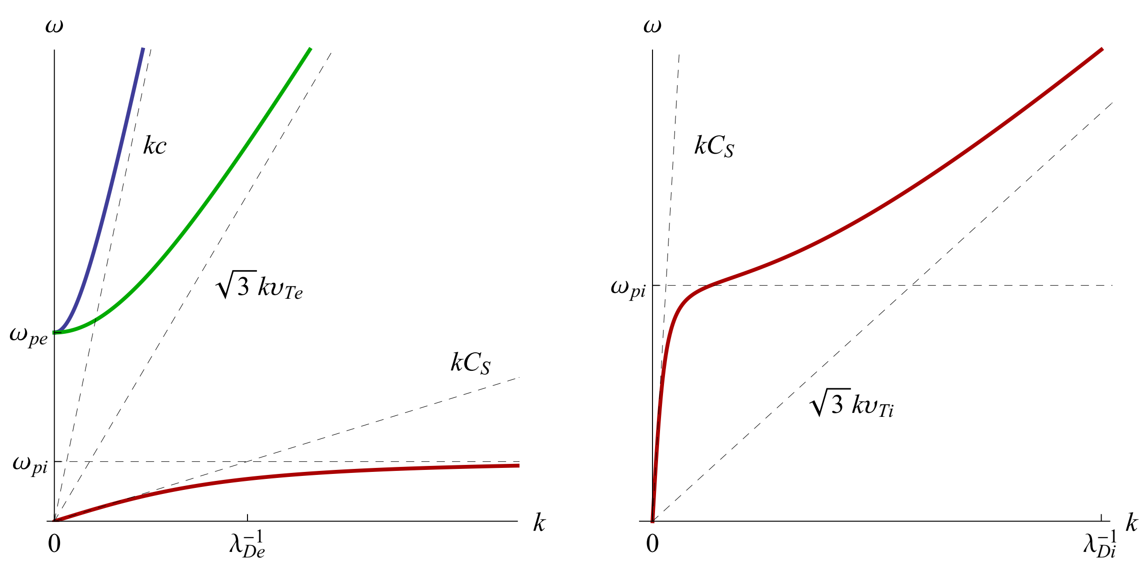


Figure 7.2: Left – the same as Fig. 7.1 but in coordinates $\omega(k)$. Right – a close-up of the left figure focusing on the IAW/IPW branch and extending to larger k.

7.2.3 High-frequency oscillations: Langmuir waves

First, let us consider longitudinal waves at high (electron) frequencies, such that the ion contribution is negligible. Then, Eq. (7.19) becomes

$$0 = 1 - \frac{\omega_{pe}^2}{\omega^2 - 3k^2 v_{Te}^2}, \tag{7.23}$$

or equivalently (Figs. 7.1 and 7.2),

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_{Te}^2. \tag{7.24}$$

This is the dispersion relation of electron Langmuir waves, or electron plasma waves (EPW), in warm plasma (cf. Sec. 2.2.3). Keep in mind that, by adopting $\gamma_e = 3$, we have restricted our model to the high-frequency limit, $\omega \gg k v_{Te}$. Then, we can as well use

$$\omega = \sqrt{\omega_{pe}^2 + 3k^2 v_{Te}^2} \approx \omega_{pe} + \frac{3k^2 v_{Te}^2}{2\omega_{pe}} \tag{7.25}$$

(for $\omega > 0$). Although small, the thermal corrections are important in that they make the group velocity nonzero:

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{3\boldsymbol{k} v_{Te}^2}{\omega}. \tag{7.26}$$

Notice also that the phase and group velocities are connected by the following relation:

$$\boldsymbol{v}_g \cdot \boldsymbol{v}_p = 3 v_{Te}^2. \tag{7.27}$$

7.2.4 Low-frequency oscillations: Debye shielding and ion sound

If *all* species are hot, then the dispersion relation becomes

$$0 = 1 + \sum_s \frac{1}{k^2 \lambda_{Ds}^2} \equiv 1 + \frac{1}{k^2 \lambda_D^2}. \tag{7.28}$$

The solution to this, $k = \pm i \lambda_D^{-1}$ (where the sign must be chosen such that it agrees with boundary conditions), describes Debye shielding,

$$|\tilde{\boldsymbol{E}}| \propto |e^{ikz}| = e^{-k_i z} = e^{\pm z/\lambda_D}. \tag{7.29}$$

Another notable regime is when electrons are hot while ions are coldish. For simplicity, let us consider a plasma with only one type of ions. Then, the corresponding dispersion relation is

$$0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - 3k^2 v_{Ti}^2} + \frac{1}{k^2 \lambda_{De}^2}. \tag{7.30}$$

Its solution is readily found to be (Figs. 7.1 and 7.2)

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{1}{k^2 \lambda_{De}^2}} + 3k^2 v_{Ti}^2 = \omega_{pi}^2 \frac{k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} + 3k^2 v_{Ti}^2. \tag{7.31}$$

In particular, at $k\lambda_{De} \ll 1$, Eq. (7.31) becomes

$$\omega^2 \approx C_S^2 k^2, \tag{7.32}$$

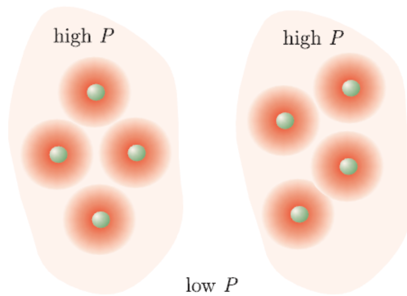


Figure 7.3: Schematic of the electron-pressure oscillations that accompany oscillations of the cold-ion density. Green are the ions; red are their Debye spheres, which contain electron pressure.

where C_S, called the ion sound speed, is given by

$$C_S^2 \doteq \omega_{pi}^2 \lambda_{De}^2 + 3v_{Ti}^2 = \frac{Z_i T_{0e} + 3T_{0i}}{m_i}. \tag{7.33}$$

(Here, $Z_i \doteq |e_i/e_e|$ is the ion charge state, and we have used $n_{i0}e_i + n_{e0}e_e = 0$.) These waves are known as ion acoustic waves (IAW), or ion sound waves. Notably, they exist even in the limit of zero T_{i0}. This is understood by the fact that each ion carries a Debye cloud of electrons. Even when there is no pressure associated with ions *per se*, there is an electron pressure associated with each Debye cloud. Oscillations of the ion density cause oscillations of this electron pressure. The latter creates a restoring force density $-\nabla P_e$, which leads to sound-like oscillations (Fig. 7.3).

At $k \gg \lambda_{De}^{-1}$, Eq. (7.31) leads to

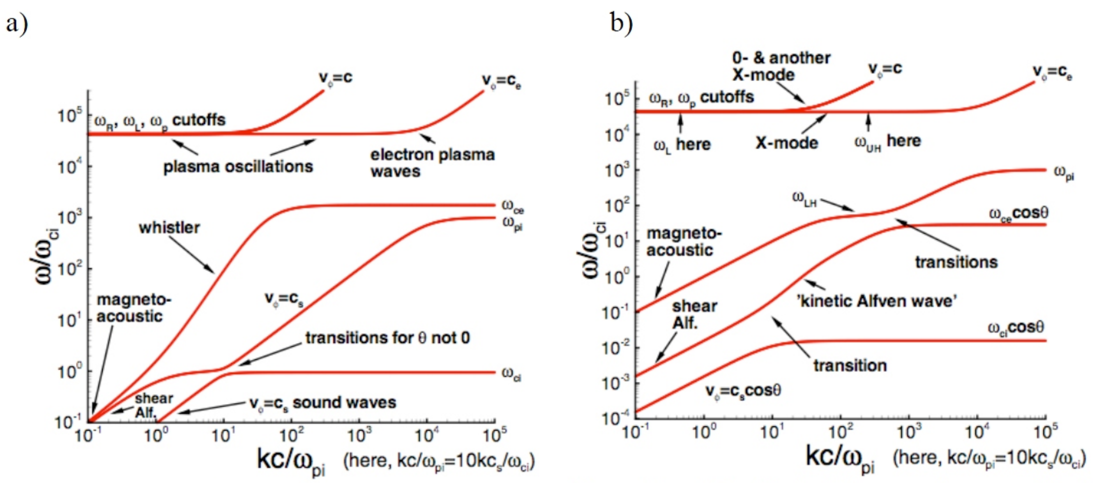
$$\omega^2 \approx \omega_{pi}^2 + 3k^2 v_{Ti}^2. \tag{7.34}$$

These are called ion plasma waves (IPW). The electron properties do not enter Eq. (7.34) explicitly because sufficiently hot electrons do not contribute to $\epsilon_\parallel$ and are important for IPW only as a homogeneous neutralizing background. Also notably, the regime $\lambda_{De}^{-1} \ll k \ll \lambda_{Di}^{-1}$ corresponds to an approximately constant frequency, $\omega_p^2 \approx \omega_{pi}^2$.

7.3 Magnetized plasma

To understand how the dispersion curves are affected by nonzero temperature in magnetized plasma, let us start with the limit when the angle θ between the wave vector and the dc magnetic field is small. Electron waves are not affected significantly in this limit, but at low frequencies, a new, ion-sound branch appears. Because the magnetic Lorentz force is small at small θ, this branch will be similar to that in Figs. 7.1 and 7.2 except where it crosses other branches, i.e., resonates with other waves. In those regions, frequency gaps form much like it was discussed in Sec. 6.4.5. The corresponding mode structure is shown in Fig. 7.4(a). As θ increases, the figure transforms continually into the one shown in Fig. 7.4(b). For details see Refs. [3, 33] (Exercise 7.1).

Exercise 7.1: Explore the figures $\omega(k)$ in Ref. [33] and explain, qualitatively, how they relate to the corresponding figures for cold plasma from Lecture 6. Plot the corresponding $N(\omega)$.



Dispersion relation for the full warm plasma model with the parameters of Swanson's Fig. 3.1. V_A/c_s=10, c/V_A=1000, m_i/m_e=1836, and ω_{pe}/ω_{ce}=23.3, except θ=π/10 for nearly parallel propagation. Electron fluid pressure effects are used without further approximation. Here, p_{i0}=0 and γ_e=1.

Dispersion relation for the full warm plasma model with the parameters of Swanson's Fig. 3.1. V_A/c_s=10, c/V_A=1000, m_i/m_e=1836, and ω_{pe}/ω_{ce}=23.3, except θ=0.495π for nearly perpendicular propagation. Electron fluid pressure effects are used without further approximation. Here, p_{i0}=0 and γ_e=1.

Figure 7.4: From Ref. [35]. For $\omega(k)$ at even larger k, see Refs. [3, 33].

Problems for Part III

PIII.1 Methods of cold-plasma diagnostics

Here, you are asked to explore basic methods of CCS-plasma diagnostics that use electromagnetic waves. Assume that: (i) the wave frequencies ω are large enough such that the ion response is negligible; (ii) the wavevectors $\boldsymbol{k}$ are parallel to the electron-density gradient, so the wave propagation can be considered one-dimensional; and (iii) the inhomogeneity scales are much larger than the wavelength λ, so the GO approximation can be used. Correspondingly, the phase shift that a wave gains by propagating from location z_1 to location z_2 can be approximated by[3]

$$\theta = \int_{z_1}^{z_2} dz\, k(z). \tag{7.35}$$

(a) *Interferometry.* — Assuming nonmagnetized plasma, calculate θ as a functional of the electron density, $n_e(z)$. Show that, to the lowest order in n_e, the phase shift acquired by the wave due to the plasma is proportional to the space-averaged density. Can this method be applied "as is" in magnetized plasma? If yes, what wave should one use? Estimate θ acquired by such a wave with $\lambda \sim 1\,\mu$m in a typical tokamak.

(b) *Reflectometry.* — For the same type of waves, calculate the phase shift that a wave gains if it starts at $z = 0$, reflects from a cutoff region, and returns back to $z = 0$. (Obviously, the small-density approximation cannot be used here.) Express this shift $\theta(\omega)$ as a functional of $z(\omega_p)$ (this may require integration by parts), where $z(\omega_p)$ is a function inverse to $\omega_p(z)$. Express $z(\omega_p)$ as a *functional* of $\theta'(\omega)$ using the Abel transform, defined as follows [36]:

$$g \mapsto f: \quad f(x) = \int_0^x dt\, \frac{g(t)}{(x-t)^\alpha}, \tag{7.36a}$$

$$f \mapsto g: \quad g(t) = \frac{\sin(\pi\alpha)}{\pi}\left[\int_0^t dx\, \frac{f'(x)}{(t-x)^{1-\alpha}} + \frac{f(0)}{t^{1-\alpha}}\right]. \tag{7.36b}$$

Show that $\theta'(\omega)$ is the propagation time. Then, explain how the above result can be used to infer $n(z)$ from experimental measurements. What do you think are the limitations of this diagnostic?

(c) *Faraday rotation.* — Show that, to the lowest nonvanishing order in ω^{-1}, a linearly-polarized wave propagating in plasma along a dc magnetic field experiences rotation of its polarization angle ϑ at the rate

$$\frac{d\vartheta}{dz} = \frac{\omega_p^2 |\Omega_e|}{2\omega^2 c}. \tag{7.37}$$

This is called the Faraday effect, which is commonly used to measure the magnetic-field strength by measuring the rotation angle $\Delta\vartheta$ experimentally.

[3] Reflection points introduce additional order-one phase shifts [cf. Eq. (2.38)], but those can be ignored here.

PIII.2 Wave transformations in the ionosphere

Here, you are asked to study wave propagation in the Earth's ionosphere.[4] Assume that the ionospheric plasma is collisionless, ions are immobile, and the Earth's curvature is negligible.

(a) Assume the cold-plasma approximation and neglect the Earth's magnetic field. At some altitude (~ 300 km), the electron density has the maximum value $n_{\max} = 10^6$ cm^{-3}. Suppose there is an antenna on the ground emitting radiation with frequency $f = 12$ MHz at some angle α with respect to the vertical. At what α will this radiation be reflected back to the Earth?

> **Hint:** Reflection occurs when the *vertical* group velocity becomes zero. How does the horizontal wave number and the frequency evolve along the rays?

Now, consider the influence of the Earth's magnetic field $\boldsymbol{B}_0$, assuming it is homogeneous and parallel to the ground. Suppose $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$, where x is the vertical axis and the z axis is along $\boldsymbol{B}_0$.

(b) In magnetized electron plasma, waves can experience reflection at three different locations corresponding to three different values of $X \doteq \omega_p^2/\omega^2$. Find these values $X_{1,2,3}$ from Eq. (6.30) as functions of $Y \doteq |\Omega_e|/\omega$ and of $N_\parallel$, assuming $N_\parallel \neq 0$. Assuming also that $0 < Y < 1$ and $N_\parallel^2 \neq 1$, show that there is exactly one value of $N_\parallel^2$, denoted $\bar{N}_\parallel^2(Y)$, at which two of the reflection points coincide. Calculate $\bar{N}_\parallel^2(Y)$ and the corresponding value of X.

(c) Show that $N_\perp$ satisfies $\mathsf{a}N_\perp^4 + \mathsf{b}N_\perp^2 + \mathsf{c} = 0$, where a, b, and c depend on X. (The dependence on Y and $N_\parallel$ is also assumed.) Find $\mathsf{a}(X)$ and outline how to find $\mathsf{b}(X)$ and $\mathsf{c}(X)$. Without solving this equation, sketch $N_\perp^2(X)$ at fixed $N_\parallel^2$ for $N_\parallel^2 \gtrless \bar{N}_\parallel^2$ and $N_\parallel^2 = \bar{N}_\parallel^2$. You may consider it known (or show it yourself) that

$$\mathsf{b}^2 - 4\mathsf{ac} = \left(\frac{XY}{1-Y^2}\right)^2 \left[Y^2(N_\parallel^2 - 1)^2 - 4(X-1)N_\parallel^2\right]. \tag{7.38}$$

> **Hint:** Consider $N_\parallel = 0$ first, for which case $N_\perp^2(X)$ should be easy to find. (What are the two modes in this regime?) Then, consider how the plot is modified for $N_\parallel \neq 0$ by analyzing $N_\perp^2(0)$, cutoffs, resonance(s), and the number of real roots for $N_\perp^2$.

(d) Now consider $N_\perp$ in regions where it is real. Sketch $N_\perp(X)$ corresponding to your sketches of $N_\perp^2(X)$ in part (c). (Remember to plot both $N_\perp > 0$ and $N_\perp < 0$.) Using these results, explain the dependence of the field pattern on the launch angle α in Fig. 7.5. (Ignore the specific numbers and focus on qualitative physics.)

PIII.3 MHD waves

In this problem, you are asked to derive the low-frequency modes of magnetized plasma from the single-fluid ideal MHD model with nonzero temperature. Adopt the following model equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0, \tag{7.39a}$$

$$\rho_m \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = \frac{(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}}{4\pi} - \frac{\nabla B^2}{8\pi} - \nabla P, \tag{7.39b}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{7.39c}$$

[4] Similar effects are important in magnetically confined fusion plasmas, as to be discussed in Part IV.

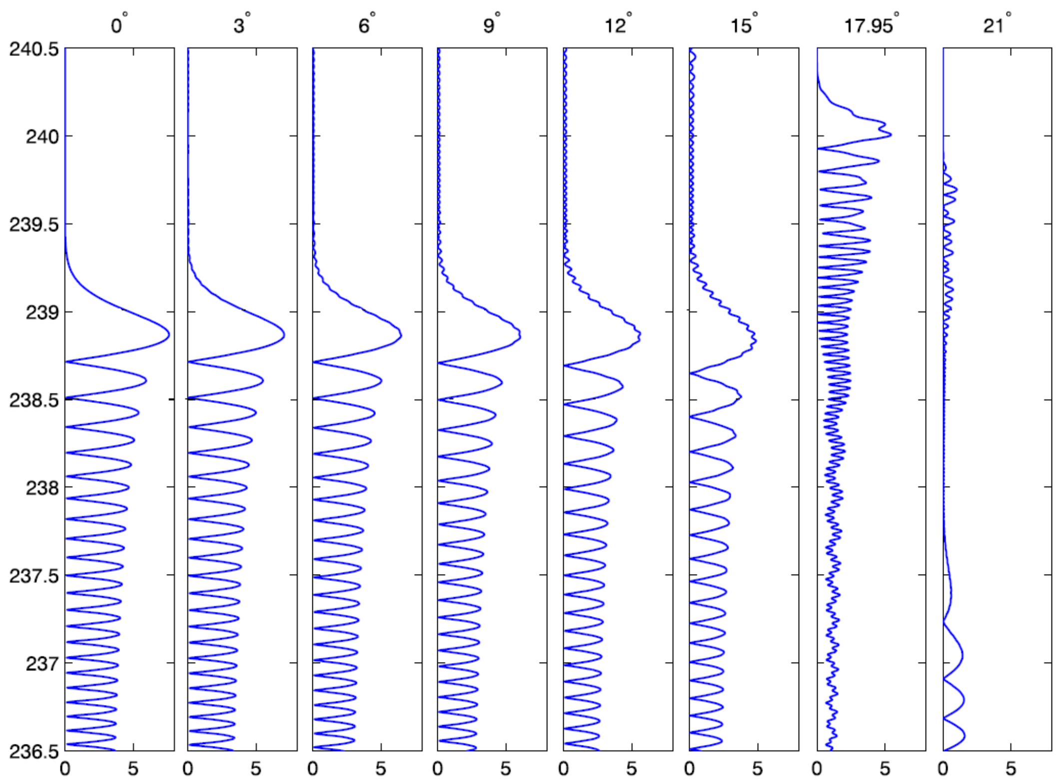


Figure 7.5: The absolute value of the wave electric field $|\tilde{\boldsymbol{E}}(x)|$ (horizontal axis) versus the altitude x (vertical axis, in km) for a standing wave launched from the ground ($x = 0$) at different angles α (numbers on top) between $\boldsymbol{k}$ and the vertical. The figure is adapted from Ref. [37].

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\left(\frac{P}{\rho_m^\gamma}\right) = 0, \tag{7.39d}$$

where ρ_m is the mass density, P is pressure, and γ is a constant polytropic index. As usual, assume that the background plasma is homogeneous and has no average velocity. Also assume homogeneous stationary magnetic field $\boldsymbol{B}_0$. For simplicity, you may also adopt the coordinate system such that $\boldsymbol{B}_0 = B_0\bar{\boldsymbol{e}}_z$ and $\boldsymbol{k} = k_\perp\bar{\boldsymbol{e}}_x + k_\parallel\bar{\boldsymbol{e}}_z$. (Here, $\bar{\boldsymbol{e}}_a$ is a unit vector along ath axis.)

(a) Linearize Eqs. (7.39) and derive a PDE for the fluid velocity; then use it to derive the dispersion relation by adopting $\partial_t = -i\omega$ and $\nabla = i\boldsymbol{k}$. Show that this dispersion relation can be represented in the form

$$(\omega^2 - k_\parallel^2 V_A^2)\big[\omega^4 - \omega^2 k^2 (V_A^2 + V_S^2) + V_S^2 V_A^2 k^2 k_\parallel^2\big] = 0, \tag{7.40}$$

where $V_S \doteq (\gamma P_0/\rho_{m0})^{1/2}$ and the index 0 denotes unperturbed quantities.

(b) Plot *in polar coordinates* the phase speed as a function of the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$ for all branches. Which of these waves can propagate along $\boldsymbol{B}_0$? Which of these waves can propagate perpendicularly to $\boldsymbol{B}_0$?

(c) Approximate ω^2 for $V_S \ll V_A$. What is the physical mechanism of each branch in this limit?

(d) Calculate the energy density of an Alfvén wave in a cold plasma with negligible $k_\parallel$ in the limit $\gamma_A \gg 1$. (The answer is the same for both types of Alfvén waves.)

PIII.4 Alfvén resonance

Consider a cold stationary electron–ion plasma with one type of ions and a background magnetic field $\boldsymbol{B}_0$ along the z axis. The plasma is homogeneous along the y and z axes, but the background density and B_0 depend on x. In this plasma, consider stationary waves with electric field $\tilde{\boldsymbol{E}} = \boldsymbol{\mathcal{E}}(x)e^{-i\omega t + ik_z z}$ with $k_z > 0$ and $0 < \omega/\Omega_i \ll 1$. Use that

$$\boldsymbol{\epsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}, \qquad S \approx \frac{\omega_{pi}^2}{\Omega_i^2} \gg 1, \qquad D \approx -\frac{\omega}{\Omega_i} S \ll S. \tag{7.41}$$

(a) Assume that P is large enough such that $\mathcal{E}_z$ is negligible. Starting from Maxwell's equations and assuming the notation $N_z \doteq ck_z/\omega$, show that $\mathcal{E}_y$ satisfies

$$\left(\frac{c^2}{\omega^2}\frac{d^2}{dx^2} + S - N_z^2 - \frac{D^2}{S - N_z^2}\right)\mathcal{E}_y = 0. \tag{7.42}$$

(b) Assume $S = (1 + x/L)N_z^2$, so that $x = 0$ corresponds to the so-called Alfvén resonance, where $S = N_z^2$. Assume $L > 0$. Argue that Eq. (7.42) can be written as

$$\left(x\frac{d^2}{dx^2} + \frac{x^2 - 1}{2\alpha^2}\right)\mathcal{E}_y = 0, \tag{7.43}$$

where x has been appropriately rescaled and α is a constant that you are asked to find. Plot the GO dispersion curves $k_x(x) \lessgtr 0$ corresponding to Eq. (7.43). Also find and plot the inverse function, which has the form $x(k_x) = \bar{x}(k_x) \pm \Delta(k_x)$.

(c) In the GO limit, calculate the x-component of the group velocity at $S - N_z^2 \gg D$ (for $x > 1$) and $S - N_z^2 \ll D$ (for $-1 < x < 0$) to determine the direction of the action flows along the dispersion curves. Identify the branches at $x \to +\infty$ and at $x \to 0-$. Describe what happens to a wave launched toward the Alfvén resonance.

(d) Consider $g_1(k_x) \doteq e^{i\Theta(k_x)} \int \mathcal{E}_y(x) e^{ik_x x}\, dx$, where $\Theta(k_x) \doteq \int \bar{x}(k_x)\, dk_x$ and $\bar{x}$ is the same as in problem (b). Take for granted (or prove it yourself) that g_1 and $g_2 \doteq -isg_1 - g_1'$ satisfy

$$i\,\frac{d}{dk_x}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} s & -i \\ i & -s \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \qquad s \doteq \alpha^2 k_x^2. \tag{7.44}$$

Assume that $g_{1,2} \propto e^{-i\theta(k_x)}$, where $\zeta \doteq \theta'(k_x)$ changes slowly with k_x compared with θ. Derive the two corresponding GO dispersion branches $\zeta_{1,2}(k_x)$ and explain how they are related to $x(k_x)$ in problem (b). By considering the "frequency gap" $|\zeta_1 - \zeta_2|$, estimate the parameter that determines the mode-conversion efficiency. Is energy deposition at the Alfvén resonance larger at small α or at large α?

Part IV

Waves in plasmas: kinetic theory

In this part of the course, we extend our previous models of plasma waves by including kinetic effects.

Lecture 8

Introduction to kinetic theory of plasma waves

Fluid theory used in the previous lectures is only a rough reduction of the complete (Klimontovich) description that accounts for each individual particle in the plasma. The next best approach is the kinetic approach, in which plasma is considered as a fluid in phase space. This lecture is intended as an introduction into kinetic theory and its applications to the modeling of plasma waves.

8.1 Introduction

8.1.1 Distribution function

Suppose that the motion of a single particle is fully described by some set of phase-space variables Γ, for example, $\Gamma = (\boldsymbol{x}, \boldsymbol{v})$ or $\Gamma = (\boldsymbol{x}, \boldsymbol{p})$. Then, one can define the particle phase-space density $\mathcal{F}(\Gamma)$ in space Γ, independently for each given species (Box 8.1). Let us consider two different sets of such variables, assuming that they are connected by some invertible function Q,

$$Q : \Gamma_1 \mapsto \Gamma_2. \tag{8.1}$$

First, suppose the phase-space density corresponding to a single particle with some coordinate Γ_{20}. Such phase-space density is given by $\delta(\Gamma_2 - \Gamma_{20})$. In terms of the particle coordinate Γ_{10} in the Γ_1 space, the same function can be expressed as $\delta[\Gamma_2 - Q(\Gamma_{10})]$. In the case of multiple particles, the corresponding phase-space density in Γ_2 must be averaged over all Γ_{10}, or in other words, integrated over their phase-space density $\mathcal{F}_1$. This gives the following general rule for mapping the phase-space density $\mathcal{F}_1$ in Γ_1 to the phase-space density $\mathcal{F}_2$ in Γ_2:

$$\mathcal{F}_2(\Gamma_2) = \int d\Gamma_1\, \delta[\Gamma_2 - Q(\Gamma_1)]\mathcal{F}_1(\Gamma_1) = \mathcal{F}_1[Q^{-1}(\Gamma_2)] \left|\frac{\partial \Gamma_1}{\partial \Gamma_2}\right|, \tag{8.2}$$

where the latter ratio denotes the Jacobian of the corresponding variable transformation and Q^{-1} is the function inverse to Q.

Among all possible variable transformations, there are so-called *canonical* transformations, which are special. They correspond to unit Jacobians, so the canonical phase-space densities (which we denote with F as opposed to the general phase-space densities $\mathcal{F}$) are transformed simply as

$$\mathsf{F}_2(\Gamma_2) = \mathsf{F}_1(Q^{-1}(\Gamma_2)). \tag{8.3}$$

In this sense, the canonical phase-space density F is an invariant with respect to canonical transformations.

Box 8.1: Kinetic modeling of quantum plasmas and general broadband waves

Here, we discuss only classical kinetic theory. For quantum particles, phase-space coordinates are operators and there is no such thing as the phase-space density, so the theory has to be formulated differently. For a system described by a state function $|\psi\rangle$, one starts by introducing the density operator $\hat{\varrho} \doteq |\psi\rangle\langle\psi|$ and its Weyl symbol ϱ, which is known as the Wigner function (or Wigner tensor, if $\langle \boldsymbol{x}|\psi\rangle$ is a vector). The Wigner function satisfies the Moyal equation $\partial_t \varrho = \{\{H, \varrho\}\}$, where H is the Weyl symbol of the particle Hamiltonian and $\{\{\ldots,\ldots\}\}$ are the so-called Moyal brackets [38]. This equation is a generalization of the kinetic equation that is derived below. (The classical distribution function is the coarse-grained limit of the Wigner function ϱ up to a constant factor.) The same formalism is applicable to classical waves [39, 40].

8.1.2 Liouville's theorem

It is well known that the system evolution in time,

$$Q_t : \Gamma_0 \mapsto \Gamma_t, \tag{8.4}$$

can be considered as a canonical transformation. (For example, see Ref. [12, Sec. 45]. Here, t is time and Q_t can be understood as the evolution operator.) This gives an equation describing the evolution of the canonical phase-space density of any given species s:

$$\mathsf{F}_s[Q_t(\Gamma_0)] = \mathsf{F}_{0s}(\Gamma_0). \tag{8.5}$$

More explicitly, this can also be written as

$$\mathsf{F}_s[t, \boldsymbol{x}(t, \boldsymbol{x}_0, \boldsymbol{p}_0), \boldsymbol{p}(t, \boldsymbol{x}_0, \boldsymbol{p}_0)] = \mathsf{F}_{0s}(\boldsymbol{x}_0, \boldsymbol{p}_0), \tag{8.6}$$

where $(\boldsymbol{x}, \boldsymbol{p})$ are some canonical coordinates, $(\boldsymbol{x}_0, \boldsymbol{p}_0)$ are their initial values. By differentiating this equality with respect to time, one obtains

$$0 = \frac{d}{dt}\, \mathsf{F}_s[t, \boldsymbol{x}(t, \boldsymbol{x}_0, \boldsymbol{p}_0), \boldsymbol{p}(t, \boldsymbol{x}_0, \boldsymbol{p}_0)] = \frac{\partial \mathsf{F}_s}{\partial t} + \frac{d\boldsymbol{x}}{dt} \cdot \frac{\partial \mathsf{F}_s}{\partial \boldsymbol{x}} + \frac{d\boldsymbol{p}}{dt} \cdot \frac{\partial \mathsf{F}_s}{\partial \boldsymbol{p}}, \tag{8.7}$$

which is known as Liouville's theorem (Box 8.2).

In nonrelativistic plasma physics, it is customary to work with the particle phase-space density $f(t, \boldsymbol{x}, \boldsymbol{v})$ in the $(\boldsymbol{x}, \boldsymbol{v})$ space instead of the canonical phase-space density $\mathsf{F}(t, \boldsymbol{x}, \boldsymbol{p})$. The relation between f and F is obtained using Eq. (8.2):

$$f_s(t, \boldsymbol{x}, \boldsymbol{v}) = \mathsf{F}_s[t, \boldsymbol{x}, \boldsymbol{p}(t, \boldsymbol{x}, \boldsymbol{v})] \left| \frac{\partial(\boldsymbol{x}, \boldsymbol{p})}{\partial(\boldsymbol{x}, \boldsymbol{v})} \right|. \tag{8.8}$$

For nonrelativistic plasmas, which we consider below, one can take

$$\boldsymbol{p}(t, \boldsymbol{x}, \boldsymbol{v}) = m_s \boldsymbol{v} - \frac{e_s}{c}\, \boldsymbol{A}(t, \boldsymbol{x}), \tag{8.9}$$

where $\boldsymbol{A}$ is the electromagnetic vector potential. The corresponding Jacobian is

$$\frac{\partial(\boldsymbol{x}, \boldsymbol{p})}{\partial(\boldsymbol{x}, \boldsymbol{v})} = m_s^3, \tag{8.10}$$

so f_s and F_s differ only by a constant factor. Therefore, they satisfy the same equation,

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \nabla f_s + \frac{\boldsymbol{F}_s}{m_s} \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = 0, \tag{8.11}$$

where $\boldsymbol{F}_s = m_s \dot{\boldsymbol{v}}$ is the force.

Box 8.2: Intuitive interpretation of Liouville's theorem

Liouville's theorem can also be interpreted as follows. Consider a small fluid element in phase space. The shape of this element may become distorted with time but its volume $d\Gamma$ is conserved, because the transformation (8.4) is canonical. The number of particles inside this volume is conserved too, by definition, so $\mathsf{F}_s\, d\Gamma = \text{const}$. Then, $\mathsf{F}_s = \text{const}$ along the trajectory followed by each given phase-space element.

8.2 Vlasov equation

8.2.1 Macroscopic fields and collision operator

In general, both f_s and $\boldsymbol{F}_s$ can be split into macroscopic parts, which we denote with the index "m", and microscopic fluctuations, which we denote with the prefix δ. Then, after averaging over fluctuations, one obtains

$$\frac{\partial f_{s,\mathrm{m}}}{\partial t} + \boldsymbol{v}\cdot\nabla f_{s,\mathrm{m}} + \frac{\boldsymbol{F}_{s,\mathrm{m}}}{m_s}\cdot\frac{\partial f_{s,\mathrm{m}}}{\partial \boldsymbol{v}} = -\left\langle \frac{\delta \boldsymbol{F}_s}{m_s}\cdot\frac{\partial(\delta f_s)}{\partial \boldsymbol{v}}\right\rangle \equiv C_s, \tag{8.12}$$

where C_s is called the collision operator. If C_s is large compared to the term $(\boldsymbol{F}_{s,\mathrm{m}}/m_s)\cdot\partial_{\boldsymbol{v}} f_{s,\mathrm{m}}$ that describes long-range interactions, then Eq. (8.12) is reduced to

$$\frac{\partial f_{s,\mathrm{m}}}{\partial t} + \boldsymbol{v}\cdot\nabla f_{s,\mathrm{m}} \approx C_s, \tag{8.13}$$

which is called the Boltzmann equation. This equation adequately describes neutral gases, where particles interact mainly in pairs, via short-range forces. In contrast, plasmas are, by definition, ensembles where each particle interacts with many neighbors simultaneously (because plasmas are defined as ensembles where the number of particles in the Debye sphere is large). In this case, $\delta\boldsymbol{F}_s$ caused by particle discreteness is almost zero, so C_s is small. Below, we will mostly consider the limit of *ideal* plasma, where the effect of the fluctuations is completely negligible. Then, one obtains

$$\frac{\partial f_{s,\mathrm{m}}}{\partial t} + \boldsymbol{v}\cdot\nabla f_{s,\mathrm{m}} + \frac{\boldsymbol{F}_{s,\mathrm{m}}}{m_s}\cdot\frac{\partial f_{s,\mathrm{m}}}{\partial \boldsymbol{v}} = 0, \tag{8.14}$$

or with $\boldsymbol{F}_{s,\mathrm{m}}$ explicitly written as the Lorentz force,

$$\frac{\partial f_{s,\mathrm{m}}}{\partial t} + \boldsymbol{v}\cdot\nabla f_{s,\mathrm{m}} + \frac{e_s}{m_s}\left(\boldsymbol{E}_{s,\mathrm{m}} + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}_{s,\mathrm{m}}\right)\cdot\frac{\partial f_{s,\mathrm{m}}}{\partial \boldsymbol{v}} = 0. \tag{8.15}$$

This is known as the (collisionless) Vlasov equation. Below, we adopt Eq. (8.15) as our primary model of plasma dynamics. The index m will be omitted for brevity.

