Lewis on Set Theory

David Lewis in the short monograph *Parts of Classes* (Lewis 1991, henceforth PC) undertakes a fundamental re-examination of the relationship between merelogy, the general theory of parts, and set theory, the general theory of collections. He assumes a certain minimum background familiarity with both subjects, and limitations of space make the same assumption inevitable in the present account. Varzi (2011) is recommended as a clear, concise survey of mereology in general. Karen Bennett's chapter in the present volume discusses Lewis's version of mereology in particular. Among the asumptions surveyed by Varzi, Lewis's are about the strongest, including *unrestricted composition*, the claim that any things whatsoever have a *fusion*, but leaving open whether everything is a fusion of *atoms*, or things with no parts other than themselves. For set theory in the form in which it will be considered here, Boolos (1989) (with Boolos (1971) in the background), to which reference will in any case be essential, should suffice.

Lewis became aware of the possibility of certain technical improvements too late in the production process for his book to incorporate them except as an appendix to PC (Burgess & al 1991, henceforth PC*), co-authored with A. P. Hazen and the present writer and including also a contribution by W. V. Quine. Lewis became aware of the possibility of certain further technical improvements too late to incorporate them in the book at all, whence the follow-up paper, "Mathematics is Megethology" (Lewis 1993, henceforth MM), which Lewis described in its introductory session as "an abridgement of parts of [PC] not as it is, but as it would have been had I known sooner what I know now."

A half-dozen themes pursued in the book and article: (1) formulation of theses on how the mereological notion of part applies to classes; (2) restatement of traditional mereologists' complaints about set theory, concentrating on the notion of singleton sets, and motivating a "structuralist" approach to be further explained below; (3) defense beyond what is already found in (Lewis 1986) of the background assumptions of mereology, including controversial theses on "ontological innocence" and "composition as identity"; (4) elaboration of a framework combining mereology with the plural quantification of Boolos (1984 and 1985); (5) consideration of how to simulate within such a framework, using assumptions related to the axiom of choice, quantification over relations; (6) development of the structuralist treatment of set theory using such simulated quantification, showing that beyond the framework the only assumptions needed are about how many atoms there are, whence the identification of mathematics with "megethology," the theory of size.

I will take up the six themes in the order listed. The most important sources are as follows: PC chapter 1 for (1); PC chapter 2 for (2); PC §3.6 for (3); PC chapter 3 for (4); PC* plus Hazen (1997) for (5); and MM plus Boolos (1989) for (6).

§1. Parts of Classes

The notion of "part" is applied in ordinary language to entities of many different sorts. As the introductory portion of Varzi (2011) makes plain, however, the word is not, in ordinary language, always applied in the same sense. To claim logic-like universality for mereology — such a claim is not fully explicit in Lewis, though his thought tends in that direction — is to claim there is a core sense of "part" that can be applied univocally to entities of any sort whatsoever, including classes, if such there be. It is the applicability of the notion of and the mereological theory of "parthood" to sets and classes that is Lewis's most fundamental claim.

The form of the theory of sets and classes of concern to Lewis is a versions of what is variously called "second-order Zermelo-Frankel set theory with choice" (henceforth second-order ZFC) or "Morse-Kelly set theory" (henceforth MK) to be met with in the literature. Both versions can be developed to as to admit individuals (German: Urelemente), items that are neither sets nor classes but can belong to such collections. In the version followed by Lewis, that of the *lumpers* as opposed to the *splitters*, sets are classes of a kind, the "small" ones and equivalently the ones that can be members of other classes; those of the other kind, the "large" ones and equivalently the ones that cannot be members of other classes, are called *proper classes*.

Lewis has one main thesis concerning mereology and classes, with one outstanding corollary. His main thesis follows immediately from two subordinate theses. The second of these is deduced from three yet further subordinate theses.

