Plasma is turbulence, and turbulence is plasma

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One of the important challenges of plasma theory is searching for new ways of thinking about turbulence, which remains notoriously complicated notwithstanding decades of research. This paper puts forth an approach that can help make turbulence theory more intuitive. Generic wave turbulence can be modeled as *effective quantum plasma* where wave quanta serve as particles and coherent fields mediate their collective interactions. The dynamics of this effective plasma is governed by the Wigner–Moyal equation (WME), which is a "full-wave" generalization of the wave kinetic equation. The WME does not assume scale separation and retains phase correlations, so coherent structures can be modeled properly. A similar approach is already used *ad hoc* in applications to specific systems such as optical turbulence, but the systematic theory makes the WME more flexible. For example, the WME has been recently applied to drift-wave and Navier–Stokes turbulence, and similar applications to magnetohydrodynamic turbulence are envisioned. Since different systems correspond to different effective particles, there is more than one "plasma physics of turbulence", which is intellectually stimulating.

Introduction. — Turbulence is a ubiquitous phenomenon that is responsible for many aspects of plasma dynamics, particularly, structure formation and transport of particles, momentum, and energy. Understanding these effects is of tremendous importance in various areas of plasma physics, including fusion science, planetary science, and astrophysics. Nevertheless, turbulence remains notoriously difficult to model, and *ab initio* simulations are often considered as the only feasible option. In this situation, searching for new ways of thinking about turbulence is an important challenge of fundamental plasma theory, perhaps even more important than understanding specific effects driven by turbulence.

This paper puts forth an intuitive and arguably promising way of thinking about turbulence, especially inhomogeneous wave turbulence. Adopting this approach will not solve all problems in turbulence theory, but it can make some of them more tractable.

Basic idea. — To motivate the basic idea, let us start with the following observation. One area where turbulence studies have been particularly fruitful is plasma kinetic theory. It is not commonly viewed as a turbulence theory *per se*, but it *is* one in the sense that any, even regular, dynamics of plasma can be considered as turbulent dynamics of quantum matter waves. Kinetic theory hides the complexity of the quantum field by describing only the Fourier spectrum of its two-point correlation function, or the Wigner function W [1]. In the geometrical-optics (classical) limit, when the de Broglie wavelengths and quantum correlations are negligible, W can be interpreted as the particle distribution function and satisfies a Liouville-type equation (e.g., Vlasov equation). Then, quantum-turbulence theory becomes an intuitive Hamiltonian theory of incompressible flows in phase space.

Due to the success of this reduced theory of quantum turbulence (a.k.a. plasma kinetic theory), it is tempting to describe classical turbulence in the same way. Indeed, Liouville-type "wave kinetic equations" (WKE) for nonlinear classical waves are widely known. However, since the wavelengths of classical fluctuations are typically much larger than those of quantum fluctuations, the geometrical-optics approximation underlying the WKE is fragile. For example, the WKE is often inadequate for modeling structure formation, because the characteristic scales are often determined by diffraction, which the WKE neglects along with other full-wave effects and phase information in general. Because of this, practical applications of the WKE have been limited, and direct numerical simulations are often preferred instead.

Here, we argue that this can be fixed. Even without scale separation, wave turbulence can be modeled intuitively as effective plasma, except waves must be treated as *quantum* particles. To see this, let us return to the analogy with plasma kinetic theory. When the particle de Broglie wavelength and (or) quantum correlations are non-negligible, W is no longer governed by the Liouville equations but satisfies a more general Wigner–Moyal equation (WME) [2]. Although the WME is a *pseudo*-differential equation (see below), it largely retains the Hamiltonian structure of the Liouville equation and the associated conservation laws, so quantum turbulence generally remains tractable. For example, nonlinear modulational instabilities of quantum matter can be viewed as linear plasma instabilities, and their saturation can be understood from properties of the particle Hamiltonian.

We propose that classical turbulence be modeled similarly with the WME for the Wigner function (matrix) on the ray phase space. No scale separation is needed in this case, and yet the dynamics remains tractable. This idea was already applied in the past to *manifestly quantumlike* systems, for example, optical turbulence governed by the nonlinear Schrödinger equation [3]. Here, we propose to apply it to a broader class of equations, including those governing magnetized-plasma turbulence. We also argue that, by following the systematic approach outlined below, one can significantly reduce the amount of calculations compared to that needed in *ad hoc* formulations.

Wigner–Moyal formulation. — The approach itself is described as follows. Consider the governing equation for some turbulent field ψ in the form

$$i\partial_t \psi = \hat{H}\psi. \tag{1}$$

The variable t can be the true time or an extended time introduced artificially for uniformity. (In the latter case, ψ does not actually depend on t, so ∂_t can as well be omitted.) The field ψ will be called a wave for brevity, but it does not have to be an oscillating field. In particular, the operator $\hat{H}$, termed Hamiltonian, may or may not be Hermitian. In general, $\hat{H}$ can depend on ψ, $\psi^\dagger$, and some coherent fields, which may be governed by some additional equations.