8.2.2 Linearized Vlasov equation

As earlier, we will assume weak oscillating fields, so $f_s = n_{0s} f_{0s} + \tilde{f}_s$, where $\tilde{f}_s$ is linear in the wave field and f_{0s} is the unperturbed distribution. The coefficient n_{0s} is introduced so that f_{0s} be normalized to unity,

$$\int_{-\infty}^{\infty} d\boldsymbol{v}\, f_{0s}(\boldsymbol{v}) = 1. \tag{8.16}$$

Then, $\tilde{f}_s$ satisfies

$$\frac{\partial \tilde{f}_s}{\partial t} + \boldsymbol{v}\cdot\nabla\tilde{f}_s + \boldsymbol{v}\times\boldsymbol{\Omega}_s\cdot\frac{\partial\tilde{f}_s}{\partial\boldsymbol{v}} \approx -\frac{e_s n_{0s}}{m_s}\left(\tilde{\boldsymbol{E}} + \frac{1}{c}\,\boldsymbol{v}\times\tilde{\boldsymbol{B}}\right)\cdot\frac{\partial f_{0s}}{\partial\boldsymbol{v}} \equiv \mathsf{R}_s, \tag{8.17}$$

where terms quadratic in the wave field have been dropped and $\boldsymbol{\Omega}_s \doteq e_s \boldsymbol{B}_0/(m_s c)$, as usual. (Unlike in the fluid description that we studied earlier, the velocity is an independent variable here; thus, $\boldsymbol{v} \cdot \nabla \tilde{f}_s$ is a *linear* term and must be retained.) The first-order magnetic field that enters Eq. (8.17) can be expressed through $\tilde{\boldsymbol{E}}$ using Faraday's law:

$$\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}^{(\mathrm{f})} + (c\hat{\boldsymbol{k}}/\hat{\omega}) \times \tilde{\boldsymbol{E}}, \tag{8.18}$$

where $\tilde{\boldsymbol{B}}^{(\mathrm{f})}$ is independent of time. Hence,

$$\begin{aligned} \mathsf{R}_s &= \mathsf{R}_s^{(\mathrm{f})} - \frac{e_s n_{0s}}{m_s} \frac{\partial f_{0s}}{\partial \boldsymbol{v}} \cdot \left[\tilde{\boldsymbol{E}} + \boldsymbol{v} \times \left(\frac{\hat{\boldsymbol{k}}}{\hat{\omega}} \times \tilde{\boldsymbol{E}}\right)\right] \\ &= \mathsf{R}_s^{(\mathrm{f})} - \frac{e_s n_{0s}}{m_s} \frac{\partial f_{0s}}{\partial v_c} \left[\tilde{\boldsymbol{E}} + \frac{1}{\hat{\omega}}\, \hat{\boldsymbol{k}}(\boldsymbol{v} \cdot \tilde{\boldsymbol{E}}) - \frac{(\boldsymbol{v} \cdot \hat{\boldsymbol{k}})}{\hat{\omega}}\, \tilde{\boldsymbol{E}}\right]_c \\ &= \mathsf{R}_s^{(\mathrm{f})} - \frac{e_s n_{0s}}{m_s} \frac{\partial f_{0s}}{\partial v_c} \left[\delta_{cb}\left(1 - \frac{\boldsymbol{v} \cdot \hat{\boldsymbol{k}}}{\hat{\omega}}\right) + \frac{v_b \hat{k}_c}{\hat{\omega}}\right] \tilde{E}_b, \end{aligned} \tag{8.19}$$

where $\mathsf{R}_s^{(\mathrm{f})}$ is due to $\tilde{\boldsymbol{B}}^{(\mathrm{f})}$, so it is independent of $\tilde{\boldsymbol{E}}$. This means that one can express $\tilde{f}_s$ as

$$\tilde{f}_s = \tilde{f}_s^{(\mathrm{f})} + \tilde{f}_s^{(\mathrm{i})}, \qquad \tilde{f}_s^{(\mathrm{i})} = \hat{\boldsymbol{\alpha}}_s^{\dagger} \tilde{\boldsymbol{E}}, \tag{8.20}$$

where $\tilde{f}_s^{(\mathrm{f})}$ in independent of $\tilde{\boldsymbol{E}}$ and $\hat{\boldsymbol{\alpha}}_s^{\dagger}$ is a linear operator. (Because $\hat{\boldsymbol{\alpha}}_s^{\dagger} \tilde{\boldsymbol{E}}$ is a scalar, the coefficient between $\tilde{f}_s^{(\mathrm{i})}$ and $\tilde{\boldsymbol{E}}$ is a covector, so it is introduced with †.) This yields the induced current density

$$\tilde{\boldsymbol{j}}_s^{(\mathrm{i})}(t, \boldsymbol{x}) = e_s \int d\boldsymbol{v}\, \boldsymbol{v} \tilde{f}_s^{(\mathrm{i})}(t, \boldsymbol{x}, \boldsymbol{v}), \tag{8.21}$$

whence the conductivity operator can be inferred:

$$\hat{\boldsymbol{\sigma}}_s = e_s \int d\boldsymbol{v}\, \boldsymbol{v}\, \hat{\boldsymbol{\alpha}}_s^{\dagger}. \tag{8.22}$$

For simplicity, we will limit our consideration to homogeneous plasmas and to waves of the form $\propto \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. Then, $\hat{\boldsymbol{k}}$ can be replaced with $\boldsymbol{k}$. However, the time dependence has to be handled in a more subtle manner, as will be discussed in Lecture 9.

8.3 Phase mixing

Before we proceed to the general kinetic description of plasma waves, let us discuss some paradigmatic effects within the simple model of electrostatic interactions in nonmagnetized plasma. Because the plasma is assumed homogeneous, the linearized Vlasov equation (8.17) can be simplified by application of the spatial Fourier transform as follows:

$$\frac{\partial \tilde{f}_s}{\partial t} + ikv_x \tilde{f}_s = -\frac{e_s n_{0s}}{m_s}\, \tilde{E}\, \frac{\partial f_{0s}}{\partial v_x}, \tag{8.23}$$

where $\tilde{f}_s$ and $\tilde{E}$ are now independent of x but depend on $\boldsymbol{k}$, and the x axis is chosen parallel to $\boldsymbol{k}$. This equation is similar, for example, to Eqs. (2.5) and (6.4) and can be solved in the same way:

$$\tilde{f}_s = \tilde{f}_s^{(\mathrm{f})}(\boldsymbol{v}) \mathrm{e}^{-ikv_x t} + \tilde{f}_s^{(\mathrm{i})}(t, \boldsymbol{v}), \tag{8.24}$$

$$\tilde{f}_s^{(\mathrm{i})} = -\frac{e_s n_{0s}}{m_s} \frac{\partial f_{0s}(\boldsymbol{v})}{\partial v_x} \int_0^t dt'\, \mathrm{e}^{-ikv_x(t-t')} \tilde{E}(t'), \tag{8.25}$$

where $\tilde{f}_s^{(\mathrm{f})}$ is determined by initial conditions but not by the field.

For a delta-shaped field, $\tilde{E}(t) = \delta(t-0)$, the induced perturbation is

$$\tilde{f}_s^{(\mathrm{i})}(t,\boldsymbol{v}) = -\frac{e_s n_{0s}}{m_s}\frac{\partial f_{0s}(\boldsymbol{v})}{\partial v_x}\, \mathsf{e}^{-ikv_x t}. \tag{8.26}$$

(This is understood as the Green's function of the linearized Vlasov equation.) The corresponding induced current $e_s \int_{-\infty}^{\infty} d\boldsymbol{v}\, v_x \tilde{f}_s^{(\mathrm{i})}(t,\boldsymbol{v})$ is, by definition, the conductivity in the $(t,\boldsymbol{k})$ representation [which is the Fourier image of the function $\bar{\Sigma}_s(t,x)$ that we used earlier]:

$$\bar{\Sigma}_{s,k}(t) = -\frac{e_s^2 n_{0s}}{m_s}\int_{-\infty}^{\infty} d\boldsymbol{v}\, v_x\, \frac{\partial f_{0s}(\boldsymbol{v})}{\partial v_x}\, \mathsf{e}^{-ikv_x t}. \tag{8.27}$$

We now change the notation as follows:

$$\int_{-\infty}^{\infty} dv_y\, dv_z\, f_{0s}(\boldsymbol{v}) \to f_{0s}(v_x), \tag{8.28}$$

so $f_{0s}(v)$ is understood as the distribution of the velocity components parallel to $\boldsymbol{k}$, and $\int_{-\infty}^{\infty} dv\, f_{0s}(v) = 1$ [as opposed to $\int_{-\infty}^{\infty} d\boldsymbol{v}\, f_{0s}(\boldsymbol{v}) = 1$ that we used previously]. Then,

$$\bar{\Sigma}_{s,k}(t) = -\frac{\omega_{ps}^2}{4\pi}\int_{-\infty}^{\infty} dv\, v f_{0s}'(v)\, \mathsf{e}^{-ikvt}. \tag{8.29}$$

It is also instructive to express this function as follows:

$$\begin{aligned}
\bar{\Sigma}_{s,k}(t) &= -\frac{\omega_{ps}^2}{4\pi}\left(\frac{1}{-\mathsf{i}t}\right)\partial_k \int_{-\infty}^{\infty} dv\, f_{0s}'(v)\, \mathsf{e}^{-ikvt} \\
&= \frac{\omega_{ps}^2}{4\pi}\left(\frac{1}{-\mathsf{i}t}\right)\partial_k \int_{-\infty}^{\infty} dv\, f_{0s}(v)\partial_v(\mathsf{e}^{-ikvt}) \\
&= \frac{\omega_{ps}^2}{4\pi}\,\partial_k\left[k\int_{-\infty}^{\infty} dv\, f_{0s}(v)\, \mathsf{e}^{-ikvt}\right] \\
&\equiv \frac{\omega_{ps}^2}{4\pi}\,\partial_k\left[k\mathcal{F}_{0s}(-kt)\right],
\end{aligned} \tag{8.30}$$

where $\mathcal{F}_{0s}$ is the Fourier image of f_{0s}. In the case when the background plasma is cold and stationary, i.e., $f_{0s}(v) = \delta(v)$, one has $\mathcal{F}_{0s} = 1$, so

$$\bar{\Sigma}_{s,k}(t) = \frac{\omega_{ps}^2}{4\pi}, \tag{8.31}$$

in agreement with our earlier result (2.9). But the cold-plasma approximation is only a crude model. In reality, $f_{0s}(v)$ is always a smooth function with some finite width v_{Ts}. Then, $\mathcal{F}_{0s}$ has a characteristic scale v_{Ts}^{-1}, meaning $\mathcal{F}_{0s}$ is small when its argument is much larger than v_{Ts}^{-1}. (In fact, the Fourier image of a smooth function decreases exponentially when the argument tends to infinity.) Therefore, $|\bar{\Sigma}_{s,k}(t)| \to 0$ on the time scale $(kv_{Ts})^{-1}$.[1] This effect is known as *phase mixing*, because it is caused by destructive interference of the contributions to $\bar{\Sigma}_{s,k}$ produced by $\tilde{f}_s$ at different velocities.

Note also that the decay of the plasma cumulative response, which is described by $\bar{\Sigma}_{s,k}(t)$, does not mean that oscillations of $\tilde{f}_s$ are attenuated. Once perturbed, the distribution function continues to oscillate [Eq. (8.26)] until collisions come into play. In other words, the *microscopic* state of the plasma can store information even after *macroscopic* currents have dissipated.[2] One interesting effect that results from this is the so-called plasma-wave echo [41].

[1] In Sec. 11.1, we will explicitly derive $\bar{\Sigma}_{s,k}$ for Maxwellian species.
[2] For a mechanical analogy, see http://www.youtube.com/watch?v=8V6kc0PQa14.

Lecture 9

Eigenmodes in kinetic theory

Here, we discuss the concepts of eigenmodes in kinetic theory and dispersion relation in kinetic theory, which are more subtle than those in fluid theory.

9.1 Case–van Kampen modes

Phase mixing can be interpreted as destructive interference of microscopic eigenmodes of the plasma. Let us discuss these modes in detail. For simplicity, we will limit our consideration to the special case when ions are immobile and the electron distribution f_{0e} is isotropic, so it can be expressed as a function of $\mathcal{E} \doteq m_e v^2/2T_e$; i.e., $f_{0e}(\boldsymbol{v}) = \bar{f}(\mathcal{E})$. (The constant T_e has units of energy and can be considered as the electron temperature, but in general, f_{0e} may be non-Maxwellian.) Also, for simplicity, we will assume that $\bar{f}$ is monotonically decreasing, so

$$\partial_{\boldsymbol{v}} f_{0e} = \frac{m_e \boldsymbol{v}}{T_e}\,\bar{f}'(\mathcal{E}) = -\frac{m_e \boldsymbol{v}}{T_e}\,|\bar{f}'(\mathcal{E})|. \tag{9.1}$$

Then, Eq. (8.23) can be written as follows:

$$i\partial_t \tilde{f}_e = k v_x \tilde{f}_e + \frac{i e_e n_{0e}}{T_s}\,|\bar{f}'(\mathcal{E})|\, v_x \tilde{E}. \tag{9.2}$$

Let us complement this equation with Ampere's law. For electrostatic oscillations, it can be written as $\partial_t \tilde{E} = -4\pi \tilde{j}_x$, or equivalently,

$$i\partial_t \tilde{E} = -4\pi e_e i \int d v_x\, v_x \tilde{f}_e. \tag{9.3}$$

By using a variable transformation $\tilde{f}_e \to i\alpha \tilde{f}_e$ with an appropriate real α, one can bring this set of equations to the form

$$i\partial_t \tilde{f}_e = k v_x \tilde{f}_e + v_x \tilde{E}, \qquad i\partial_t \tilde{E} = \textstyle\int d v_x\, v_x \tilde{f}_e. \tag{9.4}$$

Let us discretize the velocity space into N chunks of the size Δv centered around $v_x = v_a$, $a = 1, 2, \dots, N$. Let us also replace $\tilde{f}_e \to (\Delta v)^{-1/2} \tilde{f}$, and introduce $g_a \doteq v_a \sqrt{\Delta v}$. This brings Eq. (9.4) to the Schrödinger form

$$i\partial_t \boldsymbol{\psi} = \boldsymbol{H}\boldsymbol{\psi}, \tag{9.5}$$

Box 9.1: Graph representation of kinetic Hamiltonians

Hermitian Hamiltonians are often represented as graphs. Graph's nodes correspond to the components of the state vector $\boldsymbol{\psi}$ and edges denote nonzero elements of the Hamiltonian matrix. In particular, each diagonal element connects the corresponding node with itself, so is it represented as a loop. The Hamiltonian (9.6) corresponds to a graph that is a "star with loops": the node that represents $\tilde{E}$ is connected with each node representing $\tilde{f}_a$, but no $\tilde{f}_a$ and $\tilde{f}_b$ are connected directly with each other unless $a = b$. Such topology of Hamiltonian graphs is a signature feature of collisionless-plasma models and mean-field theories in general.

where $\boldsymbol{\psi} = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_N, \tilde{E})^\intercal$ is a column vector and (Box 9.1)

$$\boldsymbol{H} = \begin{pmatrix} kv_1 & 0 & 0 & \dots & 0 & g_1 \\ 0 & kv_2 & 0 & \dots & 0 & g_2 \\ 0 & 0 & kv_3 & \dots & 0 & g_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & kv_N & g_N \\ g_1 & g_2 & g_3 & \dots & g_N & 0 \end{pmatrix}. \tag{9.6}$$

Let us search for eigenmodes of this system in the form $\psi \propto e^{-i\omega t}$. The corresponding frequencies ω satisfy the following equation (Exercise 9.1):

$$0 = D(\omega) = \omega - \sum_{a=1}^{N} \frac{g_a^2}{\omega - kv_a}. \tag{9.7}$$

Exercise 9.1: Derive Eq. (9.7) by proving the following equality for general ϑ_a and g_a:

$$\det \begin{pmatrix} \vartheta_1 & 0 & 0 & \dots & 0 & g_1 \\ 0 & \vartheta_2 & 0 & \dots & 0 & g_2 \\ 0 & 0 & \vartheta_3 & \dots & 0 & g_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \vartheta_N & g_N \\ g_1^* & g_2^* & g_3^* & \dots & g_N^* & \vartheta \end{pmatrix} = \left(\vartheta - \sum_{a=1}^{N} \frac{|g_a|^2}{\vartheta_a} \right) \prod_{b=1}^{N} \vartheta_b.$$

As an algebraic equation of order $N+1$, Eq. (9.7) has $N+1$ solutions for ω. All these solutions are real because $\boldsymbol{H}$ is Hermitian (Exercise 9.2). The fact that Eq. (9.7) always has $N+1$ real solutions can also be seen graphically as shown in Fig. 9.1. The corresponding eigenvectors satisfy

$$\tilde{f}_a = \frac{g_a}{\omega - kv_a} \tilde{E}. \tag{9.8}$$

Exercise 9.2: Show that if $\bar{f}(\mathcal{E})$ is nonmonotonic, then $\boldsymbol{H}$ is not Hermitian and that the shape of $D(\omega)$ changes qualitatively such that complex roots in general become possible.

In the limit $N \to \infty$, these modes are known as Case–van Kampen modes [42, 43].[1] The mode spectrum becomes continuous in this case, so *any* ω is an eigenfrequency. Strictly speaking, the concept

[1]Here, we consider only a special case of Case–van Kampen modes that corresponds to monotonic $\bar{f}(\mathcal{E})$. One can also generalize these modes by adding ion motion and background magnetic field.

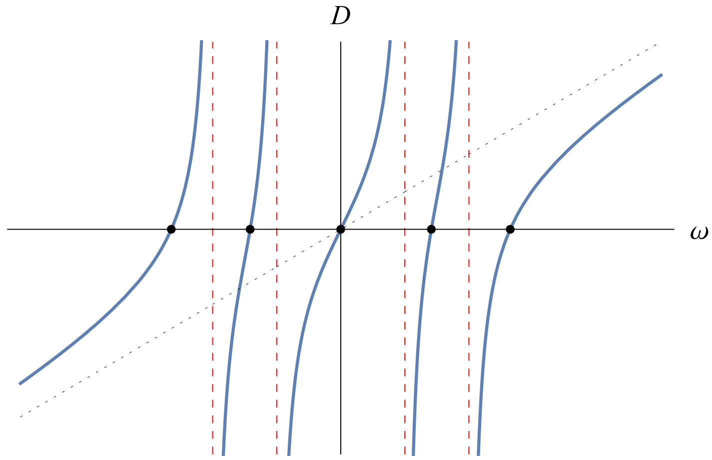


Figure 9.1: A graphical solution of Eq. (9.7) for $N = 4$. The solutions (black points) correspond to the crossings of D of the horizontal axis. The vertical dashed lines correspond to $\omega = kv_a$, the oblique line represents the asymptotic $D(\omega) = \omega$.

of a dispersion relation becomes irrelevant then. In fact, for any bounded absolutely integrable function u, one can find initial conditions such that $\tilde{E}(t) = u(t)$ [44]. The only subtlety is that such initial conditions are typically not analytic [because the eigenvectors (9.8) are not analytic] and thus are not realized naturally. To study what happens for "natural" initial conditions, one should consider the general initial-value problem. This is discussed below.

9.2 Initial-value problem

As discussed in Lecture 8, the general solution of the Vlasov equation for each $\tilde{f}_s$ consists of $\tilde{f}_s^{(\mathrm{f})}$, which is due to the free thermal motion, and the induced perturbation $\tilde{f}_s^{(\mathrm{i})}$, which is a functional of $\tilde{\boldsymbol{E}}$. Correspondingly, the current density is $\tilde{\boldsymbol{j}} = \tilde{\boldsymbol{j}}^{(\mathrm{f})} + \tilde{\boldsymbol{j}}^{(\mathrm{i})}$, where $\tilde{\boldsymbol{j}}^{(\mathrm{f})}$ is determined by all $\tilde{f}_s^{(\mathrm{f})}$ and $\tilde{\boldsymbol{j}}^{(\mathrm{i})}$ is determined by all $\tilde{f}_s^{(\mathrm{i})}$. We have been ignoring $\tilde{\boldsymbol{j}}^{(\mathrm{f})}$ until now, but in general, this current density is not negligible and determines the source term S in the wave equation [see Eq. (1.21)], which we assume here in the form

$$\hat{\boldsymbol{D}}_E \tilde{\boldsymbol{E}} = \mathsf{S}, \qquad \hat{\boldsymbol{D}}_E = \frac{c^2}{\hat{\omega}^2}\,(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger - \mathbf{1}\,\hat{k}^2) + \hat{\boldsymbol{\epsilon}}. \tag{9.9}$$

9.2.1 Equation for the field spectrum

Equation (9.9) is generally an integral equation of the form

$$\int_{t_0}^{\infty} dt' \int_{-\infty}^{\infty} d\boldsymbol{x}'\, \boldsymbol{D}_E(t - t', \boldsymbol{x} - \boldsymbol{x}')\tilde{\boldsymbol{E}}(t', \boldsymbol{x}') = \mathsf{S}, \tag{9.10}$$

where the system is assumed homogeneous. It can be converted into a simpler, algebraic, equation by applying the Fourier transform in $\boldsymbol{x}$ and the Laplace transform in time. Originally, the Laplace transform of a given function F is defined as

$$\mathcal{L}F: \quad F(t) \mapsto \bar{F}(s) \doteq \int_0^{\infty} dt\, \mathrm{e}^{-st} F(t), \tag{9.11}$$

Box 9.2: Convolution theorem for the Laplace transform

The convolution theorem for the Laplace transform is valid only if the corresponding functions are sufficiently well behaved. The dispersion operator $\hat{\boldsymbol{D}}_E$ in the form (9.9) *integrates* the field, so it *is* well behaved. In contrast, *differential* operators have singular kernels and require special treatment. In particular,

$$\mathcal{L}F' = -i\omega\bar{F}(\omega) - F(t_0), \qquad \mathcal{L}F'' = -\omega^2\bar{F}(\omega) + i\omega F(t_0) - F'(t_0)$$

(and so on for higher-order derivatives), as can be seen via taking the integral in Eq. (9.13) by parts. This means that if $\hat{\boldsymbol{D}}$ is a polynomial of ∂_t, then $\hat{\boldsymbol{D}}\tilde{\boldsymbol{E}} = \mathsf{S}$ becomes $\bar{\boldsymbol{D}}(\omega, \boldsymbol{k})\bar{\boldsymbol{E}}(\omega, \boldsymbol{k}) = \bar{\mathsf{S}} + \Delta\bar{\mathsf{S}}$, where $\bar{\boldsymbol{D}}(\omega, \boldsymbol{k})$ and $\Delta\bar{\mathsf{S}}$ are polynomials of ω and $\Delta\bar{\mathsf{S}}$ is determined by $F(t_0)$, $F'(t_0)$, ... In cold-plasma problems, it is convenient to work with the field equation in the form

$$\hat{\boldsymbol{D}}\tilde{\boldsymbol{E}} = -\frac{4\pi i\hat{\omega}}{c^2}\,\tilde{\boldsymbol{j}}^{(\mathrm{f})}, \qquad \hat{\boldsymbol{D}} = \frac{\hat{\omega}^2}{c^2}\,\hat{\boldsymbol{D}}_E,$$

because $\tilde{f}_s^{(\mathrm{f})}$ are time-independent in the absence of thermal motion and thus $\hat{\omega}\tilde{\boldsymbol{j}}^{(\mathrm{f})} = 0$. In this case, the spectral representation of the field equation is simply $\bar{\boldsymbol{D}}(\omega, \boldsymbol{k})\bar{\boldsymbol{E}}(\omega, \boldsymbol{k}) = \Delta\bar{\mathsf{S}}$. But because this equation has the same form as Eq. (9.15) up to the definition of the dispersion operator and the source term, the discussion of the initial-value problem in the main text applies to such problems just as well.

where $\operatorname{Re} s$ must be large enough for the integral to converge. The inverse transform is

$$\mathcal{L}^{-1}\bar{F}: \quad \bar{F}(s) \mapsto F(t) = \frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib} ds\, e^{st}\bar{F}(s), \tag{9.12}$$

where a is a real number that can be chosen arbitrarily as long as it is large enough such that all singularities of $\bar{F}$ remain to the left from the contour $\mathcal{B}$. Here, we assume an equivalent but a slightly modified definition:

$$\bar{F}(\omega) = \int_{t_0}^{\infty} dt\, e^{i\omega t}F(t), \tag{9.13}$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty+ia}^{+\infty+ia} d\omega\, e^{-i\omega t}\bar{F}(\omega), \tag{9.14}$$

where the former integral is now taken from t_0 instead of zero and a variable transformation $s = -i\omega$ has been applied. The latter makes the Laplace transform look similar to the Fourier transform, except for the following: (i) the integral in Eq. (9.13) is taken over *half* of the time axis; (ii) ω is generally complex and must have a large enough imaginary part for the integral in Eq. (9.13) to converge; and (iii) the integral in Eq. (9.14) is taken not necessarily along the real axis but parallel to it, specifically, at a distance a such that all singularities of the integrand remain below the contour.

The Laplace transform in time and the Fourier transform in space convert the convolution integral in Eq. (9.10) into a product (Box 9.2), so one obtains

$$\tilde{\boldsymbol{E}}(\omega, \boldsymbol{k}) = \boldsymbol{D}_E^{-1}(\omega, \boldsymbol{k})\mathsf{S}(\omega, \boldsymbol{k}), \tag{9.15}$$

where the right-hand side is determined by the initial conditions. From here, the field in the $(t, \boldsymbol{k})$ space is obtained by applying the inverse Laplace transform:

$$\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) = \frac{1}{2\pi}\int_{-\infty+ia}^{+\infty+ia} d\omega\, e^{-i\omega t}\boldsymbol{D}_E^{-1}(\omega, \boldsymbol{k})\mathsf{S}(\omega, \boldsymbol{k}). \tag{9.16}$$

9.2.2 Dispersion relation and quasimodes

Let us *analytically continue* $\boldsymbol{D}_E$ into the whole complex-ω plane (how to do this explicitly in practical calculations will be discussed in Lecture 10) and consider *bending* the integration contour. Specifically, instead of integrating along a straight line parallel to the real axis at large enough $\operatorname{Im}\omega$, let us shift the contour downward. The integral will remain unaffected if the contour sticks to singular points of the integrand. Now, let us adopt several assumptions.

> **Assumption 1:** Let us assume that S, which is determined by the initial conditions, is an entire function of ω (i.e., a function that is analytic everywhere possibly excluding $\omega = \infty$).

Then, the contour sticks only to singularities of $\boldsymbol{D}_E^{-1}$. By definition of an inverse matrix, it follows that such singularities are possible, in particular, when

$$\det \boldsymbol{D}_E(\omega, \boldsymbol{k}) = 0. \tag{9.17}$$

If $\det \boldsymbol{D}_E$ is analytic near such points, it can be Taylor-expanded. Then, the corresponding singularities of $\boldsymbol{D}_E^{-1}$ are poles. We denote them as $\omega_q(\boldsymbol{k})$.

> **Assumption 2:** Let us assume that $\boldsymbol{D}_E^{-1}$ has no singularities other than poles.

Note now that, as we are shifting the contour further toward $\operatorname{Im}\omega \to -\infty$, the exponent under the integral approaches zero (at $t > 0$), because

$$e^{-i\omega t} \sim e^{(\operatorname{Im}\omega)t} = e^{-|\operatorname{Im}\omega|t}. \tag{9.18}$$

> **Assumption 3:** Let us assume that $\boldsymbol{D}_E^{-1}\mathsf{S}$ does not grow *too* rapidly (if at all) at $\operatorname{Im}\omega \to -\infty$, so the smallness of the aforementioned exponent is enough to ensure that the integral over the horizontal part of the contour vanishes.

Under these assumptions, the whole integral can be expressed as the sum over the contributions of the aforementioned poles only, i.e.,[2]

$$\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) = \sum_q \boldsymbol{R}_q(\boldsymbol{k}) e^{-i\omega_q(\boldsymbol{k})t}, \tag{9.19}$$

where the coefficients $\boldsymbol{R}_q$ are determined by the initial conditions, or more specifically, proportional to the corresponding residues of $\boldsymbol{D}_E^{-1}(\omega, \boldsymbol{k})\mathsf{S}(\omega, \boldsymbol{k})$ at $\omega = \omega_q(\boldsymbol{k})$. In principle, one can choose initial conditions such that all but one of these coefficients be zero; then, the resulting field is

$$\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) = \boldsymbol{R}_q(\boldsymbol{k}) e^{-i\omega_q(\boldsymbol{k})t}. \tag{9.20}$$

Because this solution has a well-defined complex frequency, $\omega_q(\boldsymbol{k})$, it can be considered as an eigenmode. Also, the mode polarization $\boldsymbol{R}_q$ can in general be found as follows. By applying the Laplace transform to Eq. (9.20), we obtain

$$\tilde{\boldsymbol{E}}(\omega, \boldsymbol{k}) = \frac{i\boldsymbol{R}_q(\boldsymbol{k})}{\omega - \omega_q(\boldsymbol{k})}. \tag{9.21}$$

By substituting this into Eq. (9.15), we also obtain

$$\boldsymbol{D}_E(\omega, \boldsymbol{k})\boldsymbol{R}_q(\boldsymbol{k}) = i[\omega_q(\boldsymbol{k}) - \omega]\mathsf{S}(\omega, \boldsymbol{k}). \tag{9.22}$$

[2] Here, assume that all poles of $\boldsymbol{D}_E^{-1}$ are simple. For nth-order poles, additional terms like $\propto t^m e^{-i\omega_q(\boldsymbol{k})t}$ appear in Eq. (9.19), with $m = 1, 2, \dots (n-1)$. This case will not be considered because it is not typical. In particular, the presence of high-order zeros of $\det \boldsymbol{D}_E$ does not necessarily imply the presence of high-order poles in $\boldsymbol{D}_E^{-1}$.

At $\omega = \omega_q(\boldsymbol{k})$, this gives

$$\boldsymbol{D}_E[\omega_q(\boldsymbol{k}), \boldsymbol{k}]\boldsymbol{R}_q(\boldsymbol{k}) = 0. \tag{9.23}$$

This shows that the polarization vectors of complex-frequency eigenmodes, $\boldsymbol{e}_q(\boldsymbol{k}) \doteq \boldsymbol{R}_q(\boldsymbol{k})/R_q(\boldsymbol{k})$ can be found as the zero eigenvectors of $\boldsymbol{D}_E\,(\omega_q(\boldsymbol{k}), \boldsymbol{k})$, much like in the case of real-frequency waves.

Finally, notice the following. If a plasma is described by (a finite number of) fluid equations, then S is a polynomial in ω and $\boldsymbol{D}_E$ is a rational matrix function of ω; i.e., each of its elements is representable as $p(\omega)/q(\omega)$, where p and q are polynomials (cf. Box 9.2). In this case, all the assumptions in the above derivation are satisfied automatically. This means that the eigenmodes as defined above are the same as those one would find by simply replacing ∇ with $i\boldsymbol{k}$ and ∂_t with $-i\omega$. Such eigenmodes are the true modes of the system. If $\boldsymbol{D}_E$ is not a rational function of ω, though, the eigenmodes as defined here are not actually the true modes [unlike the Case–van Kampen modes (Sec. 9.1)]; for example, they might not form a complete basis. However, such "quasimodes" are relevant in that they correspond to smooth initial conditions and thus are typically representative of plasma oscillations. Because of that, we will refer to $\det \boldsymbol{D}_E(\omega, \boldsymbol{k}) = 0$ as "the" dispersion relation, and we will refer to quasimodes as "the" plasma modes. However, "non-modal" effects are also, in principle, possible.

Lecture 10

Dispersion properties of nonmagnetized plasma

Here, we apply the results of the previous lectures to study the dispersion properties of nonmagnetized plasma.

10.1 Dielectric properties

10.1.1 Susceptibility at $\operatorname{Im}\omega > 0$

A simple way to calculate the conductivity at $\operatorname{Im}\omega > 0$ is to calculate the plasma response to a monochromatic field as discussed in Sec. 1.2.1. Then, one readily obtains

$$\tilde{f}_s^{(\mathrm{i})} = -\frac{i e_s n_{0s}}{m_s(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}\frac{\partial f_{0s}}{\partial v_c}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{k}}{\omega}\right) + \frac{v_b k_c}{\omega}\right]\tilde{E}_b. \tag{10.1}$$

The corresponding current density (8.21) can be written as follows:

$$\tilde{\boldsymbol{j}}_s^{(\mathrm{i})} = -\frac{i\omega_{ps}^2}{4\pi}\int_{-\infty}^{\infty} d\boldsymbol{v}\,\frac{\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right) + \frac{k_c v_b}{\omega}\right]\frac{\partial f_{0s}}{\partial v_c}\,\tilde{E}_b. \tag{10.2}$$

Hence, the spectral conductivity at $\operatorname{Im}\omega > 0$ is

$$(\boldsymbol{\sigma}_s)_{ab}(\omega, \boldsymbol{k}) = -\frac{i\omega_{ps}^2}{4\pi}\int_{-\infty}^{\infty} d\boldsymbol{v}\,\frac{v_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right) + \frac{k_c v_b}{\omega}\right]\frac{\partial f_{0s}}{\partial v_c}, \tag{10.3}$$

and the corresponding susceptibility can be expressed as

$$(\boldsymbol{\chi}_s)_{ab}(\omega, \boldsymbol{k}) = \frac{\omega_{ps}^2}{\omega}\int_{-\infty}^{\infty} d\boldsymbol{v}\,\frac{v_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right) + \frac{k_c v_b}{\omega}\right]\frac{\partial f_{0s}}{\partial v_c}. \tag{10.4}$$

10.1.2 Susceptibility at any $\operatorname{Im}\omega$: Landau's rule

In order to analytically continue these results to $\operatorname{Im}\omega \leq 0$, let us start with proving two auxiliary statements.

Lemma 1

Consider an integral of the form

$$\mathcal{I}_C(u) \doteq \int_C dv\, \frac{h(v)}{v-u}, \tag{10.5}$$

where C is some contour in the complex-v space and h is an arbitrary function. Assuming that h is absolutely integrable on C, it is easy to see that $\mathcal{I}_C(u)$ is well-defined for all $u \notin C$, and so are all its derivatives. Therefore, $\mathcal{I}_C(u)$ is analytic for all $u \notin C$. ■

Lemma 2

Suppose that u is initially in the region A_1 that is above a given contour C. Consider moving u to the region A_2 that is on the other side of the contour C. When u crosses C, the function $\mathcal{I}_C(u)$ may become singular. But let us consider the integral $\mathcal{I}_\mathsf{L}(u)$ taken over contour L that is the same as C except it is bended to remain below the pole at all v. The is called a Landau contour. Then, according to Lemma 1, the function $\mathcal{I}_\mathsf{L}(u)$ remains analytic. Because one also has $\mathcal{I}_\mathsf{L}(u) = \mathcal{I}_C(u)$ in the whole region A_1, the function $\mathcal{I}_\mathsf{L}(u)$ represents the analytic continuation of $\mathcal{I}_C(u)$ to the region A_2 (and remember that the analytic continuation is unique). By splitting the integral into the principal-value part and the pole contribution, one finds that the expression

$$\int_\mathsf{L} dv\, \frac{h(v)}{v-u} = \fint_C dv\, \frac{h(v)}{v-u} + i\pi h(u) \times \begin{cases} 0, & u \in A_1, \\ 1, & u \in C, \\ 2, & u \in A_2 \end{cases} \tag{10.6}$$

represents the analytic continuation of $\mathcal{I}_C(u)$. [Both terms in Eq. (10.6) are discontinuous but their sum is continuous.] Note that unless $u \in C$, one can replace $\fint$ with $\int$, because then the principal value of the integral coincides with the integral itself. Also note that if the initial region A_1 corresponds to the region below C, then $i\pi$ in the above formula must be replaced with $-i\pi$. ■

By Lemma 2, the analytic continuation of Eq. (10.4)

$$(\boldsymbol{\chi}_s)_{ab}(\omega, \boldsymbol{k}) = \frac{\omega_{ps}^2}{\omega} \int_\mathsf{L} d\boldsymbol{v}\, \frac{v_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \left[\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right) + \frac{k_c v_b}{\omega}\right] \frac{\partial f_{0s}}{\partial v_c}. \tag{10.7}$$

This result may seem surprising in the following sense. The same formula for $\operatorname{Im}\omega > 0$ is readily obtained from the linearized Vlasov equation if one simply adopts $\partial_t = -i\omega$. But for strictly monochromatic oscillations of the distribution function, $\partial_t \tilde{f}_s = -i\omega \tilde{f}_s$ is a precise equality irrespective of the sign of $\operatorname{Im}\omega$, so why should one expect a different result? The explanation is as follows. Remember that $\tilde{f}$ consists of two terms: the field-induced term $\tilde{f}_s^{(\mathrm{i})} \propto \exp(-i\omega t)$ and the term $\tilde{f}_s^{(\mathrm{f})}$ that is determined by free oscillations, which has the time dependence of the form $\tilde{f}_s^{(\mathrm{f})} \propto \exp(-ikvt)$. At $\operatorname{Im}\omega > 0$, $\tilde{f}_s^{(\mathrm{i})}$ eventually becomes much larger than $\tilde{f}_s^{(\mathrm{f})}$, so the initial conditions do not matter. (More precisely, the moment t_0 at which the initial conditions are prescribed can be shifted to $-\infty$, so $\tilde{f}_s = \tilde{f}_s^{(\mathrm{i})}$ at any finite t.) But at $\operatorname{Im}\omega < 0$, $\tilde{f}_s^{(\mathrm{i})}$ is essential, and

$$\partial_t \tilde{f}_s \approx -i\omega \tilde{f}_s^{(\mathrm{i})} - ikv \tilde{f}_s^{(\mathrm{f})} \neq -i\omega \tilde{f}_s. \tag{10.8}$$

Although such $\tilde{f}_s$ is not monochromatic, integrals of this function (such as the current density) can be monochromatic, which is how plasma supports the quasimodes that were introduced in Lecture 9.

