AXIOMS OF INFINITY
AS THE STARTING POINT FOR
RIGOROUS MATHEMATICS

ABSTRACT: A more than sufficient starting point for the development of all currently accepted rigorous mathematics is provided by a pair of assumptions about how many individuals there are: the relative assertion that there are as many of them as there are small classes of them, and the absolute assertion that there are indescribably many of them. But explaining what the foregoing assertion *means* is more work than establishing that it is *true*.

§1 WHAT IS REQUIRED OF A STARTING POINT FOR RIGOROUS MATHEMATICS?

Mathematical results are discovered by a mix of direct intuitive insight, generalization or extrapolation from special cases verified by computation, analogy with earlier results, and other heuristic processes. Historically, some though not all mathematical cultures have made a distinction between what is admissible (indeed, indispensable) in the context of discovery and what is admissible in the context of justification. They have made a distinction between conjecture and theorem, and required that a newly-discovered result be ranked only a conjecture and not a theorem until it has been obtained rigorously, without appeal to intuition or extrapolation or analogy, from previously admitted results. What is required for rigorous justification is that any new notions must be logically defined in terms of previously admitted notions, and any new results logically deduced from previously admitted results together with these new definitions. This requirement will be rather pointless unless the previously admitted notions and results were themselves obtained rigorously, and on pain of circularity or infinite regress, rigorous mathematics must ultimately have a starting point in certain first principles, which is to say, in primitives or undefined notions and postulates or unproved assumptions. The traditional assumption was that the primitives should be self-evidently meaningful and the postulates self-evidently true, and so the first principles would provide a genuine "foundation" for the mathematics obtained from them by definition and deduction.

Modern mathematics inherited in Euclid's *Elements* a model of this distinctive form of organization, an impressive if imperfect partial realization of the ideal of rigor for one branch of mathematics, plane geometry. But for a long time this model was more admired than imitated, and indeed it is hard to imagine the mathematics which the seventeenth and eighteenth centuries contributed to the rapidly-advancing physics of the period having been developed if Euclidean rigor had been insisted upon too strictly. However, it is perhaps equally difficult to imagine some of the mathematics the twentieth century contributed to physics ever having been developed if standards of rigor had never been raised beyond the level at which they stood circa 1800. What led to the insistence on higher standards during the nineteenth century is a complicated question, but one partial answer is easily enough identified. For it was during the nineteenth century that mathematical investigations increasingly came to adopt a strategy of indirection that practically demands rigor.

The indirect approach to dealing with a question about some traditional mathematical structure, such as the natural number system, is to (i) introduce certain auxiliary structures, such as the systems of algebraic integers in various extensions of the rational field of finite degree, then (ii) recognize these auxiliary structures as belonging to certain broad general types, such as commutative rings with unity, and then (iii) apply general results about these types to the auxiliary structures, and thereby obtain information about the original structure of interest. Because the auxiliary structures are generally unfamiliar, one cannot in investigating them confidently rely on intuitions derived from more familiar structures, and because the results invoked about broad types of structures are supposed to be perfectly general, one needs to insist that these results be based on nothing more than what is explicit in the definitions characterizing the type. Herein lies one source of the demand for rigor. But what is demanded is not quite the organization of mathematical science suggested by ancient ideals.

For one thing, since the indirect approach frequently involves bringing geometric ideas into the investigation of analysis, algebraic ideas into the investigation of geometry, and so on, one cannot be content with separate starting points for the different branches of mathematics: geometric primitives and postulates for geometry, arithmetic primitives and postulates for arithmetic, and so on, as the traditional models of rigor may have suggested. A single, unified "foundation" is called for. For another thing, the ancient ideal that the meaning of the primitives and the truth of the postulates should be immediately evident, has turned out to be too much to hope for. Though the word "foundations" is still used, the more modest "starting point" would really be more appropriate to the modern situation. In general, one must allow the chosen primitives and postulates from which one starts to demonstrate their worth not by their immediate evidence but by what they eventually allow to be derived from them: "By their fruits ye shall know them." At the outset one can demand no more than consistency, that the "fruits" include no contradictions, and even for that Gödel has taught us that we cannot seek an irrefragable proof, but at most some reasonable assurance.

