# *Web-based Technical Appendix for*
# "Unequal Effects of Trade on Workers with Different Abilities"


Elhanan Helpman
Harvard University and CIFAR

Oleg Itskhoki
Princeton University

Stephen Redding
London School of Economics


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## A.1 Problem of the Firm

As discussed in detail in Helpman, Itskhoki and Redding (2008; henceforth HIR), the problem of the firm is given by:

$$\pi(\theta) \equiv \max_{\substack{n \geq 0, \\ a_c \geq a_{\min}, \\ I_x \in \{0,1\}}} \left\{ \frac{1}{1+\beta\gamma} r(n, a_c, I_x) - bn - \frac{c}{\delta} a_c^{\delta} - f_d - I_x f_x \right\}, \tag{A1}$$

subject to revenues from sales in the domestic and foreign market given by[1]

$$r(n, a_c, I_x) = \left[ 1 + I_x \tau^{-\frac{\beta}{1-\beta}} \left( \frac{A^*}{A} \right)^{\frac{1}{1-\beta}} \right]^{1-\beta} A y(n, a_c)^{\beta}$$

and output given by

$$y(n, a_c) = \left( \kappa_y \theta n^{\gamma} a_c^{1-\gamma k} \right),$$

as explained in the text of the paper. The firm's variables of optimization are the measure of workers to match with ($n$), the screening cutoff ($a_c$) and export status ($I_x$ with $I_x = 1$ if the firm exports and $I_x = 0$ otherwise). The firm will stay in the industry if and only if its operating profit is non-negative, $\pi(\theta) \geq 0$. Note that the operating profit in (A1) equals the revenues

[1] Total revenues equal $r(q_d, q_x) = A q_d^{\beta} + \tau^{-\beta} A^* q_x^{\beta}$, where $\tau$ is the iceberg trade cost, $q_d$ is sales at home and $q_x$ are shipments abroad so that total output $y = q_d + q_x$. Maximizing over $q_d$ and $q_x$ yields the expression in the text.

that accrue to the firm as a result of bargaining with the workers (a share $1/(1+\beta\gamma)$ of total revenues) minus all labor market costs (search and screening) and fixed costs of production and exporting.

The firm's first-order conditions for the choice of $n$ and $a_c$ are given by:

$$\frac{\beta\gamma}{1+\beta\gamma} r(\theta) = bn(\theta),$$
$$\frac{\beta(1-\gamma k)}{1+\beta\gamma} r(\theta) = ca_c(\theta)^{\delta},$$

where $r(\theta)$ denotes the optimal revenues of a firm with productivity $\theta$. These first-order conditions imply that more productive firms have higher revenues, sample more workers and have higher screening thresholds. Using the expression for revenues as a function of $(n, a_c, I_x)$ we can solve for closed-expressions for $n(\theta)$ and $a_c(\theta)$ given the export status $I_x$. Firm employment is obtained as $h(\theta) = n(\theta)\left[a_{\min}/a_c(\theta)\right]^k$ and it is also increasing in firm productivity as long as $\delta > k$.

Firm profits are given by

$$\pi(\theta) = \frac{\Gamma}{1+\beta\gamma} r(\theta) - f_d - I_x(\theta) f_x,$$

where $\Gamma \equiv 1 - \beta\gamma - \beta(1-\gamma k)/\delta > 0$. The zero-profit cutoff productivity is then defined by $\pi(\theta_d) = 0$ with $I_x(\theta_d) = 0$. The exporting cutoff productivity is defined by $\pi(\theta_x^-) = \pi(\theta_x^+)$ given that $I_x(\theta_x^-) = 0$ and $I_x(\theta_x^+) = 1$. The explicit conditions for $\theta_d$ and $\theta_x$ are provided in HIR. We focus on the range of parameter values for which $\theta_x > \theta_d$.

As a result of bargaining, the wage rate equals a fraction $\beta\gamma/(1+\beta\gamma)$ of average revenues per worker hired, as shown in HIR. Using the firm's first-order conditions this implies:

$$w(\theta) = \frac{\beta\gamma}{1+\beta\gamma} \frac{r(\theta)}{h(\theta)} = \frac{b}{h(\theta)/n(\theta)} = b\left(\frac{a_c(\theta)}{a_{\min}}\right)^k. \tag{A2}$$

Therefore, the wage rate is also increasing in the productivity of the firm.

These relationships together with the zero-profit cutoff condition $\pi(\theta_d) = 0$ allow us to obtain the closed-form expressions for the firm-specific variables provided in (1) in the paper.