10.1.3 General dielectric tensor

Finally, let us present the general expression for the dielectric tensor. Assuming the usual notation

$$f_0(\boldsymbol{v}) \doteq \frac{1}{\omega_p^2}\sum_s \omega_{ps}^2 f_{0s}(\boldsymbol{v}), \qquad \omega_p^2 \doteq \sum_s \omega_{ps}^2, \tag{10.9}$$

and using $\epsilon_{ab}(\omega,\boldsymbol{k}) = \delta_{ab} + \sum_s [\chi_{ab}(\omega,\boldsymbol{k})]_s$ with Eq. (10.7), one obtains

$$\epsilon_{ab}(\omega,\boldsymbol{k}) = \delta_{ab} + \frac{\omega_p^2}{\omega}\int_{\mathsf{L}} d\boldsymbol{v}\,\frac{v_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right) + \frac{k_c v_b}{\omega}\right]\frac{\partial f_0}{\partial v_c}. \tag{10.10}$$

Because

$$\int_{\mathsf{L}} d\boldsymbol{v}\,\frac{v_a}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\delta_{cb}\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)\frac{\partial f_{0s}}{\partial v_c} = \frac{1}{\omega}\int_{-\infty}^{\infty} d\boldsymbol{v}\, v_a\frac{\partial f_{0s}}{\partial v_b} = -\frac{\delta_{ab}}{\omega}\int_{-\infty}^{\infty} d\boldsymbol{v}\, f_{0s} = -\frac{\delta_{ab}}{\omega}, \tag{10.11}$$

one can also present this in the following alternative form, which will be useful below:

$$\epsilon_{ab}(\omega,\boldsymbol{k}) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ab} + \frac{\omega_p^2}{\omega^2}\int_{\mathsf{L}} d\boldsymbol{v}\,\frac{k_c v_a v_b}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\frac{\partial f_0}{\partial v_c}. \tag{10.12}$$

10.2 Dielectric properties of isotropic plasma

10.2.1 Dielectric tensor

Let us now consider an important special case of isotropic plasma,[1] that is, plasma where $f_0(\boldsymbol{v}) = \bar{f}_0(v)$, with $v \doteq (v_x^2 + v_y^2 + v_z^2)^{1/2}$. Then,

$$\frac{\partial}{\partial v_a}\, f_0(\boldsymbol{v}) = \bar{f}_0'(v)\,\frac{\partial}{\partial v_a}\sqrt{v_x^2 + v_y^2 + v_z^2} = \frac{v_a}{v}\,\bar{f}_0'(v), \tag{10.13}$$

and it is easy to see that that $\boldsymbol{\epsilon}$ acquires the following diagonal form:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_\parallel & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\perp \end{pmatrix}, \tag{10.14}$$

assuming that the x axis is chosen along $\boldsymbol{k}$. Correspondingly, the dispersion matrix is as follows:

$$\boldsymbol{D}_E(\omega,\boldsymbol{k}) = \begin{pmatrix} \epsilon_\parallel(\omega,\boldsymbol{k}) & 0 & 0 \\ 0 & \epsilon_\perp(\omega,\boldsymbol{k}) - N^2 & 0 \\ 0 & 0 & \epsilon_\perp(\omega,\boldsymbol{k}) - N^2 \end{pmatrix}, \tag{10.15}$$

where $N \doteq kc/\omega$. This means that like in fluid plasma, waves can be of two types: transverse electromagnetic waves and longitudinal electrostatic waves.

10.2.2 Transverse waves

The transverse waves satisfy

$$0 = \epsilon_\perp(\omega,\boldsymbol{k}) - (kc/\omega)^2. \tag{10.16}$$

[1] A more general case is addressed in a homework.

This is similar to the electromagnetic-wave dispersion that we studied earlier except $\epsilon_\perp(\omega, \boldsymbol{k})$ is somewhat different from its cold limit $\epsilon_{\text{cold}}(\omega) \doteq 1 - \omega_p^2/\omega^2$. From Eq. (10.12), one has

$$\epsilon_{ab}(\omega, \boldsymbol{k}) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ab} + \frac{\omega_p^2}{\omega^2}\int_{\mathsf{L}} d\boldsymbol{v}\, \frac{k v_a v_b}{\omega - k v_x}\frac{\partial f_{0s}}{\partial v_x}, \tag{10.17}$$

so the thermal corrections can be estimated as follows:

$$\epsilon_\perp(\omega, \boldsymbol{k}) - \epsilon_{\text{cold}}(\omega) = \frac{\omega_p^2}{\omega^2}\int_{\mathsf{L}} d\boldsymbol{v}\, \frac{k v_y^2}{\omega - k v_x}\frac{\partial f_{0s}}{\partial v_x} \sim \frac{\omega_p^2}{\omega^2}\frac{k^2 v_T^2}{\omega^2} \ll \frac{\omega_p^2}{\omega^2}. \tag{10.18}$$

(Here, we have assumed that $\omega/k \gtrsim c$ by analogy with waves in cold plasma, and we have also assumed $v_T \ll c$, because our theory is nonrelativistic to begin with.) Then,

$$\epsilon_\perp(\omega, \boldsymbol{k}) \approx 1 - \omega_p^2/\omega^2 + \mathcal{O}(v_T^2/c^2). \tag{10.19}$$

This indicates that the electromagnetic waves in isotropic nonmagnetized plasma are not significantly affected by (nonrelativistic) temperature; i.e., their dispersion relation is approximately the same as in cold plasma,

$$\omega^2 \approx \omega_p^2 + k^2 c^2. \tag{10.20}$$

10.2.3 Longitudinal waves

The longitudinal waves satisfy

$$\epsilon_\parallel(\omega, \boldsymbol{k}) = 0, \qquad \epsilon_\parallel = 1 + \sum_s \chi_{s,\parallel}, \tag{10.21}$$

and

$$\chi_{s,\parallel}(\omega, \boldsymbol{k}) = \frac{\omega_{ps}^2}{\omega}\int_{\mathsf{L}} d\boldsymbol{v}\, \frac{v_x}{\omega - k v_x}\left[\delta_{xx}\left(1 - \frac{k v_x}{\omega}\right) + \frac{k v_x}{\omega}\right]\frac{\partial f_{0s}}{\partial v_x}. \tag{10.22}$$

Let us integrate over v_y and v_z and change the notation as in Eq. (8.28). Then,

$$\chi_{s,\parallel}(\omega, \boldsymbol{k}) = \frac{\omega_{ps}^2}{\omega}\int_{\mathsf{L}} dv\, \frac{v}{\omega - k v}\frac{\partial f_{0s}}{\partial v}, \tag{10.23}$$

where v now denotes v_x. Also notice that

$$\frac{v}{\omega - kv} = \frac{1}{k}\frac{(kv - \omega) + \omega}{\omega - kv} = -\frac{1}{k} - \frac{\omega}{k^2}\frac{1}{v - \omega/k}. \tag{10.24}$$

Then,

$$\int_{\mathsf{L}} dv\, \frac{v f_{0s}'(v_x)}{\omega - k v_x} = \int_{\mathsf{L}} dv \left(-\frac{1}{k} - \frac{1}{k^2}\frac{\omega}{v - \omega/k}\right) f_{0s}'(v) = -\frac{\omega}{k^2}\int_{\mathsf{L}} dv\, \frac{f_{0s}'(v)}{v - \omega/k}, \tag{10.25}$$

where we used that $\int_{-\infty}^{\infty} dv\, f_{0s}'(v) = f_{0s}(+\infty) - f_{0s}(-\infty) = 0$. This leads to the following expression for the susceptibility (Exercise 10.1):

$$\chi_{s,\parallel}(\omega, \boldsymbol{k}) = -\frac{\omega_{ps}^2}{k^2}\int_{\mathsf{L}} dv\, \frac{f_{0s}'(v)}{v - \omega/k}, \tag{10.26}$$

Exercise 10.1: Show that Eq. (10.26) can as well be derived using $\sigma_s(\omega, \boldsymbol{k}) = \int_0^\infty dt\, e^{i\omega t}\Sigma_{s,k}(t)$, with $\Sigma_{s,k}$ taken from Eq. (8.29).

or more explicitly,

$$\chi_{\parallel,s}(\omega, \boldsymbol{k}) = -\frac{\omega_{ps}^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f'_{0s}(v)}{v - \omega/k} - i\pi \frac{\omega_{ps}^2}{k|k|}\, f'_{0s}\left(\frac{\omega}{k}\right) \times \begin{cases} 0, & \operatorname{Im}\omega > 0, \\ 1, & \operatorname{Im}\omega = 0, \\ 2, & \operatorname{Im}\omega < 0. \end{cases} \tag{10.27}$$

Because v and the phase velocity ω/k can be comparable, the influence of kinetic effects on the susceptibility can be strong, so let us discuss longitudinal waves in more detail. For simplicity, let us focus on waves with small ω_i and $\epsilon_\parallel(\omega, \boldsymbol{k})$ that is smooth enough. Then, Eq. (10.21) can be simplified as follows:

$$0 = \epsilon_\parallel(\omega_r + i\omega_i, \boldsymbol{k}) \approx \epsilon_\parallel(\omega_r, \boldsymbol{k}) + i\omega_i \partial_\omega \epsilon_\parallel(\omega_r, \boldsymbol{k}). \tag{10.28}$$

Because $\epsilon_\parallel$ on the right-hand side is evaluated at ω_r, it can be calculated using Eq. (10.27) with $\operatorname{Im}\omega = 0$:

$$\epsilon_\parallel(\omega_r, \boldsymbol{k}) = \epsilon_r(\omega_r, \boldsymbol{k}) + i\epsilon_i(\omega_r, \boldsymbol{k}), \tag{10.29a}$$

$$\epsilon_r(\omega_r, \boldsymbol{k}) \doteq 1 - \frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f'_0(v)}{v - \omega_r/k}, \tag{10.29b}$$

$$\epsilon_i(\omega_r, \boldsymbol{k}) \doteq -\pi \frac{\omega_p^2}{k^2}\, f'_0\left(\frac{\omega_r}{k}\right). \tag{10.29c}$$

For small enough ϵ_i, one can further simplify the dispersion relation as follows:

$$\epsilon_r(\omega_r, \boldsymbol{k}) + i\left[\omega_i \partial_\omega \epsilon_r(\omega_r, \boldsymbol{k}) + \epsilon_i(\omega_r, \boldsymbol{k})\right] \approx 0. \tag{10.30}$$

The real and imaginary part of this complex equation give equations for ω_r and ω_i; specifically,[2]

$$\epsilon_r(\omega_r, \boldsymbol{k}) = 0, \tag{10.31a}$$

$$\omega_i = -\frac{\epsilon_i(\omega_r, \boldsymbol{k})}{\partial_\omega \epsilon_r(\omega_r, \boldsymbol{k})}. \tag{10.31b}$$

In Lecture 11, we will apply these results to study longitudinal waves in Maxwellian plasma.

10.3 Stability of electrostatic oscillations

As shown above, the dispersion relation for electrostatic oscillations in nonmagnetized plasma can be written as

$$\epsilon(\omega) = 0, \tag{10.32}$$

where the index $_\parallel$ and the argument $\boldsymbol{k}$ are omitted for brevity and

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{k^2} \int_{\mathsf{L}} dv\, \frac{f'_0(v)}{v - \omega/k}, \qquad f_0(v) \doteq \sum_s \frac{\omega_{ps}^2}{\omega_p^2}\, f_{0s}(v). \tag{10.33}$$

[2]Remember that this model holds only for smooth enough $\epsilon_\parallel(\omega, \boldsymbol{k})$. Distributions with narrows beams can be unstable even when $\epsilon_i = 0$; for example, see Problem PI.3.

At $\operatorname{Im}\omega > 0$, the Landau contour L in Eq. (10.33) can be replaced with the real axis:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv\, \frac{f_0'(v)}{v - \omega/k}, \tag{10.34}$$

where the argument k is omitted for brevity. If f_0' is absolutely integrable, the right-hand side is analytic, which is proven like Lemma 1 in Lecture 10. At $\operatorname{Im}\omega = 0$, one has

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f_0'(v)}{v - \omega/k} - i\pi \frac{\omega_p^2}{k|k|}\, f_0'\left(\frac{\omega}{k}\right). \tag{10.35}$$

This function is also analytic provided that f_0 is a physical distribution, which are always smooth on the real axis.[3] Thus, $\epsilon(\omega)$ is analytic at $\operatorname{Im}\omega \geq 0$. This property allows one to assess plasma stability without actually solving Eq. (10.32), specifically, as follows.

10.3.1 Nyquist theorem

Consider the following integral over real ω:

$$\bar{N} \doteq \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \frac{\epsilon'(\omega)}{\epsilon(\omega)}. \tag{10.36}$$

At large enough ω, one has $\sigma \propto \omega^{-1}$ (Appendix AI.1), so $\epsilon'(\omega) \propto \omega^{-3}$. Thus one can just as well write Eq. (10.36) as

$$\bar{N} = \frac{1}{2\pi i} \oint_C d\omega\, \frac{\epsilon'(\omega)}{\epsilon(\omega)}, \tag{10.37}$$

where C is the closed contour that includes the real axis $\mathbb{R}$ and a semi-circle of infinite radius at $\operatorname{Im}\omega > 0$. By analyticity of ϵ in the area encircled by C, the function $\epsilon'(\omega)/\epsilon(\omega)$ can have singularities within C only at $\omega = \omega_q$ that satisfy $\epsilon(\omega_q) = 0$. Also due to analyticity of ϵ, one has $\epsilon(\omega) = \alpha_q(\omega - \omega_q)^{n_q}$ in the vicinity of these points, where α_q is some constant, and n_q is some positive integer. Then, the integral over C can be expressed as a sum of residues at ω_q:

$$\begin{aligned}
\bar{N} &= \sum_q \frac{1}{2\pi i} \oint_{C_q} d\omega\, \frac{\epsilon'(\omega)}{\epsilon(\omega)} \\
&= \sum_q \frac{1}{2\pi i} \oint_{C_q} d\omega\, \frac{n_q \alpha_q (\omega - \omega_q)^{n_q - 1}}{\alpha_q\, (\omega - \omega_q)^{n_q}} \\
&= \sum_q \frac{n_q}{2\pi i} \oint_{C_q} d\omega\, \frac{1}{\omega - \omega_q} \\
&= \sum_q n_q.
\end{aligned} \tag{10.38}$$

In the case of simple poles ($n_q = 1$), the right-hand side equals the sum of zeros of ϵ in the upper half of the complex-ω plane, i.e., the number of unstable modes. One can also extend this statement to general n_q assuming the convention that each n_qth-order zero counts as n_q zeros.

Notice now that $\bar{N}$ can also be calculated differently:

$$\bar{N} = \frac{1}{2\pi i} \int_{\mathbb{R}} d\omega\, \frac{\epsilon'(\omega)}{\epsilon(\omega)} = \frac{1}{2\pi i} \int_{\epsilon(\mathbb{R})} \frac{d\epsilon}{\epsilon} = \frac{1}{2\pi i} [\ln \epsilon(\omega \to +\infty) - \ln \epsilon(\omega \to -\infty)], \tag{10.39}$$

[3] In contrast, away from the real axis, $f_0(v)$ is not necessarily analytic (cf. Problem PIV.2), so this argument is not extendable to $\operatorname{Im}\omega < 0$.

where $\epsilon(\mathbb{R})$ is the projection of the real axis in the ω space to the ϵ space. By definition of the complex logarithm, $\ln \epsilon = \ln |\epsilon| + i\vartheta$, where $\vartheta \doteq \arg \epsilon$. Also, $|\epsilon(\omega \to \pm\infty)| = 1$, so

$$\bar{N} = \frac{1}{2\pi i}[i\vartheta(\omega \to +\infty) - i\vartheta(\omega \to -\infty)] = \frac{\Delta\vartheta}{2\pi}. \tag{10.40}$$

The expression on the right-hand side equals the number of times that $\epsilon(\mathbb{R})$ encircles the origin. In combination with Eq. (10.38), we then arrive at the following theorem, known as the Nyquist theorem:

> **Theorem:** The number of unstable modes equals the number of times the contour $\epsilon(\mathbb{R})$ encircles the origin.

10.3.2 Single-peak distributions

The Nyquist theorem shows that single-peak distributions f_0 cannot support unstable modes.[4] This is seen as follows. On the contour $\mathbb{R}$, which is the real axis in the complex-ω plane, one has

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f_0'(v)}{v - \omega/k} - i\pi \frac{\omega_p^2}{k^2} f_0'\left(\frac{\omega}{k}\right). \tag{10.41}$$

Let us assume that, when projected to the complex-ϵ plane, this contour encircles the origin. For this to occur, there must be some $\omega = \omega_*$ such that $\operatorname{Re}\epsilon(\omega_*) < 0$ and $\operatorname{Im}\epsilon(\omega_*) = 0$. Since ω_* is real by definition of $\mathbb{R}$, this requires the following two conditions be satisfied simultaneously:

$$1 - \frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f_0'(v)}{v - \omega_*/k} < 0, \tag{10.42a}$$

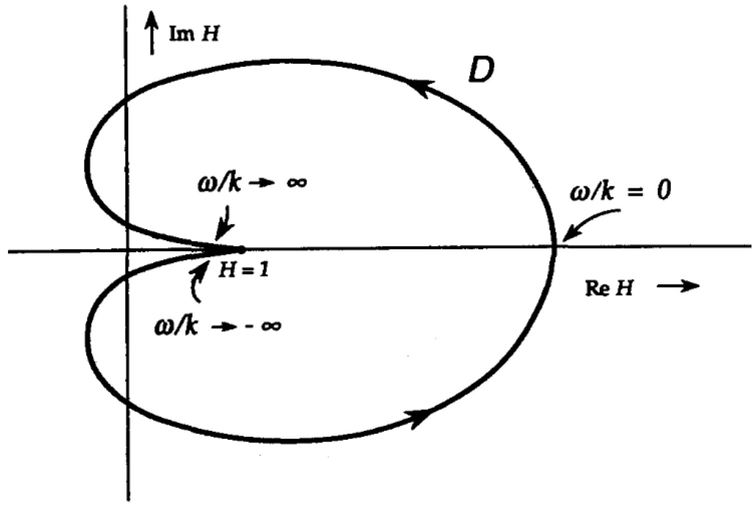
$$\pi \frac{\omega_p^2}{k^2} f_0'\left(\frac{\omega_*}{k}\right) = 0. \tag{10.42b}$$

Suppose that $f_0(v)$ has only one peak, say, at some $v = v_*$. Then, Eq. (10.42b) implies $\omega_*/k = v_*$, so the inequality (10.42a) can be expressed as follows:

$$\frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} dv\, \frac{f_0'(v)}{v - v_*} > 1. \tag{10.43}$$

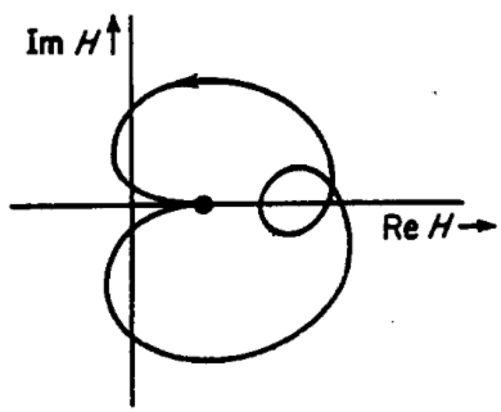
But because the signs of $f_0'(v)$ and of $(v - v_*)$ are opposite at all v, this integral is negative; hence, the inequality (10.43) cannot be satisfied. This means that our assumption regarding the existence of ω_* is invalid, so $\epsilon(\mathbb{R})$ cannot encircle the origin. This means that by the Nyquist theorem, single-peak distributions cannot support unstable modes.

[4] In the figures below, which are taken from Ref. [1], H is the same as our ϵ.



10.3.3 Double-peak distributions

Suppose now that f_0 has two peaks. Then there are three extrema: a maximum v_1, a minimum v_2, and another maximum v_3. Then, much like in the previous case, there are generally three points where $\epsilon(\mathbb{R})$ crosses the real axis in the complex-ϵ plane, namely at $\omega_j = kv_j$. This implies a *loop*:



However, as long as v_1 and v_3 are not too far from each other, the loop is small and thus does not encircle the origin. This means that by the Nyquist theorem, the appearance of a second peak does *not* immediately lead to an instability. Having an instability requires the peaks to be sufficiently far from each other.

Lecture 11

Electrostatic waves in isotropic Maxwellian plasma

In this lecture, we apply the results of the previous lectures to study basic electrostatic waves in isotropic Maxwellian plasma. (Because *only* such waves will be considered, the index $_\parallel$ will be omitted.) First, we present two alternative but equivalent calculations of the Maxwellian-species susceptibility. Then, we discuss asymptotic approximations and, finally, apply them to explicitly calculate an approximate dispersion relation for Langmuir waves and ion acoustic waves in Maxwellian electron plasma.

11.1 Susceptibility of Maxwellian plasma

11.1.1 Plasma dispersion function

Let us consider species s with the Maxwellian distribution

$$f_{0s}(v) = \frac{1}{\sqrt{2\pi}v_{Ts}} \exp\left(-\frac{v^2}{2v_{Ts}^2}\right). \tag{11.1}$$

From Eq. (8.30), one obtains

$$\begin{aligned}
\bar{\Sigma}_{s,k}(t) &= \frac{\omega_{ps}^2}{4\pi}\frac{\partial}{\partial k}\left[k\int_{-\infty}^{\infty} \mathsf{d}v\, \frac{1}{\sqrt{2\pi}v_{Ts}} \exp\left(-\frac{v^2}{2v_{Ts}^2} - ikvt\right)\right] \\
&= \frac{\omega_{ps}^2}{4\pi}\frac{\partial}{\partial \alpha}\left[\alpha\int_{-\infty}^{\infty} \mathsf{d}z\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - i\alpha z\right)\right] \\
&= \frac{\omega_{ps}^2}{4\pi}\frac{\partial}{\partial \alpha}[\alpha I(\alpha)],
\end{aligned} \tag{11.2}$$

where $\alpha \doteq kv_{Ts}t$ and

$$\begin{aligned}
I(\alpha) &\doteq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathsf{d}z\, \exp\left(-i\alpha z - \frac{z^2}{2}\right) \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathsf{d}y\, \exp\left(-\frac{2yi\alpha}{\sqrt{2}} - y^2\right) \\
&= \frac{\mathsf{e}^{-\alpha^2/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathsf{d}y\, \exp\left[-\left(y + \frac{i\alpha}{\sqrt{2}}\right)^2\right] = \mathsf{e}^{-\alpha^2/2}.
\end{aligned} \tag{11.3}$$

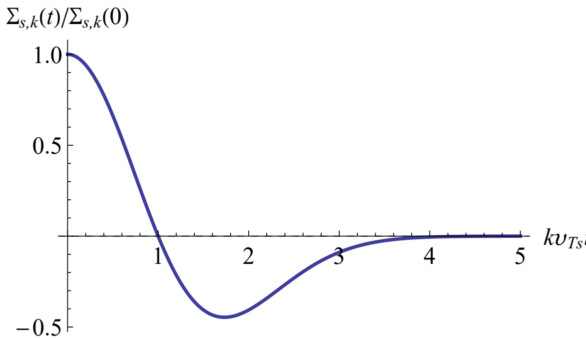


Figure 11.1: $\bar{\Sigma}_{s,k}(t)$ for Maxwellian plasma.

Using

$$\frac{\partial}{\partial\alpha}\left[\alpha I(\alpha)\right] = \frac{\partial}{\partial\alpha}\left(\alpha e^{-\alpha^2/2}\right) = (1-\alpha^2)e^{-\alpha^2/2}, \tag{11.4}$$

one obtains

$$\bar{\Sigma}_{s,k}(t) = \frac{\omega_{ps}^2}{4\pi}\left[1-(kv_{Ts}t)^2\right]\exp\left[-\frac{(kv_{Ts}t)^2}{2}\right]. \tag{11.5}$$

It is seen then that the phase mixing occurs on the time scale $(kv_{Ts})^{-1}$, as anticipated (Fig. 11.1). Also, using Eq. (11.5), one can readily calculate the conductivity in the spectral representation,

$$\sigma_s(\omega,k) = \frac{\omega_{ps}^2}{4\pi}\int_0^\infty dt\left[1-(kv_{Ts}t)^2\right]\exp\left[i\omega t - \frac{1}{2}(kv_{Ts}t)^2\right]. \tag{11.6}$$

It is also common to express this result as follows. Let us introduce the so-called *plasma dispersion function*[1] (Fig. 11.2)

$$Z(\zeta) \doteq i\int_0^\infty dz\,\exp\left(i\zeta z - \frac{z^2}{4}\right) = i\sqrt{\pi}e^{-\zeta^2} - 2\mathcal{S}(\zeta), \tag{11.7}$$

where $\mathcal{S}(\zeta)$ is the Dawson function,

$$\mathcal{S}(\zeta) \doteq e^{-\zeta^2}\int_0^\zeta dy\,e^{y^2} \equiv \frac{1}{2}\,e^{-\zeta^2}\sqrt{\pi}\,\mathrm{erfi}(\zeta). \tag{11.8}$$

Then, σ_s can be expressed as (Exercise 11.1)

$$\sigma_s(\omega,k) = \frac{\omega_{ps}^2}{4\pi\omega}\,i\zeta_s^2 Z'(\zeta_s), \qquad \zeta_s \doteq \frac{\omega}{kv_{Ts}\sqrt{2}}. \tag{11.9}$$

The corresponding susceptibility is given by

$$\chi_s(\omega,k) = \frac{4\pi i\sigma_s(\omega,k)}{\omega} = -\frac{\omega_{ps}^2}{\omega^2}\,\zeta_s^2 Z'(\zeta_s) = -\frac{Z'(\zeta_s)}{2k^2\lambda_{Ds}^2}, \tag{11.10}$$

where $\lambda_{Ds} \doteq v_{Ts}/\omega_{ps}$ is the Debye length of species s. Because Z is an entire function of its argument, the expression on the right-hand side of Eq. (11.10) is an entire function of ω. In particular, this means that this expression can be used at any $\mathrm{Im}\,\omega$.

[1] We assume $k > 0$ for simplicity. For the general case and additional details, see Sec. 8.14 in Ref. [1]. Some properties of the plasma dispersion function are also summarized in Appendix AIV.1.

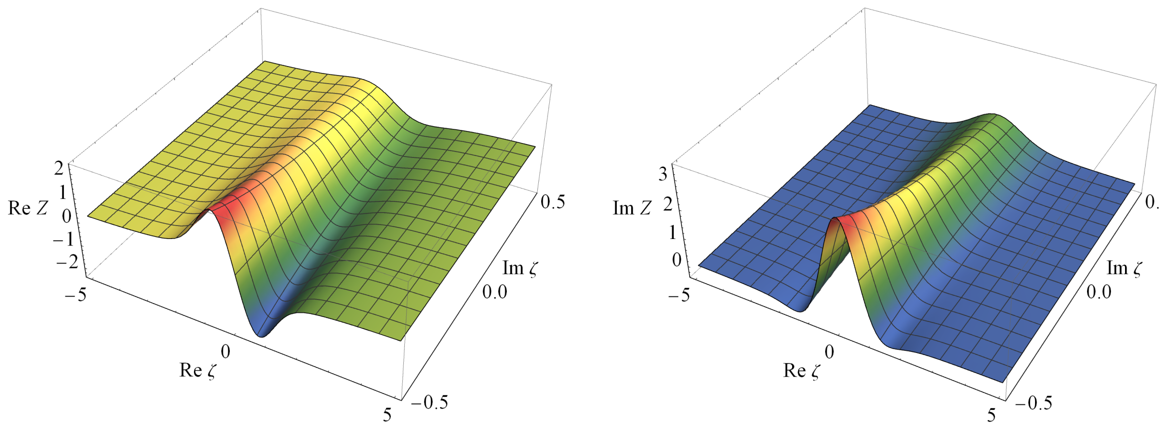


Figure 11.2: $\operatorname{Re} Z(\zeta)$ (left) and $\operatorname{Im} Z(\zeta)$ (right) as functions of $(\operatorname{Re}\zeta, \operatorname{Im}\zeta)$.

Exercise 11.1: Derive Eq. (11.9).

11.1.2 An alternative derivation using Landau's rule

Now, let us derive an alternative representation of χ_s for Maxwellian species starting from Eq. (10.26). First, note that

$$f'_{0s}(v) = \frac{1}{\sqrt{2\pi} v_{Ts}} \left(-\frac{v}{v_{Ts}^2}\right) \exp\left(-\frac{v^2}{2v_{Ts}^2}\right), \tag{11.11}$$

so

$$\chi_s(\omega, k) = \frac{\omega_{ps}^2}{k^2 v_{Ts}^2} \frac{1}{\sqrt{2\pi}} \int_{\mathsf{L}} du\, \frac{u e^{-u^2/2}}{u - \frac{\omega}{k v_{Ts}}} = -\frac{1}{2k^2 \lambda_{Ds}^2} \frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, \frac{-2z e^{-z^2}}{z - \zeta_s}. \tag{11.12}$$

Note that

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, \frac{-2z e^{-z^2}}{z - \zeta_s} &= \frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, \frac{1}{z - \zeta_s} \left(\frac{d e^{-z^2}}{dz}\right) \\
&= -\frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, e^{-z^2} \frac{d}{dz} \left(\frac{1}{z - \zeta_s}\right) \\
&= \frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, e^{-z^2} \frac{d}{d\zeta_s} \left(\frac{1}{z - \zeta_s}\right) \\
&= \frac{1}{\sqrt{\pi}} \frac{d}{d\zeta_s} \int_{\mathsf{L}} dz\, e^{-z^2} \left(\frac{1}{z - \zeta_s}\right).
\end{aligned} \tag{11.13}$$

Then, finally,

$$\chi_s(\omega, k) = -\frac{\tilde{Z}'(\zeta_s)}{2k^2 \lambda_{Ds}^2}, \qquad \tilde{Z}(\zeta) \doteq \frac{1}{\sqrt{\pi}} \int_{\mathsf{L}} dz\, \frac{e^{-z^2}}{z - \zeta}. \tag{11.14}$$

This result is identical to Eq. (11.10) except the plasma dispersion function Z is replaced with $\tilde{Z}$. Thus, $\tilde{Z}$ given by (11.14) is just an alternative representation of Z.

11.1.3 General susceptibility

The results of the previous sections are summarized as follows. Maxwellian species have the susceptibility

$$\chi_s(\omega, k) = -\frac{Z'(\zeta_s)}{2k^2\lambda_{Ds}^2}, \qquad \zeta_s = \frac{\omega}{kv_{Ts}\sqrt{2}}. \tag{11.15}$$

Here, Z is the plasma dispersion function, which is given by Eq. (11.7) but can also be expressed as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{\mathsf{L}} dz\, \frac{e^{-z^2}}{z-\zeta}. \tag{11.16}$$

For real ζ, this gives

$$Z_r(\zeta) = \frac{1}{\sqrt{\pi}} \fint_{-\infty}^{\infty} dz\, \frac{e^{-z^2}}{z-\zeta}, \qquad Z_i(\zeta) = \sqrt{\pi}\, e^{-\zeta^2}. \tag{11.17}$$

Then,

$$\epsilon_r(\omega_r, k) = 1 - \sum_s \left(\frac{Z_r'(\zeta_r)}{2k^2\lambda_D^2}\right)_s, \tag{11.18a}$$

$$\epsilon_i(\omega_r, k) = -\sum_s \left(\frac{Z_i'(\zeta_r)}{2k^2\lambda_D^2}\right)_s = \sqrt{\pi}\sum_s \left(\frac{\zeta_r\, e^{-\zeta_r^2}}{k^2\lambda_D^2}\right)_s, \tag{11.18b}$$

where $\zeta_r \doteq \omega_r/(kv_T\sqrt{2})$. The index r in ζ_r is henceforth dropped for brevity.

11.2 Asymptotics

11.2.1 Warm species

Let us search for waves in plasma with "warm" species, i.e., such that ζ is large but finite. Then, due to the exponential factor, the integral is mainly determined by the integrand at $z \ll \zeta$. Then,

$$\frac{1}{z-\zeta} = -\frac{1}{\zeta(1-z/\zeta)} \approx -\frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{z^n}{\zeta^n}. \tag{11.19}$$

This leads to

$$Z_r(\zeta) = -\frac{\zeta^{-1}}{\sqrt{\pi}} \fint_{-\infty}^{\infty} dz\, e^{-z^2}\sum_{n=0}^{\infty}\frac{z^n}{\zeta^n} = -\frac{\zeta^{-1}}{\sqrt{\pi}} \fint_{-\infty}^{\infty} dz\, e^{-z^2}\sum_{n=0}^{\infty}\frac{z^{2n}}{\zeta^{2n}}, \tag{11.20}$$

where we used the fact that the integrals with odd powers of z are zero. Hence,

$$Z_r(\zeta) \approx -\frac{1}{\zeta}\left[J(0) + \frac{1}{\zeta^2}J(1) + \dots\right], \qquad J(n) \doteq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dz\, z^{2n} e^{-z^2}. \tag{11.21}$$

This integral can be expressed through the gamma function. Alternatively, notice that

$$J(n) = (-1)^n \bar{J}^{(n)}(1), \tag{11.22}$$

where $^{(n)}$ denotes the $n$the-order derivative and

$$\bar{J}(\beta) \doteq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz\, e^{-\beta z^2} = \frac{1}{\sqrt{\pi\beta}} \int_{-\infty}^{\infty} dz\, e^{-z^2} = \frac{1}{\sqrt{\beta}}. \tag{11.23}$$

Then, one readily obtains

$$J(0) = \bar{J}(1) = 1, \qquad J(1) = -\bar{J}'(1) = 1/2, \tag{11.24}$$

and so on. Hence,

$$Z_r(\zeta) = -\frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \ldots\right) \approx -\frac{1}{\zeta} - \frac{1}{2\zeta^3}. \tag{11.25}$$

Using Eq. (11.15), one then obtains the real part of the susceptibility in the form

$$\begin{aligned} \chi_{s,r}(\omega_r, k) &\approx -\frac{1}{2k^2\lambda_{Ds}^2}\left(\frac{1}{\zeta_s^2} + \frac{3}{2\zeta_s^4}\right) \\ &= -\frac{\omega_{ps}^2}{2k^2 v_{Ts}^2}\frac{2k^2 v_{Ts}^2}{\omega_r^2}\left(1 + \frac{3}{2\zeta_s^2}\right) \\ &= -\frac{\omega_{ps}^2}{\omega_r^2}\left(1 + \frac{3k^2 v_{Ts}^2}{\omega_r^2}\right) \end{aligned} \tag{11.26}$$

and the imaginary part of the susceptibility in the form

$$\begin{aligned} \chi_{s,i}(\omega_r, k) &\approx \sqrt{\pi}\,\frac{\zeta_s e^{-\zeta_s^2}}{k^2\lambda_{Ds}^2} \\ &= \sqrt{\frac{\pi}{2}}\frac{\omega_r}{k v_{Ts}}\frac{\omega_{ps}^2}{k^2 v_{Ts}^2}\, e^{-\zeta_s^2} \\ &= \sqrt{\frac{\pi}{2}}\frac{\omega_r}{\omega_{ps}}\frac{e^{-\zeta_s^2}}{(k\lambda_{Ds})^3}. \end{aligned} \tag{11.27}$$

11.2.2 Hot species

To approximate $Z(\zeta)$ for hot species ($\zeta \ll 1$), notice that

$$S(\zeta) = \exp(-\zeta^2)\int_0^{\zeta} dy\, \exp(y^2) \approx \int_0^{\zeta} dy = \zeta, \tag{11.28}$$

so one readily obtains[2]

$$Z(\zeta) \approx ie^{-\zeta^2}\sqrt{\pi} - 2\zeta, \qquad Z'(\zeta) \approx -2i\zeta e^{-\zeta^2}\sqrt{\pi} - 2 \approx -2(1 + i\zeta\sqrt{\pi}). \tag{11.29}$$

It is also instructive to notice that $\operatorname{Re} Z(\zeta) = -2\zeta$ for real ζ is seen from the following:

$$\begin{aligned} \frac{1}{\sqrt{\pi}} ⨍_{-\infty}^{\infty} dz\, \frac{e^{-z^2}}{z - \zeta} &= \frac{1}{\sqrt{\pi}} ⨍_{-\infty}^{\infty} du\, \frac{e^{-(\zeta+u)^2}}{u} \\ &\approx \frac{e^{-\zeta^2}}{\sqrt{\pi}} ⨍_{-\infty}^{\infty} du\, \frac{e^{-u^2} e^{-2\zeta u}}{u} \end{aligned}$$

[2] The second term on the right-hand side in Eq. (11.29) is much smaller than the first term and may, in fact, be comparable to the corrections that we neglected in the Dawson function approximation. However, this term is more important as it is the primary cause of damping, whereas corrections to the Dawson function would only slightly affect the real part of the dispersion relation.

$$
\begin{aligned}
&\approx \frac{1}{\sqrt{\pi}} \fint_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u} (1 - 2\zeta u) \\
&= \frac{1}{\sqrt{\pi}} \fint_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u} - \frac{2\zeta}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-u^2} \\
&= -2\zeta. \qquad (11.30)
\end{aligned}
$$

11.3 Waves

Now let us use the above results to derive the dispersion relations for Langmuir wave and ion acoustic from kinetic theory.