Main Thesis: *The parts of a class are precisely its subclasses.*

First Thesis: *One class is part of another if and only if it is a subclass.*

Second Thesis: *Any part of a class is a class.*

Division Thesis: *There are only individuals, classes, and fusions thereof.*

Priority Thesis: *No class is a part of any individual.*

Fusion Thesis: *Any fusion of individuals is an individual.*

Corollary: *Singletons are atoms.*

Besides a deduction of the Second Thesis from Division and Priority and Fusion, Lewis offers various motivating considerations and heuristic arguments in favor of these theses, but the justification of some ultimately remains largely pragmatic: The assumptions together lead to a powerful and attractive theory.

It is a consequence of these theses and mereology that the principle of the universality of set theory, according to which any condition determines a class whose members are all only those things for which the condition holds, bar proper classes, must fail, at least assuming for nontriviality that there is at least one individual and therefore at least one class. For that principle implies a strengthened version of division according to which there are only individuals and classes, whereas the fusion of an individual with a class cannot be a class by the Second Thesis, since it has an individual as a part, and cannot be an individual by the Priority Thesis, since it has a class as a part. Thus mereology and set theory cannot both be universal, and Lewis opts for mereology.

In practice, Lewis generally ignores fusions of individuals and classes, in effect tacitly assuming one is quantifying only over individuals and classes. For the most part, he also in effect assumes one is quantifying only over atoms (including singletons) and fusions thereof (including classes), to the exclusion of what he calls atomless gunk, if such there be. He sketches an adaptation of his results the case where there is gunk. (If *all* there is is gunk, the assumption that there are lots of nonoverlapping globs of it will do in place of the assumption that there are lots of atoms in the overall construction.)

§2. Objections to Singletons

Mereology was founded in Poland by the nominalist Stanisław Leśniewski and popularized in the West by the nominalist Nelson Goodman precisely as a partial *substitute* for the set theory that they as nominalists rejected. Though Lewis is, by contrast, anything but a nominalist, still he does very much sympathize with traditional nominalistically-inclined mereologists' complaints about set theory.

The background is as follows. Cantor's work on trigonometric series led him to move from thinking of the points (plural) where a function misbehaves to thinking the set (singular) of points of misbehavior as a single object, to which operations can be applied, notably the operation of throwing away isolated points. Repeated application of this operation may have the result that only one point remains or none at all, and so it is a simple and natural step to admit singleton or unit sets and a null or empty set as "ideal elements" or limiting or degenerate cases, though in fact singletons and the empty set only really came to play an important role in set theory with Zermelo's axiomatization of the subject.

There is a certain kind of philosopher addicted to quibbling and querulous objections to mathematicians' habitual practice of rounding out systems of entites by positing "ideal objects," and more generally of counting in limiting or degenerate cases. Lewis gives a good impersonation of such a philosopher in some of his remarks on set theory, when he observes that Cantor's definition of set, which in free translation runs "any collection into a whole of definite, well-distinguished sensible or intelligible objects," is difficult to reconcile with reckoning in singletons, and impossible to reconcile with reckoning in an empty set.

Lewis considers a counterargument along the following lines. Suppose we have a stamp collection, complete with a catalogue, and then, finding ourselves in reduced circumstances, have to begin selling off our stamps, deleting their listings from the catalogue. This may go on until we have only one stamp left or none at all, leaving us with a singleton collection or an empty one. We need not even have a physical collection, but just a catalogue, if we switch the example from stamp collections to Facebook selections of "favorites" in one or another category, if as fans we are fickle. Lewis rejects the argument on the grounds that all this talk of "collecting" is merely metaphorical. Well, of course it is: Cantor's notion of *set* was new — he did not suppose that what appears on the surface to be plural talk of points was deep down secretly singular talk of point-sets all along, and that he was merely making explicit what was already implicit —and a new notion can only be introduced by heuristic metaphors.

One would hardly expect such quibbling over minor ontological assumptions from, of all persons, David Lewis, he of the incredulous-stare-inducing ontology of real, concrete possible worlds. And the querulous objection is in any case pointless, since set theory with the limiting or degenerate cases can easily be interpreted in set theory without them. (As Lewis surely knew, at least by the time of MM, from Hazen 1991.) Indeed, assuming the existence of at least two individuals *a* and *b*, the "pure" sets of ordinary set theory can be mapped one-to-one onto those sets *x* such that *x* itself, all sets that are elements of *x*, all sets that are elements of elements of *x*, and so on, have among their elements *a*, *b*, and *no other* individuals.