Finally, we briefly discuss the determination of labor market tightness ($x$) and the search cost ($b$) given expected worker income ($\omega$). Detailed discussion is provided in HIR. From (A2), expected worker income upon being matched with a $\theta$-firm equals the search cost and is independent of $\theta$:

$$w(\theta)\frac{h(\theta)}{n(\theta)} = b.$$

Labor market tightness $x$ corresponds to the matching probability for a worker. Therefore, expected worker income from job search in the industry equals $bx$. In order for a worker to choose to search for a job in this industry, it has to equal expected income from his outside option, $\omega$:

$$bx = \omega.$$

Finally, using a Cobb-Douglas matching function and a cost of posting vacancies for firms, we can show that the search cost is an increasing power function of labor market tightness (and parameters of the matching technology):

$$b = \alpha_0 x_1^{\alpha}.$$

Combining these two expressions allows us to solve for both $x$ and $b$ as functions of $\omega$ only. Therefore, holding expected worker income $\omega$ constant implies constant values of labor market tightness $x$ and the search cost $b$.

## A.2 Proof of Result 2

Consider the ability-specific hiring rate in the open economy relative to autarky, $\sigma^T(a)/\sigma^A(a)$, where the expressions for $\sigma^T(a)$ and $\sigma^A(a)$ are given in the paper. For the range of abilities $a \in \left[a_d, a_x^-\right]$, we have a constant relative hiring rate:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \frac{1}{1 + \rho^{z - \frac{\beta}{\Gamma}}\left[\Upsilon_x^{\frac{1-\beta}{\Gamma}} - 1\right]} \equiv \chi_0 < 1.$$

For ability $a \in \left(a_x^-, a_x^+\right)$, this ratio is decreasing and strictly below $\chi_0$:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \chi_0 \frac{1 - \left(a_d/a_x^-\right)^{k/\mu}}{1 - \left(a_d/a\right)^{k/\mu}} < \chi_0 < 1,$$

where we used the facts that $a_d/a_x^- = \rho^{\beta/(\delta\Gamma)}$ and $\mu = \beta k/[\delta(z\Gamma - \beta)]$. Finally, for abilities $a \geq a_x^+$, the relative hiring rate increases monotonically from below $\chi_0$ to 1 as $a \to \infty$:

$$\frac{\sigma^T(a)}{\sigma^A(a)} = \frac{1 - \chi_1\left(a_d/a\right)^{k/\mu}}{1 - \left(a_d/a\right)^{k/\mu}} \leq 1,$$

where $\chi_1 = \chi_0 \Upsilon_x^{z(1-\beta)/\beta} > 1$ since $\rho < 1$, $\Upsilon_x > 1$, and we assume $z\Gamma > \beta$. One can verify directly using the definitions of $a_x^-$ and $a_x^+$ that $\sigma^T(a)/\sigma^A(a)$ is a continuous function of $a$ on $[a_d, \infty)$. Therefore, there exists $\hat{a}_x > a_x^+$ such that:

$$\frac{\sigma^T(a)}{\sigma^A(a)} < \chi_0 \qquad \forall a \in \left(a_x^-, \hat{a}_x\right) \qquad \text{and} \qquad \frac{\sigma^T(a)}{\sigma^A(a)} > \chi_0 \qquad \forall a > \hat{a}_x.$$

Specifically, $\hat{a}_x$ solves

$$\frac{1-\chi_1\big(a_d/\hat{a}_x\big)^{k/\mu}}{1-\big(a_d/\hat{a}_x\big)^{k/\mu}}=\chi_0 \qquad \Leftrightarrow \qquad \left(\frac{a_d}{\hat{a}_x}\right)^{k/\mu}=\frac{1-\chi_0}{\chi_1-\chi_0}.$$

Using the expressions for $\chi_0$, $\chi_1$ and the definition of $a_x^+$, one can verify directly that $\hat{a}_x > a_x^+$.

Recall that the ability-specific unemployment rate is given by $u(a) = 1 - x\sigma(a)$. Under the assumption that expected worker income $\omega$ is constant, labor market tightness $x$ is also constant, as discussed in Appendix A.1 above. Therefore, the normalized change in the unemployment rate equals one minus the relative hiring rate:

$$\frac{u^T(a)-u^A(a)}{1-u^A(a)}=1-\frac{\sigma^T(a)}{\sigma^A(a)}.$$

Result 2 then follows directly from the properties of $\sigma^T(a)/\sigma^A(a)$ discussed above and illustrated in Figure 1 in the paper. ■

## A.3 Conditional Wage Distributions

Consider a wage distribution conditional on ability $a \geq a_d$. As explained in the paper, workers with ability $a$ receive wages in the range $\big[w_d, w\big(\theta_c(a)\big)\big]$, where $w(\theta)$ is a firm-specific wage defined in (1) in the paper and $\theta_c(a)$ is the most productive firm that hires workers with ability $a$.

