LOGIC IN THE SECOND HALF OF THE TWENTIETH CENTURY

Modern logic emerged in the period from 1879 to the Second World War. In the post-war period what we know as classical first-order logic largely replaced traditional syllogistic logic in introductory textbooks, but the main development has been simply enormous growth: The publications of the Association for Symbolic Logic, the main professional organization for logicians, became ever thicker. While 1950 saw volume 15 of the *Journal of Symbolic Logic*, about 300 pages of articles and reviews and a 6-page member list, 2000 saw volume 65 of that journal, over 1900 pages of articles, plus volume 6 of the *Bulletin of Symbolic Logic*, 570 pages of reviews and a 60-page member list. Of so large a field, the present survey will have to be ruthlessly selective, with no coverage of the history of informal or inductive logic, or of philosophy or historiography of logic, and slight coverage of applications. Remaining are five branches of pure, formal, deductive logic, four being the branches of mathematical logic recognized in Barwise 1977, first of many handbooks put out by academic publishers: set theory, model theory, recursion theory, proof theory. The fifth is philosophical logic, in one sense of that label, otherwise called non-classical logic, including extensions of and alternatives to textbook logic. For each branch, a brief review of pre-war background will be followed by a few highlights of subsequent history. The references will be mix primary and secondary sources, landmark papers and survey articles.

SET THEORY

Set theory began in the 1870s when Georg Cantor applied to infinite sets the criterion for having the same cardinal number that we apply to finite sets: the cardinals of two sets are the same when their elements can be put into one-to-one correspondence. He showed there was a non-trivial subject here by showing to begin with that there are at least two infinite cardinals, *countable* and *continuum*, represented by the sets of natural and of real numbers. He formulated the conjecture, known as the *continuum hypothesis* (CH), that there are no intermediates. For the rest, set theory divides into three areas: (I) *descriptive set theory*, the theory of definable sets of real numbers; (II) the theory arbitrary sets of real numbers, with CH is the central conjecture; (III) *infinitary combinatorics*, the theory of arbitrary sets of arbitrary numbers, and the arithmetic of infinite cardinals. After 1900, Cantor's theory was put on an axiomatic basis by Ernst Zermelo, who made explicit an assumption, the *axiom of choice*, earlier left implicit. Zermelo's work was amended by Abraham Fraenkel to produce the system known as ZFC. By the middle 1930s a group writing under the collective pseudonym *Bourbaki* was advocating something like ZFC as a framework for rigorous development of all modern abstract mathematics. By mid-century their point of view had prevailed.

Now Kurt Gödel had famously established in 1931 that any system of axioms sufficient to develop basic number theory will be *incomplete*, leaving some statement formulable in its language *undecided*: neither *refutable* nor *provable* from the axioms; or equivalently, both *consistent* with but *independent* of those axioms. (There are systems, such as the axioms of the algebra of the real numbers, that are complete, but these are not sufficient to develop basic number theory, and Gödelian examples cannot be formulated in their languages.) ZFC certainly suffices for basic number theory, and so Gödel's incompleteness theorem guarantees there are statements formulable in its language that are undecidable by it. Thus with ZFC accepted as a framework, there are mathematical statements left undecided by the whole of mathematics as we know it. The examples produced by Gödel's proof are rather contrived, raising the question whether more natural examples can be found. Gödel himself suspected that CH was one. (Gödel 1948 expresses the view that if it is, that does not show that CH is meaningless, but rather that mathematics as we know it needs further axioms beyond ZFC.) Using what are called *constructible sets*, Gödel 1940 established the positive half of the conjecture: CH is consistent with ZFC. For more pre-war background, see Kanamori 2012a, in addition to the general source van Heijenoort 1967 for developments down through the Gödel incompleteness theorems.

The 1940s and 1950s were comparatively dull for set theory. Then Paul Cohen announced in 1963 and published in 1966 a proof, by a method called *forcing*, of the negative result Gödel had anticipated: CH is independent of ZFC. This work inaugurated a golden age, as Cohen's methods, alongside Gödel's, were taken up and adapted and elaborated. The names of Robert Solovay and Ronald Jensen appear as leading figures in the section headings in the survey of this period in Kanamori 2012b; the names of other contributors too numerous to list here can be found in the text of those sections. Through the 1960s and 1970s and beyond, such logicians attacked problems in all areas (I), (II), (III) above, showing many undecidable in ZFC, explaining why they had theretofore remained unsolved.