Even with the demand for self-evidence dropped, it is not so easy to locate a set of first principles adequate to serve as a starting point for the development of the whole range of diversified yet interconnected branches that constitute modern mathematics. The work of Zermelo, amended by Frankel, though originally intended as a “foundation” or starting point only for one novel branch of mathematics, set theory, eventually turned out to suffice for every other branch as well, though it was not until nearly the middle of the last century that this was widely recognized, and even today the Zermelo-Frankel system of axiomatic set theory has at best a semi-official status. Something like it (though there are some slips in matters of detail) was indeed adopted by the Bourbaki group, whose encyclopedic *Éléments de mathématique* represents the modern ideal of rigor in much the way that Euclid’s *Elements* represented the ancient ideal; but even adoption by the Bourbaki group is not quite the same thing as official adoption by the international mathematical community as a whole.

There is, in fact, something in the Bourbaki codification that is untrue to the way mathematics — and I mean pure mathematics, not applied mathematics, in the context of justification, not of discovery — is practiced, and inevitably so. For in deriving mathematics rigorously from a chosen starting point, one must adopt definitions for all notions not listed among the first principles chosen; but while it is thus inevitable that some definition or other be chosen for each notion, for a given notion there is in general no one definition such that it is inevitable that *it* should be the chosen one. One might, for instance, define the real numbers system **R** in Dedekind’s way, as a system of pairs of sets of rational numbers, and the base of the natural logarithms e as $\lim_{n\to\infty} (1 + n^{-1})^n$. But one might equally well define the real number system **R** in Cantor's way, as a system of equivalence classes of sequences of rational numbers, and the base of the natural logarithms e as $1/0! + 1/1! + 1/2! + 1/3! + \ldots$, and the mathematical community as a whole has no preference between the alternatives. Whatever approach a group like the Bourbakists adopts, one can be sure that there is another that, from the standpoint of the mathematical community as a whole, would have been no more and no less legitimate.

For in contrast to the situation in some of the empirical sciences, where international bodies fix official definitions of key notions, there are in mathematics no official definitions of **R** or e. The mathematician who wishes to observe the requirements of rigor will take care to give logical definitions of any new notions involved, in terms of notions already to be found in the earlier literature, and logical deductions of any new results claimed, again in terms of results already to be found in the earlier literature, but in generally will be indifferent to what definitions were used in order to get from first principles to the notions and results in the immediately preceding literature being relied upon. Further, the working mathematician is in general indifferent to what the first principles themselves are supposed to be. Surely only a smallish minority would even be able to state the Zermelo-Frankel axioms — or any others, for that matter. One can at most set certain bounds to the *logical strength* of what is being assumed.

What the working mathematician does assume is that the first principles, whatever they may be, permit the kinds of constructions of auxiliary structures as products, quotients, and so forth, of already admitted structures that are commonly used. And since, as specialists though not ordinary mathematicians are aware, the Zermelo-Frankel axioms in fact do little more than assert the universal admissibility of such constructions, what the working mathematician needs is that Zermelo-Frankel set theory should at least be *interpretable* in the system first principles, whatever exactly those may be. That is to say, the primitives and postulates taken as a starting point need not be set-theoretic, but it must be possible, through appropriate definitions and deductions, to recover the first principles of Zermelo-Frankel set theory from whatever starting point has been chosen. (We do not need to worry about the famous axiom of choice, which together with the Zermelo-Frankel axioms gives the system called ZFC, since Gödel established that if the Zermelo-Frankel axioms *without* choice are interpretable, so are the Zermelo-Frankel axioms *with* choice; similarly for the much less famous axiom of foundation, which some do, and some don't count as part of the Zermelo-Frankel system.) Perhaps the interpretability of something a little less than the full list of Zermelo-Frankel axioms would do; perhaps Zermelo set theory is enough, though making do with it might require a little more in the way of bookkeeping, or keeping track of one's assumptions, than mathematicians are accustomed to undertaking.

Inversely, if one wants a reasonable assurance of consistency, the first principles had better be at least interpretable in Zermelo-Frankel set theory or something close to it. For instance, if someone advocated that the "foundation" for mathematics should be some theory about a "category of all categories," in the sense of a literal category or literally all categories, and hence of a category that was one of its own objects, there would be the strongest grounds for suspicion of inconsistency until some interpretation in something close to Zermelo-Frankel set theory is found. For no other system has been so very closely studied by logicians, and most of them seem to agree there is a definite intuitive picture behind the formal list of Zermelo-Frankel axioms, giving grounds for confidence in its consistency beyond mere inductive inference from the failure of anyone to stumble on a contradiction so far. Perhaps the same picture can justify a little more, but not too *much* more.