In the closed economy, for any $\tilde{w} \in \big[w_d, w\big(\theta_c(a)\big)\big]$ there exists $\tilde{\theta} \geq \theta_d$ such that $w(\tilde{\theta}) = \tilde{w}$. Denote the inverse of $w(\cdot)$ by $\theta_w(\cdot)$ so that $\theta_w(\tilde{w}) = \tilde{\theta}$. Then workers with ability $a$ are hired by all firms with $\theta \in [\theta_d, \theta_c(a)]$, while those hired by $\theta \in [\theta_d, \theta_w(w)]$ receive wages in $[w_d, w]$. A firm with productivity $\theta$ in these ranges hires workers with ability $a$ proportional to $n(\theta)$ since matching is random.[2] Therefore, we can express the conditional wage distribution as:

$$F_w^A(w|a)=\frac{\int_{\theta_d}^{\theta_w(w)} n(\theta)\mathrm{d}G_\theta(\theta)}{\int_{\theta_d}^{\theta_c(a)} n(\theta)\mathrm{d}G_\theta(\theta)} \qquad \text{for} \quad w\in\big[w_d, w\big(\theta_c(a)\big)\big]. \tag{A3}$$

Using the firm-specific solution for $n(\theta)$ and the Pareto productivity distribution, we can integrate to obtain expression (4) in the paper, where note from (1) in the paper that $w\big(\theta_c(a)\big) = w_d \cdot \big(a/a_d\big)^k$. We reproduce the closed economy wage distribution here:

$$\forall a \geq a_d \qquad F_w^A(w|a)=\frac{1-\big(w_d/w\big)^{1/\mu}}{1-\big(a_d/a\big)^{k/\mu}} \qquad \text{for} \quad w_d \leq w \leq w_d\big(a/a_d\big)^k.$$

[2] This means that the ratio of $a$-workers in firms $\theta'$ and $\theta'' \in [\theta_d, \theta_c(a)]$ equals $n(\theta')/n(\theta'')$.

Now consider the open economy. For workers with ability $a \in [a_d, a_x^-]$ employed only by non-exporting firms, the same logic as above applies. Therefore, we have

$$F_w^T(w|a) = F_w^A(w|a) \qquad \text{for} \quad a_d \le a \le a_x^- \quad \text{and} \quad w_d \le w \le w_d \left(a/a_d\right)^k.$$

Workers with ability $a \in (a_x^-, a_x^+)$ experience the same labor market outcomes (in terms of the wage distribution and unemployment rate) as workers with ability $a = a_x^-$. Therefore,

$$F_w^T(w|a) = F_w^T\left(w|a_x^-\right) \qquad \text{for} \quad a_x^- < a < a_x^+ \quad \text{and} \quad w_d \le w \le w_x^- = w_d\left(a_x^-/a_d\right)^k.$$

Finally, consider workers with abilities $a \ge a_x^+$ employed both by non-exporters and some exporters. These workers receive wages on $[w_d, w_x^-] \cup [w_x^+, w(\theta_c(a))]$, where $w_x^- = w_d \rho^{-\beta k/(\delta\Gamma)}$ and $w_x^+ = w_x^- \Upsilon_x^{(1-\beta)k/(\delta\Gamma)}$ are the wages paid by most productive non-exporter and least productive exporter respectively. We can still define $\theta_w(w)$ as the productivity of firm paying wage $w$ with the additional requirement that $\theta_w(w) = \theta_x$ for $w \in [w_x^-, w_x^+]$. Then the characterization in (A3) also applies for the open economy conditional wage distribution. Computing the integrals in (A3) using the solution for firm-specific allocations and the productivity distribution, we obtain:

$$\forall a \ge a_x^+ \qquad F_w^T(w|a) = \begin{cases} \dfrac{1-\left(w_d/w\right)^{1/\mu}}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_d \le w \le w_x^-, \\[2ex] \dfrac{1-\rho^{z-\beta/\Gamma}}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_x^- < w < w_x^+, \\[2ex] 1-\dfrac{\Upsilon_x^{z(1-\beta)/\beta}\left[\left(w_d/w\right)^{1/\mu}-\left(a_d/a\right)^{k/\mu}\right]}{1+\rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma}-1\right]-\Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu}}, & w_x^+ \le w \le w_d\left(a/a_d\right)^k. \end{cases}$$

