Modeling drift-wave turbulence as quantumlike plasma

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Introduction. — Drift-wave (DW) turbulence can generate banded sheared flows, called zonal flows (ZFs), which can significantly influence turbulent transport. Due to the importance of this effect, physics of DW–ZF interactions has been attracting attention. As a common model, the wave-kinetic equation (WKE) has been used [1], which assumes that characteristic wavelengths of DWs are negligible compared to ZF scales. However, such "geometrical-optics" (GO) description misses essential physics, which stimulated formulations of more general, "full-wave" models, e.g., the so-called CE2. (For an overview, see Ref. [2].) But the full-wave models developed so far are not very intuitive, and their relation to the WKE is not obvious. Thus, developing robust intuitive understanding of DW–ZF interactions has been an open problem.

Here, we present a *quantumlike* approach to inhomogeneous DW turbulence [3–9]. Basically, we show how to extend the WKE-based approach beyond GO by treating DWs as effective quantum (finite-wavelength) particles. We also apply this approach to analytic and numerical modeling and explain some basic effects in DW turbulence in collisionless plasma.

Basic equations. — As a governing model, we assume the modified Hasegawa–Mima equation (mHME), which has been widely used in basic-physics studies of DW turbulence. It has been seen numerically, also in our work [8,9], that eddy–eddy interactions can often be ignored when modeling DW–ZF interactions. (This is known as the quasilinear approximation.) Then, the mHME can be expressed in the following dimensionless form [3,4]:

$$i\partial_t \tilde{w} = \hat{H}\tilde{w}, \quad \hat{H} = \hat{p}_x U - \hat{p}_x(\beta - U'')\hat{\bar{p}}^{-2}, \quad \partial_t U = \partial_y \langle (\hat{p}_y \hat{\bar{p}}^{-2}\tilde{w})(\hat{p}_x \hat{\bar{p}}^{-2}\tilde{w})^* \rangle. \tag{1}$$

We assume slab geometry with uniform magnetic field along z and constant density gradient β along y (radial coordinate in toroidal plasmas). The zonal velocity $\vec{U}(t,y)$ is along x (poloidal coordinate); $\tilde{w}$ is the DW part of the generalized vorticity, or the perturbation of the ion guiding-center density; also, $\hat{\vec{p}} \doteq -i\nabla$, $\hat{\bar{p}}^2 \doteq 1 + \hat{p}^2$, $U'' \doteq \partial_y^2 U$, $\langle \ldots \rangle$ denotes zonal averaging (i.e., averaging over x), and $\doteq$ denotes definitions. Dissipation and forcing could be added [3] but are not discussed below. Since the equation for $\tilde{w}$ has the form of a Schrödinger equation (up to $\hbar$), $\tilde{w}$ can be viewed as a state function of a DW as an effective quantum particle, or "drifton".

Linear DWs. — The drifton dynamics in prescribed ZFs can be studied like the dynamics of quantum particles in prescribed fields. However, notice the following. Although Eqs. (1) are conservative overall [3], the drifton Hamiltonian $\hat{H}$ has a nonzero anti-Hermitian part, because U'' and $\hat{\bar{p}}^{-2}$ do not commute. This means that unless U''' is negligible, driftons are not conserved. For example, driftons can be spontaneously generated by intense ZFs, which is known as the Kelvin–Helmholtz tertiary instability (TI). In Ref. [6], we showed that a laminar sinusoidal ZF with $U = U_0 \cos qy$ is unstable with respect to DW-like Floquet modes $\tilde{w} = \mathrm{Re}\,[\tilde{w}(y)e^{ik_x x}]$ if $q^2 > 1 + k_x^2$ and $|U''|_{\max} > \beta$; otherwise, all drifton states have real frequencies. This is related to the fact that $\hat{H}$ is *pseudo-Hermitian* at $\partial_t U = 0$. Specifically, in this case, $\hat{H}\hat{Q} = \hat{Q}\hat{H}^\dagger$, where $\hat{Q} \doteq \beta - U''$ is Hermitian, so $|U''|_{\max} = \beta$ is the threshold at which pseudo-Hermiticity is broken and complex frequencies emerge if q is large enough. If a primary instability and dissipation are added, Kelvin–Helmholtz modes can become unstable also at smaller q, albeit the modes are significantly different then. They become localized near extrema of U, so $\tilde{w}$ satisfies the equation of a quantum harmonic oscillator with complex frequency. We also show that these localized modes are exactly the cause of the Dimits shift [to be published].