Let us promote ψ to an abstract vector $|\psi\rangle$, as commonly done in quantum mechanics, and rewrite the above equation in the invariant vector form:

$$i\partial_t |\psi\rangle = \hat{H} |\psi\rangle . \tag{2}$$

By projecting Eq. (2) on the eigenvectors $|\mathbf{x}\rangle$ of the coordinate operator $\hat{\mathbf{x}}$, one recovers the original equation for $\langle \mathbf{x}|\psi\rangle \equiv \psi$. By projecting Eq. (2) on the eigenvectors $|\mathbf{k}\rangle$ of the wave-vector operator $\hat{\mathbf{k}} \doteq -i\nabla$, one obtains the equation for $\langle \mathbf{k}|\psi\rangle$, which is the Fourier transform of ψ. (Here, the symbol $\doteq$ denotes definitions and Euclidean coordinates are assumed. For generalizations to non-Euclidean coordinates, see Ref. [4].) But we are interested in a *phase-space* formulation of turbulence, so we seek to project the governing equation on *both* $|\mathbf{x}\rangle$ and $|\mathbf{k}\rangle$. A vector equation does not allow for such double projection, so we need to switch to operator equations. For that, let us introduce the "density operator" $\hat{W} = |\psi\rangle \langle\psi|$. (If ψ has multiple components, we introduce a matrix of operators instead, namely, $\hat{W}_{\alpha\beta} = |\psi_\alpha\rangle \langle\psi_\beta|$; see Refs. [5, 6].) As usual, it is governed by the von Neumann equation,

$$i\partial_t \hat{W} = [\hat{H}_H, \hat{W}]_- + i[\hat{H}_A, \hat{W}]_+. \tag{3}$$

Here, the indices $_H$ and $_A$ denote the Hermitian and anti-Hermitian parts and the brackets denote a commutator and anti-commutator, respectively.

Let us consider the Weyl transform $\mathscr{W}$, which in a sense, is the most natural projection of a given operator to a function on the $(\mathbf{x}, \mathbf{k})$ space [7]; for example, it maps Hermitian operators to real functions (Hermitian matrices [6]). By applying the Weyl transform to Eq. (3), one obtains the following WME:

$$\partial_t W = \{\{H_H, W\}\} + [[H_A, W]]. \tag{4}$$

Here $H_{H,A} \doteq \mathscr{W}[\hat{H}_{H,A}]$, and $W \doteq \mathscr{W}[\hat{W}]$ is the Wigner function, which is a real function (or Hermitian matrix, if ψ has multiple components). Also, $\{\{A, B\}\} \doteq 2A \sin(\hat{\mathcal{L}}/2)\, B$ and $[[A, B]] \doteq 2A \cos(\hat{\mathcal{L}}/2)\, B$ are called the Moyal brackets, and $\hat{\mathcal{L}} \doteq \{\overleftarrow{\cdot}, \overrightarrow{\cdot}\}$ is the canonical Poisson bracket. The arrows show the directions in which the derivatives act; for example, $A\hat{\mathcal{L}}B = \{A, B\}$.

Now, let us suppose that we can define some average $\overline{(\ldots)}$, which can be, for example, the average over some insignificant variable or the ensemble average. Let us split W into its average $\bar{W}$ and fluctuations $\tilde{W}$, and similarly for $H_{H,A}$. By averaging Eq. (4), one obtains, by properties of the Weyl transform, that

$$\partial_t \bar{W} = \{\{\bar{H}_H, \bar{W}\}\} + [[\bar{H}_A, \bar{W}]] + C, \tag{5}$$

where we introduced

$$C \doteq \overline{\{\{\tilde{H}_H, \tilde{W}\}\}} + \overline{[[\tilde{H}_A, \tilde{W}]]}. \tag{6}$$

Typically, $\bar{H}_{H,A}$ depend on the coherent fields U, whose generic equations can be written as follows:

$$\hat{\mathcal{D}}U = F[\bar{W}]. \tag{7}$$

Here, $\hat{\mathcal{D}}$ is some evolution operator and $F[W]$ is some functional of the Wigner function. [For example, in the case of the nonlinear Schrödinger equation with cubic nonlinearity, one can adopt $U = \overline{|\psi|^2} = \int \bar{W}(t, \mathbf{x}, \mathbf{k})\, d^3k$, in which case $\hat{\mathcal{D}} = 1$ and $F[\cdot] = \int (\cdot)\, d^3k$. As another example, consider drift-wave turbulence within the Hasegawa–Mima model. In this case, U can be the zonal-flow velocity, and the explicit form of Eq. (7) is derived in Ref. [8]. Also, for the case of Navier–Stokes turbulence, see Ref. [5].] In contrast with many other theories of inhomogeneous turbulence, no cumbersome calculations are needed here; instead, one makes use of the known theorems of the Weyl-transform theory, or Weyl calculus. Also advantageously, the WME is closely linked with the WKE, which is obtained in the geometrical-optics limit via $\{\{H_H, W\}\} \to \{H_H, W\}$ and $[[H_H, W]] \to 2H_A W$.