The real objection lies elsewhere. Nominalists have traditionally objected less to the part of Cantor's definition quoted so far, than to the additional clause "which are called the *elements* of the set," with its implication that set-formation is less a process of merger, like that by which Italy was formed from various minor states, than a process of federation, by which thirteen colonies became the United States. The implication, to be more explicit, is that even after the many have been collected together into a one, it is still discernible *which* many they were: that just as the set is determined by its elements, so also the elements are determined by the set. Mereological fusion, by contrast, obliterates the separate identities of the fused: A single whole can be taken apart in many ways, and there is no one way of taking it apart of which it can be said that the genuine parts of which the set is composed are just those pieces into which it is disassembled when taken apart in that way and no other.

Given Lewis's theses, to be an element of a set or member of class is just to have a singleton that is a part thereof. Grant the notion of singleton, and you have granted the notion of element or member, and the traditionally objectionable part of Cantor's definition. *That* is the real source of Lewis's objection to singletons, or rather, that together with the observation that many of the categories of metaphysics in the Australian style do not apply in any obvious way to singletons. (Is the relation of a singleton to its single member and internal or an external relation?) But though Lewis grouses and kvetches about such matters almost as much as a Leśniewskian or a Goodmanian might, to the point that one is expecting his discussion to issue in a proposal that if not literally nominalist would at least be in spirit nominalistical, on the contrary he ends by affirming that we must accept set theory *like it or not*.

For in the most memorable passage in all this material (PC 59, MM 15), Lewis writes that he laughs to think how *presumptuous* it would be to reject mathematics for philosophical reasons, and goes on the review the "great discoveries" of philosophy in the past, beginning with the proof of the impossibility of motion. This is perhaps a bit unfair, in that natural science after all emerged from natural philosophy; but the point stands that it is comically immodest for the part of philosophy that is still struggling, and therefore still called "philosophy," to seek to "correct" the part of philosophy that has succeeded, and is now called "science," and especially for anything as soft as philosophy to seek to "correct" mathematics, the hardest of the hard sciences. At any rate, with a forceful profession of faith in mathematics — he heads the relevant section "Credo" and might almost have followed the Tertullianists in adding "quia absurdum" — Lewis renounces renunciation of mathematics.

He remains tempted not by renunciation but by reinterpretation of certain kind. The reinterpretation in question is generally known in the contemporary literature as "structuralism," though it goes back (strictly speaking only in the case of arithmetic, though that case is easily adaptable others) to Benacerraf (1965), rather than to any Parisian theoretician fashionable during in the sixties, apart perhaps from a very tenuous link to

Bourbaki. The "structuralist" idea, which for Lewis is inspired by his reading of Ramsey, would be this, that instead of accepting a specific singleton-forming function, of philosophically inscrutible nature, simply to posit that there exists at least one function having the properties orthordox set theory ascribes to the singleton function. Lewis fears that even this degree of departure from strict and literal acceptance of set theory might constitute an unacceptable philosophical revisionism, but he perhaps need not have worried so much, for there are historical precedents.

In the seventeenth and eighteenth centuries, leading mathematicians (among them Descartes and Newton) had a more or less definite idea what (positive) real numbers were: ratios of magnitudes, such as lengths. In the nineteenth century, however, mathematicians came to feel that this geometric conception of the continuum needed to be replaced by something more purely arithmetic, and the constructions of Dedekind (his "cuts") and Cantor (equivalence classes of Cauchy sequences) eventually emerged. By the early twentieth century such constructions were beginning to appear in undergraduate textbooks. G. H. Hardy, in his Cambridge freshman calculus textbook *Pure Mathematics* (in the second edition of 1914 and all subsequent ones), expounds Dedekind's construction, and then remarks that alternatives are possible, and that no great importance should be attached to the particular form of definition he as just finished presenting. He formulates — and quotes Russell as endorsing — the general principle that in mathematics it matters that our symbols should be susceptible to *some* interpretation, but that if several are possible, it does not matter which we choose. Hardy's principle would seem to be just as applicable to set theory as to the calculus, and if so one has it on very high mathematical authority that there is nothing objectionable in the course that so tempts Lewis. Let us, in any case, see what that course involves.