Gödel had shown that neither ZFC nor any consistent extension thereof can prove its own consistency, but in some cases ZFC *can* prove that *if* ZFC is consistent, *then* ZFC+B is consistent; this is a more precise statement of what Gödel and Cohen proved for B = CH and B = not-CH, respectively. When ZFC can prove that *if* ZFC+A is consistent, *then* ZFC+B is consistent, usually it turns out that a much weaker system can prove the same, and we say that the *consistency strength* of ZFC+B (or of B, for short) is less than or equal that of ZFC+A (or of A, for short). By this standard, ZFC and ZFC+CH and ZFC+not-CH are of the same strength. But there are many hypotheses that are known to be of higher and higher strength — or, since consistency strength might equally well have been called *inconsistency risk*, known to be riskier and riskier.

These are the *large cardinal axioms*. Such a hypothesis asserts the existence of a cardinal that is in some sense — a different sense for each axiom — very much larger than any smaller cardinal. The weakest asserts the existence of an *inaccessible*, unreachable by any of the processes Cantor used to obtain larger cardinals from smaller ones. (Inaccessibles have occasionally been used outside pure set theory, in category theory.) The consistency strengths of the many large cardinal axioms that have been proposed all form a linearly ordered scale; moreover, when other hypotheses are proved undecidable, their consistency strengths generally turn out to fit somewhere in this scale. This fact is noteworthy since it is easy enough to contrive artificial examples of incomparable consistency strength.

By the end of the 1980s, the work of a group of set-theorists called the “cabal” (Alexander Kechris, D. A. Martin, Yannis Moschovakis, John Steel, Hugh Woodin) had shown, using game-theoretic methods, that large cardinals high enough up the scale give satisfying solutions to virtually all problems of area (I). Moreover, the conjunction of the theorems stating these solutions is of about the same consistency strength as the axioms used to prove them. There has been no subsequent breakthrough of comparable magnitude, though Harvey Friedman (see Harrington et al. 1985) has found results more concrete-looking than those of area (I) that are also of about the same consistency strength as certain large cardinal axioms. Gödel had hoped large cardinal axioms might be of help also in area (II); but it transpired that they leave undecided even the most basic problem there, CH.

There *is* an alternative to CH, of no greater consistency strength, that *does* settle in an interesting way a number of problems in area (II). But though it is called Martin’s *axiom*, it does not have much support among set theorists as a candidate new axiom, in the way some of the large cardinal axioms do. And there are limits to what even powerful extensions of Martin’s axiom (as in Shelah 1982) tell us about area (III).

We are left with a picture of a sequence of extensions of ZFC of greater and greater consistency strength, and a varied range of statements more and more of which become provable as we move higher and higher on the scale, but also with a range of problems unaffected by these developments, including the oldest problem of all, CH. For else was going on as this picture was emerging, see the introduction to Foreman and

Kanamori 2010.

MODEL THEORY

The rudiments of model theory are covered in the courses on logic commonly taken by philosophers, so will be only briefly recalled here. First, "syntactic" notions: A *first-order language* has the logical predicate of identity = and various other predicates *P*, *Q*, *R*, each with a fixed number of places, giving rise to *atomic formulas* such as $x = y$ and *Px*, *Qxy*, *Rxyz*. From these, compound formulas can be produced using logical operations of negation ¬, conjunction ∧, disjunction ∨, the indicative conditional →, and quantification universal and existential, ∀ and ∃. *Sentences* are formulas in which every occurrence of a variable is "bound" by a quantifier, like $\forall x \exists y Qxy$ and unlike *Qxy*. A *theory* is a set of sentences. Next, "semantic" notions: An *interpretation* provides a *universe* of objects for variables *x*, *y*, *z* to range over, and for each predicate an associated relation on that universe, of the right number of places. The notion of *truth* of a sentence under an interpretation is then defined by recursion on the complexity of the sentences: for instance, a conjunction is true if the conjuncts both are. (Quantification is a longer story, told in textbooks.) A *model* of a theory is an interpretation in which all sentences of the theory are true. The *cardinal* of a model is that of its universe; the theory of models of finite cardinality has a special flavor, and pure model theory largely concentrates on infinite models.

The foregoing notions have roots in the work of Alfred Tarski in the 1930s, and were further developed with student such as Robert Vaught in the 1950s, when a wide range of model-building techniques were introduced (see Vaught 1974). A number of model-theoretic results had been established as far back as the 1920s, before the Tarskian definitions were formulated, and hence before those results could be fully rigorously stated. Examples include the celebrated *Löwenheim-Skolem theorems*, saying that if a theory has a model of one infinite cardinal, it has models of all infinite cardinals.