Below, we focus on the DW dynamics in a self-consistent U generated from noise in the absence of dissipation. Using Eqs. (1), we construct a theory of DW turbulence as collisionless quantumlike plasma of driftons which interact via U as a collective field. (Eddy–eddy interactions can be added as collisions [7].) Several derivative models and effects will be discussed.

Weak quasimonochromatic DWs. — Suppose $\tilde{w}$ is weak and quasimonochromatic, $\tilde{w} = e^{ik_x x + ik_y y}\psi(t,y)$, where ψ is a slow envelope. Since $\partial_t \approx -v_g\partial_y$, where v_g is the linear DW group velocity, the equation for U readily yields a local an "equation of state" $U = U(|\psi|)$ and the equation for $\tilde{w}$ becomes a nonlinear Schrödinger equation (NLSE) with a local nonlinearity:

$$U \approx -\langle|\psi|^2\rangle/(4\beta), \qquad i(\partial_t + v_g\partial_y)\psi \approx -\chi\partial_y^2\psi - k_x\langle|\psi|^2\rangle/(4\beta)\psi, \tag{2}$$

where $v_g = 2\beta k_x k_y/\bar{k}^4$, $\chi = (1 - 4k_y^2/\bar{k}^2)\beta k_x/\bar{k}^4$, and $\bar{k}^2 \doteq 1 + k^2$. (This NLSE was also derived in the past from other considerations [1].) We showed numerically that Eqs. (2) indeed approximate the mHME adequately at small ψ [8]. They correctly predict the modulational instability (MI) of a monochromatic short-wavelength DW, and the zonal structures generated in this case are, basically, NLSE solitons propagating in y at the linear group velocity. In the presence of background shear flows, NLSE solitons deteriorate but can be reinstated if a primary instability is added [to be published]. This may explain why in weak-turbulence simulations, propagating structures are often seen instead of stationary ZFs, which appear only at large enough $|\tilde{w}|$.

Statistical model for general turbulence. — More general (strong, broadband) turbulence can be described within a statistical model. We promote $\tilde{w}$ to an abstract state vector $|\tilde{w}\rangle$, introduce the density operator $\hat{W} \doteq |\tilde{w}\rangle\langle\tilde{w}|$, which satisfies $\partial_t \hat{W} = \hat{H}\hat{W} - \hat{W}\hat{H}^\dagger$, and project the latter equation on the phase space (x, y, p_x, p_y) using the Wigner–Weyl transform. This leads to [3]

$$\partial_t \bar{W} = \{\{H_H, \bar{W}\}\} + [\![H_A, \bar{W}]\!], \qquad \partial_t U = (2\pi)^{-2} \partial_y \int \mathrm{d}^2 p\, \bar{p}^{-2} \star p_x p_y \bar{W} \star \bar{p}^{-2}. \tag{3}$$

Here, $\bar{W}(t, y, \vec{p}) \doteq \int \mathrm{d}^2 s\, e^{-i\vec{p}\cdot\vec{s}} \langle \tilde{w}(t, \vec{x} + \vec{s}/2) \tilde{w}(t, \vec{x} - \vec{s}/2) \rangle$ is the zonally-averaged Wigner function (Weyl symbol of $\hat{W}$), $H_H = p_x U - \beta p_x / \bar{p}^2 + {}^1\!/_2\, [\![U'', p_x \bar{p}^{-2}]\!]$ and $H_A = {}^1\!/_2\, \{\{U'', p_x \bar{p}^{-2}\}\}$ are the Weyl images of the Hermitian and anti-Hermitian parts of $\hat{H}$, $\{\{\cdot,\cdot\}\}$ and $[\![\cdot,\cdot]\!]$ are the so-called Moyal brackets, and $\star$ is the Moyal product known from quantum mechanics. Equations (3) are similar to the equations of quantum-plasma kinetic theory, with $\bar{W}$ serving as the (drifton) phase-space distribution and U serving as the electric field; thus, they can be analyzed in a similar manner. For example, the MI can be understood as a linear instability of drifton plasma, and its growth rate can be calculated accordingly for any background distribution $\bar{W}_0(\vec{p})$ [3, 8]. One can also integrate Eqs. (3) numerically. The first Wigner–Moyal simulations of DW turbulence were done in Refs. [4, 8, 9]. Comparison with nonlinear direct numerical simulations and other analytic theories shows that Eqs. (3) are an adequate tool for modeling many key effects in DW turbulence, including the MI, the TI, and the ZF saturation, as also discussed below.