§3. Protestations of Innocence

Lewis's mereological assumptions amount to what mathematicians call the theory of a *complete Boolean algebra*, or ignoring gunk, a *complete* atomic *Boolean algebra*, consisting of a number of atoms — just how many will turn out to be the great issue — and arbitrary fusions thereof. For what mathematicians call completeness amounts to what mereologists call unrestricted composition, permitting arbitrary fusions. There is just one departure from the usual mathematical approach, the dropping of the assumption of a null item in the algebra, which is one instance where Lewis *does* object to introducing an "ideal element" to round out a system. The minor complications this course involves him in will be ignored in the exposition below.

By completeness or unrestricted composition, absolutely abitrary unions or joins are possible, as are absolutely arbitrary intersections or meets except those that would turn out to be null, and a complement exists for anything except the universal item or fusion of all things, whose complement would be null. Lewis, by the way, calls that universal fusion "Reality," and idiosyncratically calls two things whose intersection would be null — in other words, two things that are nonoverlapping or disjoint — "distinct," a word more normally used to mean nonidentical.

Despite upholding unrestricted composition, yielding fusions of scattered, heterogeneous parts — fusions critics have considered monstrous and mythical, or in a word, chimeras — Lewis wishes to claim a kind of ontological innocence, comparable to that of first-order logic, for mereology. The claim of ontological innocence is largely based on a variant of Donald Baxter's thesis of "composition as identity," the claim that when one thing is the fusion of many things, "They are it and it is them."

Lewis takes this identity thesis somewhat less than literally, claiming that the relation of things to the fusion thereof is, though not strictly speaking identity, at any rate "analogous" to identity. It is difficult, however, to see how anything less than literal identity could suffice for ontological innocence. It is difficult, also, to see how Lewis can be acquitted of question-begging when he argues that one respect in which there is analogy is in ontological innocence.

For Lewis, the plural includes the singular: He does not object to the kind of counting in of limiting or degenerate cases involved in reading "there are some things" as "there are one or more things" (though he would object to reading it as "there are zero or more things"). The relation he finds analogous to identity amounts to the relation some things, the *x*s, bear to other things, the *ys*, just in case (i) the fusion of the *x*s is identical with the fusion of the *y*s, and (ii) either there is just one single *x* or there is just one single *y* or both. This includes identity of the usual kind, between one single *x* and one single *y*, as a special case.

Apart from symmetry, however, the relation in question lacks the usual formal properties of identity. It is not reflexive, since some two or more things, the *x*s, never stand in this relation to themselves, or to any other two or more things, the *y*s. And though it is transitive in the sense that when a single *x* bears this relation to some *ys* and those *y*s bear the same relation to a single thing *z*, then *x* is identical with *z*, it is intransitive in the sense that even when some two or more things, the *x*s, bear this relation to as single thing *y* and this *y* bears the same relation to some two or more things, the *z*s, the *x*s still do not bear this relation to the *z*s.

Nor need the *x*s then be identical with the *z*s as plural identity is usual understood. For the usual understanding requires that each single thing among the *x*s be also among the *z*s, whereas the eight ranks of a chessboard bear the Lewis relation to that chessboard, and the chessboard bears the Lewis relation to its eight files, while the ranks are not the files. Above all, as Lewis acknowledges, the indiscernibility of identicals fails utterly for plural things and their single fusions, since they are many while it is one. In our example, the ranks are horizontal while the files are vertical; and though there are eight ranks and eight files, there are sixty-four squares, whose fusion is again the same old chessboard.

It may be that in some sense the fusion is nothing over and above the things it is the fusion of, as Lewis asserts; but the things seem to be something over and above their fusion, consisting of that fusion plus a particular mode of division. Needless to say, the "plus" here is not *mereological* summation or fusion. In subsequent discussion on "composition as identity" — see Sutton (2008) for an overview — critics have outnumbered defenders. Yet even on this, his weakest point, Lewis's discussion, in his inimitable style, remains well worth reading.