A crucial notion throughout mathematics is that of isomorphism. Consider two interpretations of the same language, having universes U_1 and U_2 and associating relations Q_1 and Q_2 with each predicate *Q*. A one-to-one correspondence between U_1 and U_2 is an *isomorphism* if for all *Q*, any given elements of U_1 stand in relation Q_1 if and only if the corresponding elements of U_2 stand in relation Q_2. If there is an isomorphism, the interpretations are said to be of the same *isomorphism type*. It can be shown that the same sentences are true in models of the same isomorphism type. Since isomorphism requires one-to-one correspondence and hence identity of cardinality, the Löwenheim-Skolem theorems imply that if a theory has any infinite model at all, it has infinite models of different isomorphism types: a first-order theory can never be *categorical* in the sense of characterizing a model "uniquely up to isomorphism." At most it may happen for a given infinite cardinal that all models of a theory *of that cardinal* are isomorphic, in which case we say the theory is categorical *in that cardinal*.

There are several logics allowing more operators than first-order logic, but working with the same kind of models, that escape this limitation. *Second-order* logic allows quantification over subsets *X* as well as over elements *x* of the universe of a model. It goes all the way back to Gottlob Frege (see the first selection in van Heijenoort). *Infinitary* logic allows conjunctions and disjunctions of infinite lists of formulas. *Generalized quantifier* logic allows, in addition to "for some *x*" and "for all *x*" also "for infinitely many *x*" and "for all but finitely many *x*." Both infinitary and generalized quantifier logics, which go back to Tarski 1958 and Mostowski 1957, were intensively investigated through the 1970s and beyond, and generalized quantifiers remain of interest in linguistics. All three logics mentioned allow the natural number system and some other structures to be characterized uniquely up to isomorphism. The price is the failure of various theorems that hold for first-order logic, beginning with Löwenheim-Skolem.

Returning to first-order logic, the *spectrum* of a theory is the function assigning each infinite cardinal the number of isomorphism types of models of the theory with that cardinal. The *spectrum problem* is to characterize the possible spectra for complete theories, of which abstract algebra provides many specific examples. As a first step, Morley 1965 showed that a complete theory categorical in one uncountable cardinal is categorical in all uncountable powers. An almost total solution to the spectrum problem was worked out by the end of the last century, mainly by Saharon Shelah, who discovered a "main gap" between theories where the number of isomorphism types of models increases rapidly with cardinal, and theories where it increases only rather slowly (see Shelah 1986). Left unsolved is the status of *Vaught's conjecture*, to the effect that even if CH is false, the number of isomorphism types of countable models cannot be intermediate between countable and continuum.

Model theory has many applications to abstract algebra and other areas of core mathematics, but apart perhaps from the work of Robinson 1966 reconstructing something like Leibniz's infinitesimal calculus, such applications seem to hold little interest for philosophers. So though the center of gravity of model theory post-Shelah has lain in algebraic applications, they will be left aside here.

RECURSION THEORY

The proposal has been gaining ground to replace the long-established label at the head of this section with *computability theory*. Soare 1996 argues the case on historical grounds, and is a convenient source for background on the work of Alonzo Church, S. C. Kleene, and Alan Turing in the 1930s that gave rise to this subject, whatever one calls it. From this background only a few points need be recalled. (1) Workers such as those mentioned were seeking to give rigorous analyses of what it is for a function on the natural numbers to be effectively calculable. (2) All approaches had to allow *partial* functions, which may be undefined for some numbers as inputs, giving no outputs. (3) All approaches ultimately led to same class of functions. (4) The thesis that this class is indeed the classes of effectively calculable functions is now the consensus. If we reserve *effectively calculable* for the intuitive pre-theoretic notion, what term shall we use for the rigorous technical notion so many writers have characterized in so many equivalent ways? Fairly early *recursive* won out, though *computable* is now a challenger.

A set *A* of natural numbers is recursively *decidable* if its characteristic function —

$a(n) = 0$ or 1 according as n isn't or is in A — is recursive, and recursively *enumerable* if it is the range of some recursive partial function — the set of all outputs $b(n)$ for some such b. Any recursively decidable set is recursively enumerable — let $b(n) = n/a(n)$ — but there is a stock example of a recursively enumerable but recursively undecidable set, the *halting* set H, obtained as follows. There is a way to assign natural number indices to one-place recursive partial functions so that the following is itself a two-place recursive partial function: $t(m, n)$ = the output of the partial function with index m for input n. Then H is the set of m such that $t(m,m)$ is defined. For a positive calculability result one need only describe the calculation procedure and show that it works; it is for a negative incalculability result that one needs some kind of analysis of what calculability amounts to, and the most famous results from the 1930s are indeed negative ones, the recursive undecidability of H being perhaps the most basic.