## A.4 Proof of Result 3

Expression (4) in the paper for the autarky conditional wage distribution immediately implies that for any $a_2 > a_1 \ge a_d$, $F_w^A(w|a_2) < F_w^A(w|a_1)$ for all $w \in \left(w_d, w_d(a_2/a_d)^k\right)$. Therefore, the wage distribution for workers with higher ability first-order stochastically dominates the wage distribution for workers with lower abilities. A direct implication of this is that the conditional average wage $\bar{w}(a)$ increases with $a \ge a_d$. A closed-form expression for the conditional average wage can be computed as:

$$\bar{w}^A(a) = \frac{w_d}{1-\mu}\frac{1-\left(a_d/a\right)^{k(1-\mu)/\mu}}{1-\left(a_d/a\right)^{k/\mu}}, \tag{A4}$$

which can be verified to increase in $a \geq a_d$. In Helpman, Itskhoki and Redding (2008) we additionally compute the Coefficient of Variation and the Theil Index of wage inequality for $F_w^A(w|a)$ and show that they both increase in $a$. ■

Similar results obtain for the open economy conditional wage distribution, $F_w^T(w|a)$, derived in Appendix A.3. First of all, as long as $a_1 < a_2$ and both $a_1$ and $a_2$ do not belong to $\left[a_x^-, a_x^+\right]$, $F_w^T(w|a_2) < F_w^T(w|a_1)$, so that there is a similar first-order stochastic dominance property for workers of higher ability. This again implies that the conditional average wage increases (weakly) in $a$. The closed-form expression for the average wage in the open economy is:

$$\bar{w}^T(a) = \frac{w_d}{1-\mu} \begin{cases} \frac{1-\left(a_d/a\right)^{k(1-\mu)/\mu}}{1-\left(a_d/a\right)^{k/\mu}}, & a_d \leq a \leq a_x^-, \\ \frac{1-\rho^{z-\beta(1-k/\delta)/\Gamma}}{1-\rho^{z-\beta/\Gamma}}, & a_x^- < a < a_x^+, \\ \frac{1-\rho^{z-\beta(1-k/\delta)/\Gamma}+\Upsilon_x^{\frac{z(1-\beta)}{\beta}}(a_d/a_x)^{\frac{k(1-\mu)}{\mu}}\left(1-(a_x^+/a)^{k(1-\mu)/\mu}\right)}{1-\rho^{z-\beta/\Gamma}+\Upsilon_x^{\frac{z(1-\beta)}{\beta}}(a_d/a_x)^{\frac{k}{\mu}}\left(1-(a_x^+/a)^{k/\mu}\right)}, & a \geq a_x^+. \end{cases} \tag{A5}$$

We use (A4) and (A5) to construct Figure 2 in the paper.

One can also calculate in closed-form the coefficient of variation and the Theil index for the open economy conditional wage distribution.

## A.5 Proof of Result 4

Consider the autarky and open economy conditional wage distributions derived in Appendix A.3, $F_w^A(w|a)$ and $F_w^T(w|a)$. Part (i) follows immediately. Part (ii) follows from Result 3 and the fact that $F_w^T(w|a) = F_w^A\left(w|a_x^-\right)$ for $a \in \left[a_x^-, a_x^+\right]$.

To prove part (iii) we need to show that there exists $\tilde{a}_x$ such that $F_w^T(w|a) < F_w^A(w|a)$ for all $w > w_d$ when $a > \tilde{a}_x$. Consider the autarky and open economy wage distributions for a worker of ability $a$ over the interval $\left[w_d, w_x^-\right]$. From the expressions for $F_w^A(w|a)$ and $F_w^T(w|a)$ above, these distributions take the same value when:

$$1 + \rho^{z-\beta/\Gamma}\left[\Upsilon_x^{(1-\beta)/\Gamma} - 1\right] - \Upsilon_x^{z(1-\beta)/\beta}\left(a_d/a\right)^{k/\mu} = 1 - \left(a_d/a\right)^{k/\mu}.$$