§4. Background on Mereoplethynticology

In the mathematical literature, the completeness assumption for a

Boolean algebra is generally stated in second-order terms: For any class of things, there is a fusion thereof. In the context of Lewis's project this will not do. Instead he draws on the resources of Boolos's plural quantification, which the present writer has elsewhere called *plethynticology*, and says simply that for any things (plural) there is a fusion (singular) thereof.

Technically, the only substantive assumption of plethynticology is that for any condition (that holds of at least one thing) there are some things such that they are precisely the things for which the conditions hold. (And which things are these? Those for which the condition holds, obviously.) Philosophically, Lewis joins Boolos in emphatically rejecting the suggestion that plural quantification over things secretly involves singular quantification over pluralities of things, construed as single things of some extraordinary set- or class-like kind, and inveighs at some length over such "singularism." For a fuller polemic even than Lewis's against that misguided view, see McKay (2006).

The effect is that Lewis has the resources to interpret *monadic third-order logic*. In the terminology of Hazen (1997) for reading the formalism of that logic, a first-order variable x is said to range over *individuals*, a second-order variable X is said to range over *species*, and a third-order variable X is said to range over *genera*. For Lewis, the "individuals" are atoms, the "species" are fusions of atoms, and singular quantification over "genera" would be replaced by plural quantification over such fusions of atoms. Likewise *couples*, or species with just two individuals belonging to them, for Lewis are *diatoms*, or fusions of just two atoms. Having noted that this is Lewis's official interpretation, for the purely formal developments to follow it will be convenient to fall in with Hazen's terminology, leaving the reader to work out the translation.

Following Lewis (though speaking like Hazen), one would like at this point to formulate the assumption that there exists function for which certain conditions hold, characteristic of the singleton function, in terms of which we can define a binary relation on individuals for which certain other conditions hold, characteristic of the membership relation. But once we have gone structuralist, we may as well just assume the existence of such a binary relation. What we want is a binary relation such that the following hold:

(i) For any y only a few x stand in the given relation to y.

(ii) For any small species X there is a unique individual y such that the individuals x standing in the given relation to y are precisely those belonging to X.

We can then call those y for which there is at least one x standing in the given relation to *y* our nonempty sets, and the call the relation in question set-membership or elementhood — that so doing may have the consequence that Julius Caesar turns out to be a set would have troubled Frege, but does not trouble Lewis, who does not count intuitions to the effect that sets are nonphysical and nonmental and nonspatial and nontemporal as official parts of set theory — and try to see what further assumptions are needed to recover the usual axioms of first-order ZFC, and thence of second-order ZFC, letting fusions of many sets play the role of proper classes. Note that we are going to need quantification over relations anyhow, in order to define what it is for a species to be small (there is no one-to-one correlation between its individuals and *all* individuals), or for the individuals for which some condition holds to be few (the species of such individuals is small).

But while in monadic third-order logic we have variables x and X and X for individuals and species and genera, we have no variables for Ξ for binary relations on individuals. This brings us to the technical point on which there was to be progress between the first draft of PC and the published version including PC*: the simulation of quantification over binary relations Ξ on individuals. Two methods (a) and (b) of simulation were suggested in PC*, a hybrid (ab) between them was suggested in MM, while a new modification (b') of (b) will be suggested below.

Note that in the framework as described so far, we do have available unordered pairs and therewith *symmetric relations*, genera with only couples belonging to them. Moreover there is a general result in mathematical logic (see Boolos et al. 2007, §21.3) for coding any finite number of relations of any finite number of places by a single binary relation. Unfortunately, the method requires that one have available, in addition to the domain of individuals in which one is primarily interested, a further domain of auxiliary individuals, which is not an assumption likely to tempt Lewis.

All the alternatives, making do with a single domain of individuals, have in common that they involve the axiom of choice (henceforth AC). The use of AC in one way or another is inevitable, according to the results of Hazen (1997) — though in the same paper Hazen also indicated that special features of set theory would have made it possible for Lewis to formulate a structuralist version of *it* without general quantification over relations, and without AC.