The first half and more of the 1950s was dominated by a single problem. An effective procedure for deciding membership in a set involves no use of external sources of information. But suppose we do have an external source of information that for any n we ask about will tell us whether or not it is in a certain set A. If there is a procedure for deciding membership in the set B that uses such an external source but is otherwise effective, then we say B is *reducible to* A, or *of degree of recursive undecidability less than or equal to that of* A. All recursive sets have the same degree, the least degree, called $\mathbf{0}$. The degree of H, called $\mathbf{0'}$, is greater, and the degree of any other recursively enumerable set is less than or equal to $\mathbf{0'}$. The dominant problem was this: does there exist a recursively enumerable set of degree strictly between **0** and **0'**? The question was independently answered, in the affirmative, by Albert Muchnik and Richard Friedberg (in Russian and English, respectively) in 1956-57, using what is called the *priority* method. As with forcing, so with priority, the introduction of a new technique was followed by a flurry of activity lasting through the 1960s and 1970s at least. The structure of the system of degrees (especially recursively enumerable ones) and the reducibility order on them was intensively investigated, almost as if it were as natural an object of study as the natural number system.

The kind of relative decidability involved in degree theory is only one direction of generalization of the original notion. Others led to the developments of so-called *higher-type* and *ordinal* recursion theories. For workers in other areas of logic, however, recursion theory was of interest more for its applications to their specialties. For instance, the notion of recursiveness is needed to state Gödel's incompleteness theorem in its most general form, besides being a tool in the proof of even its simplest form. But indeed each of set theory and model theory and recursion theory has uses in the other two. In retrospect, however, arguably the most important development in recursion theory during our period of interest lay on the borders with computer science.

Turing's well-known approach to characterizing effective calculability, in terms of the operations of his idealized machines, though it dates from before there were any real electronic digital computers, remains relevant to the theory of such technology. For unlike the rival approaches of Church, Kleene, and others, it provides not only a formal notion of calculability, but also a way of counting how many steps are needed in a given calculation, which is important since a procedure that makes it possible in principle to calculate a function may still leave it infeasible in practice, if the time demands are too great. Such issues are pursued in *complexity theory*, which makes distinctions among recursive functions based on how rapidly time requirements for producing an output rise as a function of the size of the input. Though foreshadowed in a long-neglected 1956 letter of Gödel to John von Neumann, complexity theory began in a serious way with work of Stephen Cook and others in the early 1970s, from which emerged the famous *P vs NP* problem. Of this problem there exist many semi-popular accounts, but a special interest attaches to Cook 2006, officially announcing a million-dollar prize for its solution, something not on offer for any other problem in or near logic.

PROOF THEORY

A first-order sentence is *valid* if it is true in all interpretations. A *proof-procedure* is a method intended to be usable to demonstrate that a sentence is valid, generally by producing a derivation of the sentence from sentences known to be valid by rules known to preserve validity. The procedure is *sound* if every derivable sentence is indeed valid, and *complete* if every valid sentence has a derivation. There are many sound and complete proof procedures for first-order logic — but none for second-order, infinitary, generalized-quantifier, or other such logics. Proof theory in a narrow sense has been concerned with two special types of proof procedures, and centers around two theorems, Gerhard Gentzen's *cut-elimination* for *sequent calculus* and Dag Prawitz's *normalization* for *natural deduction*, dating from the 1930s and 1960s respectively. For the pre-war philosophical history and background in the 1920s debates between David Hilbert and L. E. J. Brouwer, see van Plato 2014. Both central theorems show how, in general at the cost of great lengthening, "indirect" derivations can be replaced by more "direct" ones.

Such technical results, with bells and whistles, have endless consequences, and can be used to establish how the scale of consistency strength, showing how power to derive more results is balanced against risk of collapsing in contradiction, which is marked in set theory by large cardinal axioms as benchmarks, can be developed for much weaker theories. These extend right down to so-called *Robinson arithmetic* **Q**, which cannot even prove that the function $exp(n) = 2^n$ is defined for all n (see Hajek and Pudlak 1993). The positions as to the scope and limits of legitimate mathematics that were advocated by dissidents opposing the rise of the now-established set-theoretic mathematics — the small critical schools called *finitist*, *constructivist*, *intuitionist*, *predicativist*, and so on — show up as points on this scale. As Simpson 1982 illustrates, when a mathematical theorem cannot be proved at one level on the scale but can be proved one level up, often adding it to the lower level lets one to deduce the characteristic axiom that distinguishes the higher level. Friedman calls such proving axioms from theorems "reverse mathematics," and it arguably has been the most active area of proof theory in a broad sense in recent decades.

