MAT 307: Combinatorics

Lecture 7: Extremal combinatorics

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1 Relationship between Ramsey theory and extremal theory

Consider the following theorem, which falls within the framework of Ramsey theory.

Theorem 1 (Van der Waerden, 1927). *For any $k \geq 2, \ell \geq 3$, there is n such that any k-coloring of $[n]$ contains a monochromatic arithmetic progression of length ℓ: $\{a, a+b, a+2b, \ldots, a+(\ell-1)b\}$.*

This is a classical theorem which predates even Ramsey's theorem about graphs. We are not going to present the proof here. Note, however, that in order to prove such a statement, it would be enough to show that for a sufficiently large $[n]$, any subset of at least n/k elements contains an arithmetic progression of length ℓ. This is indeed what Szemerédi proved, much later and using much more involved techniques.

Theorem 2 (Szemerédi). *For any $\delta > 0$ and $\ell \geq 3$, there is n_0 such that for any $n \geq n_0$ and any set $S \subseteq [n], |S| \geq \delta n$, S contains an arithmetic progression of length ℓ.*

It can be seen that this implies Van der Waerden's theorem, since we can set $\delta = 1/k$ and for any k-coloring of $[n]$, one color class contains at least δn elements. Szemerédi's theorem is an *extremal* type of statement - stating, that any object of sufficient size must contain a certain structure.

2 Bipartite graphs

Definition 1. *A graph G is called bipartite, if the vertices can be partitioned into V_1 and V_2, so that there are no edges inside V_1 and no edges inside V_2.*

Equivalently, G is bipartite if its vertices can be colored with 2 colors so that the endpoints of every edge get two different colors. (The 2 colors correspond to V_1 and V_2.) Thus, bipartite graphs are called equivalently *2-colorable*.

We also have the following characterization, which is useful to know.

Lemma 1. *G is bipartite, if and only if it does not contain any cycle of odd length.*

Proof. Suppose G has an odd cycle. Then obviously it cannot be bipartite, because no odd cycle is 2-colorable.

Conversely, suppose G has no odd cycle. Then we can color the vertices greedily by 2 colors, always choosing a different color for a neighbor of some vertex which has been colored already. Any additional edges are consistent with our coloring, otherwise they would close a cycle of odd length with the edges we considered already. □

The easiest extremal question is about the maximum possible number of edges in a bipartite graph on n vertices.

Lemma 2. *A bipartite graph on n vertices can have at most $\frac{1}{4}n^2$ edges.*

Proof. Suppose the bipartition is (V_1, V_2) and $|V_1| = k$, $|V_2| = n - k$. The number of edges between V_1 and V_2 can be at most $k(n-k)$, which is maximized for $k = n/2$. □

3 Graphs without a triangle

Let us consider Ramsey's theorem for graphs, which guarantees the existence of a monochromatic triangle for an arbitrary coloring of the edges. An analogous *extremal question* is, what is the largest number of edges in a graph that does not have any triangle? We remark that this is not the right way to prove Ramsey's theorem - even for triangles, it is not true that for any 2-coloring of a large complete graph, the larger color class must contain a triangle.
Exercise: what is a counterexample?

The question how many edges are necessary to force a graph to contain a triangle is very old and it was resolved by the following theorem.

Theorem 3 (Mantel, 1907)**.** *For any graph G with n vertices and more than $\frac{1}{4}n^2$ edges, G contains a triangle.*

Proof. Assume that G has n vertices, m edges and no triangle. Let d_x denote the degree of $x \in V$. Whenever $(x,y) \in E$, we know that x and y cannot share a neighbor (which would form a triangle), and therefore $d_x + d_y \leq n$. Summing up over all edges, we get

$$mn \geq \sum_{(x,y)\in E} (d_x + d_y) = \sum_{x \in V} d_x^2.$$

On the other hand, applying Cauchy-Schwartz to the vectors $(d_1, d_2, \ldots, d_n)$ and $(1, 1, \ldots, 1)$, we obtain

$$n \sum_{x\in V} d_x^2 \geq \left(\sum_{x \in V} d_x \right)^2 = (2m)^2.$$

Combining these two inequalities, we conclude that $m \leq \frac{1}{4}n^2$. □

We remark that the analysis above can be tight only if for every edge, any other vertex is connected to exactly one of the two endpoints. This defines a partition $V_1 \cup V_2$ such that we have all edges between V_1 and V_2, i.e. a complete bipartite graph. When $|V_1| = |V_2|$, this is the unique extremal graph without a triangle, containing $\frac{1}{4}n^2$ edges.