11.3.1 Langmuir waves

For Langmuir waves, the electron contribution is much larger than the ion contribution. Assuming that electrons are warm, one finds

$$\epsilon_r(\omega_r, k) \approx 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3k^2 v_{Te}^2}{\omega_r^2}\right), \qquad \epsilon_i(\omega_r, k) \approx \sqrt{\frac{\pi}{2}} \frac{\omega_r}{\omega_{pe}} \frac{e^{-\zeta_e^2}}{(k\lambda_{De})^3}. \qquad (11.31)$$

Equation (10.31a) leads to

$$0 = \epsilon_r(\omega_r, k) \approx 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3k^2 v_{Te}^2}{\omega_r^2}\right), \qquad (11.32)$$

so one obtains $\omega_r^2 = \omega_{pe}^2 + \mathcal{O}\left(k^2 v_{Te}^2\right)$. The second term on the right-hand side is small, so to the lowest (zeroth) order in the temperature, one has $\omega_r^2 \approx \omega_{pe}^2$, as expected. Then, to the next (first) order in the temperature, one has

$$0 = 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3k^2 v_{Te}^2}{\omega_{pe}^2}\right). \qquad (11.33)$$

This leads to

$$\omega_r^2 = \omega_{pe}^2\left(1 + \frac{3k^2 v_{Te}^2}{\omega_{pe}^2}\right) = \omega_{pe}^2 + 3k^2 v_{Te}^2, \qquad (11.34)$$

or in other words,

$$\omega_r \approx \pm\omega_{pe}\left(1 + \frac{3}{2}\, k^2 \lambda_{De}^2\right). \qquad (11.35)$$

Equation (10.31b) leads to

$$\frac{\omega_i}{|\omega_r|} = -\frac{1}{\omega_r} \frac{\epsilon_i(\omega_r, k)}{\partial_\omega \epsilon_r(\omega_r, k)} \approx -\sqrt{\frac{\pi}{2}} \frac{|\omega_r|^3}{2\omega_{pe}^3} \frac{e^{-\zeta_e^2}}{(k\lambda_{De})^3}, \qquad (11.36)$$

where we have substituted

$$\frac{\partial \epsilon_r(\omega_r, k)}{\partial \omega} \approx \frac{\partial}{\partial \omega}\left(1 - \frac{\omega_{pe}^2}{\omega_r^2}\right) = \frac{2\omega_{pe}^2}{\omega_r^3}. \qquad (11.37)$$

We can also use the approximation $|\omega_r| \approx \omega_{pe}$ but ζ_e in $e^{-\zeta_e^2}$ must be calculated more accurately:

$$\zeta_e^2 = \frac{\omega_r^2}{2k^2 v_{Te}^2} = \frac{\omega_{pe}^2 + 3k^2 v_{Te}^2}{2k^2 v_{Te}^2} = \frac{1}{2\kappa^2} + \frac{3}{2}, \qquad (11.38)$$

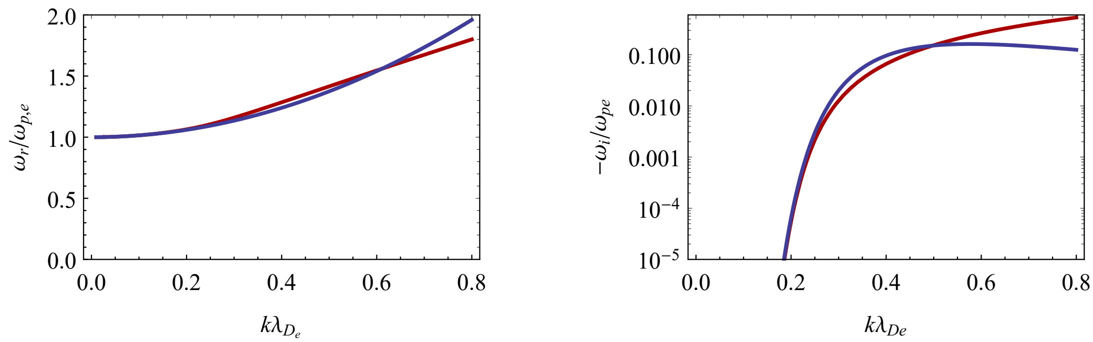


Figure 11.3: ω_r/ω_{pe} (left) and $-\omega_i/\omega_{pe}$ (right) as functions of $k\lambda_{De}$ for Langmuir waves in Maxwellian electron plasma: red – numerical solution, blue – Eqs. (11.35) and (11.39).

where $\kappa \doteq k\lambda_{De}$. This leads to

$$\frac{\omega_i}{|\omega_r|} \approx -\sqrt{\frac{\pi}{8}}\,\kappa^{-3}\exp\left(-\frac{1}{2\kappa^2}-\frac{3}{2}\right). \tag{11.39}$$

Also note that in application to Langmuir waves, our assumption that $\zeta_e \gg 1$ can be expressed simply as $\kappa \ll 1$, because $\omega^2 \approx \omega_{pe}^2$. In this case, $\kappa^{-3}\exp(-\kappa^{-2}/2) \ll 1$, so $\omega_i \ll \omega_r$; otherwise, our theory is inapplicable. A comparison with the numerical solution of the dispersion relation is shown in Fig. 11.3.

Note that the fluid calculation of the Langmuir wave dispersion relation in Sec. 7 properly captured ω_r but not ω_i. The kinetic calculation shows that in warm plasma, Langmuir waves have $\omega_i < 0$ and therefore dissipate. This effect is known as Landau damping. It was predicted theoretically in Ref. [45], and the first experimental observation was reported in Ref. [46]. The physical mechanism of Landau damping will be discussed in Lecture 12.

11.3.2 Ion acoustic waves

Assuming that electrons are hot, one obtains

$$\chi_e = -\frac{1}{2k^2\lambda_{D,e}^2}\,Z'(\zeta_e) \approx \frac{1}{k^2\lambda_{D,e}^2} + \frac{i\zeta_e\sqrt{\pi}}{k^2\lambda_{D,e}^2}. \tag{11.40}$$

Assuming that ions are cold, one obtains

$$\begin{aligned}\chi_i &= -\frac{1}{2k^2\lambda_{D,i}^2}\,Z'(\zeta_i)\\ &\approx -\frac{1}{2k^2\lambda_{D,i}^2}\left[-2ie^{-\zeta_i^2}\sqrt{\pi} + \frac{1}{\zeta_i^2}\left(1+\frac{3}{2\zeta_i^2}\right)\right]\\ &= \frac{ie^{-\zeta_i^2}\sqrt{\pi}}{k^2\lambda_{D,i}^2} - \frac{\omega_{p,i}^2}{\omega^2}\left(1+\frac{3}{2\zeta_i^2}\right)\\ &\approx -\frac{\omega_{p,i}^2}{\omega^2}.\end{aligned} \tag{11.41}$$

Hence, the dielectric function in the form

$$\epsilon_r(\omega,k) = 1 + \frac{1}{k^2\lambda_{D,e}^2} - \frac{\omega_{p,i}^2}{\omega^2}, \qquad \epsilon_i(\omega,k) = \frac{\zeta_e\sqrt{\pi}}{k^2\lambda_{D,e}^2}. \tag{11.42}$$

Equation (10.31a) leads to

$$1 + \frac{1}{k^2\lambda_{D,e}^2} - \frac{\omega_{p,i}^2}{\omega_r^2} = 0, \tag{11.43}$$

so one obtains

$$\omega_r^2 = \omega_{p,i}^2\left(1 + \frac{1}{k^2\lambda_{D,e}^2}\right)^{-1} = \frac{\omega_{p,i}^2 k^2\lambda_{D,e}^2}{1 + k^2\lambda_{D,e}^2} = \frac{k^2 C_s^2}{1 + k^2\lambda_{D,e}^2}, \tag{11.44}$$

where C_s is the ion sound speed given by

$$C_s^2 = \omega_{p,i}^2\lambda_{D,e}^2 = \frac{\omega_{p,i}^2}{\omega_{p,e}^2}\,v_{T,e}^2 = \frac{n_i e_i^2}{n_e e_e^2}\frac{T_e}{m_i} = \frac{e_i}{|e_e|}\frac{T_e}{m_i} \equiv \frac{Z_i T_e}{m_i}. \tag{11.45}$$

In particular, let us consider the limit of small $k\lambda_{\mathrm{De}}$, when

$$\frac{\partial\epsilon_r(\omega,k)}{\partial\omega} \approx \frac{2\omega_{p,i}^2}{\omega^3} \approx \frac{2}{\omega_r}\frac{1}{k^2\lambda_{D,e}^2}. \tag{11.46}$$

Then, Eq. (10.31b) leads to

$$\omega_i = -\frac{\sqrt{\pi/2}}{k^2\lambda_{D,e}^2}\frac{\omega_r}{kv_{T,e}}\frac{\omega_r}{2}k^2\lambda_{D,e}^2 = -\omega_r\sqrt{\frac{\pi}{8}}\frac{\omega_r}{kv_{T,e}} = -\omega_r\sqrt{\frac{\pi}{8}\frac{Z_i m_e}{m_i}} \ll \omega_r. \tag{11.47}$$

This shows that ions also experience Landau damping. Unlike for Langmuir waves, this damping is not *exponentially* small at small $k\lambda_{D,e}$; rather, it is small due to the smallness of m_e/m_i. The reason for this different scaling will become clear after we discuss the physical mechanism of Landau damping in the next lecture.

Lecture 12

Landau damping and kinetic instabilities

In this lecture, we discuss the physical mechanism of Landau damping and the associated nonlinear effects. Although our discussion is limited to electrostatic interactions in nonmagnetized plasma, the qualitative effects to be considered are relevant also in more general settings.

12.1 Passing and trapped particles

First, let us discuss the single-particle motion in a prescribed sinusoidal wave with a fixed amplitude E_0. In the frame of reference traveling with the phase velocity ω/k, the wave potential is stationary and the particle coordinate $\bar{x} \doteq x - (\omega/k)t$ is governed by a nonlinear-pendulum equation

$$\ddot{\bar{x}} = \frac{e_s}{m_s}\, E_0 \sin(k\bar{x}), \tag{12.1}$$

with equilibria at $k\bar{x} = 2\pi n$. (Here, n is an integer, and we will assume $e_s E_0 < 0$ and $k > 0$, but the signs can always be changed by a variable transformation $\bar{x} \to \bar{x} + \pi/k$.) Let us consider the vicinity of an equilibrium with, say, $n = 0$. Close enough to the equilibrium, Eq. (12.1) can be approximated with the equation of a harmonic oscillator,

$$\ddot{\bar{x}} + \Omega_{b0}^2 \bar{x} = 0, \tag{12.2}$$

whose frequency, called the bounce frequency, is given by

$$\Omega_{b0} \doteq \sqrt{\frac{|e_s E_0| k}{m_s}}. \tag{12.3}$$

More generally, one obtains from the energy conservation that

$$\bar{v}^2 - \frac{1}{2}\, v_t^2 [\cos(k\bar{x}) - 1] = v_0^2, \tag{12.4}$$

or, equivalently,

$$\bar{v}^2 + v_t^2 \sin^2(k\bar{x}/2) = v_0^2, \tag{12.5}$$

where $\bar{v} \doteq v - \omega/k$ is the velocity in the moving frame, $v_t \doteq 2\Omega_{b0}/k$, and $v_0 \doteq |\bar{v}(\bar{x} = 0)|$ is a constant determined by the initial conditions. The trajectory with $v_0 = v_t$ is a separatrix, it separates passing

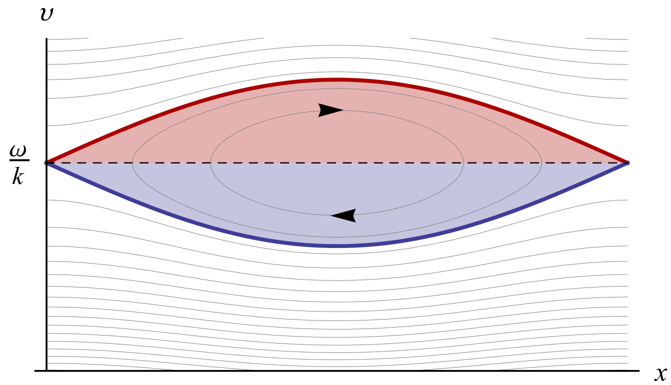


Figure 12.1: Trajectories of passing particles (non-shaded region) and trapped particles (shaded region). The arrows indicate the direction of the bounce motion. The dashed line corresponds to $v = \omega/k$.

(unbounded) trajectories from trapped (bounded) trajectories (Fig. 12.1). Passing trajectories, which correspond to $v_0 > v_t$, extend from $-\infty$ to $+\infty$; particles on these trajectories have a nonzero average velocity. In contrast, trapped trajectories, which have $v_0 < v_t$, are confined to a single wave period and have zero average velocity in the moving frame, or the average velocity ω/k in the laboratory frame.

The bounce frequency of trapped particles equals Ω_{b0} at $v_0/v_t \ll 1$ (even though Ω_{b0} is often called "the" bounce frequency), but generally, it is found as follows. From Eq. (12.5), one obtains

$$\frac{d\bar{x}}{dt} = \pm\sqrt{v_0^2 - v_t^2 \sin^2(k\bar{x}/2)}. \tag{12.6}$$

Then, the oscillation period $T = \oint dt$ can be expressed as follows:

$$T = 4\int_0^{x_*} \frac{d\bar{x}}{\sqrt{v_0^2 - v_t^2 \sin^2(k\bar{x}/2)}}, \tag{12.7}$$

where x_* is the right stopping point. Let us adopt a new variable θ such that

$$\sin\theta = \frac{v_t}{v_0}\sin(k\bar{x}/2), \qquad \cos\theta\, d\theta = \frac{kv_t}{2v_0}\cos(k\bar{x}/2)\, d\bar{x}. \tag{12.8}$$

Then, on $x \in (0, x_*)$, one has

$$d\bar{x} = \frac{2v_0}{kv_t}\frac{\cos\theta}{\sqrt{1-\sin^2(k\bar{x}/2)}}\, d\theta = \frac{2}{kv_t}\frac{\sqrt{v_0^2 - v_t^2\sin^2(k\bar{x}/2)}}{\sqrt{1-(v_0^2/v_t^2)\sin^2\theta}}\, d\theta. \tag{12.9}$$

This leads to

$$T_b = \frac{4K(r)}{\Omega_{b0}}, \qquad r \doteq \frac{v_0^2}{v_t^2}, \tag{12.10}$$

where K is the complete elliptic integral of the first kind,

$$K(r) \doteq \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r\sin^2\theta}}. \tag{12.11}$$

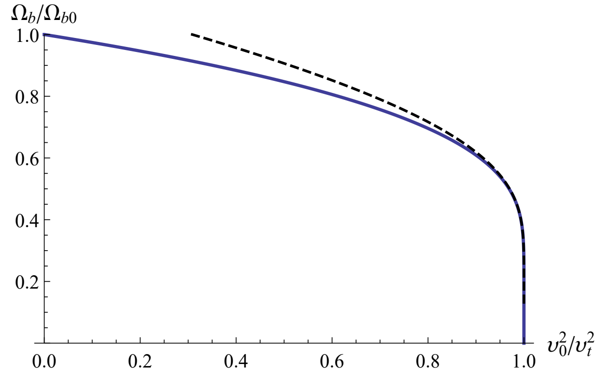


Figure 12.2: Ω_b/Ω_{b0} vs. v_0^2/v_t^2: solid blue – exact formula (12.12); dashed black – asymptotic (12.13).

The corresponding bounce frequency is

$$\Omega_b \doteq \frac{2\pi}{T_b} = \frac{\pi\Omega_{b0}}{2K(r)}. \tag{12.12}$$

As seen in Fig. 12.2, $\Omega_b \sim \Omega_{b0}$ for all trapped particles except those very close to the separatrix, where

$$\frac{\Omega_b(r \to 1)}{\Omega_{b0}} \approx \frac{\pi}{\ln[16/(1-r)]} \to 0. \tag{12.13}$$

12.2 Wave–particle energy exchange

Now, let us discuss the energy exchange between particles (electrons or ions) and the wave.

12.2.1 Direct Landau damping: wave dissipation

Particles with $|\bar{v}| \gg v_t$ do not significantly participate in the resonant energy exchange with the wave, so let us focus on trapped particles. Let us also split the trapped population into two groups: particles that initially have $v < \omega/k$ and those that initially have $v > \omega/k$. In Fig. 12.1, those are marked with blue and red colors, respectively. At $t > 0$, the "blue" particles will, on average, go up in velocity; thus, they will increase their kinetic energy in the laboratory frame at the expense of the electric-field energy. In contrast, the "red" particles will on average decrease their energy and thus give energy to the field. If the velocity distribution has a negative slope near the resonance [$f_0'(\omega/k) < 0$], then the number of blue particles is larger than the number of red particles, so overall, the wave experiences damping. This is the linear stage of Landau damping, which is described by the equations derived (for the Maxwellian distribution) in Lecture 11.

If linear damping is strong enough, the wave dissipates before nonlinear effects become important. Otherwise, Landau damping eventually enters a nonlinear stage [47, 48], when, loosely speaking, the blue particles exchange places with the red ones. When that happens, the plasma starts to give energy back to the field. But the system does not *quite* return to its original state after the bounce period, because particles with different energies have different bounce periods. By the time π/Ω_{b0}, particles with small v_0 will have performed half-rotation around the trapping island but particles with larger v_0 will have not. This means that the absolute difference between the number of particles above and below the resonance will be less than that in the initial state [Fig. 12.3(a)]. This difference is attenuated further at the next half-rotation, and so on. Eventually, one ends up with a (quasi)stationary structure where the density profile across the island is completely flattened [Fig. 12.4(a)], at least to the extent

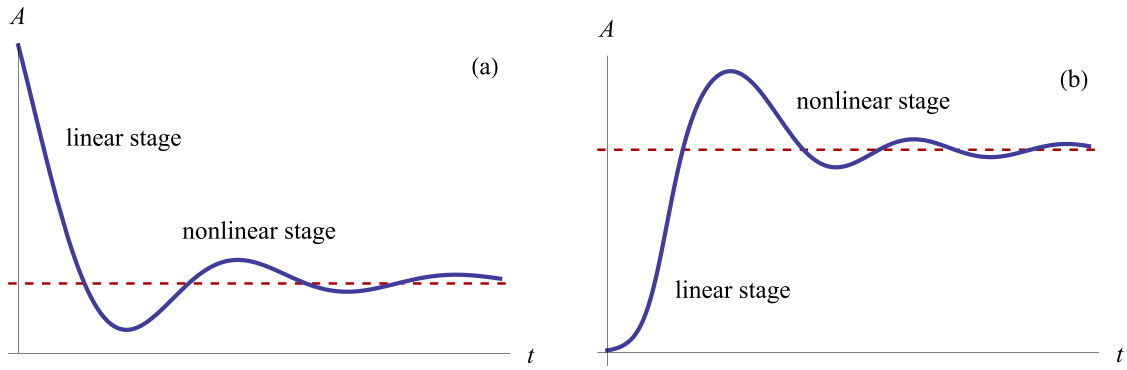


Figure 12.3: A schematic of the evolution of the wave amplitude $A(t)$ resulting from: (a) Landau damping and (b) inverse Landau damping. The linear stage corresponds to $A \propto e^{\omega_i t}$ predicted by linear theory (Lecture 11). The dashed lines mark the asymptotic values at $t \to \infty$.

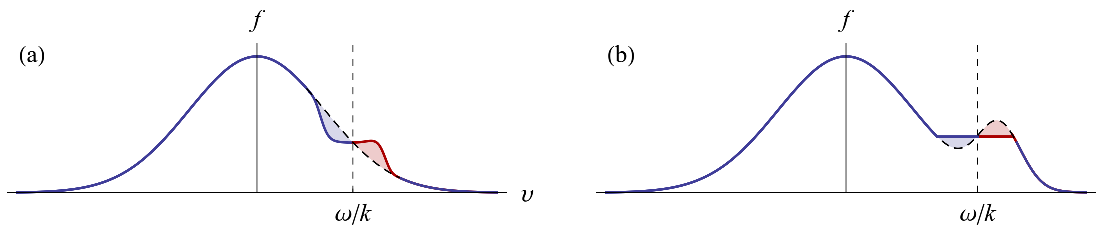


Figure 12.4: Flattening of the distribution function near the resonance $v = \omega/k$: dashed – initial distribution, solid – distribution after saturation: (a) Landau damping – particles from the blue-shaded region end up in the red-shaded region; (b) inverse Landau damping – particles from the red-shaded region end up in the blue-shaded region.

that $f_0''(\omega/k)$ is negligible. After that, individual particles continue to bounce back and forth, but the macroscopic current density remains zero, so the electrostatic energy saturates at a constant level.[1]

These nonlinear effects limit the applicability of the linear Landau-damping theory presented in Lecture 11. However, this theory can still be applicable at $t \gtrsim \Omega_{b0}^{-1}$ if a plasma is collisional. If the collision rate ν_s exceeds Ω_{b0}, then the distribution function is kept close to Maxwellian at all times and nonlinear flattening does not occur. That said, ν_s cannot be too large either. It should remain small compared to kv_{Ts}, because otherwise collisional effects overshadow kinetic effects and the wave–particle interaction ceases to be resonant, eliminating Landau damping. In summary then, for the linear theory of collisionless Landau damping to apply at $\omega_i \lesssim \Omega_{b0}$, one must have

$$\Omega_{b0} \ll \nu_s \ll kv_{Ts}. \tag{12.14}$$

12.2.2 Inverse Landau damping: wave amplification

Similar considerations apply when the slope of the distribution near the resonance is positive [$f_0'(\omega/k) > 0$]. In this case, Landau damping is replaced with *inverse* Landau damping, which causes wave amplification [Fig. 12.3(b)]. This can result in instabilities, such as the bump-on-tail instability [Fig. 12.4(b)] and the two-stream instability (Problem PIV.3). However, keep in mind that depending on the boundary conditions, inverse Landau damping may produce a continuous spectrum of waves, in which case

[1]The effect can be understood as phase mixing. It is *not* the same phase mixing as in the linear problem (Sec. 8.3), but it is similar conceptually: each particle in the trapping island acts as an oscillator, and the macroscopic current is determined by the average displacement of these oscillators from the equilibrium.

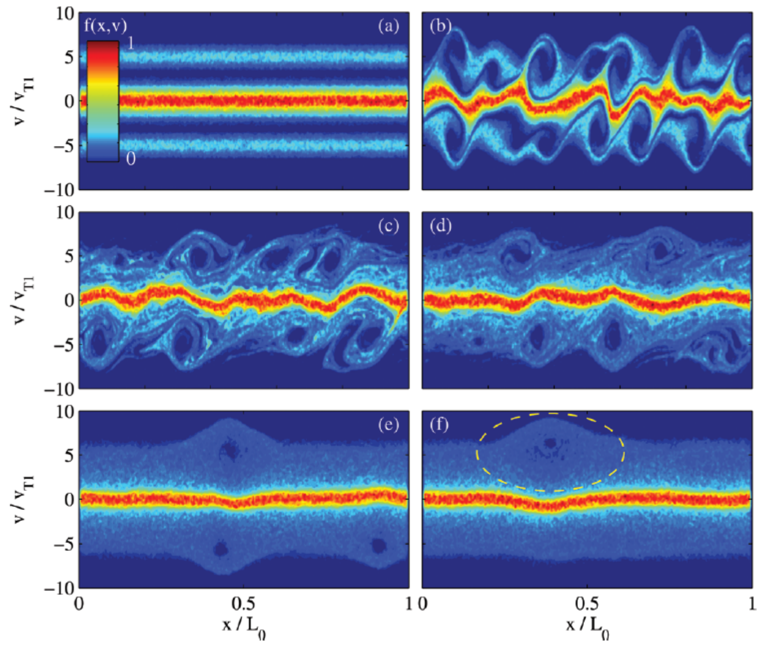


Figure 12.5: Results of particle-in-cell simulations illustrating the development of the two-stream instability (from Ref. [49]).

the nonlinear saturation is different. (This will be discussed at the end of the course.) Also note that unlike direct Landau damping, inverse Landau damping necessarily yields nonlinear structures (Fig. 12.5). We will briefly describe such structures in the next section.

12.2.3 Saturated states, BGK waves

Nonlinear waves produced as a result of Landau damping (direct or inverse) are close to so-called Bernstein–Greene–Kruskal (BGK) waves [50], which are exact stationary solutions of the nonlinear Vlasov equation. Such waves have zero linear damping rate but exhibit a nonlinear frequency shift $\delta\omega \doteq \omega - \omega_L$ [here ω_L is the frequency that satisfies the linear dispersion relation $\epsilon(\omega_L, k) = 0$], which can be calculated as follows. For simplicity, suppose that trapped particles are well separated in the velocity space from the bulk plasma (passing particles), which then can be described by real linear $\epsilon(\omega, k)$. For simplicity, assume that trapped particles are electrons. By treating them as external charges with some density $n_t(x)$, one obtains from Gauss's law that

$$k^2\epsilon(\omega, k)\varphi_k = 4\pi e_e n_{t,k}, \tag{12.15}$$

where the index k denotes the corresponding Fourier harmonics. Using the notation $U_k \doteq e_e\varphi_k$ for the particle potential energy, one can also rewrite this as follows:

$$\frac{4\pi e_e^2 n_{t,k}}{k^2 U_k} = \epsilon(\omega, k) \approx \epsilon(\omega_L, k) + \frac{\partial\epsilon(\omega_L, k)}{\partial\omega}\,\delta\omega = \frac{\partial\epsilon(\omega_L, k)}{\partial\omega}\,\delta\omega \approx \frac{2\omega_p^2}{\omega_L^3}\,\delta\omega \approx \frac{2\delta\omega}{\omega_p}. \tag{12.16}$$

where we assumed $\epsilon \approx 1 - \omega_p^2/\omega^2$. Assuming also that trapped particles are accumulated at bottoms of the wave troughs (this corresponds to a delta-shaped spatial distribution), one has

$$\frac{n_{t,k}}{k^2 U_k} \sim -\frac{\bar{n}_t}{k^2 U} = -\frac{\bar{n}_t}{|e_e k E|}. \tag{12.17}$$

(The minus is due to the fact that the density has a maximum where the potential energy has a minimum, and $\bar{n}_t$ denotes the spatial average of n_t, i.e., the number of trapped particles per wavelength.) This shows that the frequency shift is *negative* and is given by

$$\frac{\delta\omega}{\omega_p} \sim -\frac{4\pi e_e^2 \bar{n}_t}{|e_e k E|} = -\frac{\omega_t^2}{\Omega_b^2}, \tag{12.18}$$

where ω_t is the plasma frequency associated with the density $\bar{n}_t$; i.e.,

$$\omega_t \doteq \sqrt{\frac{4\pi e_e^2 \bar{n}_t}{m_e}}. \tag{12.19}$$

Also notice the unusual scaling $|\delta\omega| \sim E^{-1}$. This scaling changes if trapped particles are spread out across the island. In fact, if the distribution is smooth, one can show that $|\delta\omega| \sim \sqrt{E}$, which is indeed seen in laser–plasma interactions. For further details on dispersion and adiabatic dynamics of BGK-like waves, see, for example, Refs. [51,52]. In particular, note that due to the nonlinear frequency shift $\delta\omega = \delta\omega(|E|)$, BGK-like waves with trapped particles can be subject to nonlinear instabilities [53–56].

Lecture 13

Dispersion and dissipation in magnetized plasma

In this lecture, we extend our kinetic calculation of the plasma dielectric properties to three-dimensional electromagnetic waves and magnetized plasma.

13.1 General dispersion operator from kinetic theory

Calculating the dielectric tensor of magnetized plasma requires a more general approach than the one we have been using so far. Let us return to the linearized Vlasov equation (8.17) and consider its characteristics, i.e., phase-space trajectories $(\boldsymbol{x}'(t'), \boldsymbol{v}'(t'))$ that particles would follow in the absence of a wave:

$$\frac{d\boldsymbol{x}'}{dt'} = \boldsymbol{v}'(t'), \qquad \frac{d\boldsymbol{v}'}{dt'} = \frac{1}{m_s}\,\boldsymbol{F}_{0s}[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')]. \tag{13.1}$$

(Here, $\boldsymbol{F}_{0s}$ is the zeroth-order Lorentz force, if any. The primes denote the fact that these trajectories are different from the true ones, which account for both zeroth *and* first-order fields.) The full time derivative of $\tilde{f}_s$ along such a trajectory is given by

$$\begin{aligned}\frac{d}{dt'}\,\tilde{f}_s[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')] &= \left(\frac{\partial \tilde{f}_s}{\partial t} + \boldsymbol{v}\cdot\nabla\tilde{f}_s + \frac{\boldsymbol{F}_{0s}}{m_s}\cdot\frac{\partial \tilde{f}_s}{\partial \boldsymbol{v}}\right)_{[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')]}\\ &= R_s[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')],\end{aligned} \tag{13.2}$$

where we substituted Eq. (8.17). Assuming the initial condition

$$\boldsymbol{x}'(t'=t) = \boldsymbol{x}, \qquad \boldsymbol{v}'(t'=t) = \boldsymbol{v}, \tag{13.3}$$

one then obtains a solution for $\tilde{f}_s$ at any given $(t, \boldsymbol{x}, \boldsymbol{v})$:

$$\tilde{f}_s(t, \boldsymbol{x}, \boldsymbol{v}) = \tilde{f}_s[t, \boldsymbol{x}'(t), \boldsymbol{v}'(t)] = \tilde{f}_s^{(0)} + \int_{t_0}^{t} dt'\, R_s[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')], \tag{13.4}$$

where the integral depends on $\boldsymbol{x}$ and $\boldsymbol{v}$ through Eqs. (13.3) and $\tilde{f}_s^{(0)}$ is an integration "constant". Using Eq. (8.19) for R_s, one obtains

$$\tilde{f}_s^{(\mathrm{i})} = -\int_{t_0}^{t} dt'\, \left\{\frac{e_s n_{0s}}{m_s}\frac{\partial f_{0s}}{\partial v_c}\left[\delta_{cb}\left(1 - \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{k}}}{\hat{\omega}}\right) + \frac{v_b \hat{k}_c}{\hat{\omega}}\right]\tilde{E}_b\right\}_{[t', \boldsymbol{x}'(t'), \boldsymbol{v}'(t')]}. \tag{13.5}$$

From here, one can find the induced current density (8.21), infer the conductivity (8.22), and the dielectric tensor. For the details, see the separate slides (copied from Ref. [1]) and Problem PIV.5.

In particular, for collisionless, nonrelativistic, isotropic, Maxwellian plasma without average flows, one finds that (Exercise 13.1)

$$\boldsymbol{\epsilon} = \mathbf{1} + \sum_s \boldsymbol{\chi}_s, \qquad \boldsymbol{\chi}_s = \left(\frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \boldsymbol{Y}_n \right)_s, \tag{13.6}$$

where the matrices $\boldsymbol{Y}_n$ are given by

$$\boldsymbol{Y}_n = \begin{pmatrix} \dfrac{n^2 I_n}{\lambda} A_n & -in\,(I_n - I_n')\,A_n & \dfrac{k_\perp}{\Omega}\dfrac{nI_n}{\lambda} B_n \\ in\,(I_n - I_n')\,A_n & \left(\dfrac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n'\right) A_n & \dfrac{ik_\perp}{\Omega}(I_n - I_n')B_n \\ \dfrac{k_\perp}{\Omega}\dfrac{nI_n}{\lambda} B_n & -\dfrac{ik_\perp}{\Omega}(I_n - I_n')B_n & \dfrac{2(\omega - n\Omega)}{k_\parallel w^2} I_n B_n \end{pmatrix}. \tag{13.7}$$

Here, $I_n = I_n(\lambda)$ are modified Bessel functions of the first kind,

$$A_n = \frac{Z(\xi_n)}{k_\parallel w}, \qquad B_n = -\frac{Z'(\xi_n)}{2k_\parallel}, \qquad \lambda = \frac{k_\perp^2 w^2}{2\Omega^2}, \qquad \xi_n = \frac{\omega - n\Omega}{k_\parallel w}, \tag{13.8}$$

Z is the plasma dispersion function,

$$Z(\xi) = -2\mathcal{S}(\xi) + i\sqrt{\pi}e^{-\xi^2}, \tag{13.9}$$

and $\mathcal{S}$ is the Dawson function (Lecture 11).

Exercise 13.1: Identify the limit in which Eqs. (13.6)–(13.8) reproduce the cold-plasma dielectric tensor (6.13).

13.2 Power absorption in Maxwellian plasma

In particular, let us discuss what the above results entail for the power absorption.

13.2.1 Basic formulas

As shown in Lecture 5, the wave power dissipated per unit volume is given by

$$\mathscr{P}_{\text{abs}} = \frac{\omega}{8\pi} \boldsymbol{\mathcal{E}}^\dagger \boldsymbol{\epsilon}_{\text{A}}(\omega, \boldsymbol{k}) \boldsymbol{\mathcal{E}}, \tag{13.10}$$

where $\boldsymbol{\epsilon}_{\text{A}}$ is the anti-Hermitian part of the dielectric tensor and ω and $\boldsymbol{k}$ are the GO frequency and wavevector, which are real by definition. Because ω and $\boldsymbol{k}$ in Eq. (13.10) are real, only the imaginary parts of A_n and B_n contribute to $\boldsymbol{\epsilon}_{\text{A}}$. But ξ_n are also real, so $\operatorname{Im} A_n$ and $\operatorname{Im} B_n$ are determined entirely by the second term in Eq. (13.9):

$$\operatorname{Im} A_n = \frac{\sqrt{\pi}}{k_\parallel w} e^{-\xi_n^2}, \tag{13.11a}$$

$$\operatorname{Im} B_n = -\frac{1}{2k_\parallel}\frac{d}{d\xi_n}\left(\sqrt{\pi}e^{-\xi_n^2}\right) = \frac{\sqrt{\pi}}{k_\parallel}\xi_n e^{-\xi_n^2} = (\xi_n w)\,(\operatorname{Im} A_n). \tag{13.11b}$$

This leads to

$$\boldsymbol{\epsilon}_{\mathrm{A}} = \sum_s (\boldsymbol{\chi}_s)_{\mathrm{A}}, \qquad (\boldsymbol{\chi}_s)_{\mathrm{A}} = \left(\frac{\sqrt{\pi}}{k_\parallel w} \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\xi_n^2} e^{-\lambda} \bar{\boldsymbol{Y}}_n \right)_s, \tag{13.12}$$

where the matrices $\bar{\boldsymbol{Y}}_n$ are given by

$$\bar{\boldsymbol{Y}}_n = \begin{pmatrix} \dfrac{n^2 I_n}{\lambda} & -in(I_n - I_n') & \dfrac{k_\perp w}{\Omega} \dfrac{n I_n}{\lambda} \xi_n \\ in(I_n - I_n') & \dfrac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n' & \dfrac{i k_\perp w}{\Omega} (I_n - I_n') \xi_n \\ \dfrac{k_\perp w}{\Omega} \dfrac{n I_n}{\lambda} \xi_n & -\dfrac{i k_\perp w}{\Omega} (I_n - I_n') \xi_n & 2 I_n \xi_n^2 \end{pmatrix}. \tag{13.13}$$

In a sufficiently cold plasma, ξ_n are large everywhere except at

$$\omega \approx n\Omega_s, \qquad n = 0, \pm 1, \pm 2, \dots \tag{13.14}$$

Then, collisionless dissipation is noticeable only at these frequencies.

13.2.2 Interpretation based on single-particle dynamics

Let us try to understand the above results, particularly the resonance condition, within the single-particle picture. Consider a particle with charge e_s that follows a given trajectory $\boldsymbol{x}(t)$. The power absorbed by this particle on average over the field oscillations is given by

$$\mathsf{P} = e_s \left\langle \dot{\boldsymbol{x}}(t) \cdot \tilde{\boldsymbol{E}}[t, \boldsymbol{x}(t)] \right\rangle. \tag{13.15}$$

In the presence of a strong magnetic field $\boldsymbol{B}_0 = \bar{\boldsymbol{e}}_z B_0$, the particle trajectory can be locally expressed as

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \bar{\boldsymbol{e}}_z v_\parallel t + \tilde{\boldsymbol{x}}_\perp(t), \tag{13.16}$$

where $v_\parallel$ is the average velocity parallel to $\boldsymbol{B}_0$, and $\tilde{\boldsymbol{x}}_\perp$ is the transverse quiver displacement of the particle from a field line that the particle follows on average, $\langle \tilde{\boldsymbol{x}}_\perp \rangle = 0$. The constant $\boldsymbol{x}_0$ can always be made zero by redefining the origin. Then, assuming the notation

$$\tilde{\boldsymbol{E}}(t, \boldsymbol{x}) = \mathrm{Re}\left(\boldsymbol{\mathcal{E}}\, e^{-i\omega t + i k_\parallel z + i \boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} \right), \tag{13.17}$$

one can rewrite Eq. (13.15) as follows:

$$\mathsf{P} = \mathrm{Re} \left\langle g(t)\, e^{-i(\omega - k_\parallel v_\parallel) t} \right\rangle, \tag{13.18}$$

where g is given by

$$g(t) \doteq e_s \left[\dot{\tilde{\boldsymbol{x}}}_\perp(t) \cdot \boldsymbol{\mathcal{E}}_\perp + v_\parallel \mathcal{E}_z \right] e^{i \boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{x}}_\perp(t)}. \tag{13.19}$$

Because g is periodic in time with period $2\pi/\Omega_s$, where Ω_s is particle's gyrofrequency, this function can be represented as a Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} g_n\, e^{i n \Omega_s t} \tag{13.20}$$

with some coefficients g_n. Then,

$$\mathsf{P} = \mathrm{Re} \sum_{n=-\infty}^{\infty} g_n \left\langle e^{-i(\omega - n\Omega_s - k_\parallel v_\parallel) t} \right\rangle, \tag{13.21}$$

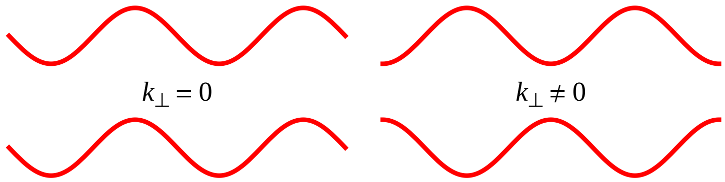


Figure 13.1: A schematic of magnetic field lines in magnetized plasma perturbed by an electromagnetic wave: left $k_\perp = 0$, right – $k_\perp \neq 0$. The x axis is vertical, and the z axis is horizontal.

so substantial energy exchange is possible when there exist an integer n such that

$$\omega \approx k_\parallel v_\parallel + n\Omega_s. \tag{13.22}$$

This is called resonant absorption at the nth harmonic. Different particles have different $v_\parallel$; however, assuming that the distribution of $v_\parallel$ is centered around zero (i.e., there are no average flows) and Ω_s are much larger than $k_\parallel w_s$, plasma as a whole can effectively absorb the wave power only if Eq. (13.14) is satisfied for some species s.