Denote the $a$ which solves this equation by $\tilde{a}_x$. Using the definitions of $a_x^-$ and $a_x^+$, one can solve for:

$$\left(\frac{\tilde{a}_x}{a_x^+}\right)^{k/\mu} = \frac{\Upsilon_x^{\frac{z(1-\beta)}{\beta}} - 1}{\Upsilon_x^{\frac{z(1-\beta)}{\beta}} - \Upsilon_x^{\frac{1-\beta}{\beta}\frac{z\Gamma-\beta}{\Gamma}}} > 1,$$

since $z\Gamma > \beta$. On the range $[w_d, w_x^-]$, the autarky wage distribution lies below that in the open economy for $a < \tilde{a}_x$ and above that in the open economy for $a > \tilde{a}_x$. Therefore, for the open economy wage distribution to first-order stochastically dominate that in autarky, a necessary condition is that $a > \tilde{a}_x$. We now show that this is also sufficient. For $w \in [w_x^-, w_x^+]$, the open economy wage cdf is flat while the autarky wage cdf is increasing. Therefore, we only need to verify that for any $a > \tilde{a}_x$ and for all $w > w_x^+$ the open economy wage cdf still lies below that in autarky. This is straightforward to establish since both cdfs reach 1 at $w\big(\theta_c(a)\big) = w_d\big(a/a_d\big)^k$ and the slope of the open economy cdf is always steeper in $w$ for $w_x^+ < w < w_d\big(a/a_d\big)^k$.[3] This establishes the first-order stochastic dominance of the open economy wage distribution for $a \geq \tilde{a}_x > a_x^+$. Another corollary of this argument is that for $a \in (a_x^+, \tilde{a}_x)$, there is no first-order stochastic ordering of the two distributions. ■

An immediate corollary of Result 4 is that $\bar{w}^T(a) = \bar{w}^A(a)$ for $a_d \leq a \leq a_x^-$; $\bar{w}^T(a) < \bar{w}^A(a)$ for $a_x^- < a < a_x^+$ and $\bar{w}^T(a) > \bar{w}^A(a)$ for $a > \tilde{a}_x$. Moreover, there exists $a_x^+ < \tilde{\tilde{a}}_x < \tilde{a}_x$ such that $\bar{w}^T(a) - \bar{w}^A(a)$ switches sign from negative to positive at $a = \tilde{\tilde{a}}_x$. These features are illustrated in Figure 2.

## A.6 Construction of Figure 3

To construct Figure 3 we pick a quantile $q$ of the aggregate wage distribution and find the corresponding wage rate $w_q$. Formally, we have $[0, 1] \ni q = F_w^j(w_q^j)$, where $j \in \{A, T\}$ and $F_w(w)$ represents the unconditional industry wage distribution. Given $w_q^j$ we can recover the abilities which are consistent with this wage rate. Specifically, only workers with ability above $a_c\big(\theta_w^j(w_q^j)\big)$ can receive a wage $w_q^j$, where $\theta_w^j(w)$ denotes the productivity of a firm which pays wage rate $w$ (in autarky and open economy respectively), as defined in Appendix A.3. Therefore, the distribution of ability consistent with wage rate $w_q^j$ is a Pareto with shape parameter $k$ and cutoff $a_c\big(\theta_w^j(w_q^j)\big)$. Higher $q$ corresponds to a higher $w_q^j$ which also corresponds to a higher ability cutoff. With the obtained distribution of ability for each quantile, we can compute expected unemployment rates $\bar{u}^j(q) \equiv \mathbb{E}\left\{u^j(a) | a \geq a_c\big(\theta_w^j(w_q^j)\big)\right\}$, using the expressions for $u^j(a)$ and

[3] To see this, rewrite the autarky conditional wage distribution for comparability as

$$F_w^A(w|a) = 1 - \frac{\left[\big(w_d/w\big)^{1/\mu} - \big(a_d/a\big)^{k/\mu}\right]}{1 - \big(a_d/a\big)^{k/\mu}}.$$

Then for any $a$, $z\Gamma > \beta$ immediately implies a larger coefficient (in absolute value) in front of $\left[\big(w_d/w\big)^{1/\mu} - \big(a_d/a\big)^{k/\mu}\right]$ for the open economy wage distribution. In other words, the open economy wage distribution is always steeper for $w > w_x^+$ and has to lie below that in autarky.

$\sigma^j(a)$ provided in the text of the paper. We then plot $\bar{u}^j(q)$ against $q$ in Figure 3.

## References


[1] Helpman, Elhanan, Oleg Itskhoki and Stephen Redding (2008) "Wages, Unemployment and Inequality with Heterogeneous Firms and Workers," *NBER Working Paper No. 14122.*

[2] Helpman, Elhanan, Oleg Itskhoki and Stephen Redding (2009) "Inequality and Unemployment in a Global Economy," *CEPR Discussion Paper No. 7353.*