4 Graphs without a clique K_{t+1}

More generally, it is interesting to ask how many edges G can have if G does not contain any clique K_{t+1}. An example of a graph without K_{t+1} can be constructed by taking t disjoint sets of vertices, $V = V_1 \cup \ldots \cup V_t$, and inserting all edges between vertices in different sets. Now, obviously there is no K_{t+1}, since any set of $t+1$ vertices has two vertices in the same set V_i. The number of edges in such a graph is maximized, when the sets V_i are as evenly sized as possible, i.e. $|V_i| - |V_j| \in \{-1, 0, +1\}$ for all i, j. We call such a graph on n vertices the *Turán graph* $T_{n,t}$. Turán proved in 1941 that this is indeed the graph without K_{t+1} containing the maximum number of edges. Note that the number of edges in $T_{n,t}$ is $\frac{1}{2}(1 - \frac{1}{t})n^2$, assuming for simplicity that n is divisible by t.

Theorem 4 (Turán, 1941)**.** *Among all K_{t+1}-free graphs on n vertices, $T_{n,t}$ has the most edges.*

Proof. Let G be a graph without K_{t+1} and v_m a vertex of maximum degree d_m. Let S be the set of neighbors of v_m, $|S| = d_m$, and $T = V \setminus S$. Note that by assumption, S has no clique of size t.

We modify the graph into G' as follows: we keep the graph inside S, we include all possible edges between S and T, and we remove all edges inside T. For each vertex, the degree can only increase: for vertices in S, this is obvious, and for vertices in T, the new degrees are at least d_m, i.e. at least as large as any degree in G. Thus the total number of edges can only increase.

By induction, we can prove that $G[S]$ can be also modified into a union of $t-1$ disjoint independent sets with all edges between them. Therefore, the best possible graph has the structure of a Turán graph.

To prove that the Turán graph is the *unique extremal graph*, we note that if G had any edges inside T, then we strictly gain by modifying the graph into G'. □

We present another proof of Turán's theorem, which is probabilistic. Here, we only prove the quantitative part, that $\frac{1}{2}(1-\frac{1}{t})n^2$ is the maximum number of edges in a graph without K_{t+1}.

Proof. Let's consider a probability distribution on the vertices, $p_1, \ldots, p_n$ such that $\sum_{i=1}^n p_i = 1$. We start with $p_i = 1/n$ for all vertices. Suppose we sample two vertices v_1, v_2 independently according to this distribution - what is the probability that $\{v_1, v_2\} \in E$? We can write this probability as

$$\Pr[\{v_1, v_2\} \in E] = \sum_{i,j:\{i,j\} \in E} p_i p_j.$$

At the beginning, this is equal to $\frac{2}{n^2}|E|$.

Now we modify the distribution in order to make $\Pr[\{v_1, v_2\} \in E]$ as large as possible. We claim that the probability distribution that maximizes this probability is uniform on some maximal clique. We proceed as follows: If there are two non-adjacent vertices i, j such that $p_i, p_j > 0$, let $s_i = \sum_{k:\{i,k\} \in E} p_k$ and $s_j = \sum_{k:\{j,k\} \in E} p_k$. If $s_i \geq s_j$, we set the probability of vertex i to $p_i + p_j$ and the probability of vertex j to 0 (and conversely if $s_i < s_j$). It can be verified that this increases $\Pr[\{v_1, v_2\} \in E]$ by $p_j(s_i - s_j)$ or $p_i(s_j - s_i)$, respectively.

Eventually, we reach a situation where there are no two non-adjacent vertices of positive probability, i.e. the distribution is on a clique Q. Then,

$$\Pr[\{v_1, v_2\} \in E] = \Pr[v_1 \neq v_2] = 1 - \sum_{i \in Q} p_i^2.$$

By Cauchy-Schwartz, this is maximized when p_i is uniform on Q, i.e.

$$\Pr[\{v_1, v_2\} \in E] \leq 1 - \frac{1}{|Q|} \leq 1 - \frac{1}{t}$$

assuming that there is no clique larger than t. Recall that the probability we started with was $\frac{2}{n^2}|E|$ and we never decreased it in the process. Therefore,

$$|E| \leq \left(1 - \frac{1}{t}\right) \frac{n^2}{2}.$$

□