13.2.3 Landau damping

Let us discuss qualitative physical mechanisms that contribute to $\mathscr{P}_{\text{abs}}$, starting with Landau damping (LD). Electrostatic waves polarized along $\boldsymbol{B}_0$ are similar to the electrostatic waves in nonmagnetized plasma (Problem PIV.6). Then, by analogy with Lecture 12, one can identify the $\mathcal{E}_z$-driven absorption at the zeroth harmonic ($n = 0$) as LD. The corresponding absorption power density is given by

$$\mathscr{P}_{\text{LD}} = \frac{\omega}{8\pi}\,(\epsilon_{\text{A},zz})_{n=0}\,|\mathcal{E}_z|^2. \tag{13.23}$$

The corresponding power absorbed by species s is

$$\begin{aligned}\mathscr{P}_{\text{LD},s} &= \frac{\omega}{8\pi}\,(\chi_{\text{A},zz})_{s,n=0}\,|\mathcal{E}_z|^2\\ &= \frac{\omega}{8\pi}\left[\frac{\sqrt{\pi}}{k_\parallel w}\,\frac{\omega_p^2}{\omega}\,\mathsf{e}^{-\xi_0^2}\mathsf{e}^{-\lambda}2I_0(\lambda)\xi_0^2\right]_s|\mathcal{E}_z|^2\\ &= \frac{\omega|\mathcal{E}_z|^2}{4\sqrt{\pi}}\left\{\frac{\omega_p^2}{\omega^2}\left[\mathsf{e}^{-\lambda}I_0(\lambda)\right]\left(\xi_0^3\mathsf{e}^{-\xi_0^2}\right)\right\}_s.\end{aligned} \tag{13.24}$$

Note that $\mathscr{P}_{\text{LD},s}$ is maximized at $\lambda_s \to 0$, in which case it is given by

$$\mathscr{P}_{\text{LD},s} = \frac{\omega|\mathcal{E}_z|^2}{4\sqrt{\pi}}\left[\frac{\omega_p^2}{\omega^2}\,h(\xi_0)\right]_s, \qquad h(\xi_0) \doteq \xi_0^3\mathsf{e}^{-\xi_0^2}. \tag{13.25}$$

In a cold plasma, ξ_0 is large, so $h(\xi_0)$ is exponentially small; then because $\xi_s \propto w_s^{-1} \propto m_s^{1/2}$, the electron contribution dominates. In hot plasma, ξ_0 is small, so $h(\xi_0) \approx \xi_0^3$, and $\mathscr{P}_{\text{LD},s} \propto m_s^{1/2}$; in this case, the ion contribution dominates.

13.2.4 Transit-time magnetic pumping

For electromagnetic waves, there is an additional dissipation mechanism that is similar to Landau damping but determined by $\epsilon_{\text{A},yy}$. It can be understood as follows. Linear waves with nonzero $k_\perp$ produce oscillations of the *magnitude* of the total field $B \doteq |\boldsymbol{B}_0 + \tilde{\boldsymbol{B}}|$ (Fig. 13.1):

$$\tilde{B} = \sqrt{\tilde{B}_\perp^2 + (\boldsymbol{B}_0 + \tilde{\boldsymbol{B}})^2} - B_0$$

$$\begin{aligned}
&\approx \sqrt{B_0^2 + 2\boldsymbol{B}_0 \cdot \tilde{\boldsymbol{B}}} - B_0 \\
&\approx \tilde{B}_z = \frac{c}{\omega}\,(\boldsymbol{k} \times \tilde{\boldsymbol{E}})_z = \frac{c}{\omega}\,k_\perp \tilde{E}_y.
\end{aligned} \tag{13.26}$$

For simplicity, suppose that $\omega \ll \Omega_s$. Then, the oscillations of B produce an oscillating diamagnetic force $\tilde{F} \approx -\mu\,\partial_z \tilde{B}$ on each particle along the z axis, where $\mu \approx m_s v_\perp^2/2B_0$ is particle's magnetic moment. In the complex form, one can write the ensemble-average of this force as

$$\langle \tilde{F} \rangle_s = -ik_\parallel \langle \mu \rangle_s \tilde{B}_z = -ik_\parallel \frac{T_s}{B_0}\frac{c}{\omega}\,k_\perp \tilde{E}_y, \tag{13.27}$$

which can be viewed as an effective longitudinal electric field, $\tilde{E}_{\text{eff},s} = \langle \tilde{F} \rangle_s / e_s$. Like the true longitudinal electric field, $\tilde{E}_{\text{eff},s}$ produces Landau damping, which is known as transit-time magnetic pumping (TTPM). Accordingly,

$$\begin{aligned}
\mathscr{P}_{\text{TTMP},s} &\sim \mathscr{P}_{\text{LD,eff},s} \\
&= \frac{\omega|\tilde{E}_{\text{eff},s}|^2}{4\sqrt{\pi}}\left(\frac{\omega_p^2}{\omega^2}\,\xi_0^3 e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{E}_y|^2}{4\sqrt{\pi}}\,k_\parallel^2 k_\perp^2 \left(\frac{c^2 T^2}{e^2 B_0^2 \omega^2}\,\frac{\omega_p^2}{\omega^2}\,\frac{\omega^2}{k_\parallel^2 w^2}\,\frac{\omega}{k_\parallel w}\,e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{E}_y|^2}{4\sqrt{\pi}}\left(\frac{T^2}{m^2 w^2}\,\frac{k_\perp^2}{\Omega^2}\,\frac{\omega_p^2}{\omega^2}\,\frac{\omega}{k_\parallel w}\,e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{E}_y|^2}{8\sqrt{\pi}}\left(\lambda\,\frac{\omega_p^2}{\omega^2}\,\xi_0 e^{-\xi_0^2}\right)_s.
\end{aligned} \tag{13.28}$$

In particular, notice that TTMP is more efficient in high-β plasmas, because

$$\begin{aligned}
\mathscr{P}_{\text{TTMP},s} &\sim \frac{|\tilde{E}_y|^2}{8\sqrt{\pi}}\left(\lambda\,\frac{\omega_p^2}{\omega}\,\frac{\omega}{k_\parallel w}\,e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{B}_z|^2}{8\sqrt{\pi}}\left(\frac{T}{mc^2\Omega^2}\,\frac{4\pi n_0 e^2}{m}\,\xi_0 e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{B}_z|^2}{8\sqrt{\pi}}\left(\frac{4\pi n_0 T}{B_0^2}\,\xi_0 e^{-\xi_0^2}\right)_s \\
&= \frac{\omega|\tilde{B}_z|^2}{8\sqrt{\pi}}\,(\beta \xi_0 e^{-\xi_0^2})_s.
\end{aligned} \tag{13.29}$$

The maximum of this function is achieved at $\xi_0 \sim 1$, so $\mathscr{P}_{\text{TTMP, max}} \sim \beta\omega|\tilde{B}_z|^2/(8\sqrt{\pi})$.

A more rigorous expression for the TTMP comes from the general formula (13.10), where it corresponds to the contribution of $\mathcal{E}_y$ and $n = 0$:

$$\mathscr{P}_{\text{TTMP}} = \frac{\omega}{8\pi}\,(\epsilon_{\text{A},yy})_{n=0}|\mathcal{E}_y|^2, \tag{13.30}$$

which leads to a result consistent with Eq. (13.28). That said, generally, one should also take into account the "cross terms" $\mathscr{P}_{yz,n=0}$ and $\mathscr{P}_{zy,n=0}$, which can be of the same order as $\mathscr{P}_{yy,n=0}$ and $\mathscr{P}_{zz,n=0}$ [57]. In this sense, Landau damping and TTMP are not always separable.

13.2.5 Cyclotron damping

Another distinctive mechanism of collisionless dissipation is cyclotron damping (CD), which corresponds to $|n| \geq 1$. The basic features of cyclotron damping can be understood within a simple model when $\tilde{E}_z$ is negligible and λ_s is small. Then the power absorbed through CD by species s per unit volume is given by (Problem PIV.8)

$$\mathscr{P}_{\text{CD},s} \approx \sqrt{\pi}\, \frac{\omega_{ps}^2}{k_\parallel w_s} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \left(\frac{\lambda_s}{2}\right)^{n-1} \left\{ e^{-\xi_n^2} \frac{|\tilde{E}_+|^2}{16\pi} + e^{-\xi_{-n}^2} \frac{|\tilde{E}_-|^2}{16\pi} \right\}_s . \tag{13.31}$$

Note that the coefficients decrease rapidly with n, so the absorption at high-order resonances is relatively weak. Also, efficient heating is possible only when $(\xi_{\pm n})_s$ are small enough, i.e., $\omega = \pm n\Omega_s$, and the field is polarized such that its corresponding component $\tilde{E}_\pm \doteq \tilde{E}_x \pm i\tilde{E}_y$ does not vanish.

Like Landau damping, cyclotron heating can saturate through nonlinear effects. To understand these effects qualitatively, let us consider a simple model in which a transverse wave propagates strictly parallel to $\boldsymbol{B}_0$, i.e., $\boldsymbol{k} = k_\parallel \bar{\boldsymbol{e}}_z$. The wave electric field $\tilde{\boldsymbol{E}}'$ in the frame moving at the phase velocity $\boldsymbol{v}_p = (\omega/k_\parallel)\bar{\boldsymbol{e}}_z$ can be expressed through the wave fields in the laboratory frame using the well known relativistic transformation [9]. Specifically, one finds that

$$\begin{aligned}
\gamma_p^{-1} \tilde{\boldsymbol{E}}' &= \tilde{\boldsymbol{E}} + \frac{\boldsymbol{v}_p}{c} \times \tilde{\boldsymbol{B}} \\
&= \tilde{\boldsymbol{E}} + \frac{\omega}{ck} \left[\bar{\boldsymbol{e}}_z \times \left(\frac{c}{\omega} \boldsymbol{k} \times \tilde{\boldsymbol{E}} \right) \right] \\
&= \tilde{\boldsymbol{E}} + \bar{\boldsymbol{e}}_z \times (\bar{\boldsymbol{e}}_z \times \tilde{\boldsymbol{E}}) \\
&= \tilde{\boldsymbol{E}} + \bar{\boldsymbol{e}}_z (\bar{\boldsymbol{e}}_z \cdot \tilde{\boldsymbol{E}}) - \tilde{\boldsymbol{E}} (\bar{\boldsymbol{e}}_z \cdot \bar{\boldsymbol{e}}_z) \\
&= 0,
\end{aligned} \tag{13.32}$$

where γ_p is the Lorentz factor associated with $\boldsymbol{v}_p$. Then, the particle energy in this frame is conserved,

$$(v'_\perp)^2 + (v'_\parallel)^2 = \text{const.} \tag{13.33}$$

(From now on, we assume that both v' and v_p are nonrelativistic.) In the laboratory-frame variables, this can be expressed as follows:

$$(v_\perp)^2 + (v_\parallel - \omega/k_\parallel)^2 = \text{const.} \tag{13.34}$$

This describes a (semi-)circle in the $(v_\parallel, v_\perp)$ plane shifted by $\omega/k_\parallel$ along $v_\parallel$ axis (Fig. 13.2). Depending on the initial phase at which a particle enters the wave, the particle may gain or lose energy from the wave and thus can move up or down such a circle. This causes diffusion of the particle distribution in the velocity space along the circles (13.34), which are hence called *diffusion paths*. The diffusion coefficient depends on the distance from the resonance (13.22) and is maximized at

$$v_\parallel = (\omega - n\Omega_s)/k_\parallel. \tag{13.35}$$

To better understand how individual particles interact with the wave, let us differentiate Eq. (13.34) with respect to time; then, one obtains

$$\begin{aligned}
\frac{dv_\parallel}{dt} &= -\frac{1}{v_\parallel - \omega/k_\parallel} \frac{d}{dt} \left(\frac{v_\perp^2}{2} \right) \\
&= \frac{k_\parallel}{\omega(1 - kv_\parallel/\omega)}\, \boldsymbol{v}_\perp \cdot \frac{d\boldsymbol{v}_\perp}{dt} \\
&= \frac{k_\parallel}{\omega(1 - kv_\parallel/\omega)}\, \boldsymbol{v}_\perp \cdot \frac{e_s}{m_s} \left(\tilde{\boldsymbol{E}} + \frac{1}{c} \boldsymbol{v} \times \tilde{\boldsymbol{B}} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}_0 \right)_\perp .
\end{aligned} \tag{13.36}$$

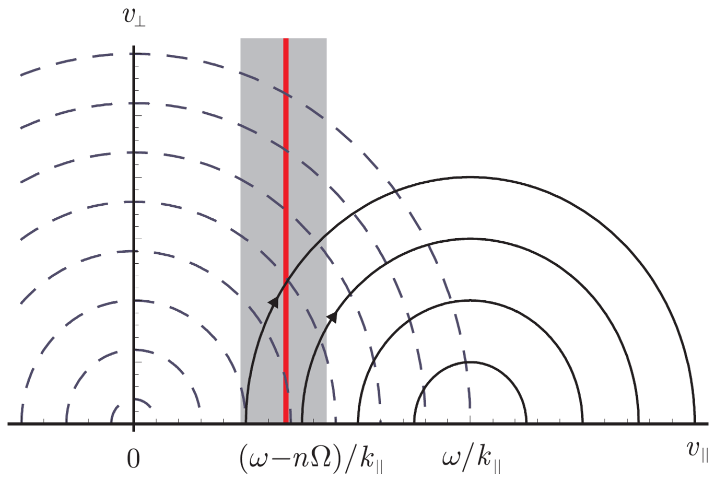


Figure 13.2: A schematic of the diffusion paths (sold curves) given by Eq. (13.34). The vertical red line denotes the resonance (13.35). Also shown are isosurfaces of the Maxwellian distribution (dashed curves). At sufficiently high $\omega/(k_\parallel w_s)$, the wave heats particles with large $v_\parallel^2$, and the majority of those have low $v_\perp^2$ (because the distribution function falls off as $\sim \exp[-(v_\parallel^2 + v_\perp^2)/w_s^2]$), so on average, particles are heated up. The shaded region corresponds to the trapping area.

Notice that

$$\begin{aligned}\left(\tilde{\boldsymbol{E}} + \frac{1}{c}\boldsymbol{v} \times \tilde{\boldsymbol{B}}\right)_\perp &= \left[\tilde{\boldsymbol{E}} + \frac{1}{\omega}\boldsymbol{v} \times (\boldsymbol{k} \times \tilde{\boldsymbol{E}})\right]_\perp \\ &= \left[\tilde{\boldsymbol{E}} + \frac{\boldsymbol{k}(\boldsymbol{v}\cdot\tilde{\boldsymbol{E}})}{\omega} - \tilde{\boldsymbol{E}}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega}\right]_\perp \\ &= \tilde{\boldsymbol{E}}\left(1 - \frac{kv_\parallel}{\omega}\right) + \boldsymbol{k}_\perp \frac{(\boldsymbol{v}\cdot\tilde{\boldsymbol{E}})}{\omega} \\ &= \tilde{\boldsymbol{E}}\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right) \end{aligned} \tag{13.37}$$

(remember that $\boldsymbol{k}_\perp = 0$ has been assumed) and

$$\boldsymbol{v}_\perp \cdot (\boldsymbol{v} \times \boldsymbol{B}_0)_\perp = \boldsymbol{v}_\perp \cdot (\boldsymbol{v}_\perp \times \boldsymbol{B}_0)_\perp = (\boldsymbol{v}_\perp \times \boldsymbol{v}_\perp) \cdot \boldsymbol{B}_0 = 0. \tag{13.38}$$

Hence, we obtain

$$\frac{dv_\parallel}{dt} = \frac{k_\parallel}{\omega(1 - k_\parallel v_\parallel/\omega)}\,\boldsymbol{v}_\perp \cdot \frac{e_s}{m_s}\tilde{\boldsymbol{E}}\left(1 - \frac{kv_\parallel}{\omega}\right) = \frac{e_s}{m_s}\,\frac{k_\parallel \boldsymbol{v}_\perp}{\omega}\cdot\tilde{\boldsymbol{E}}. \tag{13.39}$$

Suppose that the field is circularly polarized,

$$\tilde{\boldsymbol{E}} = \operatorname{Re}\left[(\bar{\boldsymbol{e}}_x \mp i\bar{\boldsymbol{e}}_y)\mathcal{E}e^{-i\omega t + ikz}\right]. \tag{13.40}$$

Then,

$$\boldsymbol{v}_\perp \cdot \tilde{\boldsymbol{E}} = |\mathcal{E}|\operatorname{Re}\left[(\boldsymbol{v}_x \mp i\boldsymbol{v}_y)e^{-i\omega t + ik_\parallel z + i\arg\mathcal{E}}\right] \tag{13.41}$$

$$= v_\perp|\mathcal{E}|\sin[k_\parallel z - (\omega \mp \Omega_s)t + \text{const}], \tag{13.42}$$

where we used that $v_x \mp iv_y \propto e^{\pm i\Omega_s t}$ (cf. Sec. 6.1). Substituting this into Eq. (13.39) leads to

$$\frac{d^2z'}{dt} = \frac{e_s}{m_s}\,E_0^{(\text{eff})}\sin(k_\parallel z'), \tag{13.43}$$

where

$$z' \doteq z - \frac{\omega \mp \Omega_s}{k_\parallel}\, t + \text{const}, \qquad E_0^{(\text{eff})} \doteq \frac{k_\parallel v_\perp}{\omega}\, |\mathcal{E}|, \tag{13.44}$$

and $k_\parallel > 0$ will be assumed for clarity. This is similar to Eq. (12.1) that we used to explore the nonlinear saturation of Landau damping earlier; the only difference is that the resonance is now shifted by Ω_s and E_0 is replaced with $E_0^{(\text{eff})}$. This means that the nonlinear saturation mechanism for cyclotron damping is similar to that of Landau damping, except that the trapping width (Fig. 13.2) is now of order $(e_s E_0^{(\text{eff})}/m_s k_\parallel)^{1/2}$ and the corresponding bounce time is

$$T \sim \sqrt{\frac{m_s}{|e_s E_0^{(\text{eff})}| k_\parallel}} = \sqrt{\frac{m_s}{|e_s E_0| k_\parallel} \left(\frac{\omega}{k_\parallel v_\perp}\right)}. \tag{13.45}$$

If $v_\perp$ is small, reaching the nonlinear stage through cyclotron damping takes longer than at Landau damping (for given $k_\parallel e_s E_0/m_s$). This is understood from the fact that a resonant particle with small $v_\perp$ travels primarily upward in Fig. 13.2, so it takes longer for $v_\parallel$ to become nonresonant.

This picture can be significantly modified, though, when waves have a broad spectrum. A particle interacting with multiple waves that have different phase velocities is not necessarily constrained to a single diffusion path but can diffuse in wider regions of the velocity space. Then, the nonlinear saturation of the wave absorption is governed by very different equations. This regime will be discussed, within a simpler model, in Lecture 15.

Lecture 14

Waves in magnetized plasma: kinetic theory

In this lecture, we apply the results of Lecture 13 to explore kinetic waves in magnetized isotropic Maxwellian plasma without flows.

14.1 Basic equations

Let us assume a homogeneous magnetized isotropic Maxwellian plasma without flows, with geometry as in Lecture 6. As shown in Lecture 13, the dielectric tensor of such plasma can be written as[1]

$$\boldsymbol{\epsilon} = \mathbf{1} + \sum_s \boldsymbol{\chi}_s, \qquad \boldsymbol{\chi}_s = \left(\frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \boldsymbol{Y}_n \right)_s, \tag{14.1}$$

where the matrices $\boldsymbol{Y}_n$ are given by

$$\boldsymbol{Y}_n = \begin{pmatrix} \dfrac{n^2 I_n}{\lambda} A_n & -in\,(I_n - I_n')\,A_n & \dfrac{k_\perp}{\Omega}\dfrac{nI_n}{\lambda} B_n \\ in\,(I_n - I_n')\,A_n & \left(\dfrac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n'\right) A_n & \dfrac{ik_\perp}{\Omega}(I_n - I_n')B_n \\ \dfrac{k_\perp}{\Omega}\dfrac{nI_n}{\lambda} B_n & -\dfrac{ik_\perp}{\Omega}(I_n - I_n')B_n & \dfrac{2(\omega - n\Omega)}{k_\parallel w^2} I_n B_n \end{pmatrix}, \tag{14.2}$$

$I_n = I_n(\lambda)$ are modified Bessel functions of the first kind, and

$$A_n = \frac{Z(\xi_n)}{k_\parallel w}, \qquad B_n = -\frac{Z'(\xi_n)}{2k_\parallel}, \qquad \lambda = \frac{k_\perp^2 w^2}{2\Omega^2}, \qquad \xi_n = \frac{\omega - n\Omega}{k_\parallel w}, \tag{14.3}$$

where Z is the plasma dispersion function introduced in Lecture 11.

As usual, the dispersion relation and the equation for the field polarization are

$$\det \boldsymbol{D}_E(\omega, \boldsymbol{k}) = 0, \qquad \boldsymbol{D}_E(\omega, \boldsymbol{k})\boldsymbol{h} = 0, \tag{14.4}$$

with the dispersion matrix given by

$$\boldsymbol{D}_E = \begin{pmatrix} \epsilon_{xx} - N_\parallel^2 & \epsilon_{xy} & \epsilon_{xz} + N_\perp N_\parallel \\ \epsilon_{yx} & \epsilon_{yy} - N^2 & \epsilon_{yz} \\ \epsilon_{zx} + N_\perp N_\parallel & \epsilon_{zy} & \epsilon_{zz} - N_\perp^2 \end{pmatrix}. \tag{14.5}$$

Below, we will focus on waves propagating perpendicularly to the magnetic field ($N_\parallel \to 0$). For parallel propagation ($N_\perp \to 0$), see Problem PIV.6.

[1] For an alternative representation of $\boldsymbol{\epsilon}$ that does not involve infinite series, see Ref. [58].

14.2 Perpendicular propagation: general considerations

At $k_\parallel \to 0$, one has $\xi_n \to \infty$, so $Z(\xi_n) \approx -\xi_n^{-1}$ [Eq. (11.25)]. This leads to

$$A_n \to \frac{1}{k_\parallel w}\left(-\frac{k_\parallel w}{\omega - n\Omega}\right) = -\frac{1}{\omega - n\Omega}, \tag{14.6a}$$

$$B_n \to -\frac{1}{2k_\parallel}\left(\frac{k_\parallel w}{\omega - n\Omega}\right)^2 = -\frac{k_\parallel w^2}{2(\omega - n\Omega)^2} \to 0. \tag{14.6b}$$

However, $(Y_n)_{zz}$ remains nonzero, because

$$(Y_n)_{zz} = \frac{2(\omega - n\Omega)}{k_\parallel w^2}\, I_n B_n \to -\frac{2(\omega - n\Omega)}{k_\parallel w^2}\, I_n \,\frac{k_\parallel w^2}{2(\omega - n\Omega)^2} = \frac{I_n}{\omega - n\Omega}. \tag{14.7}$$

Then, the polarization equation (14.4) can be written as

$$\begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} - N^2 & 0 \\ 0 & 0 & \epsilon_{zz} - N^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = 0, \tag{14.8}$$

so there are two types of modes. Ordinary (O) modes, which are polarized along the z axis, satisfy

$$\epsilon_{zz} - N^2 = 0 \tag{14.9}$$

and subsume the cold O wave as a special case. Extraordinary (X) modes, which are polarized in the plane transverse to the z axis, satisfy

$$\det \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} - N^2 \end{pmatrix} = 0 \tag{14.10}$$

and subsume the cold X wave as a special case. The effects introduced by kinetic corrections for O and X waves are similar, but Eq. (14.10) is somewhat richer than Eq. (14.9), so in this lecture we will focus on X waves. For a more comprehensive discussion, see Refs. [59, 60].

14.3 Waves in the upper-hybrid frequency range

14.3.1 Electrostatic approximation

According to cold-plasma theory (Lecture 6), X waves can be adequately described within the electrostatic approximation at least at some frequencies, so let us assume that first. Within the electrostatic approximation, the polarization vector $\boldsymbol{h}$ is parallel to $\boldsymbol{k}$, and $\boldsymbol{k}$ is parallel to the x axis by our convention. Then, from the x component of

$$\begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} - N^2 \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix} = 0, \tag{14.11}$$

one obtains, in the limit $h_y \to 0$, that (Exercise 14.1)

$$\epsilon_{xx}(\omega, k) = 0. \tag{14.12}$$

For clarity, let us focus on electron frequencies (of order $\omega_{\text{uh}} \sim \Omega_e$ or higher), where the ion contribution is negligible. Then, $\boldsymbol{\epsilon} = \mathbf{1} + \boldsymbol{\chi}_e$,

$$(\boldsymbol{\chi}_e)_{xx} \approx \frac{\omega_{pe}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_e}\, \frac{n^2 I_n(\lambda_e)}{\lambda_e}\, A_n$$

Exercise 14.1: For an x-polarized wave, the y component of Eq. (14.11) is $\epsilon_{yx}(\omega, k) = 0$, which is inconsistent with Eq. (14.12). Explain why Eq. (14.12) should be preferred over this equation.

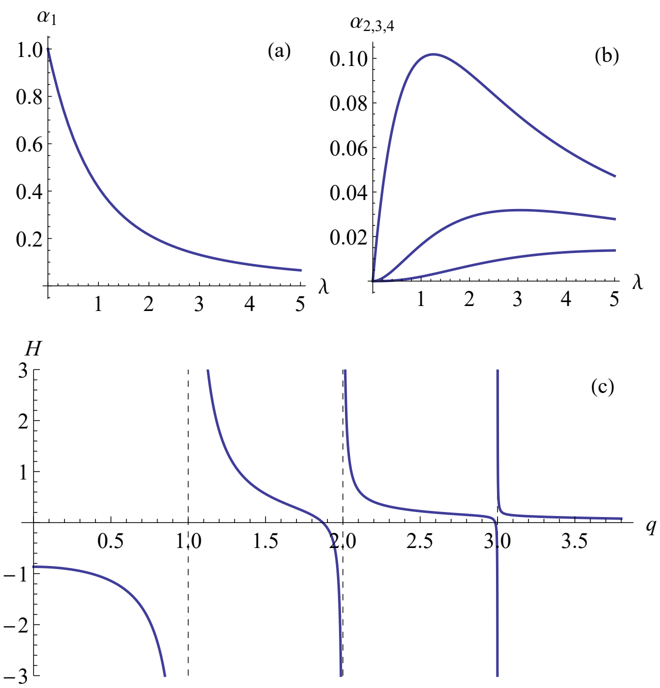


Figure 14.1: Upper row: $\alpha_n(\lambda)$ for various n. Lower row: $H(q, \lambda)$ at $\lambda = 0.2$.

$$
\begin{aligned}
&\approx -\frac{\omega_{pe}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_e} \frac{n^2 I_n(\lambda_e)}{\lambda_e} \frac{1}{\omega - n\Omega_e} \\
&= -\frac{\omega_{pe}^2}{\omega} \sum_{n=1}^{\infty} e^{-\lambda_e} \frac{n^2 I_n(\lambda_e)}{\lambda_e} \left(\frac{1}{\omega - n\Omega_e} + \frac{1}{\omega + n\Omega_e} \right) \\
&= -\frac{\omega_{pe}^2}{\omega} \sum_{n=1}^{\infty} e^{-\lambda_e} \frac{n^2 I_n(\lambda_e)}{\lambda_e} \frac{2\omega}{\omega^2 - (n\Omega_e)^2} \\
&= -\frac{\omega_{pe}^2}{\Omega_e^2} \sum_{n=1}^{\infty} e^{-\lambda_e} \frac{2 I_n(\lambda_e)}{\lambda_e} \frac{n^2}{(\omega/\Omega_e)^2 - n^2},
\end{aligned}
\tag{14.13}
$$

where we used $I_n(\lambda) = I_{-n}(\lambda)$. Let us introduce

$$
q \doteq \frac{\omega}{\Omega_e}, \qquad \beta^2 \doteq \frac{\omega_{pe}^2}{\Omega_e^2}, \qquad \alpha_n(\lambda) \doteq \frac{2 I_n(\lambda)}{\lambda}\, e^{-\lambda}. \tag{14.14}
$$

Then, Eq. (14.12) leads to

$$
\frac{1}{\beta^2} = \sum_{n=1}^{\infty} \frac{n^2 \alpha_n(\lambda_e)}{q^2 - n^2} \equiv H(q, \lambda_e), \tag{14.15}
$$

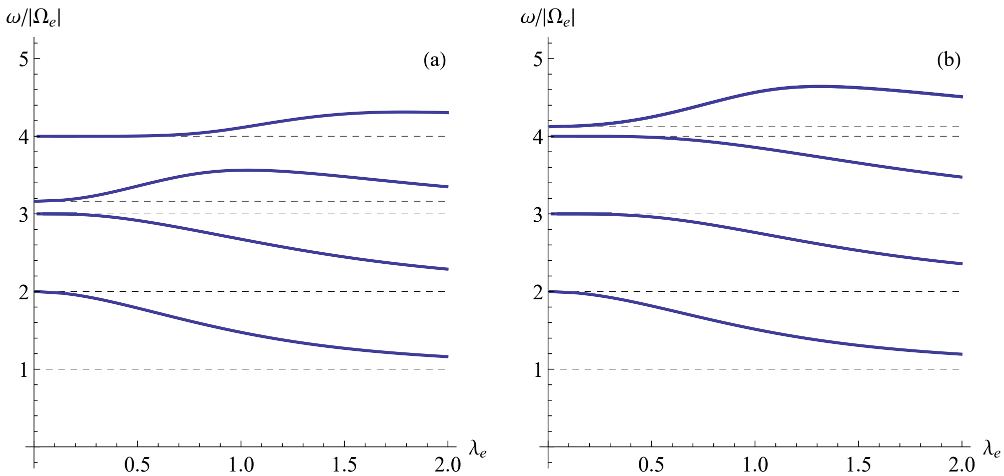


Figure 14.2: The frequencies ω of the first four EBW in overdense plasma vs. $\lambda_e = k_\perp^2 v_{Te}^2/\Omega_e^2$: left – $\beta = 3$ ($\omega_{\rm uh} \approx 3.16|\Omega_e|$); right – $\beta = 4$ ($\omega_{\rm uh} \approx 4.12|\Omega_e|$). The dashed lines correspond to $\omega = n|\Omega_e|$ with integer n and also to $\omega = \omega_{\rm uh}$.

with α_n and H illustrated in Fig. 14.1. The modes described by Eq. (14.15) are known as (electrostatic) electron Bernstein waves (EBW). Their frequencies can be approximately calculated as follows. Note that

$$\alpha_n(\lambda \to 0) \sim \left(\frac{\lambda}{2}\right)^{n-1}, \qquad \alpha_n(\lambda \to \infty) \sim \sqrt{\frac{2}{\pi}}\, \lambda^{-3/2}, \tag{14.16}$$

so there is always a mode with $\omega(\lambda_e \to 0) \to \omega_{\rm uh}$. Also, clearly, there are modes with $\omega \to n|\Omega_e|$ at both $\lambda_e \to 0$ and $\lambda_e \to \infty$. To connect these limits, let us consider EBW at $\beta \ll 1$ (very underdense plasma). In this case, $H(q, \lambda_e)$ has to be large to satisfy Eq. (14.15), so q must be close to some integer n. Then, one can retain only the nth term in the sum in Eq. (14.15), so the latter becomes

$$1 - \beta^2 \frac{n^2 \alpha_n(\lambda_e)}{q^2 - n^2} \approx 0. \tag{14.17}$$

This leads to $q^2 = n^2[1 + \beta^2 \alpha_n(\lambda_e)]$, or equivalently,

$$\omega^2 = n^2[\Omega_e^2 + \omega_{pe}^2 \alpha_n(\lambda_e)]. \tag{14.18}$$

Thus, at $n = 1$, one has

$$\omega(\lambda_e \to 0) \to \omega_{\rm uh}, \qquad \omega(\lambda_e \to \infty) \to |\Omega_e|, \tag{14.19}$$

while at $n > 1$, one has $\omega \to n|\Omega_e|$ at both $\lambda_e \to 0$ and $\lambda_e \to \infty$. The situation is somewhat different in denser plasmas, as illustrated in Fig. 14.2.

Notably, all solutions of Eq. (14.15) are real. This means EBW as considered here do not dissipate. The absence of dissipation is due to the fact that these waves propagate transversely to $\boldsymbol{B}_0$ and cannot resonantly interact with plasma particles. However, this ceases to be the case when relativistic effects are taken into account [60]. Also, $k_\parallel$ cannot be strictly zero in practice, and the ideal-plasma approximation is inapplicable at very large $k_\perp$. In real plasmas, EBW dissipate efficiently, because they have a small group velocity and thus take a long time to propagate through a plasma of a given length. This makes these waves convenient vehicles for depositing energy into plasma, especially because EBW can propagate in dense plasmas which electromagnetic waves cannot penetrate (Sec. 14.3.3). However, note that EBW cannot be launched by antennas placed outside the plasma, because EBW cannot propagate outside plasma. To understand how to launch these waves, electromagnetic effects have to be considered. Qualitatively, these effects are understood as follows.

14.3.2 Electromagnetic dispersion

As discussed in Sec. 1.2.1, the electrostatic approximation generally holds when $N^2 \gg \epsilon_{ab}$, which in our case corresponds to large enough λ_e. At small λ_e, the waves are electromagnetic and can be described as in Lecture 6 using the cold-plasma approximation. The only question then is how to connect these two limits and, in particular, what happens near cyclotron resonances, where even small thermal corrections matter. To answer this question, consider the following:

- By studying asymptotics of Eq. (14.10) at $\lambda_e \to 0$ or at $\lambda_e \to \infty$ (or numerically), one can show that there is an additional mode localized near each cyclotron resonance, $\omega \approx n|\Omega_e|$. These modes are called Dnestrovskii–Kostomarov (DK) modes.
- The dispersion curve of the cold-plasma X wave crosses the resonances at $\omega = n|\Omega_e|$, so with DK modes taken into account, one can expect the new dispersion curves to exhibit avoided crossing (Sec. 6.4.5) near the resonances. In other words, the X wave should continuously transform into DK modes.
- Homogeneous Maxwellian plasma has no free energy, so the corresponding waves are stable. By general theory of avoided crossing (Sec. 6.4.5), resonant coupling of two stable waves modifies the corresponding dispersion curves that there remain two real frequencies for each real k.

These considerations are sufficient to plot the corresponding dispersion curves unambiguously. This is illustrated in Fig. 14.3 for $n = 1$ in underdense plasma. There is no commonly accepted names for the individual branches, but one can identify them *locally* based on the closest asymptotic. Then, one can say that the electrostatic EBW continuously transforms into the cold X wave, and the latter continuously transforms into the DK mode.

14.3.3 EBW application to plasma heating

Suppose one wants to heat a magnetically confined fusion plasma with an EBW. The antenna has to be placed outside the plasma, so it launches a vacuum wave. The idea is to transform this vacuum wave into an EBW adiabatically near the upper-hybrid resonance. Coupling with the DK mode is diabatic (the corresponding frequency gap is small at nonrelativistic temperatures), so it is usually ignored. However, delivering power to the upper-hybrid resonance can be tricky. This is seen from the Clemmow–Mullaly–Allis (CMA) diagram, which shows the parameter ranges where a cold X wave can propagate (Fig. 14.4; see also Problem PIII.2). If an X wave is launched from the low-B_0 side ($|\Omega_e|/\omega < 1$, $\omega_{pe} = 0$), such wave is reflected at the low-density cutoff ($R = 0$). An X wave launched from the high-B_0 side ($|\Omega_e|/\omega > 1$, $\omega_{pe} = 0$) can enter the plasma but cannot penetrate the dense plasma core because of the high-density cutoff ($L = 0$).