To be more specific, beyond the Zermelo-Frankel system, set theorists have studied a series of stronger and stronger so-called higher axioms of infinity, also known as large cardinal axioms. Among the first few, which is to say, the weakest, such assumptions are those known as the axioms of *inaccessible* cardinals, *Mahlo* cardinals, and *indescribable* cardinals. Suitable inaccessibility assumptions would suffice to accommodate so-called Grothendieck universes, an assumption of a branch of category with significant applications to core mathematics that on its face goes beyond Zermelo-Frankel set theory, though it is generally regarded as a disposable convenience. Suitable Mahlo assumptions would suffice to decide many of the concrete mathematical statements that Harvey Friedman has in recent years been proving to be undecidable in Zermelo-Frankel set theory, but that strike many as very plausible. Suitable indescribability assumptions are central to an elegant alternative axiomatization of set theory, stronger than the Zermelo-Frankel system, but still widely regarded among set theorists as highly intuitive, that is due to Bernays (1976). Beyond that point lie many stronger larger cardinal axioms, but these are generally regarded as attractive on account of some of their consequences (in particular, for so-called descriptive set theory, the theory of Borel and projective sets of real numbers), rather than from any sense that they are intrinsically necessary, given the underlying intuitive picture of sets. (For some of these there is a kind of evidence of consistency in the form of what are called "inner models," but this is not quite the same thing.)

If the criteria of acceptability are on the one hand the derivability of permission to introduce product, quotient, and other auxiliary structures, and on the other hand a reasonable assurance of consistency based on an intuitive picture, then the starting point for rigorous mathematics *might as well be* an axiomatic set theory, one somewhere in the range from Zermelo to Bernays set theory, or thereabouts. That is why logicians can habitually move from technical results of the form "such-and-such is not provable in Zermelo-Frankel set theory or minor perturbations thereof" to conclusions of the form "such-and-such is not provable by generally accepted mathematical means" — and why these conclusions are rightly widely accepted. (For instance, all purported proofs and disproofs of the continuum hypothesis, which is known to be undecidable on the basis of the Zermelo-Frankel axioms or anything like them, resemble purported trisections of the angle and squarings of the circle in bearing all the outward and visible signs of being the productions of cranks, and on examination are found to involve fallacies, in the rare case when their reasoning is clearly enough articulated to permit logical evaluation at all.)

But if the starting point might as well be a set theory somewhere in the indicated range, it might equally well be anything else mutually interpretable with such a set theory. And this observation brings me at last to my main topic: The starting point could be taken to be simply the assumption that *there exists sufficiently many objects or individuals*, with no further assumptions about the nature of such objects or individuals or about their relations to each other, and in particular, without any assumptions about some one, specific, distinguished relation such as the elementhood relation ∈ of set theory. This is an old philosophical idea, manifested in different forms in the work of the late George Boolos (1989) and the late David Lewis (1993), with elements traced by those authors back to still earlier sources. In what follows I will be doing little more than making fully explicit what is at least implicit in the work of Boolos and Lewis, in hopes of making such results better known. The chief novelty will be that I aim towards the top of the range, with Bernays set theory, rather than towards the bottom, with Zermelo set theory, or somewhere in the middle.

§2 How can axioms of infinity provide a starting point?

The background assumption here has been that for mathematical purposes it is important that each nonlogical notion (whether expressed in symbols or in words, as **R** and *e*, or as "the real number system" and "the base of the natural logarithms") should have some interpretation, but that it is in general a matter of indifference what the definition is. By contrast, logical notions (whether expressed in symbols or in words, as ~ and ∧ and ∨ and → and ↔ and ∀ and ∃, or as "not" and "and" and "or" and "if…then" and "iff" and "all" and "some") are assumed to have their fixed, ordinary interpretations. The background assumption of Boolos and Lewis (and many of their predecessors, including most importantly in the present context Bernays), is that the logic we are working with is not just *first order* but what is called *second order*, allowing quantification not just over individuals x, but also over classes X of individuals and over two-place relations R on individuals (and in principle over three- and over higher-place relations, too, though in practice no use will be made of these). But here in speaking of quantification over classes and relations I am in fact assuming what is only one of several proposed readings of the formalism of second-order logic. (Others would speak of "concepts" or "pluralities" or "fusions" rather than "classes.") This is not the place to go into the question of alternative readings, since it is on just this point that Boolos and Lewis differ, whereas my aim here is to bring out what they have in common. For that purpose I adopt what is, I believe, the most usual reading.