§5. Axioms of Choice

Method (a), developed in PC* from a suggestion of the present writer, involves also the assumption that there are infinitely many individuals. Of course, that will follow from orthodox set theory once we are in a position to formulate it; but it may be thought undesirable to presuppose it in the very formulation of our set-theory-generating assumptions. How AC comes in with method (a) is perhaps the only point worth elaborating before moving on to alternatives. The infamous Banach-Tarski paradox (according to which a ball can be cut up into finitely many pieces, and the pieces after translation and rotation fit back together to form two balls each the size of the original) depends ultimately on a simple consequence of AC: that if a set is infinite, it can be decomposed into three nonoverlapping pieces, any one of which is equinumerous with the union of the other two. That is the assumption that is used in method (a).

Method (b) begins with what is (as was subsequently recognized) an old trick of Zermelo that makes it possible within our framework to simulate quantification over linear orderings of individuals. Such an ordering can be represented by the genus to which belong all and only those species that are initial segments in the ordering. Such a linear ordering makes it possible, in a systematic way, to distinguish given any two individuals x and y, one that is *preferred* and one that is *spurned*. These are simply the one that comes earlier and the one that comes later in the linear order.

Hazen in effect combined these considerations with another observation, that given such a way of distinguishing a preferred from a spurned individual among any pair of individuals, an arbitrary binary relation can be represented by a species, that of those individuals that are self-related, together with two symmetric relations. One is the symmetric relation that holds between two individuals if and only if the preferred one stands to the spurned one in the given relation; the other is the symmetric relation that holds between two individuals if and only if the spurned one stands in the given relation to the preferred one. In this way, a quantifier ranging over relations can be simulated by three quantifiers, one ranging over species and the other two over genera representing symmetric relations. All this, however, is provided we assume there exists at least one linear ordering of all individuals. The existence of such a linear ordering is another consequence of the axiom of choice.

Now in a set-theoretic context there are numerous equivalents, and innumerable consequences, of AC. All have historically been to a degree controversial, but the most basic formulation of AC has had wide appeal. Gödel (1947) emphatically declared it to be "evident" (given that we have clear our heads of any notion that the existence of sets somehow depends on our being able to define them). A number of early set theorists implicitly assumed it without even noticing that they were doing so. It asserts that for any set of nonempty, nonoverlapping sets, there is a set that overlaps each of them in one and only one element (or in our context, for any genus of nonempty, nonoverlapping species, there is a species that overlaps each of them in one and only on individual).

From this there follow the various equivalents and consequences, some of which (such as the Banach-Tarski result) have been declared "paradoxical" and "counterintuitive," and are admitted even by Gödel to be "surprising and unexpected." The assumption about the decomposability of any infinite set, and about the linear orderability of any set, are among such "surprising and unexpected" consequences, only to be accepted because they are derivable from the basic, "evident" version.

Unfortunately, though the derivations are, in the context of set theory, where we have ordered pairs freely available, easy enough to be included in undergraduate courses, nothing like those derivations is possible within the Lewis framework or monadic third-order logic. There is no difficulty in *stating* the basic, "evident" form of the axiom, but the derivation of the *useful* consequences, including those used in methods (a) and (b), requires that we have ordered pairs or surrogates for them available already, which we do not.

This suggests rethinking the issue, and the first thing to notice is a point already incorporated into the exposition above, that Hazen's method (b) really only requires that it should be possible, given any two individuals, to label one preferred and the other spurned, in some systematic way. The preferences need not cohere into a linear preference order: Condorcet cycles, in which the preference goes to x over y and y over z but z over x, would be no obstacle. And there is a treatment of the axiom of choice that would make such a systematic choice of a preferred one out of any two immediately available.