The view that the meaning of the logical particles should be conceived proof-theoretically rather than model-theoretically — that we should explain conjunction by saying that a conjunction is derivable from its two conjuncts and each conjunct is derivable from the conjunction, rather than by saying that a conjunction is true if and only if both its conjuncts are true — has a substantial following among philosophers. Dummett 1977 draws on such ideas to defend intuitionism. Feferman 1998 draws on other proof-theoretic notions and results to defend predicativism. But it is more common for proof theorists today to abstain from advocacy and merely lay out available trade-offs between power and risk — for instance, pinning down the exact strength of theories of truth in the spirit of Kripke 1975, or of consistent repairs of the system of Frege shown inconsistent by Russell's paradox, as surveyed in Burgess 2005.

NON-CLASSICAL LOGIC

There are too many non-classical logics even to list the names of all of them here, but a handful stand out as objects of wide and deep study. Two date from the pre-war years: (1) the *modal logic* of C. I. Lewis, which has ancient and medieval antecedents, but in its modern revival originated in the first decade of the twentieth century and assumed definite form in Lewis and Langford 1932; (2) the *intuitionistic logic* extracted for separate study from the intuitionistic mathematics of Brouwer by his disciple Arend Heyting in 1930.

Lewis confined himself to *sentential* logic, where we consider compound sentences built up by $\neg$, $\wedge$, $\vee$, $\rightarrow$, and operators of necessity and possibility from simple sentences p, q, r. Quantified modal logic was taken up only post-war and it remains a scene of controversy to this day, though indeed even in sentential there is a proliferation of systems and no consensus as to which is the right one. Today modal logic is considered a paradigmatic *extra*-classical logic, adding to classical logic the modal operators, to represent notions of interest in philosophy but not mathematics (where *all* facts are necessary). Lewis, however, seemed to see himself as opposing the logical mainstream since Bertrand Russell. Brouwer and Heyting were unequivocally dissidents, and they considered quantification from the outset: Theirs is the paradigmatic *anti*-classical logic, and the laws of classical logic they reject included not only *excluded middle* $p \vee \neg p$ and *double negation* $\neg\neg p \rightarrow p$, but also $\neg\forall xFx \rightarrow \exists x\neg Fx$.

The post-war period saw the development of new non-classical logics, notably: (3) the extra-classical *tense logic* of Arthur Prior (in Prior 1967 and elsewhere), which adds, in place of necessity and possibility modalities, past and future tense operators; (4) *counterfactual* logic, an extension of modal logic to handle subjunctive conditionals, as in Lewis 1973; (5) the anti-classical *relevance logic* of Anderson and Belnap 1975, the eldest and best-known of the family of *paraconsistent logics*, which reject the classical law $(p \wedge \neg p) \rightarrow q$. (There are rival paraconsistent logics, as in Restall 2000 and Priest 1979, that have garnered attention, but this is mainly a development of the present century.) There is a sharp difference between intuitionistic logic and other anti-classical logics: intuitionists proving "metatheorems" about their logic use intuitionistic logic in their proofs; paraconsistent logicians use classical logic. This is inevitable since — in the words of Feferman — "nothing like sustained ordinary reasoning can be carried on" in anti-classical logics other than the intuitionistic; but it raises an obvious issue justification, discussed in the literature under the rubric "classical recapture."

Though it cannot be given much space here, mention must be made also of (6) *quantum logic*, going back to Birkhoff and von Neumann 1936, but standing rather apart.

Some deep results on modal logics were obtained during the war years using algebraic methods, as in McKinsey 1941, but a revolution occurred with the introduction of *relational models*. Goldblatt 2006 examines the relevant history minutely, and identifies as the crucial contribution work of Saul Kripke dating from the late 1950s and published in the early 1960s. Kripke models variously adapted can handle not only modal but tense, counterfactual, relevance, and other logics. For instance, early in the history of modal and intuitionistic logic Gödel had announced a deep connection between them; Kripke model theory can be used to provide a proof, which Gödel did not. (Kripkean methods largely fail for quantum logic, one reason it stands apart.)

Because (1)-(5) were introduced by philosophers, sympathetic coverage of them is conveniently available on-line in the *Stanford Encyclopedia of Philosophy*. At least as far back as Pnueli 1977, however, computer scientists began to take a major interest in such logics, and the center of gravity of non-classical logic today arguably lies in applications, and so outside the scope of this survey.

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