Equations (5)–(7) are identical to the kinetic model of quantum plasma up to the Hamiltonian, with U serving as a collective field. In this sense, *any* turbulence is effectively a quantum plasma, even when it is not plasma turbulence *per se*. Also note that C serves as the wave–wave collision operator. Homogeneous-turbulence theory is recovered in the limit when the effective plasma is collisional and near-equilibrium, i.e., $\{\{\bar{H}_H, \bar{W}\}\} \to 0$ and $[[\bar{H}_A, \bar{W}]] \to 2\bar{H}_A W$. Note that most of "plasma physics of turbulence" is ignored in this case. In contrast, the commonly used [9, 10] quasilinear approximation corresponds to the "collisionless" limit, i.e., the limit when C is negligible compared with the Moyal brackets in Eq. (5).

Applications. — Equations (5) and (7) are useful both for analytic calculations and numerical modeling. In analytic theory, the concept of the Hamiltonian allows one to picture turbulent dynamics intuitively as phase-space dynamics of effective particles. One can also apply the usual methods of plasma kinetic theory to calculate modulational instabilities as *linear* instabilities of the effective plasma [11] and to figure out their saturation mechanisms, at least if $\bar{H}_A$ is small [12]. In numerical simulations, the advantages are twofold. First of all, by comparing WME simulations with WKE simulations, one can actually check whether a given effect is captured by the geometrical-optics approach. (This is possible to do because the Wigner function and the geometrical-optics density of the wave quanta belong to the same space.) Second, one can efficiently model multi-scale processes. In contrast to direct numerical simulations, which have to resolve the smallest relevant wavelength, Eq. (5) can be simulated on a much coarser grid. [It can be computationally cheaper to simulate the dynamics in the $(\mathbf{x}, \mathbf{k})$ space as opposed to the $\mathbf{x}$ space alone if the turbulence spectrum is very broad while the spatial structure of $\bar{W}$ is mainly large-scale.] Also importantly, solving Eq. (5) yields statistical predictions in a single run.

Recently, we have been pursuing analytic and numerical quantumlike modeling of drift-wave turbulence and the associated dynamics of zonal flows, both in the quasilinear approximation [11–18] and beyond [19]. As it turns out, many aspects of drift-wave turbulence that have remained obscure so far (notwithstanding extensive research) are readily understood within the quantumlike approach. For example, we have developed a systematic understanding of the secondary instability of broadband spectra, the formation of propagating and stationary solitary zonal structures [14–17], and the Kelvin–Helmholtz instability of zonal flows [13, 14], which can be understood as "vacuum breakdown" for drift waves considered as quantum particles. A particularly remarkable outcome of this research is an explanation of the mechanisms of saturation and nonlinear oscillations of collisionless zonal flows [12] and also an explanation of the so-called Dimits shift [18], which is important for fusion science [20] and fascinating as basic physics.

It has also been shown how to extend such calculations to *vector*-wave turbulence. In particular, Ref. [5] considers Navier–Stokes turbulence as an example. This opens opportunities in Wigner–Moyal modeling of magnetohydrodynamic turbulence, where the structure of the governing equations is similar. Potential applications are envisioned in theory of the magnetorotational instability and turbulent dynamo, also as a development of a related recent research [10]. This is facilitated by the fact that even the simplest quasilinear approximation ($C \approx 0$) is known to suffice these applications at least in some limits.

Discussion. — In the long run, quantumlike modeling of turbulence seems particularly promising and exciting in the following sense. On one hand, it provides a unified and intuitive high-level approach to various turbulence problems. On the other hand, since the Hamiltonians of turbulent perturbations are different in different types of turbulence, the quantumlike approach leads to more than one "plasma physics of turbulence". The situation is similar to that in condensed-matter theory, where studying effective (quasi)particles with exotic properties is a part of the routine. This fact may be appealing to those plasma theorists who have always worked on the Vlasov–Maxwell system and would like to try something new.

In summary, it appears that fundamental theory of plasma turbulence can substantially benefit from updating its toolbox. Embracing the Weyl calculus seems like an obvious thing to do [21]. It also may be helpful to switch from the traditional variables (electromagnetic fields, velocities, densities) to more-abstract, higher-level concepts (state vectors, Wigner matrices, wave Hamiltonians) in order to make calculations more systematic and general. Apart from being both convenient and intellectually stimulating, this approach would facilitate communication between plasma theory and other disciplines, for example, geophysics, where similar turbulence studies are being carried out in parallel. Finally, the visibility of plasma physics among the other physics disciplines can be improved by communicating the fact that, in the right variables, any turbulence looks like an effective quantum plasma.

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