The background is as follows. Initial opposition to AC stressed its *nonconstructive* character: Even in its basic, "evident" form it assumes the existence of a set satisfying certain conditions without giving an explicit or implicit definition of any such set. It was soon realized, however, that the nonconstructive character of orthodox mathematics by no means begins with set theory, but is to be found already its basic logic. Classical first-order logic already assumes the validity of the scheme $\exists x(\exists y A(y) \rightarrow A(x))$ for any condition expressible by a formula A, without giving any notion of how to find such an x. (The proof is simply that if there is at least one y such that $A(y)$, then any such y will do for x, while if there is no such y, then anything at all will do for x. As the heresiarch Brouwer, founder of intuitionism, noted, the nonconstructivity comes in the principle of *excluded middle*, according to which either there is or there isn't such a y.) Something like this observation lies behind David Hilbert's proposal to introduce his ε-symbol.

Given a formula A, with this symbol we can form a term $\varepsilon x A(x)$ substitutable for individual variables, and the assumption is that $\exists y A(y) \rightarrow A(\varepsilon x A(x))$. (The suggestion of Hilbert that one could in fact then *define* quantification $\exists y A(y)$ to *mean* $A(\varepsilon x A(x))$ will here be ignored.)

Mathematicians have a way of reasoning in which, having proved or assumed the existence of at least one x such that $A(x)$ they allow themselves to introduce a term, saying, "Let a be such an x," and more generally, having proved or assumed the existence for each y of at least one x such that $A(y, x)$, to introduce a notation, saying, "For each y, let a_y be such an x." The Hilbert ε-symbol is a formal representation of this way of reasoning: a is $\varepsilon x A(x)$, and a_y is $\varepsilon x A(y, x)$. The usual formal axiomatics of set theory does not allow such notation to be used in defining sets; if it is allowed, AC becomes deducible, and does not need to be assumed as an axiom. The Hilbert ε-symbol in effect builds AC into the background logic.

The ε-symbol is most useful if one makes the following additional assumption of *extensionality*:

$$\forall x(A(x) \leftrightarrow B(x)) \rightarrow \varepsilon x A(x) = \varepsilon x B(x)$$

In a context where we have species in addition to individuals, starting from an ε-symbol for which we do *not* have extensionality in this sense, we can get one for which we *do*, by defining $\varepsilon^*A(x) = \varepsilon A^*(x)$, where $A^*(x)$ says x is an individual belonging to the species of all y such that $A(y)$. So we might as well assume extensionality from the beginning. In that case we easily get, for any a and b, a preferred one of the two, namely, $\varepsilon x(x = a \vee x = b)$. Incorporating the Hilbert ε-symbol, embodying the "evident" form of AC, into the background logic, and thereafter following Hazen, constitutes method (b') for simulating quantification over relations. It provides perhaps the neatest way to surmount the obstacle to Lewis's program.

§6. Multitudes of Individuals

The reason Frege's *Grundgesetze* system is inconsistent is that his assumption that every "concept" has an "extension" (in our context, the assumption that there is a one-to-one function from *all* species to individuals) leads to the Russell paradox. Boolos (1989) considers the fallback assumption, begotten by Cantor and baptized "limitation of size" by Russell, that only "small" concepts have "extensions." Restated in Hazen's terminology rather than Frege's, the *fallback assumption* is that there are no more small species than individuals.

This is another informal wording of the same assumption displayed in formulations (i) and (ii) of §4. It represents one way of assuming that there are *many* individuals. In particular, it implies there is an infinity of them, assuming we have at least three to get going with. For even disallowing a null or empty species, given n individuals there are $2^n - 2$ small species, and therefore by the fallback assumption that many individuals: 3 gets us $2^3 - 2 = 6$, 6 gets us $2^6 - 2 = 62$, 62 gets us $2^{62} - 2 > 4{,}000{,}000{,}000{,}000{,}000{,}000$, and so on.

The very formulation of the fallback assumption requires quantification over relations on individuals, but we have just seen that there are several ways to provide a simulacrum of that, through appropriate applications of AC. Given this fallback assumption, the "structuralist" idea is then that while we allow ourselves to write, as Boolos does, as if we had in mind some fixed, specific membership relation $\in$, our real understanding is that all our assertions are tacit generalizations about *any* relation of the right kind.