Geometrical-optics limit revisited. — In collisionless plasmas, the MI's maximum growth rate is determined by diffraction, and so is the size of saturated ZFs. Hence, keeping full-wave effects in Eqs. (3) is generally necessary to avoid nonphysical results. However, *some aspects* of DW–ZF interactions can be understood in the GO limit (and the results can be checked using Wigner–Moyal simulations). Specifically, the GO limit is derived by taking $[\![A, B]\!] \to 2AB$ and $\{\{A, B\}\} \to \{A, B\}$, where $\{A, B\}$ is the Poisson bracket. Then, Eq. (3) becomes

$$\partial_t \bar{W} = \{H_H, \bar{W}\} + 2 H_A \bar{W}, \quad \partial_t U = (2\pi)^{-2} \partial_y \int \mathrm{d}^2 p\, p_x p_y \bar{W} / \bar{p}^2, \tag{4}$$

with $H_H \approx p_x U - (\beta - U'') p_x / \bar{p}^2$ and $H_A \approx -U''' p_x p_y / \bar{p}^2$. Unlike the traditional WKE [1], Eqs. (4) respect the integrals of the original mHME [3], so we call this model *improved* WKE, or iWKE. (The iWKE was also proposed, under a different name and from different considerations, in Ref. [10].) The first analysis of the iWKE phase space was done in Ref. [5]. Three types of drifton trajectories were identified, namely, passing, trapped, and runaway trajectories. The former two are similar to the trajectories of electrons in plasma waves, but the runaway trajectories are special. They are localized in y but extend to infinity in p_y; hence, runaways eventually dissipate, e.g., through viscosity. Depending on the ZF amplitude, runaway trajectories: (i) coexist with passing and trapped trajectories, (ii) coexist with just trapped trajectories, or (iii) are the only trajectories. As it turns out, transitions between these three regimes are the cause of many interesting effects in DW turbulence. Two of them are described below.

Saturation of collisionless ZFs. — ZFs can saturate monotonically or oscillate strongly. In collisionless plasmas, this is not captured by the usual predator–prey model [1] but can be explained as follows [9]. If the initial DW is weak, ZFs saturate in regime (i). In this case, driftons can hop between passing and trapped trajectories but, assuming their initial p_y are small, cannot go far in the p_y space. Then, the DW spectrum remains narrow, so the system has a low-dimensional configuration space and is conservative, and U has to oscillate. If DWs are stronger, such that the MI rate is larger than the DW characteristic frequency $\beta p_x/\bar{p}^2$, ZFs saturate in regime (ii), so many runaways are produced. The associated spectrum broadening and dissipation result in phase mixing of DW harmonics, so the saturation of U is largely monotonic.

Cross-scale interactions. — From gyrokinetic simulations, it is known that large-scale (ITG) turbulence can suppress small-scale (ETG) turbulence. However, a definitive explanation of this effect has been lacking. We have reproduced the effect within the quasilinear mHME and propose the following explanation, also tested numerically [to be published]. Large-scale DWs cause continuous merging of small-scale ZFs roughly until the ZF scale becomes comparable to the largest characteristic DW wavelength. This causes a long chain of bifurcations in the drifton phase space, each one throwing a fraction of small-scale driftons to runaway trajectories. Runaways typically dissipate before they have a chance to become trapped again; hence, almost all small-scale driftons are eventually eliminated. Including eddy–eddy interactions does not change this fact; it only adds another channel through which driftons can become runaways.

Conclusions. — In summary, the quantumlike phase-space approach is a useful tool for understanding basic physics of many key effects in DW turbulence and DW–ZF interactions.

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