To deposit power into the core then, one solution is to launch an X wave from the high-B_0 side and bounce it off the upper-hybrid resonance ($S = 0$ curve in Fig. 14.4). When the X wave approaches the upper-hybrid resonance, the wave number k continues to grow (Exercise 14.2). Then, λ_e eventually shifts to the right from the local maximum in the dispersion curve [Fig. 14.3(a)] and gradually turns into the electrostatic EBW. The group velocity changes its sign then,

$$\boldsymbol{k} \cdot \boldsymbol{v}_g = \boldsymbol{k} \cdot \frac{\partial \omega}{\partial \boldsymbol{k}} = \left(\boldsymbol{k} \cdot \frac{\partial \lambda_e}{\partial \boldsymbol{k}}\right) \frac{\partial \omega}{\partial \lambda_e} = \left(\frac{k^2 w^2}{\Omega^2}\right) \frac{\partial \omega}{\partial \lambda_e} < 0, \tag{14.20}$$

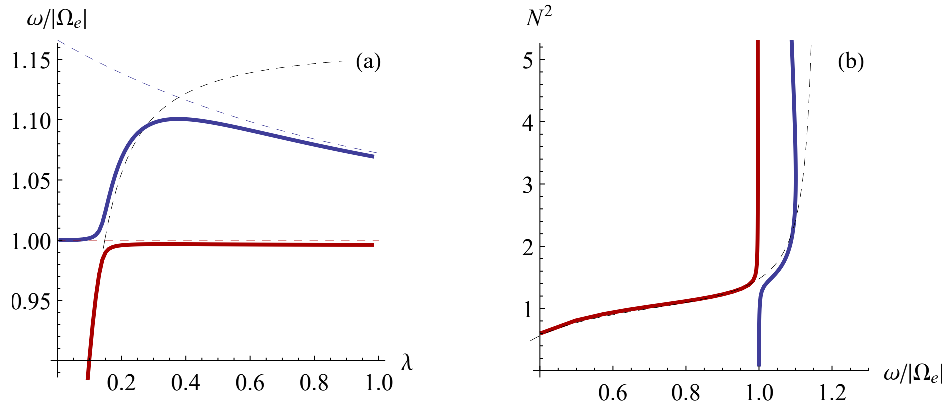


Figure 14.3: Electromagnetic dispersion curves for $\beta = 0.6$, $v_{Te}^2/c^2 = 0.1$, and ω in the vicinity of ω_{uh}: (a) – in the $(\lambda, \omega/|\Omega_e|)$ space; (b) – same in the $(\omega/|\Omega_e|, N^2)$ space. On the left: the blue dashed curve corresponds to the electrostatic approximation (14.15) for the EBW with $n = 1$; the red dashed line corresponds to $\omega = |\Omega_e|$. On both figures: the black dashed curves correspond to the cold-plasma X wave.

so the wave starts to propagate backwards. (This is called X–B conversion.) Now that the wave energy is in the EBW, it is insensitive to the cutoff at $L = 0$, so it can penetrate the dense plasma core, as desired.

Exercise 14.2: Using ray equations, explain why k continues to grow as the X wave is transforming into an EBW.

An alternative solution is to launch an O wave at a special angle that ensures complete O–X mode conversion at the location where $\omega_p^2 = \omega^2$ (Fig. 14.5). Then, the X wave propagates to the upper-hybrid resonance and transforms into the EBW as usual. (This is called O–X–B conversion.)

14.4 Waves in the lower-hybrid frequency range

14.4.1 Electrostatic dispersion relation

In the lower-hybrid frequency range, one can expect *ion* Bernstein waves (IBW), with the dispersion relation qualitatively similar to that of EBW. This is seen from the fact that the electrostatic dispersion relation is now

$$0 = \epsilon_{xx} = 1 + (\chi_e)_{xx} + \frac{\omega_{pi}^2}{\omega} \sum_{n=-\infty}^{\infty} \mathsf{e}^{-\lambda_i} \frac{n^2 I_n(\lambda_i)}{\lambda_i} A_n(\zeta_i). \tag{14.21}$$

Because

$$\frac{\lambda_e}{\lambda_i} \sim \frac{w_e^2}{w_i^2} \frac{\Omega_i^2}{\Omega_e^2} \sim \frac{m_e}{m_i} \ll 1, \tag{14.22}$$

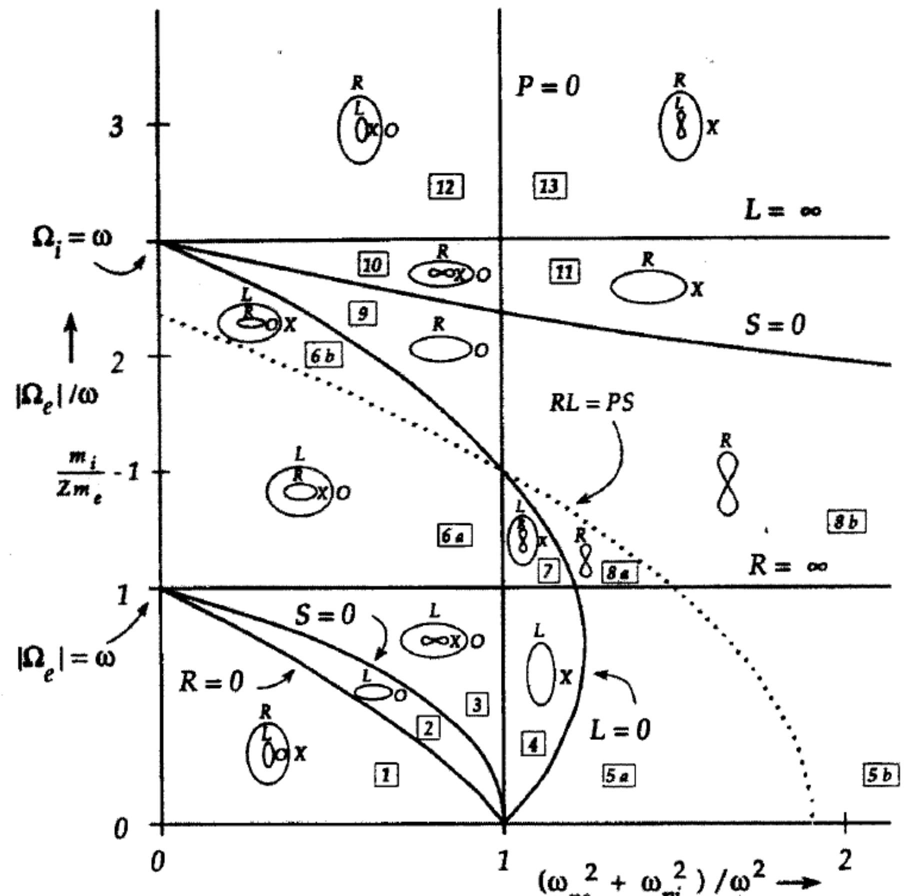


Fig. 2-1. CMA diagram for a two-component plasma. The ion-to-electron mass ratio is chosen to be 2.5. Bounding surfaces appear as lines in this two-dimensional parameter space. Cross sections of wave-normal surfaces are sketched and labeled for each region. For these sketches the direction of the magnetic field is vertical. The small mass ratio can be misleading here: the L=0 line intersects P=0 at $\Omega_i/\omega_i = 1-(Zm_e/m_i)$.

Figure 14.4: The CMA diagram for cold-plasma waves (copied from Ref. [1]). The X wave cannot propagate in the area sandwiched between the curves $R = 0$ and $S = 0$, and it also cannot propagate to the left from the curve $L = 0$.

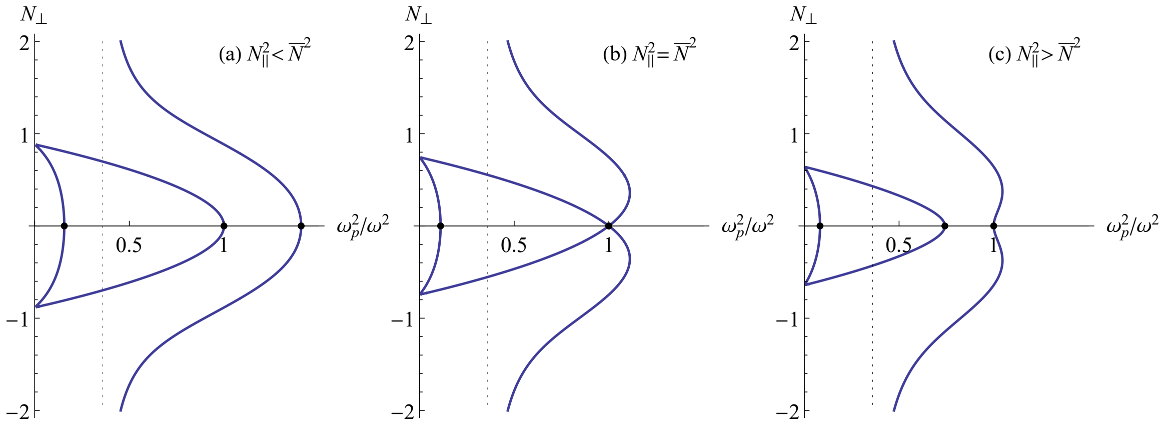


Figure 14.5: The transverse refraction index for cold-plasma waves in a homogeneous magnetic field vs. ω_p^2/ω^2. The three figures show different scenarios depending on how the initial $N_\parallel^2$ relates to the optimum value $\bar{N}_\parallel^2 = |\Omega_e|/(\omega + |\Omega_e|)$. The black points mark cutoffs, and the vertical dashed lines mark the location of the upper-hybrid resonance. See Problem PIII.2 for details.

one can treat electrons as cold for all $\lambda_i \lesssim m_i/m_e$, so $(\boldsymbol{\chi}_e)_{xx} \approx \omega_{pe}^2/\Omega_e^2$ [Eq. (14.13)]. Then, Eq. (14.21) acquires the same form, up to coefficients, as the corresponding equation for electrostatic EBW:[2]

$$1 + \frac{\omega_{pi}^2}{\left(1 + \omega_{pe}^2/\Omega_e^2\right)\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda_i} \frac{n^2 I_n(\lambda_i)}{\lambda_i} A_n(\zeta_i) = 0. \tag{14.23}$$

14.4.2 IBW application to plasma heating

The lower-hybrid frequency range, where IBW appear, is particularly important for fusion applications. Like in the case of EBW, one can excite these waves using an external antenna, but the conditions under which an external antenna can couple to lower-hybrid waves is more complicated. In principle, one can, as with EBW, start at the high-B_0 side, so the initial gyrofrequency $\Omega_{i,\text{out}}$ be not smaller than ω. But the target region corresponds to $\omega \sim \omega_{\text{lh}} \sim \sqrt{|\Omega_i \Omega_e|}$. Because ω is fixed, this requires $\Omega_{i,\text{out}} \sim \sqrt{|\Omega_i \Omega_e|}$, i.e.,

$$\frac{\Omega_{i,\text{out}}}{\Omega_i} \sim \sqrt{\frac{m_i}{m_e}} \gg 1. \tag{14.24}$$

This is impractical, so nonzero $N_\parallel$ is used instead, in which case the upper-hybrid resonance can be made accessible from vacuum (almost). For further details, see Ref. [62].

[2]If $\omega \gg \Omega_i$ and $\lambda_i \gg 1$, many terms in the sum (14.23) may be nonnegligible. Then, it may be more convenient to use the expression that does not involve infinite series [58]. Notwithstanding large λ_i, the ion susceptibility may then still be approximated well with the corresponding cold-plasma formula. For example, see Appendix B in Ref. [61].

Lecture 15

Quasilinear theory: the basics

In this lecture, we discuss the nonlinear evolution of the particle distribution driven by self-consistent waves with a sufficiently wide and dense spectrum. For simplicity, one-dimensional nonmagnetized electron plasma will be assumed.

15.1 Introduction

15.1.1 One wave: nonlinearities due to trapped particles

The linear-wave approximation used in previous lectures implies[1]

$$f_s \approx f_{s0} + f_{s1}, \qquad f_{s1} = \mathcal{O}(\tilde{E}). \tag{15.1}$$

As discussed in Lecture 12, this approximation holds in collisionless plasma only on time scales $t \lesssim \Omega_{b0}^{-1}$, where $\Omega_{b0} \propto \tilde{E}^{1/2}$ is the bounce frequency. At $t \gtrsim \Omega_{b0}^{-1}$, the distribution function acquires structure with the characteristic velocity scale equal to the trapping-island size $v_t \sim \Omega_{b0}/k$. Then, near the resonance, $\partial_v f_s$ scales as $f_s/v_t \sim \tilde{E}^{-1/2}$. This makes the third term in the Vlasov equation

$$\frac{\partial f_s}{\partial t} + v\,\frac{\partial f_s}{\partial x} + \frac{e_s}{m_s}\,\tilde{E}\,\frac{\partial f_s}{\partial v} = 0 \tag{15.2}$$

scale as $\tilde{E}\partial_v f_s \sim \tilde{E}^{1/2}$. Then, it cannot be balanced by the other terms if f_s contains only integer powers of $\tilde{E}$. This means that the standard expansion $\tilde{f}_s = \sum_n \tilde{f}_{sn}$ with $\tilde{f}_{sn} = \mathcal{O}(\tilde{E}^n)$ is inapplicable in this case, and the problem becomes complicated. Surprisingly, though, if particles interact with *many* waves simultaneously, then the problem is simplified. This is understood as follows.

15.1.2 Two waves: Chirikov criterion

First, let us consider the case when particles interact with only two waves. Individually, each wave would produce phase-space structures shown in Fig. 15.1. The two sets of trapping islands can be made stationary if one considers them in the reference frames traveling with the corresponding phase velocities, v_{p1} and v_{p2}. But in the case of two different phase velocities, there is no reference frame where both wave fields are stationary simultaneously. Hence, two different regimes can be realized depending on whether $|v_{p1} - v_{p2}|$ is larger or smaller (comparable) than the island widths, namely,

$$v_{t1} + v_{t1} \lessgtr |v_{p1} - v_{p2}|, \tag{15.3}$$

[1] Unlike in the previous lectures, here we assume the normalization for f_{s0} such that $\int f_{s0}\,dv$ equals the average density. This is done to facilitate the transition to a more general theory in Lecture 16.

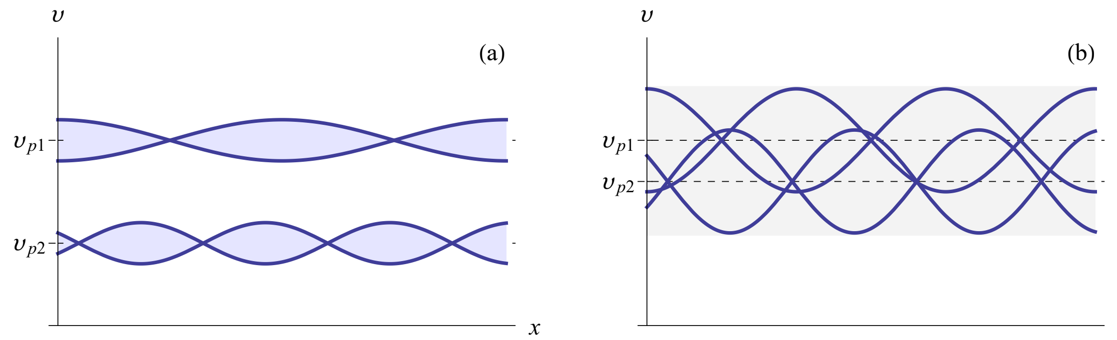


Figure 15.1: A schematic of the trapping islands produced by two waves with different phase velocities, v_{p1} and v_{p2}: (a) the Chirikov criterion is not satisfied (shaded are individual trapping islands); (b) the Chirikov criterion is satisfied (shaded is the region of stochastic motion).

which is known as the Chirikov criterion. If the Chirikov criterion is *not* satisfied ($v_{t1} + v_{t1} \ll |v_{p1} - v_{p2}|$), then any given particle can be resonant to only one of the waves. In this case, each wave saturates more or less independently from the other one, i.e., as described in Lecture 12. But if the Chirikov criterion *is* satisfied ($v_{t1} + v_{t1} \gtrsim |v_{p1} - v_{p2}|$), then the islands overlap and trapped particles no longer "belong" to a particular wave but are rather "shared" by the two waves. The trajectories of such particles are generally stochastic and extend to the whole interval $v_{p2} - v_{t2} \lesssim v \lesssim v_{p1} + v_{t1}$. Then, clearly, if the two waves saturate, they saturate *together*.

15.1.3 Many waves: statistical quasilinear approach

Now, suppose that there are not two but $N \gg 1$ waves, which create a stochastic region in the velocity space with the width $\Delta v \sim N v_t = \mathcal{O}(\tilde{E}^0)$. Such broad-band spectra are naturally formed, for instance, through bump-on-tail instabilities, with phase velocities in the range where $f_0'(v) > 0$, *assuming that the mode spectrum is sufficiently dense.*[2] Then, $\tilde{E}\partial_v f_s \sim \tilde{E} f_s/(N v_t) \propto \tilde{E} f_s$, so like in linear theory, one can search for f_s in the form (15.1). Even though the particle dynamics is complicated (stochastic) in this case, a relatively simple *statistical* theory becomes possible if $\tilde{E}$ is small enough. This theory is known as quasilinear theory.

Below, we construct quasilinear theory assuming that the Chirikov criterion is satisfied, ions are motionless, and plasma is not magnetized. (Adding magnetic field and allowing for multiple species does not affect the physics qualitatively. For example, as discussed in Sec. 13.2.5, the equations that describe cyclotron damping are similar to those describing Landau damping up to notation.) In Sec. 15.2, we will describe the "standard" version of quasilinear theory, which is relatively easy to derive but is oversimplified in some respects. A more rigorous, and more complete, version of this theory will be outlined in Lecture 16.

15.2 Basic equations

In this section, we present a simplified version of quasilinear theory [1,63,64], which uses two additional assumptions: (i) the plasma is homogeneous; (ii) the wave field evolves much faster than the linear frequencies. (The second assumption is not always articulated in literature.) For simplicity, we will also assume that the plasma is one-dimensional.

[2]Whether the spectrum is dense or not depends on the boundary conditions. In unbounded plasma, any wavenumbers are allowed, so the distribution of v_p is continuous; then, the Chirikov criterion is automatically satisfied even at vanishingly small amplitudes. In contrast, if the boundary conditions are periodic (as, for example, in a tokamak for toroidal and poloidal modes), then v_p are quantized and thus satisfying the Chirikov criterion requires finite amplitudes.

15.2.1 Equation for the distribution function

Let us start with the Vlasov equation for the electron distribution

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial f}{\partial v} = 0. \tag{15.4}$$

(Because only electrons are considered, the index $_e$ in the distribution function is omitted.). Assuming the Chirikov criterion is satisfied for many waves simultaneously, leading to stochastization of the particle dynamics in a wide range of velocities $\Delta v = \mathcal{O}(E^0)$, one can search for a solution in the form $f = f_0(t,v) + \tilde{f}(t,x,v)$, where $\tilde{f}$ is small. Then, Eq. (15.4) becomes

$$\frac{\partial f_0}{\partial t} + \frac{\partial \tilde{f}}{\partial t} + v\,\frac{\partial \tilde{f}}{\partial x} + \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial f_0}{\partial v} + \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial \tilde{f}}{\partial v} = 0. \tag{15.5}$$

By performing spatial averaging, one obtains the following equation for f_0:

$$\frac{\partial f_0}{\partial t} + \left\langle \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial \tilde{f}}{\partial v} \right\rangle = 0. \tag{15.6}$$

The equation for $\tilde{f}$ is obtained by subtracting the equation for f_0 from the original equation:

$$\frac{\partial \tilde{f}}{\partial t} + v\,\frac{\partial \tilde{f}}{\partial x} + \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial f_0}{\partial v} + \underbrace{\frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial \tilde{f}}{\partial v} - \left\langle \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial \tilde{f}}{\partial v} \right\rangle}_{\mathcal{N}} = 0. \tag{15.7}$$

When the dynamics is regular, a small field can, in principle, make $\tilde{f}$ order-one over large enough time. But here, we assume that the particle dynamics is stochastic, so correlations die out fast; then $\tilde{f}$ remains small at all times and the term $\mathcal{N}$, which is nonlinear yet has zero average, can be omitted. This is called the quasilinear approximation, and it leads to

$$\frac{\partial \tilde{f}}{\partial t} + v\,\frac{\partial \tilde{f}}{\partial x} + \frac{e_e}{m_e}\,\tilde{E}\,\frac{\partial f_0}{\partial v} \approx 0. \tag{15.8}$$

By applying the spatial Fourier transform to this equation, we also obtain

$$\frac{\partial \tilde{f}_k(t,x,v)}{\partial t} + ikv\tilde{f}_k(t,x,v) + \frac{e_e}{m_e}\,\tilde{E}_k(t)\,\frac{\partial f_0(t,v)}{\partial v} = 0. \tag{15.9}$$

As discussed earlier (Sec. 8.3), the general solution of this equation can be written as

$$\tilde{f}_k(t,v) = \tilde{f}_k(t_0,v)\,e^{-ikvt} - \frac{e_e}{m_e}\,e^{-ikvt}\int_{t_0}^{t} dt'\,e^{ikvt'}\,\tilde{E}_k(t')\,\frac{\partial f_0(t',v)}{\partial v}. \tag{15.10}$$

The function $f_0(t',v)$ in the integrand is slow, whereas $e^{ikvt'}$ and $\tilde{E}_k(t')$ are rapidly oscillating. Let us express the latter as follows:

$$\tilde{E}_k(t) = \tilde{E}_{k,0}e^{i\theta_k(t)}, \tag{15.11}$$

where $\tilde{E}_{k,0}$ is a constant, and θ_k is a complex phase, so one can introduce a local complex frequency

$$\omega_k(t) \doteq -d_t\theta_k(t) = -d_t \arg[\tilde{E}_k(t)] + i d_t \ln|\tilde{E}_k(t)| \equiv \omega_{k,r} + i\omega_{k,i}. \tag{15.12}$$

Since the actual field is real, one has

$$\tilde{E}(t,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{E}_k(t)\,e^{ikx} = \tilde{E}^*(t,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{E}_k^*(t)\,e^{-ikx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{E}_{-k}^*(t)\,e^{ikx}. \tag{15.13}$$

Box 15.1: Asymptotic representation of integrals of rapidly oscillating functions

Consider an integral of the form $\mathcal{J}(t) \doteq \int_{t_0}^{t} d\tau\, e^{i\vartheta(\tau)} F(\tau)$, where F and ϑ' are slow compared with $e^{i\vartheta}$; namely, $\varepsilon \doteq 1/(\vartheta' T_c) \ll 1$, where T_c is the characteristic time of F and ϑ'. Then,

$$\begin{aligned}
\mathcal{J}(t) &= \int_{t_0}^{t} d\tau\, \frac{F(\tau)}{i\vartheta'(\tau)} \frac{d\, e^{i\vartheta(\tau)}}{d\tau} \\
&= \frac{F(\tau)}{i\vartheta'(\tau)}\, e^{i\vartheta(\tau)}\Big|_{t_0}^{t} - \int_{t_0}^{t} d\tau\, e^{i\vartheta(\tau)} \underbrace{\frac{d}{d\tau}\left[\frac{F(\tau)}{i\vartheta'(\tau)}\right]}_{\mathcal{O}(\varepsilon F)} \\
&= \frac{F(t)}{i\vartheta'(t)}\, e^{i\vartheta(t)} + \text{const} + \mathcal{O}(\varepsilon \mathcal{J}).
\end{aligned}$$

Thus, $\tilde{E}^*_{-k} = \tilde{E}_k$. This means that and $\omega_{-k} = -\omega_k^* = -\omega_{k,r} + i\omega_{k,i}$, so

$$\omega_{-k,r} = -\omega_{k,r}, \qquad \omega_{-k,i} = \omega_{k,i}. \tag{15.14}$$

Because ω_k and f_0 are assumed slow compared to $e^{i\theta_k(t)}$, one can represent $\tilde{f}_k$ as[3] (Box 15.1)

$$\tilde{f}_k(t,v) = \tilde{g}_k(v)\, e^{-ikvt} - \frac{e_e \tilde{E}_k(t)}{im_e[kv - \omega_k(t)]} \frac{\partial f_0(t,v)}{\partial v}. \tag{15.15}$$

Assuming $\omega_{k,i} > 0$, the second term eventually dominates, so one obtains

$$\tilde{f}_k(t,v) \approx -\frac{e_e \tilde{E}_k(t)}{im_e[kv - \omega_k(t)]} \frac{\partial f_0(t,v)}{\partial v}. \tag{15.16}$$

The actual distribution $\tilde{f}(t,x,v)$ is obtained by taking the inverse Fourier transform of $\tilde{f}_k(t,v)$:

$$\tilde{f}(t,x,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{f}_k(t,v)\, e^{ikx} = -\frac{e_e}{m_e}\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\tilde{E}_k(t)\, e^{ikx}}{i[kv - \omega_k(t)]} \frac{\partial f_0(t,v)}{\partial v}. \tag{15.17}$$

Let us substitute this and the expression for $\tilde{E}(t,x)$ into the equation for f_0:

$$\begin{aligned}
\frac{\partial f_0}{\partial t} &= -\left\langle \frac{e_e}{m_e}\, \tilde{E}\, \frac{\partial \tilde{f}}{\partial v} \right\rangle \\
&= \frac{\partial}{\partial v}\left\langle \frac{e^2}{m_e^2}\, \tilde{E}(t,x) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\tilde{E}_k(t) e^{ikx}}{i[kv - \omega_k(t)]} \frac{\partial f_0(t,v)}{\partial v} \right\rangle \\
&= \frac{\partial}{\partial v}\left(\frac{e^2}{m_e^2} \left\langle \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{E}^*_{k'}(t) e^{-ik'x} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\tilde{E}_k(t) e^{ikx}}{i[kv - \omega_k(t)]} \right\rangle \frac{\partial f_0(t,v)}{\partial v} \right) \\
&= \frac{\partial}{\partial v}\left(\frac{e^2}{m_e^2} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\tilde{E}^*_{k'}(t)\tilde{E}_k(t)}{i\,[kv - \omega_k(t)]} \left\langle e^{i(k-k')x} \right\rangle \frac{\partial f_0(t,v)}{\partial v} \right) \\
&= \frac{\partial}{\partial v}\left(\frac{e^2}{m_e^2} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\tilde{E}^*_{k'}(t)\tilde{E}_k(t)}{i\,[kv - \omega_k(t)]} \frac{2\pi}{L}\, \delta(k - k')\, \frac{\partial f_0(t,v)}{\partial v} \right)
\end{aligned}$$

[3] Although Eq. (15.15) is correct as the leading-order approximation, retaining corrections $\mathcal{O}(\partial_t f_0)$ and $\mathcal{O}(\partial_t \omega_k)$ is generally necessary to keep the quasilinear model truly conservative. For Langmuir turbulence, this is not a big issue somewhat accidentally. See the footnotes below, also see Lecture 16.

$$= \frac{\partial}{\partial v}\left(\frac{e^2}{m_e^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi L}\frac{|\tilde{E}_k(t)|^2}{i[kv-\omega_k(t)]}\frac{\partial f_0(t,v)}{\partial v}\right). \tag{15.18}$$

Here, we used that the plasma length L is assumed large enough, so

$$\left\langle e^{i(k-k')x}\right\rangle = \frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{i(k-k')x} \approx \frac{2\pi}{L}\,\delta(k-k'). \tag{15.19}$$

For $\omega_{k,i} \leq 0$, the first term on the right-hand side of Eq. (15.15) may not be small compared to the second term, but we will extrapolate the final result to this case using analytic continuation, as usual. For this reason, the integration over k will, from now on, be done over the Landau contour L.

15.2.2 Diffusion coefficient

Hence, the equation for f_0 is seen to be a diffusion equation,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left[\mathsf{D}(t,v)\,\frac{\partial f_0(t,v)}{\partial v}\right], \tag{15.20}$$

where

$$\mathsf{D}(t,v) \doteq \frac{e^2}{m_e^2}\int_{\mathsf{L}}\frac{dk}{2\pi L}\frac{|\tilde{E}_k(t)|^2}{i\,[kv-\omega_k(t)]} \tag{15.21}$$

serves as the diffusion coefficient. It is also common to express D as follows:

$$\mathsf{D}(t,v) = \frac{8\pi e^2}{m_e^2}\int_{\mathsf{L}} dk\,\frac{U_k(t)}{i[kv-\omega_k(t)]}, \qquad U_k \doteq \frac{1}{2\pi L}\frac{|\tilde{E}_k(t)|^2}{8\pi}, \tag{15.22}$$

and U_k is understood as the spectral density of the electric-field energy per unit volume, because

$$\begin{aligned}
\left\langle\frac{\tilde{E}^2}{8\pi}\right\rangle &= \frac{1}{8\pi}\left\langle\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\,\tilde{E}^*_{k'}(t)e^{-ik'x}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{E}_k(t)e^{ikx}\right\rangle\\
&= \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{E}^*_{k'}(t)\tilde{E}_k(t)\,\frac{2\pi}{L}\,\delta(k-k')\\
&= \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi L}\,|\tilde{E}_k(t)|^2\\
&= \int_{-\infty}^{\infty} dk\,U_k.
\end{aligned} \tag{15.23}$$

Although Eq. (15.22) contains imaginary unit, the corresponding D is real, as easily seen from the fact that $U_k = U_{-k}$ and $\omega_{-k} = -\omega_k^*$. Furthermore, to the extent that the nonzero value of $\omega_{k,i}$ is negligible, one readily obtains, using Landau's rule, that

$$\begin{aligned}
\mathsf{D}(t,v) \approx \bar{\mathsf{D}}(t,v) &\doteq \frac{8\pi e^2}{m_e^2}\,\mathrm{Im}\int_{\mathsf{L}} dk\,\frac{U_k(t)}{kv-\omega_k(t)-i0}\\
&= \frac{16\pi^2 e^2}{m_e^2}\int_0^{\infty}\delta(kv-\omega_{k,r})\,U_k\,dk.
\end{aligned} \tag{15.24}$$

15.2.3 Field equations

To describe particles and fields self-consistently, the diffusion equation must be complemented with equations for U_k. According to the equations that we introduced earlier, one has

$$\frac{dU_k}{dt} = \frac{1}{2\pi L}\frac{1}{8\pi}\frac{d|\tilde{E}_k(t)|^2}{dt}$$

$$
\begin{aligned}
&= \frac{1}{2\pi L}\frac{1}{8\pi}\frac{d|\tilde{E}_{k,0}\,e^{i\theta_k(t)}|^2}{dt} \\
&= \frac{1}{2\pi L}\frac{|\tilde{E}_{k,0}e^{i\mathrm{Re}\,\theta_k(t)}|^2}{8\pi}\frac{de^{-2\mathrm{Im}\,\theta_k(t)}}{dt} \\
&= 2\omega_{k,i}\frac{1}{2\pi L}\frac{|\tilde{E}_{k,0}e^{i\theta_k(t)}|^2}{8\pi},
\end{aligned} \tag{15.25}
$$

or in other words,[4]

$$\frac{dU_k}{dt} = 2\omega_{k,i}U_k. \tag{15.26}$$

Since $\tilde{f}$ that we derived earlier is governed by the same equation as in stationary plasma, the frequencies $\omega_k(t)$ can be found using the GO dispersion relation:

$$1 - \frac{4\pi e^2}{m_e k^2}\int_{\mathsf{L}} dv\,\frac{\partial_v f_0(t,v)}{v-\omega_k(t)/k} = 0. \tag{15.27}$$

15.3 Properties and applications of quasilinear theory

15.3.1 Conservation laws

In summary, we have derived a closed set of equations that determine the self-consistent evolution of the particle distribution and the field spectrum:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left[\mathsf{D}(t,v)\,\frac{\partial f_0(t,v)}{\partial v}\right], \qquad \frac{dU_k}{dt} = 2\omega_{k,i}U_k, \tag{15.28}$$

where D is given by Eq. (15.22) and ω_k is found from Eq. (15.27). These equations conserve[5] the total number of particles:

$$\frac{d}{dt}\int_{-\infty}^{\infty} dv\,f_0(t,v) = \int_{-\infty}^{\infty} dv\,\frac{\partial}{\partial v}\left(\mathsf{D}\,\frac{\partial f_0}{\partial v}\right) = 0, \tag{15.29}$$

the total momentum:

$$
\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} dv\,m_e v f_0(t,v) &= \int_{-\infty}^{\infty} dv\,m_e v\frac{\partial}{\partial v}\left(\mathsf{D}\,\frac{\partial f_0}{\partial v}\right) \\
&= -\int_{-\infty}^{\infty} dv\,m_e \mathsf{D}\,\frac{\partial f_0}{\partial v} \\
&= -\int_{-\infty}^{\infty} dv\int_{\mathsf{L}} dk\,\frac{2U_k}{i(kv-\omega_k)}\frac{4\pi e^2}{m_e}\frac{\partial f_0}{\partial v} \\
&= 2i\int_{-\infty}^{\infty} dk\,U_k\,\frac{4\pi e^2}{m_e}\int_{\mathsf{L}} dv\,\frac{\partial_v f_0}{kv-\omega_k} \\
&= 2i\int_{-\infty}^{\infty} dk\,kU_k\,\frac{4\pi e^2}{m_e k^2}\int_{\mathsf{L}} dv\,\frac{\partial_v f_0}{v-\omega_k/k} \\
&= 2i\int_{-\infty}^{\infty} dk\,kU_k = 0
\end{aligned} \tag{15.30}
$$

[4] Strictly speaking, Eq. (15.26) should be the equation for the mode action $\mathcal{I}_k$ rather than for U_k. For Langmuir waves in homogeneous plasma, $\mathcal{I}_k \propto U_k/\omega_k$ and $\omega_k \approx \omega_p = \text{const}$, so the equations for $\mathcal{I}_k$ and U_k are usually not distinguished. A more rigorous approach is described in Lecture 16.

[5] As discussed in footnotes 3 and 4, the standard quasilinear theory presented here is oversimplified in two aspects. It remains conservative because the two oversimplifications cancel each other.

(where we substituted the dispersion relation and used the fact that $U_k = U_{-k}$), and the total energy:

$$\begin{aligned}
\frac{d}{dt}\left[\int_{-\infty}^{\infty} dv\, \frac{m_e v^2}{2}\, f_0(t,v) + \int_{-\infty}^{\infty} dk\, U_k(t)\right]& \\
&= \int_{-\infty}^{\infty} dv\, \frac{m_e v^2}{2}\, \partial_t f_0 + \int_{-\infty}^{\infty} dk\, \dot{U}_k \\
&= \int_{-\infty}^{\infty} dv\, \frac{m_e v^2}{2}\, \frac{\partial}{\partial v}\left(\mathsf{D}\, \frac{\partial f_0}{\partial v}\right) + \int_{-\infty}^{\infty} dk\, 2\omega_{k,i} U_k \\
&= -\int_{-\infty}^{\infty} dv\, m_e v \mathsf{D}\, \frac{\partial f_0}{\partial v} + \int_{-\infty}^{\infty} dk\, 2\omega_{k,i} U_k \\
&= -\int_{-\infty}^{\infty} dk \int_{\mathsf{L}} dv\, \frac{2U_k}{i\,(kv-\omega_k)}\, \frac{4\pi e^2 v}{m_e}\, \frac{\partial f_0}{\partial v} + \int_{-\infty}^{\infty} dk\, 2\omega_{k,i} U_k \\
&= 2i \int_{-\infty}^{\infty} dk\, (\omega_k - i\omega_{k,i}) U_k \\
&= 2i \int_{-\infty}^{\infty} dk\, \omega_{k,r} U_k \\
&= 0.
\end{aligned} \tag{15.31}$$

In the latter case, we used $\omega_{-k,r} = -\omega_{k,r}$, $U_k = U_{-k}$, and also

$$\begin{aligned}
\frac{4\pi e^2}{m_e} \int_{\mathsf{L}} dv\, \frac{v}{kv-\omega_k}\, \frac{\partial f_0}{\partial v} &= \frac{4\pi e^2}{m_e k} \int_{\mathsf{L}} dv\, \frac{kv-\omega_k+\omega_k}{kv-\omega_k}\, \frac{\partial f_0}{\partial v} \\
&= \frac{4\pi e^2}{m_e k}\left[\int_{\mathsf{L}} dv\, \frac{\partial f_0}{\partial v} + \frac{\omega_k}{k} \int_{\mathsf{L}} dv\, \frac{\partial_v f_0}{kv-\omega_k}\right] \\
&= \omega_k \left(0 + \frac{4\pi e^2}{m_e k^2} \int_{\mathsf{L}} dv\, \frac{\partial_v f_0}{v-\omega_k/k}\right) \\
&= \omega_k.
\end{aligned} \tag{15.32}$$

Let us summarize the above equations concisely:

$$\frac{d}{dt} \int dv\, f_0(t,v) = 0, \tag{15.33a}$$

$$\frac{d}{dt} \int dv\, m_e v f_0(t,v) = 0, \tag{15.33b}$$

$$\frac{d}{dt}\left[\int dv\, \frac{m_e v^2}{2}\, f_0(t,v) + \int_{-\infty}^{\infty} dk\, U_k(t)\right] = 0. \tag{15.33c}$$

Because f_0 evolves due to U_k, this theory is not entirely linear. However, since the field spectrum is determined by the local *linear* dispersion relation, this theory is called quasilinear.

15.3.2 Quasilinear evolution: broadband bump-on-tail instability

Let us return to discussing a one-dimensional broad-band bump-on-tail instability. Suppose the wave spectrum is largest initially at some k. This produces flattening at velocities close to $\omega_{k,r}/k$. But then the slope of the distribution increases at neighboring velocities. Due to this increase of f_0', the local ω_i increase too. This causes amplification of the wave field and increase of the diffusion coefficient, even in regions where it was zero initially. Hence, the flattening proceeds until the distribution becomes monotonic. This leads to formation of a "quasilinear plateau", whose height is determined by the particle conservation (Fig. 15.2).