With this understanding, Boolos (1989) in effect tells us that we automatically get, exploiting known results pertaining to von Neumann's old approach to axiomatizing set theory, all of second-order ZFC or MK (indeed, as Hazens reminds us, through plural quantification over class-surrogates, third-order ZFC or second-order MK) *except* the axioms of infinity and power set (at any rate assuming there are at least three individuals, so that two is few, an assumption needed to get the usual pairing axiom). Some of the derivations are a bit tricky: the axiom of unions requires a slight modification of a trick due to Levy, the axioms of choice and foundation a couple of tricks due to von Neumann.

Lewis, of course, with his Credo, wants all of orthodox set theory, including the infinity and power axioms, but what one needs to get these are extra assumptions easily formulable within his framework and ours, and formulable as just assumptions to the effect that there are lots of individuals. For infinity, we need merely that there should be some *small* species of individuals that is infinite; and for power, the assumption that for every small species, its subspecies are fewer than there are individuals. These are both, like the fallback assumption itself, in effect assertions about *how many* individuals there are, or in Lewis's terms, about "the size of Reality."

Set theorists have considered ever more daring extensions of the usual axioms, but these two generally take the form of "higher axioms of infinity" or "large cardinal hypotheses," which mostly lend themselves to reformulation within the frameworks under consideration as well. The mildest of these (the existence of arbitrarily large inaccessible cardinals) has even, according to McLarty (2010), been used (though really only as a dispensable convenience) in certain category-theoretic work whose results have been appealed to in works themselves in turn appealed to in the first published proof of Fermat's theorem.

That much and a bit more in the way of large cardinal theory is neatly incorporated in an extension of second-order ZFC or MK known as Bernays set theory. (See Burgess 2004 for a formulation based on plural quantification.) Bernays manages to replace most of the usual axioms of set theory by a single assumption, a so-called *reflection principle*, to the effect that there are "indescribably many" individuals. What this means is that for any statement Θ that holds of individuals and species there is a small species such that Θ continues to hold when individual variables are restricted to individuals belonging to that species, and species variables to subspecies of that species.

According to reflection, intuitively speaking, any attempt to describe how many individuals there are turns out, if true, to be an understatement, which would be equally true if one were speaking not of *all* individuals, but only of the *few* individuals belonging to some small species. Thus the statement that there exists at least one individual, in other words, that the domain over which the individual variables range is nonempty, which by courtesy is considered a principle of logic, upon reflection yields the consequence that there is a *small* species to which at least one individual belongs, and hence that there are at least *two* individuals.

Reflecting on *that* assertion yields *three*, and reflecting on *that* assertion yields *four*, and so on. Thus we have infinitely many individuals — even *without* the fallback assumption or the assumption we have at least three individuals to get going with — and reflecting on *that* conclusion yields a small species that is infinite, just what is needed for the set-theoretic infinity axiom. We do still need to make, in addition to the assumption of reflection, the fallback assumption that there are no more small species than individuals, but application of an appropriate version of reflection to *that* assumption turns out to yield just what is needed for the set-theoretic power axiom.

In the end, a framework of (i) first-order logic enriched by the Hilbert ε-symbol and (ii) Boolos's plural quantification and (iii) the assumption that the parthood relation of mereology obeys the laws of a complete atomic Boolean algebra, leaves us needing only (to revert from Hazen's to Lewis's terminology) (iv) the fallback or limitation-of-size assumption that there are no more small fusions of atoms than there are atoms, and (v) the reflection principle that there are indescribably many atoms, to get all of standard set theory, and therewith all of classical mathematics, plus the more modest of the large cardinal hypotheses to boot. The special ontological assumptions

(iv) and (v) of mathematics, beyond the framework (i)-(iii) indicated, amount in the simplest terms to just to this, that there are a lot of atoms of whatever nature around to serve as sets (or if you please, codes or surrogates for sets). This is not the place to go into technical details, and so I draw my sketch to a close. Suffice it to say that we have here, in the mereological treatment of set theory, when all the details of the outline sketched above are filled in, a very elegant application of Ludovician metaphysics.

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