The *cognoscenti* may at this point be wondering just what principles I am assuming for second-order logic, since it is known from work of Gödel that for second-order in contrast to first-order logic there is no complete proof procedure. The answer is that I will be assuming just what is called the principle of *comprehension*, though I will also have what is called the principle of *extensionality* available as well. For the non-expert, this answer may be elaborated as follows. Writing Xx for "individual x is a member of class X," we can define identity for classes as follows:

$$X = Y \qquad \text{iff} \qquad \forall x(Xx \leftrightarrow Yx)$$

The principle of *extensionality* tells us that identical classes have the same properties. More formally, it tells us that anything of the following form holds:

$$X = Y \rightarrow (\Theta(X) \leftrightarrow \Theta(Y))$$

This principle need not be taken as an unproved assumption, since given our definition of identity, it can be proved (by induction on the complexity of the condition Θ involved). What does need to be taken as an unproved assumption is the principle of *comprehension*, to the effect that every condition $\Phi(x)$ on individuals determines a class of individuals, that for every such condition there is a class whose members are precisely the individuals for which it holds. More formally, it tells us that anything of the following form holds:

$$\exists X \forall x(Xx \leftrightarrow \Phi(x))$$

While comprehension asserts the existence of *at least* one class X whose members are all and only the individuals for which $\Phi(x)$ holds, our definition of identity implies that there is *at most* one such class X. Hence there exists a unique such class, justifying the use of the definite article in the expression "*the* class of all x such that $\Phi(x)$," for which we may introduce the notation «x: $\Phi(x)$». Examples are the empty class Λ = «x: $x \neq x$», the universal class V = «x: $x = x$», and for any individual a, its unit class «a» = «x: $x = a$» and for any other individual b, the pair class «a, b» = «x: $x = a \vee x = b$». Exactly parallel assumptions apply to two-place relations.

The idea I wish to pursue is that, for mathematical purposes, all we need further, once given logic in the form described, are suitable assumptions about how many individuals there are. Given such assumptions, one can interpret axiomatic set theory, and given axiomatic set theory and mathematicians' indifference to just what definitions are used in getting from first principles to the immediate background to their own work, one can get the whole of orthodox mathematics. It is crucial for present purposes, however, that the logic be second order and not just first order, because only on this assumption can questions about “how many” even be formulated, let alone answers posited.

As to that formulation, to begin with we can define the notions of subclass, proper subclass, intersection, and union in the usual way:

$$X \subseteq Y \quad \text{iff} \quad \forall x(Xx \to Yx)$$
$$X \subset Y \quad \text{iff} \quad X \subseteq Y \wedge X \neq Y$$
$$X \cap Y \quad = \quad «z\colon Xz \wedge Yz»$$
$$X \cup Y \quad = \quad «z\colon Xz \vee Yz»$$

A partial or total function from individuals to individuals can be identified with two-place relation R on individuals of a special kind, namely one such that for every x there exists at most one or exactly one y such that Rxy, with this unique y being the value of the function for argument x. A multifunction on individuals, or function from individuals to classes of individuals, can be identified with an arbitrary relation S on individuals, with «y: Sxy» being the value of the multifunction for argument x.

Once we can thus speak of functions, we can give Cantor’s definition of what it is for the members of one class to be just as many as, no more than, or fewer than, the members of some other class (or for the former class to be the same size as, no larger than, or smaller than, the latter): They are just as many if there is a bijective function from the one to the other, no more if there is an injective function, and fewer if there is an injection in the one direction but not the other. It can then be shown that “just as many as” or “the same size as” is reflexive, symmetric, and transitive, and that “no more than” or “no larger than” is reflexive, *anti*symmetric, and transitive.