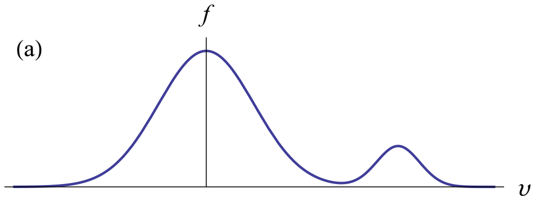


Figure 15.2: Quasilinear evolution of the distribution with a bump on tail: (a) initial distribution; (b) saturated distribution with a quasilinear plateau. The areas between the dashed curve and the solid curve in (b) are equal as required by particle conservation.

Note that the formation of a quasilinear plateau indicates that the average momentum of the resonant particles decreases. But the electrostatic field carries no momentum, so this momentum change must be absorbed by the background distribution, which apparently experiences adiabatic transformation while tail particles interact with the waves resonantly. This transformation can be explained *ad hoc* by the difference between D and $\bar{\mathsf{D}}$ (15.24) [1]. A more systematic approach is presented in Lecture 16.

Lecture 16

Quasilinear theory: resonant and adiabatic interactions

In this lecture, we amend and generalize quasilinear theory presented in Lecture 15 by accounting for nonlinear adiabatic effects caused by so-called ponderomotive forces.

16.1 Oscillation centers

A conceptually simpler quasilinear theory where the delta-approximation (15.24) *is* sufficiently accurate can be formulated using the concept of an oscillation center (OC), originally proposed in Ref. [65]. Although this concept extends to general wave–particle interactions [40], below we limit our discussion to electrostatic waves in nonrelativistic nonmagnetized plasma for simplicity.

16.1.1 OC Lagrangian

To start, let us consider an electron in a prescribed wave field $\tilde{\boldsymbol{E}} = -\nabla\tilde{\varphi}$ with local frequency ω and wavevector $\boldsymbol{k}$. The dynamics of such a particle is governed by the following Lagrangian:

$$L(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}\, m\dot{\boldsymbol{x}}^2 - e\tilde{\varphi}(t, \boldsymbol{x}). \tag{16.1}$$

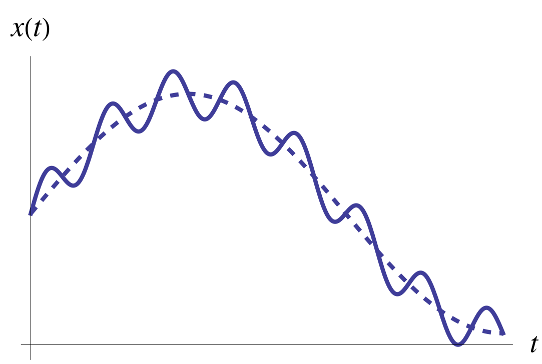


Figure 16.1: Schematic of the particle motion in a wave: particle trajectory (solid) vs. OC trajectory (dashed). The OC trajectory differs from a straight line in the presence of nonlinear effects (Sec. 16.1.2) or additional slow forces.

Let us decompose the particle coordinate $\boldsymbol{x}$ into the slow, "OC" coordinate $\boldsymbol{X} \doteq \langle \boldsymbol{x} \rangle$ and the quiver displacement $\tilde{\boldsymbol{x}} \doteq \boldsymbol{x} - \boldsymbol{X}$ (Fig. 16.1), where $\langle \ldots \rangle$ stands for the local time average. Then,

$$\boldsymbol{x}(t) = \boldsymbol{X}(t) + \tilde{\boldsymbol{x}}(t, \boldsymbol{X}, \boldsymbol{V}), \qquad \dot{\boldsymbol{x}}(t) = \boldsymbol{V}(t) + \tilde{\boldsymbol{v}}(t, \boldsymbol{X}, \boldsymbol{V}), \tag{16.2}$$

where we have also introduced the OC velocity $\boldsymbol{V} \doteq \dot{\boldsymbol{X}} = \langle \boldsymbol{v} \rangle$ and the quiver velocity $\tilde{\boldsymbol{v}} \doteq \dot{\tilde{\boldsymbol{x}}}$. Using these, one can rewrite Eq. (16.1) as follows:

$$L = \frac{1}{2}\, m\boldsymbol{V}^2 + \frac{1}{2}\, m\tilde{\boldsymbol{v}}^2 + m\tilde{\boldsymbol{v}} \cdot \boldsymbol{V} - e\tilde{\varphi}(t, \boldsymbol{X} + \tilde{\boldsymbol{x}}). \tag{16.3}$$

The Lagrangian (16.3) can be split into a slowly varying local average

$$\langle L \rangle = \frac{1}{2}\, m\boldsymbol{V}^2 + \frac{1}{2}\, m\langle \tilde{\boldsymbol{v}}^2 \rangle - e\langle \tilde{\varphi}(t, \boldsymbol{X} + \tilde{\boldsymbol{x}}) \rangle \tag{16.4}$$

and the remaining quiver part $\tilde{L} \doteq L - \langle L \rangle$. Note that the contribution of $\tilde{L}$ to the particle action $S \doteq \int_{t_1}^{t_2} L\, dt$ is bounded by $\sim \tilde{L}/\omega$, while the contribution of $\langle L \rangle$ grows, roughly linearly, with $|t_2 - t_1|$. Thus, at $\omega |t_2 - t_1| \gg 1$, one can approximate the particle action as $S \approx \int_{t_1}^{t_2} \langle L \rangle\, dt$. In other words, $\langle L \rangle \equiv \mathcal{L}$ serves as the Lagrangian of the average motion, i.e., the OC Lagrangian.

To calculate the OC Lagrangian explicitly, note that, on an oscillation period, one can adopt

$$\tilde{\boldsymbol{E}}(t, \boldsymbol{x}) \approx \operatorname{Re}\,(\boldsymbol{\mathcal{E}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}), \qquad \boldsymbol{X} \approx \boldsymbol{X}_0 + \boldsymbol{V}t. \tag{16.5}$$

Then, $\tilde{\boldsymbol{x}}$ is approximately governed by

$$\frac{d^2\tilde{\boldsymbol{x}}}{dt^2} \approx \frac{e}{m} \operatorname{Re}\,(\boldsymbol{\mathcal{E}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X}(t)}) \approx \frac{e}{m} \operatorname{Re}\,(\boldsymbol{\mathcal{E}}\, e^{-i(\omega - \boldsymbol{k}\cdot\boldsymbol{V})t + i\boldsymbol{k}\cdot\boldsymbol{X}_0}), \tag{16.6}$$

whence

$$\tilde{\boldsymbol{x}}(t, \boldsymbol{X}, \boldsymbol{V}) = -\frac{e}{m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2} \operatorname{Re}\,(\boldsymbol{\mathcal{E}}\, e^{-i(\omega - \boldsymbol{k}\cdot\boldsymbol{V})t + i\boldsymbol{k}\cdot\boldsymbol{X}_0}) = -\frac{e\tilde{\boldsymbol{E}}(t, \boldsymbol{X})}{m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2}. \tag{16.7}$$

(Here, we assume that $\omega - \boldsymbol{k}\cdot\boldsymbol{V}$ is large enough for our approximations to make sense; otherwise a different approach is needed, see Sec. 16.3.1.) In the complex representation, this gives

$$\tilde{\boldsymbol{x}}_c(t, \boldsymbol{X}, \boldsymbol{V}) = -\frac{e\tilde{\boldsymbol{E}}_c(t, \boldsymbol{X})}{m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2}, \tag{16.8}$$

$$\tilde{\boldsymbol{v}}_c(t, \boldsymbol{X}, \boldsymbol{V}) = \frac{ie\tilde{\boldsymbol{E}}_c(t, \boldsymbol{X})}{m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})}. \tag{16.9}$$

Hence,

$$\frac{1}{2}\, m\langle \tilde{\boldsymbol{v}}^2 \rangle = \frac{m}{4} \operatorname{Re}\,(\tilde{\boldsymbol{v}}_c \cdot \tilde{\boldsymbol{v}}_c^*) = \frac{e^2|\boldsymbol{\mathcal{E}}|^2}{4m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2}. \tag{16.10}$$

Also, assuming that the wave is weak, one has

$$\tilde{\varphi}(t, \boldsymbol{X} + \tilde{\boldsymbol{x}}) \approx \tilde{\boldsymbol{x}} \cdot \nabla\tilde{\varphi}(t, \boldsymbol{X}) = -\tilde{\boldsymbol{x}} \cdot \tilde{\boldsymbol{E}}(t, \boldsymbol{X}), \tag{16.11}$$

which leads to

$$e\langle \tilde{\varphi}(t, \boldsymbol{X} + \tilde{\boldsymbol{x}}) \rangle \approx -\frac{e}{2} \operatorname{Re}\,(\tilde{\boldsymbol{x}}_c \cdot \tilde{\boldsymbol{E}}_c^*(t, \boldsymbol{X})) \approx \frac{e^2|\boldsymbol{\mathcal{E}}|^2}{2m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2}. \tag{16.12}$$

Then, Eq. (16.4) gives

$$\mathcal{L} = \frac{1}{2}\, m\boldsymbol{V}^2 - \Phi(t, \boldsymbol{X}, \boldsymbol{V}), \tag{16.13}$$

where the interaction Lagrangian Φ is given by

$$\Phi(t, \boldsymbol{X}, \boldsymbol{V}) = \frac{e^2|\boldsymbol{\mathcal{E}}|^2}{4m(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2}. \tag{16.14}$$

and $\boldsymbol{\mathcal{E}}$, ω, and $\boldsymbol{k}$ may slowly depend on t and $\boldsymbol{X}$.

16.1.2 Ponderomotive energy and ponderomotive force

To understand the physical meaning of Φ, let us introduce the dipole moment caused by particle's deviation from the OC trajectory, $\tilde{\boldsymbol{d}} \doteq e\tilde{\boldsymbol{x}}$.[1] In the complex form, it is related to the wave field via

$$\tilde{\boldsymbol{d}}_c = \boldsymbol{\alpha}\tilde{\boldsymbol{E}}_c, \tag{16.15}$$

where the coefficient $\boldsymbol{\alpha}$ can be interpreted as the OC's polarizability. [In our case, $\boldsymbol{\alpha}$ is a scalar, $\alpha = -(e^2/m)(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^{-2}$, but other expressions are also possible. For example, for a nonmagnetized particle in a transverse wave, one has $\alpha = -e^2/(m\omega^2)$, which leads to a popular formula $\Phi = e^2|\boldsymbol{\mathcal{E}}|^2/(4m\omega^2)$.] Then, one can express Φ as follows:

$$\Phi = -\frac{1}{2}\,\langle\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{E}}\rangle = -\frac{1}{4}\,\boldsymbol{\mathcal{E}}^\dagger\boldsymbol{\alpha}\boldsymbol{\mathcal{E}}. \tag{16.16}$$

This shows that Φ is just the average energy of the dipole interaction between the OC and the wave.[2] The fact that the *nonlinear* term Φ is determined by the *linear* polarizability is known as the K-χ theorem. (Here, K loosely stands for Φ and χ stands for the plasma linear susceptibility; see also Sec. 16.2.1) A comprehensive discussion and key references can be found in Ref. [40].

An OC has a canonical momentum defined as usual:

$$\boldsymbol{P} \doteq \partial_{\boldsymbol{V}}\mathcal{L} = m\boldsymbol{V} - \Phi_{\boldsymbol{V}}, \tag{16.17}$$

where $\Phi_{\boldsymbol{V}} \equiv \partial_{\boldsymbol{V}}\Phi$. The fact that $\boldsymbol{P} \neq m\boldsymbol{V}$ signifies that an OC is a "dressed" particle, i.e., a dynamical object whose properties are determined not only by the particle per se but also by its interactions with the environment. The Euler-Lagrange equations for this dressed particle, $\dot{\boldsymbol{P}} = \partial_{\boldsymbol{X}}\mathcal{L}$, can be written as

$$\dot{\boldsymbol{P}} = -\partial_{\boldsymbol{X}}\Phi, \tag{16.18}$$

or equivalently, as

$$\begin{aligned} m\dot{\boldsymbol{V}} &= -\partial_{\boldsymbol{X}}\Phi + d_t\Phi_{\boldsymbol{V}} \\ &= -\partial_{\boldsymbol{X}}\Phi + \partial^2_{t\boldsymbol{V}}\Phi + \boldsymbol{V}\cdot\partial^2_{\boldsymbol{X}\boldsymbol{V}}\Phi, + \dot{\boldsymbol{V}} : \partial^2_{\boldsymbol{V}\boldsymbol{V}}\Phi. \end{aligned} \tag{16.19}$$

[1] This dipole moment naturally emerges, for example, at Taylor-expanding the single-particle charge density $\rho(t, \boldsymbol{r}) = e\delta(\boldsymbol{r} - \boldsymbol{x}(t))$ in the spatial coordinate $\boldsymbol{r}$ around $\boldsymbol{X}$ in powers of $\tilde{\boldsymbol{E}}$.

[2] For electrostatic interactions in nonmagnetized plasma considered here, Φ also happens to be equal to $m\langle\tilde{\boldsymbol{v}}^2\rangle/2$, so it is often interpreted as the energy of particle's wave-induced oscillations. However, this interpretation does not extend to more general interactions, while Eq. (16.16) does. In particular, since $\boldsymbol{\alpha}$ is not a sign-definite quantity, Φ can be positive or negative depending on what fields a particle interacts with. For example, a magnetized particle can have $\Phi < 0$ when ω is below the particle's cyclotron frequency. Also note that, in atomic physics, a term similar to Φ, called the dipole potential, emerges when one considers the interaction of an atom with an electromagnetic wave [66]. Then, $\boldsymbol{\alpha}$ is the atom's polarizability in the common sense of the word.

or equivalently, as

$$(m - \partial^2_{\boldsymbol{V}\boldsymbol{V}}\Phi) : \dot{\boldsymbol{V}} = -\partial_{\boldsymbol{X}}\Phi + \partial^2_{t\boldsymbol{V}}\Phi + \boldsymbol{V} \cdot \partial^2_{\boldsymbol{X}\boldsymbol{V}}\Phi. \tag{16.20}$$

The nonlinear terms in these equations constitute what is called the *ponderomotive force* (and the tensor $m - \partial^2_{\boldsymbol{V}\boldsymbol{V}}\Phi$ serves as the effective mass of an OC). In the cold limit, when $\omega \gg \boldsymbol{k}\cdot\boldsymbol{V}$, Φ becomes independent of $\boldsymbol{V}$, $\Phi = e^2|\boldsymbol{\mathcal{E}}|^2/(4m\omega^2)$ (same as for transverse waves), so the above equation becomes

$$m\dot{\boldsymbol{V}} \approx -\partial_{\boldsymbol{X}}\Phi(t, \boldsymbol{X}). \tag{16.21}$$

In this case, Φ is called the ponderomotive potential.[3] In the generic case, when the dependence of Φ on $\boldsymbol{V}$ is not negligible, there is no commonly accepted name for this function, but sometimes the term "ponderomotive energy" is used (also see Sec. 16.1.3).

16.1.3 OC Hamiltonian

Let us also consider the OC Hamiltonian:

$$\begin{aligned}
\mathcal{H} &\doteq \boldsymbol{P} \cdot \boldsymbol{V} - \mathcal{L} \\
&= m\boldsymbol{V}^2 - \boldsymbol{V} \cdot \Phi_{\boldsymbol{V}} - \frac{1}{2}m\boldsymbol{V}^2 + \Phi(t, \boldsymbol{X}, \boldsymbol{V}) \\
&= \frac{1}{2}m\boldsymbol{V}^2 - \boldsymbol{V} \cdot \Phi_{\boldsymbol{V}} + \Phi(t, \boldsymbol{X}, \boldsymbol{V}) \\
&= \frac{1}{2m}(m\boldsymbol{V} - \Phi_{\boldsymbol{V}})^2 - \frac{\Phi_{\boldsymbol{V}}^2}{2m} + \Phi(t, \boldsymbol{X}, \boldsymbol{V}) \\
&= \frac{\boldsymbol{P}^2}{2m} + \Phi\left(t, \boldsymbol{X}, \frac{\boldsymbol{P}}{m} + \Phi_{\boldsymbol{V}}\right) - \frac{\Phi_{\boldsymbol{V}}^2}{2m} \\
&= \frac{\boldsymbol{P}^2}{2m} + \Phi\left(t, \boldsymbol{X}, \frac{\boldsymbol{P}}{m}\right) + \mathcal{O}(\tilde{\boldsymbol{E}}^4).
\end{aligned} \tag{16.22}$$

Within the quasilinear approximation that we are interested in this lecture, terms $\mathcal{O}(\tilde{\boldsymbol{E}}^4)$ are considered negligible, so the OC Hamiltonian can be written simply as follows:

$$\mathcal{H} = \frac{\boldsymbol{P}^2}{2m} + \Phi\left(t, \boldsymbol{X}, \frac{\boldsymbol{P}}{m}\right). \tag{16.23}$$

Accordingly, Φ is sometimes called the nonlinear (second-order in $\tilde{\boldsymbol{E}}$) part of the OC Hamiltonian. The corresponding Hamilton's equations of the OC motion are

$$\frac{d\boldsymbol{X}}{dt} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{P}} = \frac{\boldsymbol{P}}{m} + \frac{\Phi_{\boldsymbol{V}}}{m}, \tag{16.24}$$

$$\frac{d\boldsymbol{P}}{dt} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{X}} = -\frac{\partial\Phi}{\partial\boldsymbol{X}}. \tag{16.25}$$

Equation (16.24) is equivalent to Eq. (16.17), and Eq. (16.25) is equivalent to Eq. (16.18).

Note that switching to the OC description of the particle motion can be understood simply as a canonical variable transformation $(\boldsymbol{x}, \boldsymbol{p}) \to (\boldsymbol{X}, \boldsymbol{P})$. Accordingly, one can introduce the OC distribution F in terms of the particle distribution f using the rules of variable transformations described in Sec. 8.1.1. This transformation is singular at $\omega \to \boldsymbol{k}\cdot\boldsymbol{V}$, so it is not directly applicable to resonant interactions, particularly interactions of plasmas with continuous wave spectra. An alternative derivation that does not have this issue will be presented in Sec. 16.3.1.

[3] In this limit, one can also easily derive Eq. (16.21) by time-averaging the particle motion equation $m\ddot{\boldsymbol{x}} = e\tilde{\boldsymbol{E}}(t, \boldsymbol{x})$ with $\tilde{\boldsymbol{E}}(t, \boldsymbol{x}) \approx \tilde{\boldsymbol{E}}(t, \boldsymbol{X}) + [\nabla\tilde{\boldsymbol{E}}(t, \boldsymbol{X})] : \tilde{\boldsymbol{x}}$.

16.2 Variational principle for nonresonant waves

16.2.1 Plasma action

Now, let us consider a self-consistent interaction between multiple electrons and a wave caused by ponderomotive forces. To do this, let us introduce the total action of the plasma, which includes the actions of the individual particles and the field action:

$$S = \int_{t_1}^{t_2} dt \sum_a \left(\frac{1}{2}\, m\boldsymbol{V}_a^2 - \Phi(t, \boldsymbol{X}_a, \boldsymbol{V}_a)\right) + \int_{t_1}^{t_2} dt \int d\boldsymbol{x}\, \frac{\langle \tilde{\boldsymbol{E}}^2 \rangle}{8\pi}, \tag{16.26}$$

where the spatial integral is taken over the whole space. (Here we consider only the average part of the plasma action for the same reason as we focused on the average action of a single particle in the previous section. Also, $\int_{t_1}^{t_2} dt$ is shortened below to $\int dt$ for brevity.) Let us introduce

$$S_{\rm OC} \doteq \int dt \sum_a \frac{1}{2}\, m\boldsymbol{V}_a^2 \tag{16.27}$$

and rewrite the above S as follows:

$$S = S_{\rm OC} + \int dt\, d\boldsymbol{x}\, \frac{|\boldsymbol{\mathcal{E}}|^2}{16\pi} - \int dt \sum_a \Phi(t, \boldsymbol{X}_a, \boldsymbol{V}_a). \tag{16.28}$$

Next, notice that

$$\begin{aligned} -\int dt \sum_a \Phi(t, \boldsymbol{X}_a, \boldsymbol{V}_a) &= -\int dt\, d\boldsymbol{x}\, d\boldsymbol{v}\, \Phi(t, \boldsymbol{x}, \boldsymbol{v}) F(t, \boldsymbol{x}, \boldsymbol{v}) \\ &= -\int dt\, d\boldsymbol{x}\, d\boldsymbol{v}\, \frac{e^2 |\boldsymbol{\mathcal{E}}|^2(t, \boldsymbol{x})}{4m(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2}\, F(t, \boldsymbol{x}, \boldsymbol{v}) \\ &\equiv \int dt\, d\boldsymbol{x}\, \frac{\mathcal{X} |\boldsymbol{\mathcal{E}}|^2}{16\pi}. \end{aligned} \tag{16.29}$$

Here,

$$\begin{aligned} \mathcal{X} &\doteq -\frac{4\pi e^2}{m} \int d\boldsymbol{v}\, \frac{F(t, \boldsymbol{x}, \boldsymbol{v})}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2} \\ &= -\frac{4\pi e^2}{mk^2}\, \boldsymbol{k} \cdot \int d\boldsymbol{v}\, F(t, \boldsymbol{x}, \boldsymbol{v})\, \frac{\partial}{\partial \boldsymbol{v}} \left(\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\right) \\ &= -\frac{4\pi e^2}{mk^2} \int d\boldsymbol{v}\, \frac{\boldsymbol{k}\cdot\partial_{\boldsymbol{v}} F(t, \boldsymbol{x}, \boldsymbol{v})}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega}. \end{aligned} \tag{16.30}$$

(The last equality is obtained using integration by parts, which is possible by our assumption that all particles are nonresonant.) Clearly, $\mathcal{X}$ is just the plasma susceptibility $\chi_\parallel$ expressed through the OC distribution.[4] Using that the field is electrostatic and

$$1 + \chi_\parallel = \epsilon_\parallel = \boldsymbol{k}^\dagger \boldsymbol{\epsilon} \boldsymbol{k} / k^2 \tag{16.31}$$

one finds from Eq. (16.28) that

$$S = S_{\rm OC} + \int dt\, d\boldsymbol{x}\, \boldsymbol{\mathcal{E}}^\dagger \boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) \boldsymbol{\mathcal{E}}, \tag{16.32}$$

[4] In linear theory, F is the same as the "unperturbed" distribution f_0 that we used earlier. Beyond linear theory, the concept of the unperturbed distribution is not well defined and one should use the OC distribution instead.

where the matrix function $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \frac{1}{16\pi}\,\boldsymbol{\epsilon}(t, \boldsymbol{x}, \omega, \boldsymbol{k}). \tag{16.33}$$

[For electromagnetic waves, one obtains the same result but with $\boldsymbol{D}$ given by Eq. (5.9).] Notice that $\boldsymbol{D}$ is necesarily Hermitian within this approach, because we had to ignore resonant interactions to arrive at this result. Furthermore, even if $\boldsymbol{D}$ had an anti-Hermitian part, its contribution to the product $\boldsymbol{\mathcal{E}}^\dagger \boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k})\boldsymbol{\mathcal{E}}$ would have been zero in any case.

16.2.2 Equations of geometrical optics from the variational principle

Combined with the least-action principle $\delta S = 0$, Eq. (16.32) can be considered as a variational principle for the wave dynamics. Indeed, let us consider variations of this action with respect to the wave variables. One of these variable is the wave phase θ, which is related to ω and $\boldsymbol{k}$ via

$$\omega = -\partial_t \theta, \qquad \boldsymbol{k} = \nabla\theta. \tag{16.34}$$

Another variable is the envelope $\boldsymbol{\mathcal{E}}$. As a complex function, it can be understood as two independent real functions:

$$\operatorname{Re}\boldsymbol{\mathcal{E}} = \frac{\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}^*}{2}, \qquad \operatorname{Im}\boldsymbol{\mathcal{E}} = \frac{\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}^*}{2i}. \tag{16.35}$$

Equivalently, one can choose the independent variables to be $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}^*$ or $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}^\dagger$. (As a reminder, the only difference between $\boldsymbol{\mathcal{E}}^*$ and $\boldsymbol{\mathcal{E}}^\dagger$ is that the former is a column vector while the latter is a row vector.) Hence, one can write (cf. Box 5.1)

$$S[\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^\dagger, \theta] = \int dt\, d\boldsymbol{x}\, \boldsymbol{\mathcal{E}}^\dagger \boldsymbol{D}(t, \boldsymbol{x}, \underbrace{-\partial_t\theta}_{\omega}, \underbrace{\nabla\theta}_{\boldsymbol{k}})\boldsymbol{\mathcal{E}}. \tag{16.36}$$

(We have omitted S_{OC} because it is independent of the wave variables and thus does not contribute to the wave equations anyway.) As usual, the assumed boundary in the least-action principle are such that the variations of the independent variables vanish at the endpoints of the time integral:

$$\delta\theta(t_1) = \delta\theta(t_2) = 0, \tag{16.37}$$

$$\delta\boldsymbol{\mathcal{E}}(t_1) = \delta\boldsymbol{\mathcal{E}}(t_2) = 0. \tag{16.38}$$

The variation of S with respect to $\boldsymbol{\mathcal{E}}^\dagger$ is

$$\delta S = \int dt\, d\boldsymbol{x}\, (\delta\boldsymbol{\mathcal{E}}^\dagger)\boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k})\boldsymbol{\mathcal{E}}, \tag{16.39}$$

so, from the least-action principle $\delta_{\boldsymbol{\mathcal{E}}^\dagger} S = 0$ one finds

$$\boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k})\boldsymbol{\mathcal{E}} = 0. \tag{16.40}$$

This is the well-known equation for the linear-wave polarization (and it also implies the linear dispersion relation $\det \boldsymbol{D} = 0$, as usual). Considering the variation of S with respect to $\boldsymbol{\mathcal{E}}$ leads to an equivalent equation that is the adjoint of Eq. (16.40). The variation of S with respect to θ gives

$$\begin{aligned}\delta S &= \int dt\, d\boldsymbol{x}\, \boldsymbol{\mathcal{E}}^\dagger[-\partial_\omega \boldsymbol{D}\, \partial_t(\delta\theta) + \partial_{\boldsymbol{k}}\boldsymbol{D}\cdot\nabla(\delta\theta)]\boldsymbol{\mathcal{E}} \\ &= \int dt\, d\boldsymbol{x}\, \left\{\partial_t[\boldsymbol{\mathcal{E}}^\dagger(\partial_\omega \boldsymbol{D})\boldsymbol{\mathcal{E}}] + \nabla\cdot[-\boldsymbol{\mathcal{E}}^\dagger(\partial_{\boldsymbol{k}}\boldsymbol{D})\boldsymbol{\mathcal{E}}]\right\}(\delta\theta),\end{aligned} \tag{16.41}$$

where we have integrated by parts. [When integrating by parts in space, we dropped the boundary term assuming that the wave field vanishes at infinity. When integrating by parts in time, we dropped the substitution at t_1 and t_2 due to Eq. (16.37).] Notice that $\boldsymbol{\mathcal{E}}^\dagger(\partial_\omega \boldsymbol{D})\boldsymbol{\mathcal{E}}$ is the wave action density $\mathcal{I}$ and $-\boldsymbol{\mathcal{E}}^\dagger(\partial_{\boldsymbol{k}} \boldsymbol{D})\boldsymbol{\mathcal{E}}$ is the action flux density $\boldsymbol{\mathcal{J}} = \boldsymbol{v}_g \mathcal{I}$. Thus, Eq. (16.41) can as well be written as

$$\delta S = \int dt\, d\boldsymbol{x} \left[\partial_t \mathcal{I} + \nabla \cdot (\boldsymbol{v}_g \mathcal{I})\right] (\delta\theta). \tag{16.42}$$

Then, the least-action principle $\delta_\theta S = 0$ yields the action conservation law,

$$\partial_t \mathcal{I} + \nabla \cdot (\boldsymbol{v}_g \mathcal{I}) = 0. \tag{16.43}$$

The above model of wave–particle interactions is quasilinear in the sense that it implies linear-GO equations for the wave [Eqs. (16.40) and (16.43)] yet also captures nonlinear dynamics of particles (OCs) determined by ponderomotive forces. We derived this model by expanding the plasma action S to the second order in the wave field [because we used linear Taylor expansion in Eq. (16.11)]. Thus, fundamentally, the quasilinear description is a description that corresponds to neglecting terms of the third and higher powers of $\tilde{\boldsymbol{E}}$ in the plasma action. Linear description is obtained from quasilinear description by artificially fixing OC velocities, which can be justified on small enough time scales.

16.3 Merging resonant and nonresonant quasilinear effects

Where is the quasilinear diffusion in the above picture and where are the ponderomotive forces in the model that was derived in Lecture 15? The answer is: nowhere, both models are incomplete. The one above assumes that particles are not resonant to the wave, so it is concerned only with nonresonant (adiabatic) interactions. Likewise, the quasilinear-diffusion model in Lecture 15 ignores ponderomotive forces in that it repeatedly ignores spacetime gradients of the field amplitudes and the average distribution without a proper justification. A more accurate model is needed that would merge resonant and nonresonant quasilinear effects within a single framework.

Such models were proposed in Refs. [65, 67]. However, those derivations are partly based on heuristic arguments, Here, we outline a more formal derivation using the same operator approach that we used throughout the course. A generalized version of this calculation (not limited to electrostatic waves in electron plasma) and a broader discussion can be found in Ref. [40].

16.3.1 Quaslinear diffusion equation for the OC distribution

As earlier, let us assume collisionless nonmagnetized electron plasma where the Chirikov criterion is satisfied but the electric field is weak and the quasilinear approximation is applicable. For generality, we will allow plasma to be spatially inhomogeneous and not necessarily one-dimensional. We will also work with the momentum distribution instead of the velocity distribution to make the equations more compact. Like in Lecture 15, we start with

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_0}{\partial \boldsymbol{x}} = -e_e \left\langle \tilde{\boldsymbol{E}} \cdot \frac{\partial \tilde{f}}{\partial \boldsymbol{p}} \right\rangle, \tag{16.44a}$$

$$\frac{\partial \tilde{f}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \tilde{f}}{\partial \boldsymbol{x}} + e_e\, \tilde{\boldsymbol{E}} \cdot \frac{\partial f_0}{\partial \boldsymbol{p}} = 0, \tag{16.44b}$$

where $\langle \ldots \rangle$ now is the statistical average. Using the same argument as in Sec. 15.2.1, let us neglect the initial conditions in Eq. (16.44b). Then, using the propagator $\hat{G}$ of the linearized Vlasov equation, the solution of Eq. (16.44b) can be written as

$$\tilde{f} = -e_e \hat{G} \left(\tilde{\boldsymbol{E}} \cdot \frac{\partial f_0}{\partial \boldsymbol{p}} \right) \equiv -e_e \hat{G} \tilde{\boldsymbol{E}} \cdot \frac{\partial f_0}{\partial \boldsymbol{p}}, \tag{16.45}$$

where $\hat{\boldsymbol{E}} \doteq \tilde{\boldsymbol{E}}(\hat{t}, \hat{\boldsymbol{x}})$ is the electric field considered as an operator. [In the coordinate representation, $\hat{E}_a$ simply performs multiplication by $\tilde{E}_a(t, \boldsymbol{x})$.] Accordingly, Eq. (16.44a) becomes

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_0}{\partial \boldsymbol{x}} = \frac{\partial}{\partial p_a}\left(\hat{D}_{ab}\,\frac{\partial f_0}{\partial p_b}\right), \tag{16.46}$$

$$\hat{D}_{ab} \doteq e^2\,\langle \hat{E}_a \hat{G} \hat{E}_b \rangle\,. \tag{16.47}$$

It is straightforward to show that the Weyl symbol of $\hat{D}_{ab}$ can be expressed through the Weyl symbol of $\hat{G}$ as follows:

$$\boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = e^2 \int d\omega'\, d\boldsymbol{k}'\, G(\omega - \omega', \boldsymbol{k} - \boldsymbol{k}')\,\langle \boldsymbol{W}_E \rangle(t, \boldsymbol{x}, \omega', \boldsymbol{k}'). \tag{16.48}$$

Here, $\langle \boldsymbol{W}_E \rangle$ is the average Wigner tensor of the electric field, i.e., the spectrum of the symmetrized two-point correlation matrix $\mathsf{C}_{ab}(t, \boldsymbol{x}, \tau, \boldsymbol{s}) \doteq \langle \tilde{E}_a(t + \tau/2, \boldsymbol{x} + \boldsymbol{s}/2)\,\tilde{E}_b(t - \tau/2, \boldsymbol{x} - \boldsymbol{s}/2)\rangle$:

$$\langle W_{E,ab} \rangle(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = \frac{1}{(2\pi)^{n+1}} \int d\tau\, d\boldsymbol{s}\, \mathsf{C}_{ab}(t, \boldsymbol{x}, \tau, \boldsymbol{s})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{s} + i\omega\tau}, \tag{16.49}$$

which is also understood as the average Weyl symbol of $\hat{\boldsymbol{W}}_E \doteq (2\pi)^{-(n+1)}\,|\tilde{\boldsymbol{E}}\rangle\,\langle \tilde{\boldsymbol{E}}|$. (Here, n is the number of spatial dimensions.) Because the field is electrostatic, it is convenient to express $\langle \boldsymbol{W}_E \rangle$ through the average Wigner function of the electrostatic potential, i.e., the spectrum of the symmetrized two-point correlation function $\mathsf{C}_\varphi(t, \boldsymbol{x}, \tau, \boldsymbol{s}) \doteq \langle \tilde{\varphi}(t + \tau/2, \boldsymbol{x} + \boldsymbol{s}/2)\,\tilde{\varphi}(t - \tau/2, \boldsymbol{x} - \boldsymbol{s}/2)\rangle$:

$$\langle W_\varphi \rangle(t, \boldsymbol{x}, \omega, \boldsymbol{k}) = \frac{1}{(2\pi)^{n+1}} \int d\tau\, d\boldsymbol{s}\, \mathsf{C}_\varphi(t, \boldsymbol{x}, \tau, \boldsymbol{s})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{s} + i\omega\tau}, \tag{16.50}$$

which is the average Weyl symbol of $\hat{W}_\varphi \doteq (2\pi)^{-(n+1)}\,|\tilde{\varphi}\rangle\,\langle\tilde{\varphi}|$. Because

$$\hat{\boldsymbol{W}}_E = \hat{\boldsymbol{k}}\,\hat{W}_\varphi \hat{\boldsymbol{k}}, \tag{16.51}$$

one readily finds, using Eqs. (3.24) and (3.25), that

$$\langle W_{E,ab} \rangle \approx k_a k_b \langle W_\varphi \rangle - \frac{i}{2}\left(k_b\,\frac{\partial \langle W_\varphi \rangle}{\partial x_a} - k_a\,\frac{\partial \langle W_\varphi \rangle}{\partial x_b}\right), \tag{16.52}$$

assuming the averaged quantities are smooth enough such that GO approximation is applicable.[5] Similarly, because f_0 is only weakly inhomogeneous, one can Weyl-expand $\hat{D}$ like in Lecture 3, that is, by using

$$\boldsymbol{D}(t, \boldsymbol{x}, \omega, \boldsymbol{k}) \approx \boldsymbol{D}(t, \boldsymbol{x}, 0, 0) + \omega\,\partial_\omega \boldsymbol{D}(t, \boldsymbol{x}, 0, 0) + (\boldsymbol{k} \cdot \partial_{\boldsymbol{k}})\boldsymbol{D}(t, \boldsymbol{x}, 0, 0) \tag{16.53}$$

and then applying the inverse Wigner–Weyl transform. After a tedious but straightforward calculation, one arrives at the following diffusion equation:

$$\frac{\partial F}{\partial t} + \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \cdot \frac{\partial F}{\partial \boldsymbol{x}} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} \cdot \frac{\partial F}{\partial \boldsymbol{p}} = \frac{\partial}{\partial p_a}\left(\bar{\mathsf{D}}_{ab}\,\frac{\partial F}{\partial p_b}\right), \tag{16.54}$$

$$\bar{\mathsf{D}}_{ab} = \pi e^2 \int d\omega\, d\boldsymbol{k}\, k_a k_b\,\langle W_\varphi \rangle(t, \boldsymbol{x}, \boldsymbol{k} \cdot \boldsymbol{v}, \boldsymbol{k}), \tag{16.55}$$

[5] Strictly speaking, W_φ can be delta-shaped [Eq. (16.59)], but we will be interested only in integrals of W_φ. Contributions of higher-order derivatives of W_φ to those integrals is small even for delta-shaped W_φ when J and $\omega_{\boldsymbol{r}}$ are smooth functions.

where nonlinearities beyond that of the second order in $\tilde{\boldsymbol{E}}$ have been neglected. (Note that W_φ is evaluated at $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$, which indicates that diffusion is governed by resonant particles.) Here, F is the OC distribution given by

$$F = f_0 + \frac{1}{2} \frac{\partial}{\partial p_a} \left(\mathcal{V}_{ab} \frac{\partial f_0}{\partial p_b} \right), \tag{16.56}$$

$$\mathcal{V}_{ab} = \frac{\partial}{\partial \vartheta} \left[\fint d\omega \, d\boldsymbol{k} \, \frac{e^2 k_a k_b \langle W_\varphi \rangle (t, \boldsymbol{x}, \omega, \boldsymbol{k})}{\omega - \boldsymbol{k} \cdot \boldsymbol{V} + \vartheta} \right]_{\vartheta = 0}. \tag{16.57}$$

Also,

$$\mathcal{H} = \frac{p^2}{2m_e} + \Phi, \qquad \Phi = \frac{\partial}{\partial \boldsymbol{p}} \fint d\omega \, d\boldsymbol{k} \, \frac{e^2 \boldsymbol{k} \langle W_\varphi \rangle (t, \boldsymbol{x}, \omega, \boldsymbol{k})}{2m_e(\omega - \boldsymbol{k} \cdot \boldsymbol{V})}, \tag{16.58}$$

which is a generalization of the OC Hamiltonian (16.23) and the ponderomotive energy (16.14) to continuous wave spectra. In particular, one can show that, for a delta-shaped $\langle W_\varphi \rangle$, the above expression for Φ reduces to Eq. (16.14).