(The only nontrivial verification is antisymmetry, the so-called Schröder-Bernstein theorem, which is ordinary set theory has several proofs, one of which, due to Zermelo, adapts to the present context of second-order logic.) We can define the members of a class to be *few* (and the class to be *small*) if they are fewer than the members of (it is smaller than) the universal class. We can also in this context give Dedekind's definition of what it is for a class to be infinite, namely, that it should be the same size as some proper subclass of itself.

With these notions in hand, we can turn to the problem of formulating assumptions about "how many" individuals there are that will suffice to permit the interpretation of set theory, and therefore will suffice as starting points for rigorous mathematics. The most important assumptions considered will imply that there are infinitely many individuals, or in other words, that the universal class is infinite, and thus will constitute "axioms of infinity" in a very general sense. The first serious attempt to provide a system of first principles that might have served as a "foundation" for all of rigorous mathematics was that of Frege. He in effect assumed the following:

> There are just as many individuals as classes.

As Russell and Zermelo each independently noted, this assumption is inconsistent, by an argument due to Cantor: Given a multifunction, identified with a relation *S*, that is an injection from individuals to classes, the class «*y*: ~*Syy*» cannot be equal to the value of this multifunction for any argument *x*, whence the multifunction in question is not a surjection and therefore not a bijection.

In place of Frege's inconsistent assumption we may consider the following restricted version, called the principle of *limitation of size*, which has a long history (roots in Cantor, adumbration by Russell, exploration by von Neumann) before it was taken up by Boolos and Lewis:

(1) There are just as many individuals as *small* classes.

Assuming (1), fixing a bijection from individuals to small classes identified with a relation E, and defining $y \in x$ to mean Exy, one at once gets the axioms of extensionality, separation, and replacement from the list of Zermelo-Frankel axioms to be found in textbooks. One also gets the axiom of union by a clever derivation due to Levy.

(What needs to be shown is that if X is a small class, and for each of its members x the class «y: Exy» is also small, then the class

$$«y\text{: } \exists x(Xx \wedge Exy)»$$

is also small. The idea is that if we had a counterexample, the indicated class would have as many members as the universal class, and we could trade our given counterexample for one in which the indicated class simply *was* the universal class. Then for any class Y, each of the classes

$$«y\text{: } Exy \wedge Yy»$$

for x a member of X, would be small, and so would be equal to «y: Ex^*y» for some x^* depending on x. The class

$$Y^* = «z: \exists x(Xx \wedge z = x^*)»$$

would then be small and could serve as a "code" for *Y*. It would follow that there are no more classes then there are small classes to serve as "codes" for them, and hence by (1) no more classes than there are individuals. But this is essentially Frege's inconsistent assumption!)

(1) is, however, only a *relative* statement. It says that there are many individuals compared to how many small classes there are, but does not imply that there are very many of either. (1) is, in fact, compatible with the assumption that there is only *one* individual, and therefore only *one* small class, the empty class Λ. To get the remaining Zermelo-Frankel axioms, pairing and infinity and power set, one would needs three additional assertions, the first two of which, at least, are *absolute* statements:

(2a) There are at least two individuals.
(2b) There is a small class that is infinite.
(2c) There are more individuals than subclasses of any one small class.

The first of this trio will eventually be seen to imply, when taken together with (1), the existence of infinitely many individuals; but it is obviously subsumed by the second of the trio.

Bernays set theory derives the pairing, infinity, and power set axioms, and more also (namely, inaccessible and Mahlo cardinals), from a single principle, formulated in a language where we have a distinguished relation ∈. What I would like to point out here is that an analogous single principle, directly and naturally formulated in pure second-order logic, *without* a distinguished relation ∈, suffices for the deduction of (2abc), and therefore in conjunction with (1) for the interpretation of Zermelo-Frankel set theory. In other words, all that is needed as a starting point for rigorous mathematics is (1) together with a single additional assumption. Whether the still stronger large cardinal assumptions that many set-theorists find attractive can similarly be directly and naturally formulated in pure second-order logic, remains a topic for future investigation.

Leaving that issue aside, the additional principle required in the present context may be cryptically formulated as follows:

(3) There are indescribably many individuals.