Notice that, unlike in the single-particle picture in Sec. 16.1.1, the singularities at $\omega - \boldsymbol{k} \cdot \boldsymbol{V}$ appear within this approach only under principal-value integrals, so the ponderomotive energy Φ and the dressing function $\mathcal{V}_{ab}$ remain well-behaved. Also, OC coordinates per se are not introduced. Instead, the distribution function is considered as a fundamental object that is transformed directly rather than inherits its transformation properties from a coordinate transformation. Also note that similar results can be obtained for plasma interaction with electromagnetic (and any other) waves, except the formulas for $\bar{\mathsf{D}}_{ab}$, $\mathcal{V}_{ab}$, and Φ are different then [40].

16.4 Interaction with on-shell waves

Finally, let us complement the equation for the distribution F with an equation for the waves. At this point, we will assume that the waves are "on-shell", constrained by some dispersion relation $\omega = \omega_r(t, \boldsymbol{x}, \boldsymbol{k})$ in the $(\omega, \boldsymbol{k})$ space. In this case, their Wigner function can be approximated as follows:

$$\langle W_\varphi \rangle \propto J(t, \boldsymbol{x}, \boldsymbol{k}) \, \delta[\omega - \omega_r(t, \boldsymbol{x}, \boldsymbol{k})]. \tag{16.59}$$

For such waves, the function J serves as the phase-space density of the wave action, if normalized properly. One can show (Box 8.1) that it satisfies the so-called wave-kinetic equation (WKE) [39,40]:

$$\frac{\partial J}{\partial t} + \frac{\partial \omega_r}{\partial \boldsymbol{k}} \cdot \frac{\partial J}{\partial \boldsymbol{x}} - \frac{\partial \omega_r}{\partial \boldsymbol{x}} \cdot \frac{\partial J}{\partial \boldsymbol{k}} = 2\omega_i J, \tag{16.60}$$

which is a generalization (and correction) of Eq. (15.26).[6] Then one has Ref. [40,65]

$$\partial_t \textstyle\int d\boldsymbol{p} \, F + \nabla \cdot \int d\boldsymbol{p} \, \boldsymbol{V} F = 0, \tag{16.61a}$$

$$\partial_t \big(\textstyle\int d\boldsymbol{p} \, \boldsymbol{p} F + \int d\boldsymbol{k} \, \boldsymbol{k} J \big) + \nabla \cdot \big(\int d\boldsymbol{p} \, \boldsymbol{p} \boldsymbol{V} F + \int d\boldsymbol{k} \, \boldsymbol{k} \boldsymbol{v}_g J \big) + \nabla \int d\boldsymbol{p} \, \Phi F = 0, \tag{16.61b}$$

$$\partial_t \big(\textstyle\int d\boldsymbol{p} \, H_0 F + \int d\boldsymbol{k} \, \omega_r J \big) + \nabla \cdot \big(\int d\boldsymbol{p} \, H_0 \boldsymbol{V} F + \int d\boldsymbol{k} \, \omega_r \boldsymbol{v}_g J \big) + \nabla \cdot \int d\boldsymbol{p} \, \boldsymbol{V} \Phi F = 0, \tag{16.61c}$$

where $H_0 \doteq p^2/(2m_e)$ is the energy of a free OC and $\boldsymbol{v}_g \doteq \partial_{\boldsymbol{k}} \omega_r$ is the group velocity. Hence, equations of quasilinear theory conserve the particle (OC) number, the total momentum, and the total energy:

$$\int d\boldsymbol{x} \, d\boldsymbol{p} \, F = \text{const}, \tag{16.62a}$$

[6] The WKE can be understood as the Vlasov equation for wave quanta. Alternatively, the Vlasov equation for plasma particles can be considered as the classical (GO) limit of the WKE derived for particles as quantum waves.

$$\int d\boldsymbol{x}\, d\boldsymbol{p}\, \boldsymbol{p} F + \int d\boldsymbol{x}\, d\boldsymbol{k}\, \boldsymbol{k} J = \text{const}, \tag{16.62b}$$

$$\int d\boldsymbol{x}\, d\boldsymbol{p}\, H_0 F + \int d\boldsymbol{x}\, d\boldsymbol{k}\, \omega_{\boldsymbol{r}} J = \text{const}. \tag{16.62c}$$

The density of the energy–momentum density of the OC distribution is different from that of the particle distribution precisely by the difference between the wave energy-momentum (Lecture 5) and the field energy-momentum (which is why electrostatic *waves* can carry momentum while electrostatic *fields* cannot):

$$\underbrace{(\text{particle energy})}_{\int dv\, {}^{1}\!/_{2}\, m_e v^2 f_0} + \underbrace{(\text{field energy})}_{\int dk\, U_k} = (\text{OC energy}) + \underbrace{(\text{wave energy})}_{\int dk\, \mathcal{U}_k}, \tag{16.63a}$$

$$\underbrace{(\text{particle momentum})}_{\int dv\, m_e v f_0} + \underbrace{(\text{field momentum})}_{\text{zero for electrostatic field}} = (\text{OC momentum}) + \underbrace{(\text{wave momentum})}_{\int dk\, \mathcal{P}_k}. \tag{16.63b}$$

In particular, as F diffuses, the OC momentum changes, but so does the wave momentum, and the total momentum $\int dv\, m_e v f_0$ remains conserved.

In summary, OC quasilinear theory subsumes both quasilinear diffusion and adiabatic ponderomotive interactions and provides a complete self-consistent framework. Although approximate, this framework honors the exact conservation laws of the original system, such as the particle, momentum, and energy conservation. Particle collisions can also be easily accommodated within this theory [40]. However, what the quasilinear approach fundamentally overlooks is wave–wave collisions and other nonlinear effects that are described by terms of the third and higher powers of $\tilde{\boldsymbol{E}}$ in the plasma action S. Such effects are beyond the scope of this course.

Appendices for Part IV

AIV.1 Plasma dispersion function

See Fig. 16.2.

Definition[16] (first form valid only for $\operatorname{Im}\zeta > 0$):

$$Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{+\infty} \frac{dt\, \exp\left(-t^2\right)}{t-\zeta} = 2i \exp\left(-\zeta^2\right) \int_{-\infty}^{i\zeta} dt\, \exp\left(-t^2\right).$$

Physically, $\zeta = x + iy$ is the ratio of wave phase velocity to thermal velocity.

Differential equation:

$$\frac{dZ}{d\zeta} = -2\left(1 + \zeta Z\right),\ Z(0) = i\pi^{1/2}; \quad \frac{d^2 Z}{d\zeta^2} + 2\zeta \frac{dZ}{d\zeta} + 2Z = 0.$$

Real argument ($y = 0$):

$$Z(x) = \exp\left(-x^2\right)\left(i\pi^{1/2} - 2\int_0^x dt\, \exp\left(t^2\right)\right).$$

Imaginary argument ($x = 0$):

$$Z(iy) = i\pi^{1/2} \exp\left(y^2\right)\left[1 - \operatorname{erf}(y)\right].$$

Power series (small argument):

$$Z(\zeta) = i\pi^{1/2} \exp\left(-\zeta^2\right) - 2\zeta\left(1 - 2\zeta^2/3 + 4\zeta^4/15 - 8\zeta^6/105 + \cdots\right).$$

Asymptotic series, $|\zeta| \gg 1$ (Ref. 17):

$$Z(\zeta) = i\pi^{1/2}\sigma \exp\left(-\zeta^2\right) - \zeta^{-1}\left(1 + 1/2\zeta^2 + 3/4\zeta^4 + 15/8\zeta^6 + \cdots\right),$$

where

$$\sigma = \begin{cases} 0 & y > |x|^{-1} \\ 1 & |y| < |x|^{-1} \\ 2 & y < -|x|^{-1} \end{cases}$$

Symmetry properties (the asterisk denotes complex conjugation):

$$Z(\zeta^*) = -\left[Z(-\zeta)\right]^*;$$

$$Z(\zeta^*) = \left[Z(\zeta)\right]^* + 2i\pi^{1/2} \exp[-(\zeta^*)^2] \quad (y > 0).$$

Two-pole approximations[18] (good for ζ in upper half plane except when $y < \pi^{1/2}x^2 \exp(-x^2),\ x \gg 1$):

$$Z(\zeta) \approx \frac{0.50 + 0.81i}{a - \zeta} - \frac{0.50 - 0.81i}{a^* + \zeta}, \quad a = 0.51 - 0.81i;$$

$$Z'(\zeta) \approx \frac{0.50 + 0.96i}{(b - \zeta)^2} + \frac{0.50 - 0.96i}{(b^* + \zeta)^2}, \quad b = 0.48 - 0.91i.$$

Figure 16.2: Properties of the plasma dispersion function Z (copied from Ref. [68]).

Problems for Part IV

PIV.1 Wave propagation: initial-value problem

Consider one-dimensional propagation of a transverse electromagnetic wave in a cold stationary collisionless nonmagnetized homogeneous electron plasma. As discussed in Lecture 2, such wave is governed by the equation of the Klein–Gordon type,

$$\partial_t^2 E - \partial_x^2 E + E = 0, \tag{16.64}$$

where E is any of the transverse components of the electric field, t is measured in units ω_p^{-1}, and x is measured in units c/ω_p. Here, you are asked to prove rigorously that signals governed by Eq. (16.64) cannot propagate faster than at the speed of light.

(a) Using the Laplace transform in t and the Fourier transform in x, show that the general solution of Eq. (16.64) can be written as follows:

$$E(t,x) = \int_{-\infty}^{\infty} dx\, \boldsymbol{G}(t, x - x') \cdot \boldsymbol{\mathcal{E}}(x'). \tag{16.65}$$

Here, the two-component vector

$$\boldsymbol{\mathcal{E}}(x) \doteq \begin{pmatrix} E(0,x) \\ \partial_t E(0,x) \end{pmatrix}$$

encodes the initial conditions, and the vector function

$$\boldsymbol{G}(t,x) \doteq \int_{\mathcal{B}} \frac{ds}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{e^{st+ikx}}{1+k^2+s^2} \begin{pmatrix} s \\ 1 \end{pmatrix} \tag{16.66}$$

serves as the Green's function. (Here, $\mathcal{B}$ is the Bromwich contour, and the bracket is a two-component vector.)

(b) Take the integral over k in Eq. (16.66). (This can be done analytically using the residue theorem.) Prove that the remaining integral is identically zero at $|x| > t > 0$.

> **Hint:** This is doable even without representing the remaining integral through elementary functions (which is also possible but not required here).

PIV.2 Longitudinal waves in Lorentzian plasma

Consider a one-dimensional electron plasma with a Lorentzian distribution,

$$f_0(v) = \frac{\Delta}{\pi} \frac{1}{v^2 + \Delta^2}, \tag{16.67}$$

where Δ is a constant.

(a) Assuming $k > 0$, calculate the response function $\sigma_k(t)$ and identify the characteristic time scale of phase mixing of such plasma. Take the Laplace transform of $\sigma_k(t)$ to obtain the spectral representation of the conductivity, $\sigma(\omega, k)$.

(b) Using the result from part (a), calculate $\chi(\omega, k)$.

(c) Show that the same result is obtained using Landau's rule and

$$\chi(\omega, k) = -\frac{\omega_p^2}{k^2} \int_{\mathsf{L}} dv \, \frac{f_0'(v)}{v - \omega/k}. \tag{16.68}$$

Plot the corresponding contours for $\operatorname{Im}\omega < 0$, $\operatorname{Im}\omega = 0$, and $\operatorname{Im}\omega > 0$.

(d) Derive the exact dispersion relation of the longitudinal oscillations in such plasma. What is their damping rate?

(e) Derive the general expression for $\sigma(t, x)$ in terms of $f_0(v)$. Plot the result for the Lorentzian plasma. Explain the plot qualitatively.

PIV.3 Two-stream instability in Lorentzian plasma

Consider a one-dimensionless electron plasma with a Lorentzian distribution with two streams.

(a) Derive the susceptibility corresponding to the electron distribution

$$f_0(v) = \frac{\Delta}{2\pi} \frac{1}{(v-u)^2 + \Delta^2} + \frac{\Delta}{2\pi} \frac{1}{(v+u)^2 + \Delta^2}. \tag{16.69}$$

(b) Show that the distribution has a single maximum if $|u| < \Delta/\sqrt{3}$ and two maxima if $|u| > \Delta/\sqrt{3}$.

(c) Plot characteristic Nyquist contours for this distribution for different u at fixed k. Argue that the instability threshold can be found from $\epsilon(0, k) = 0$. Solve this equation for k. How large should $|u|$ be for the plasma to be unstable? Can the distribution be double-peaked and yet remain stable?

PIV.4 Weibel instability

In plasmas with anisotropic temperature, transverse waves can be subject to the so-called Weibel instability. Here, you are asked to calculate this effect for nonmagnetized nonrelativistic collisionless electron plasma with motionless ions using Eq. (10.12).

(a) Assume that $f_0(\boldsymbol{v})$ is bi-Maxwellian with zero average velocity, namely,

$$f_0(\boldsymbol{v}) = \frac{1}{\pi^{3/2}\, w_\perp^2 w_\parallel} \exp\left(-\frac{v_x^2}{w_\perp^2} - \frac{v_y^2}{w_\perp^2} - \frac{v_z^2}{w_\parallel^2}\right), \tag{16.70}$$

where $w_\perp^2 = 2T_\perp/m$ and $w_\parallel^2 = 2T_\parallel/m$. Show that the dielectric tensor is diagonal. Then calculate $\epsilon_\perp$ explicitly and show that the dispersion relation of transverse waves can be written as follows:

$$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2 \frac{T_\perp}{T_\parallel} [1 + \zeta Z(\zeta)] = 0. \tag{16.71}$$

[If needed, the latter term can as well be represented in terms of $Z'(\zeta)$.]

(b) Briefly comment on how to recover the cold-wave dispersion relation (10.20) from Eq. (16.71).

(c) For the hot-plasma limit ($\zeta \ll 1$), show that Eq. (16.71) is approximately linear in ω and derive the corresponding $\omega(k)$ explicitly. (For simplicity, assume $k > 0$.) Show that, at $T_\perp > T_\parallel$, there always exist k for which waves are unstable.

(d) For the cold-plasma limit ($\zeta \gg 1$), assume that $\operatorname{Im}\zeta > 0$ and that $\operatorname{Re}\omega$ is small (which can be checked *a posteriori*); then the plasma dispersion function can be approximated as[7]

$$Z(\zeta) \approx -\frac{1}{\zeta} - \frac{1}{2\zeta^3}.$$

Assuming this asymptotic, show that Eq. (16.71) is biquadratic in ω (i.e., quadratic in ω^2). Calculate $\omega(k)$ to the first order in the temperature and show that one of the modes is unstable. Show that $T_\perp \gg T_\parallel$ is required to justify the cold-plasma approximation for this mode.

(e) Compare your analytic calculations with numerical solution of Eq. (16.71).

PIV.5 Plasma susceptibility in magnetic field

Here, you are asked to derive one of the elements of the magnetized-plasma kinetic susceptibility tensor using the following expression for the induced linear current density:

$$\boldsymbol{j}_s^{(\mathrm{i})}(t, \boldsymbol{x}) = -e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}\,\frac{m_s\omega_{ps}^2}{4\pi}\int d\boldsymbol{p}\,\boldsymbol{v}\int_0^{t-t_0} d\tau\, e^{i\beta}\Big\{\tilde{E}_x U\cos(\phi+\Omega_s\tau) + \tilde{E}_y U\sin(\phi+\Omega_s\tau) + \tilde{E}_z\left[\frac{\partial f_{0s}}{\partial p_\parallel} - V\cos(\phi+\Omega_s\tau)\right]\Big\}, \quad (16.72)$$

where the axes are chosen such that $k_y = 0$ and, accordingly,

$$\beta = -\frac{k_\perp v_\perp}{\Omega}\left[\sin(\phi+\Omega\tau) - \sin\phi\right] + (\omega - k_\parallel v_\parallel)\tau. \quad (16.73)$$

Hint: For the remaining notation, see the slides from Lecture 13. In particular, remember that $\Omega_s \doteq \Omega_{s0}/\gamma$ is the relativistic gyrofrequency and $\Omega_{s0} \doteq e_s B_0/(m_s c)$ is the nonrelativistic gyrofrequency. Also, f_{0s} is the momentum distribution, not the velocity distribution; the two are connected by the factor m^3, as explained in Sec. 8.1.2.

The derivation can be performed as explained on the slides, but you may also simplify some of the steps, because $(\boldsymbol{\chi}_s)_{xx}$ is somewhat easier to calculate than the whole tensor.

(a) Show that Eq. (16.72) leads to

$$(\boldsymbol{\chi}_s)_{xx} = \frac{\omega_{ps}^2}{\omega\Omega_{s0}}\sum_{n=-\infty}^{\infty}\int_0^\infty dp_\perp\, 2\pi p_\perp \int_{-\infty}^{\infty} dp_\parallel \left(\frac{\Omega_s}{\omega - k_\parallel v_\parallel - n\Omega_s}\right)\frac{n^2 J_n^2(z)}{z^2}\,p_\perp U. \quad (16.74)$$

(b) Assuming that f_{0s} is isotropic, nonrelativistic, Maxwellian, and has zero average velocity, show that

$$(\boldsymbol{\chi}_s)_{xx} = \frac{\omega_{ps}^2}{\omega}\sum_{n=-\infty}^{\infty}\frac{n^2 I_n(\lambda_s)}{\lambda_s}\,e^{-\lambda_s}A_n, \quad A_n = \left(\frac{Z(\xi_n)}{k_\parallel w}\right)_s. \quad (16.75)$$

[7] Note that this asymptotic is slightly different from the asymptotic of Z on the real axis derived in Sec. 11.2.1. See Appendix AIV.1 for details.

PIV.6 Kinetic waves propagating parallel to magnetic field

Here, you are asked to study waves propagating parallel to a static magnetic field in a static homogeneous anisotropic Maxwellian plasma ($T_\perp \neq T_\parallel$).

(a) Derive the limit of the dielectric tensor at $k_\perp \to 0$. Show that it has the same form as in cold plasma, except L, R, and P are replaced with, respectively,

$$\bar{L}, \bar{R} = 1 + \sum_s \left(\frac{\omega_p^2}{\omega} A_{\pm 1} \right)_s, \qquad \bar{P} = 1 + \sum_s \left(\frac{\omega_p^2}{\omega} \frac{2\omega}{k w_\parallel^2} B_0 \right)_s. \tag{16.76}$$

Present the corresponding dispersion relation and identify the corresponding waves and their polarizations.

(b) By retaining the lowest-order finite temperature corrections, show that the dispersion relation for Alfvén waves can be written as

$$\omega^2 = k_\parallel^2 V_A^2 \left[1 - \frac{p_\parallel - p_\perp}{B_0^2/(4\pi)} \right], \tag{16.77}$$

where $p_\parallel$ and $p_\perp$ are the parallel pressure and the perpendicular pressure, respectively. This dispersion relation predicts the an instability, which is known as the firehose instability. When does this instability occur? Qualitatively, what is its mechanism?

> **Hint:** See slides for the formulas for A_n and B_n. Neglect the contribution from the resonant pole but keep *two* terms when expanding the rest of the plasma dispersion function. Then simplify the result by adopting that $\omega \ll \Omega_i$. However, when expanding in ω, assume that the parameter $k w_\parallel/\omega$ is fixed and of order one.

PIV.7 Kinetic whistler waves

Here, you are asked to consider parallel propagation of high-frequency whistler waves ($\omega \gg \Omega_i$), for which the ion susceptibility is negligible.

(a) Using Eqs. (16.76) with $T_\perp = T_\parallel$, derive the whistler-wave dispersion relation. Unlike in Problem PIV.6, neglect thermal effects in $\operatorname{Re} \bar{R}$ but account for nonzero $\operatorname{Im} \bar{R}$.

(b) Present the dispersion relation for ω_r. Show that in the regime when $N^2 \gg 1$ and $\omega_r \ll |\Omega_e|$, the group velocity of these waves scales as $v_g \propto \sqrt{\omega_r}$.

(c) Suppose that a whistler wave is excited by an antenna with fixed real frequency. Estimate the distance that the wave propagates before it damps assuming that damping is weak.

> **Hint:** The idea is the same as in Sec. 10.2.3 except: (i) the dispersion function is different and (ii) complex is the wavevector rather than the frequency.

PIV.8 Cyclotron heating

(a) Consider an electromagnetic wave propagating approximately parallel to the static magnetic field $\boldsymbol{B}_0$ such that the longitudinal component of the electric field is negligible; however, allow for small nonzero $k_\perp$. Show that the wave power dissipated per unit volume is approximately given by Eq. (13.31).

(b) Suppose that the static magnetic field is inhomogeneous, $B_0 \approx \bar{B}_0(1 + z/\ell_0)$. Here, z is the coordinate along the field (the wave propagates from $z = -\infty$ to $z = +\infty$), ℓ_0 is some large enough characteristic length (you may assume that $\ell_0 > 0$ for simplicity, but this is not essential), and the constant $\bar{B}_0$ is the field strength at which the wave is in cyclotron resonance with some minority ions s. Assume that the wave frequency ω is fixed and given. Also, assume that the wave amplitude $|\tilde{E}_\pm|$ is given at $z = 0$ (and depends on z slowly enough such that its value outside the resonance region is unimportant). Calculate the *total* wave power $\int dz\, \mathscr{P}_{\mathrm{CD},s}$ that is dissipated through cyclotron damping on the minority ions per unit cross section of the wave beam using two different methods (make sure that the results are consistent):

(i) using Eq. (13.31);

(ii) using Eq. (13.10) with the cold-plasma dielectric tensor (6.13).

Hint: See footnote 1 in Lecture 6 and also Problem PII.2(e).

(c) Propose an analytic estimate for the characteristic length of the absorption region a_{abs}. Assuming parameters typical for ion cyclotron heating in tokamaks, estimate a_{abs} numerically.

Appendices

Appendix A

Abbreviations

This appendix summarizes the abbreviations used in the text.

Abbreviation	Meaning
GO	geometrical optics
CCS	collisionless cold static
DK	Dnestrovskii–Kostomarov
EBW	electron Bernstein wave(s)
EPW	electron plasma wave(s), a.k.a. (electron) Langmuir waves
IAW	ion acoustic wave(s), a.k.a. ion sound waves
IBW	ion Bernstein wave(s)
IPW	ion plasma wave(s)
MHD	magnetohydrodynamic(s)
OC	oscillation center(s)
ODE	ordinary differential equation
PDE	partial differential equation
TE, TM	transverse-electric, transverse-magnetic
WKB	Wentzel–Kramers–Brillouin
WKE	wave-kinetic equation
WWT	Wigner–Weyl transform

Appendix B

Notation

This appendix summarizes the notations that are of general importance or often used in the main text.

Symbol	Definition	Explanation
$\ldots \doteq \ldots$		definition
$\ldots \cdot \ldots$	Eq. (C.5)	scalar product of spatial or spacetime vectors
$\langle \ldots \rangle$		average over fast oscillations
$\langle \ldots \vert \ldots \rangle$		scalar product of vectors in Hilbert space
$\langle \ldots \vert$		covector in Hilbert space (bra)
$\vert \ldots \rangle$		vector in Hilbert space (ket)
$\tilde{\ldots}$		quantity of the first order in the wave amplitude
$\hat{\ldots}$		operator
$\ldots^*$		complex conjugate
$\ldots^\dagger$		Hermitian conjugate
$\ldots^\intercal$		transpose
$\ldots_{\mathrm{A}}$		anti-Hermitian part
$\ldots_{\mathrm{H}}$		Hermitian part
$[\ldots, \ldots]$		commutator
∂		partial derivative
∂_a	$\partial/\partial x^a$	partial derivative with respect to a spatial coordinate
∂_α	$\partial/\partial x^\alpha$	partial derivative with respect to a spacetime coordinate
$⨍$		principal-value integral
1		unity
$\mathbf{1}$		unit matrix
$\hat{1}$		unit operator
$\hat{\mathbf{1}}$		unit matrix operator
$\boldsymbol{\mathcal{X}}$	Eq. (1.16)	susceptibility in the coordinate representation
$\bar{\boldsymbol{\mathcal{X}}}$	Box 1.1	susceptibility in the symmetrized coordinate representation
Λ		eigenvalue of $\boldsymbol{D}_{\mathrm{H}}$
$\boldsymbol{\Lambda}$		diagonal matrix with the eigenvalues of $\boldsymbol{D}_{\mathrm{H}}$ on the diagonal
$\boldsymbol{\Theta}$	$\partial_{\boldsymbol{kk}}\omega$	photon inverse-mass tensor (times $\hbar$)
$\boldsymbol{\Sigma}$	Eq. (1.8)	conductivity in the coordinate representation
$\bar{\boldsymbol{\Sigma}}$	Box 1.1	conductivity in the symmetrized coordinate representation

$\boldsymbol{\Omega}_s$	$e_s \boldsymbol{B}_0/(m_s c)$	gyrofrequency of species s

$\boldsymbol{\chi}$		symbol of $\hat{\boldsymbol{\chi}}$, subsumes the susceptibility in the spectral representation
$\hat{\boldsymbol{\chi}}$	Eq. (1.15)	susceptibility operator
$\boldsymbol{\epsilon}$		symbol of $\hat{\boldsymbol{\epsilon}}$, subsumes the dielectric tensor
$\hat{\boldsymbol{\epsilon}}$	Eq. (1.17)	dielectric operator
ε	Eq. (3.5)	geometrical-optics parameter
γ		linear growth rate, also polytropic index
φ		electrostatic potential
λ		wavelength
λ_{Ds}	v_{Ts}/ω_{ps}	Debye length of species s
λ_s	$k_\perp^2 v_{Ts}^2/\Omega_s^2$	roughly, squared ratio of the gyroradius and the transverse wavelength
π	≈ 3.14	ratio of a circle's circumference to its diameter
θ		eikonal phase, also angle between $\boldsymbol{k}$ an $\boldsymbol{B}_0$
ρ		charge density
$\boldsymbol{\sigma}$		symbol of $\hat{\boldsymbol{\sigma}}$, subsumes the conductivity in the spectral representation
$\hat{\boldsymbol{\sigma}}$	Eq. (1.5)	conductivity operator
ω		frequency
$\bar{\omega}$	$-\partial_t \theta$	local frequency of a quasimonochromatic wave
$\hat{\omega}$	$i\partial_t$	frequency operator
ω_i	$\operatorname{Im}\omega$	imaginary part of the frequency
ω_p	Eq. (2.21)	plasma frequency
ω_{ps}	Eq. (2.6)	plasma frequency of species s
ω_r	$\operatorname{Re}\omega$	real part of the frequency

$\boldsymbol{B}, B_a$		magnetic field
$\boldsymbol{B}_0, B_0$		background magnetic field
$\boldsymbol{\mathcal{B}}, \mathcal{B}_a$		magnetic field envelope
C_S	Eq. (7.33)	ion sound speed
D	Eq. (6.14b)	element of $\boldsymbol{\epsilon}$ of CCS magnetized plasma
$\boldsymbol{D}, D_{ab}$		symbol of a dispersion operator
$\hat{\boldsymbol{D}}, \hat{D}_{ab}$		dispersion operator
$\boldsymbol{D}_\varphi$	electrostatic dispersion operator	
$\hat{\boldsymbol{D}}_E$	Eq. (1.23)	electromagnetic dispersion operator
$\boldsymbol{E}, E_a$		electric field
$\boldsymbol{\mathcal{E}}, \mathcal{E}_a$		electric field envelope
H	$\hbar\omega$	photon energy or photon Hamiltonian
I	$\int d\boldsymbol{x}\, \mathcal{I}$	wave action
$\mathcal{I}$	Sec. 5.1	wave action density
Im		imaginary part
$\boldsymbol{\mathcal{J}}, \mathcal{J}_a$	$\boldsymbol{v}_g \mathcal{I}$	action flux density
L	Eq. (6.14d)	element of $\boldsymbol{\epsilon}$ of CCS magnetized plasma
L_c		characteristics spatial scale of an envelope and (or) of a medium
L	Sec. 10.1.2	Landau contour
$\boldsymbol{N}$	$c\boldsymbol{k}/\omega$	refractive index
N		dimension of a wave field ($\mathsf{N} = 3$ for the electric field)
P	Eq. (6.14c)	element of $\boldsymbol{\epsilon}$ of CCS magnetized plasma, also pressure
$\boldsymbol{P}$	$\hat{\boldsymbol{\chi}}\boldsymbol{E}$	electric polarization
$\boldsymbol{\mathcal{P}}$	$\boldsymbol{k}\mathcal{I}$	wave momentum density
$\mathscr{P}_{\text{abs}}$		absorbed power
R	Eq. (6.14d)	element of $\boldsymbol{\epsilon}$ of CCS magnetized plasma

$Q, \boldsymbol{Q}$		Stern–Gerlach term
Re		real part
S	Eq. (6.14a)	element of $\boldsymbol{\epsilon}$ of CCS magnetized plasma
$\boldsymbol{S}$	Eq. (5.17)	Poynting vector
T_c		characteristics temporal scale of an envelope and (or) of a medium
T_s		temperature of species s
$\mathfrak{U}$	$\omega\mathcal{I}$	wave energy density
$\boldsymbol{\mathcal{V}}^\alpha$	$\partial\boldsymbol{D}/\partial k_\alpha$	k-derivative of the symbol of a dispersion operator
$\mathscr{W}, \mathscr{W}^{-1}$	Sec. 3.3	direct and inverse Wigner–Weyl transform
Z_s	e_s/e	charge state of species s
arg		argument (phase) of a complex number
c		speed of light
c.c.		complex conjugate
d		differential
d_τ	$\mathsf{d}/\mathsf{d}\tau$	full derivative with respect to τ
e	≈ 2.72	Euler's number
e	$\lvert e_e \rvert$	elementary charge
$\bar{\boldsymbol{e}}_a$		unit vector along ath axis
e_s		charge of species s
$\boldsymbol{h}$		eigenvector of $\boldsymbol{D}_{\mathrm{H}}$
$\hbar$		Planck constant
i		imaginary unit
$\boldsymbol{j}, j_a$		current density
$\boldsymbol{k}, k_a$		wavevector
$\bar{\boldsymbol{k}}, \bar{k}_a$	$\nabla\theta, \partial_a\theta$	local wavevector of a quasimonochromatic wave
$\hat{\boldsymbol{k}}, \hat{k}_a$	$-\mathsf{i}\nabla, -\mathsf{i}\partial_a$	wavevector operator
k, k_α	$(k_0, \boldsymbol{k})$	spacetime wavevector
$\hat{\mathsf{k}}, \hat{k}_\alpha$	$-\mathsf{i}\partial_\alpha$	spacetime-wavevector operator
k_0	$-\omega/c$	temporal component of the spacetime wavevector
m_s		mass of species s
n		dimension of space, $\dim \boldsymbol{x}$
n	$n+1$	dimension of spacetime, $\dim \mathsf{x}$
n_s		density of species s
$\boldsymbol{x}, x^a$		spatial location
$\hat{\boldsymbol{x}}, \hat{x}^a$	$\boldsymbol{x}, x^a$	coordinate operator
x, x^α	$\{x^0, \boldsymbol{x}\}$	location in spacetime
$\hat{\mathsf{x}}, \hat{x}^\alpha$	$\{x^0, \boldsymbol{x}\}$	spacetime-coordinate operator
x^0	ct	temporal spacetime coordinate
$\boldsymbol{v}_s$		fluid velocity of species s
$\boldsymbol{v}_g$	$\partial_{\boldsymbol{k}}\omega$	group velocity
$\boldsymbol{v}_p$	$(\boldsymbol{k}/k)(\omega/k)$	phase velocity
v_{Ts}	$\sqrt{T_{0s}/m_s}$	unperturbed thermal speed of species s
t		time
tr		trace
w_s	$v_{Ts}\sqrt{2}$	rescaled thermal speed of species s

Appendix C

Conventions

This appendix summarize the mathematical conventions assumed in the main text.

C.1 Numbers and matrices

For a given complex number z, its real part $z_r \equiv \operatorname{Re} z$ and imaginary part $z_i \equiv \operatorname{Im} z$ are defined as

$$z_r \doteq \frac{1}{2}\,(z + z^*) = z_r^*, \qquad z_i \doteq \frac{1}{2i}\,(z - z^*) = z_i^*. \tag{C.1}$$

Accordingly,

$$z = z_r + iz_i, \qquad z^* = z_r - iz_i. \tag{C.2}$$

Similarly, for any given matrix or operator M, its Hermitian part M_{H} and anti-Hermitian part M_{A} are defined as

$$M_{\mathrm{H}} \doteq \frac{1}{2}\,(M + M^\dagger) = M_{\mathrm{H}}^\dagger, \qquad M_{\mathrm{A}} \doteq \frac{1}{2i}\,(M - M^\dagger) = M_{\mathrm{A}}^\dagger. \tag{C.3}$$

Accordingly,

$$M = M_{\mathrm{H}} + iM_{\mathrm{A}}, \qquad M^\dagger = M_{\mathrm{H}} - iM_{\mathrm{A}}. \tag{C.4}$$

C.2 Geometry

Euclidean or pseudo-Euclidean coordinates are assumed. The **bold** font is used for spatial coordinates ($\boldsymbol{x} = \{x^1, x^2, ...x^n\}$, with any $n \geq 0$), vectors (covectors), and matrices. The upper and lower indices of these objects are marked with Latin indices and are interchangeable; for example, $k_a = k^a$, with $a = 1, 2, \ldots n$. For any two n-dimensional column vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, one has

$$\boldsymbol{X} \cdot \boldsymbol{Y} \doteq \boldsymbol{X}^\dagger \boldsymbol{Y} = X_a^* Y^a, \tag{C.5}$$

Here, † is a Hermitian conjugate (adjoint), or conjugate transpose, $\boldsymbol{X}^\dagger = \boldsymbol{X}^{*\intercal}$. The expression $\boldsymbol{X}^\dagger \boldsymbol{Y}$ is understood as a matrix product, with $\boldsymbol{X}^\dagger$ being a $1 \times n$ matrix and $\boldsymbol{Y}$ being a $n \times 1$ matrix. Accordingly, $\boldsymbol{X}\boldsymbol{Y}^\dagger$ is a $n \times n$ matrix with elements $X^a Y_b^*$.

For any matrix $\hat{\boldsymbol{M}}$ whose elements are operators, denoted $\hat{M}_{ab}$, the adjoint $\hat{\boldsymbol{M}}^\dagger$ is the transpose of the matrix with elements $\hat{M}_{ab}^\dagger$; i.e., $(\hat{\boldsymbol{M}}^\dagger)_{ab} = \hat{M}_{ba}^\dagger$. For example, for the column-vector operator $\hat{\boldsymbol{k}} \doteq -i\nabla$, whose elements $\hat{k}_a = -i\partial_a$ are Hermitian operators (here, $\partial_a \doteq \partial/\partial x^a$), the adjoint is the row-vector operator $\hat{\boldsymbol{k}}^\dagger = \hat{\boldsymbol{k}}^\intercal$. Then, $\hat{\boldsymbol{k}}^\dagger \hat{\boldsymbol{k}} = \hat{k}^2 = -\nabla^2$, while $\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}^\dagger$ is a matrix with elements $\hat{k}_a \hat{k}_b$.

The spacetime coordinates and spacetime vectors (covectors) are denoted with the sans-serif font in the invariant representation; but this font is used also for other objects. For example, $\mathsf{x} = \{x^0, \boldsymbol{x}\}$, and $\mathsf{n} \doteq \dim \mathsf{x} = n + 1$. The corresponding components are denoted with a regular font and Greek indices (for example, x^α), which span from 0 to n. For the spacetime, Minkowski metric is assumed in the form $g_{\alpha\beta} = g^{\alpha\beta} = \operatorname{diag}\{-1, 1, 1, \ldots\}$. For the objects in the Minkowski space, conventions are similar to those above, except the indices are raised and lowered by the Minkowski metric. For example, $k_\alpha = g_{\alpha\beta} k^\beta$, or equivalently, $k^\alpha = g^{\alpha\beta} k_\alpha$, so in particular, $k^0 = -k_0$ and $k^a = k_a$ for $a > 0$.

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