It remains to explain, adapting Bernays, how (3) is to be understood in formal terms, and to prove, again adapting Bernays, that so understood (3) implies (2abc). The basic notion we need is that of the *relativization* Θ^X of a given statement Θ to a given class X. This is the result of replacing, throughout the given statement, first-order quantifications "for all individuals" and "for some individual" by "for all members of X" and "for some member of X," and at the same time replacing second-order quantifications "for all classes" and "for some class" by "for all subclasses of X" and "for some subclass of X," and analogously for relations. Whatever Θ says about the macrocosm of all individuals, and all classes thereof, Θ^X says about the microcosm of all members of X, and all subclasses thereof.

We can now begin to explain the terminology in (3) by indicating that to say that it is *indescribable* how many individuals there are is to say that, whatever statement Θ one may make on that issue, it is if true an *understatement* that fails to describe fully how many individuals there are, in the sense that would be equally true if one were talking not about the genuine macrocosm, but about some lesser microcosm. The formal statement of (3) is given by what is called a *reflection principle*, which in its basic or absolute version simply says that anything of the following form may be taken as axiomatic:

(3a) $\Theta \rightarrow \exists(X \text{ is small} \wedge \Theta^X)$

There is also an extended or relativized version, covering also statements about specific small classes:

(3b) $\forall U[U \text{ small} \wedge \Theta(U) \rightarrow \exists(X \text{ is small} \wedge U \subseteq X \wedge \Theta^X(U))]$

What remains to be done here is to show how (1) and (3ab) yield (2abc).

Derivation of (2a): Taking for Θ a trivial logical truth such as $\forall x(x = x)$, (3a) at least tells us that there exists a small class X such $\forall x(x = x)^X$ holds, and hence that there exists a small class X. Even if this X is just the empty class Λ, than which there can be none smaller, the fact that Λ is small implies that there exists at least *one* individual, so we have $\exists x(x = x)$. (This is in any case usually assumed by convention as a part of the background logic.) Now, using a trick applied by Bernays over and over in his original work, as soon as we have this new result $\exists x(x = x)$ we can take *it* for Θ, and (3a) tells us that there exists a small class X such that $\exists x(x = x)^X$ holds, or in other words, a small class having at least one member. Even if this X has no more than one member, and is just a unit class «a» for some individual a, the fact that «a» is small tells us that there are at least *two* individuals, which was to be proved.

Derivation of (2b): Similar "bootstraps" reasoning establishes that there are at least three individuals, at least four, and so on, so that, intuitively speaking, there are infinitely many. However, officially the statement that there are infinitely many individuals means that there is a bijection between the class of all individuals and some proper subclass, and this remains to be proved. The proof is not difficult once we have (2a), and therewith the conclusion that any unit class is small. For assuming (1) and fixing a bijection identified with a two-place relation E as before, it is then clear that there is a bijection between the class of all individuals and the proper subclass thereof having as members all and only those x for which «y: Exy» is a unit class. (This cannot be *all* x, since there must be an x for which «y: Exy» is the empty class.) Now that we have established the statement that there are infinitely many individuals, taking Θ as this statement, (3a) tells us that there is a small class X such that Θ^X holds, which means a small class with infinitely many members, as was to be proved.

Derivation of (2c): Given any small class U, we must show that it has no more subclasses than there are members in some small class. To this end, let $\Gamma(U)$ be the statement that U is small, let Δ be the statement (1), and let $\Theta(U)$ be the conjunction $\Gamma(U) \wedge \Delta$. Since the relativization of a conjunction is just the conjunction of the relativizations of its conjuncts, (3b) tells us that there exists a small class X such that (i) U is a subclass of X, whence $U = U \cap X$, that (ii) $\Gamma(U)^X$ holds, and that (iii) Δ^X holds. A little thought reveals that (ii) means that $U \cap X$ is smaller than X, and so together with (i) implies that U is smaller than X, whence all subclasses of U are smaller than X also, while (iii) means that there are as many members of X as there are subclasses of X smaller than X, whence there are no more subclasses of U than there are members of X, which was to be proved.

Thus (1) and (3) together yield (2). This completes the explanation and establishment of the assertion with which we began: "A more than sufficient starting point for the development of all currently accepted rigorous mathematics is provided by a pair of assumptions about how many individuals there are: the relative assertion that there are as many of them as there are small classes of them, and the absolute assertion that there are indescribably many of them." Those with a taste for grandiose slogans may reword this as: "Mathematics is the science of infinity."

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John P. Burgess
Department of Philosophy
Princeton University
Princeton, NJ 08544-1006, USA
jburgess@princeton.edu