

TWO UNDECIDABLE QUESTIONS ABOUT GROUP ACTIONS

ABSTRACT

It is shown that for invariance under the action of special groups the statements "Every invariant PCA is decomposable into \aleph_1 invariant Borel sets" and "Every pair of invariant PCA is reducible by a pair of invariant PCA sets" are independent of the axioms of set theory.

John P. Burgess
Department of Philosophy
Princeton University
Princeton, NJ 08544-1006, USA
jburgess@princeton.edu

TWO UNDECIDABLE QUESTIONS ABOUT GROUP ACTIONS

BACKGROUND

Ashok Maitra came as a visiting professor to the statistics department at Berkeley in the early 1970s, and there taught a course that introduced graduate students from several disciplines to classical descriptive set theory. His lectures perfectly equipped students to participate in two overlapping research programs of the period: the expansion of measurable selection theory, to which he and his students back at the Indian Statistical Institute were later in the decade to make the contributions that figure so prominently in the survey [4]; and the emergence, from work by Robert Vaught [3] in the Berkeley mathematics department and logic group, of invariant descriptive set theory, which has since developed into the large field surveyed in [2]. Contributions by students from the course to the latter development began with [1]; in this note I would like to revisit a couple of questions left open in that paper.

Invariant descriptive set theory began with the study of the equivalence relation induced on a Polish space X by the continuous action of a Polish topological group G , but some of the more elementary results follow just from one or the other of two weaker assumptions: (1) that the equivalence is induced by a continuous action of a topological group whose topology is *special*, in the sense of having a countable weak basis and satisfying the Baire category theorem, or (2) that the equivalence relation, considered as a subset of the product space X^2 , is analytic. The general theory of analytic equivalence relations has been developed to a much greater extent than the general theory of special group actions, mainly because in the latter one all too soon encounters questions that are independent of the usual axioms of set theory (ZFC) in the same manner as the continuum hypothesis (CH). Some independence results were indicated in [1], but the status of other basic questions was left open. Here two more principles will be treated, and two questions still left open will be noted.

The following invariant PCA decomposition theorem (A) was proved by Vaught for Polish group actions. It follows immediately from the lemmas (A1), which holds for all analytic equivalences, and (A2), which holds for all special actions:

- (A) Every invariant PCA set is a union of \aleph_1 invariant Borel sets.
- (A1) Every invariant PCA set is a union of \aleph_1 invariant analytic sets.
- (A2) Every invariant analytic set is a union of \aleph_1 invariant Borel sets.

(A2) and hence (A) do not hold in general for analytic equivalence relations. It will be shown here that whether (A1) and hence (A) holds in general for special group actions is undecidable in ZFC. It will also be shown that it is undecidable in ZFC whether following invariant PCA reduction principle holds:

- (B) If A and B are invariant PCA sets, then there exist invariant PCA sets C and D that reduce A and B .

What reducing means is that $C \subseteq A$ and $D \subseteq B$ and $C \cap D = \emptyset$ and $C \cup D = A \cup B$.

ON THE NEGATIVE SIDE

It will be shown that the existence of counterexamples to (A) and (B) follows from the Axiom of Constructibility ($V = L$). $V = L$ will be used only through its consequence that there exists a strong Σ_1 -well-ordering of the hereditarily countable sets (HC): a well-ordering \angle in order type the first uncountable ordinal Ω such that the function $f(x) = \{y: y \angle x\}$ (and hence the order \angle itself) is Σ_1 -definable (hence Δ_1 -definable) with parameters over HC. (Recall that for sets of reals, being thus Σ_1 -definable over HC is equivalent to being PCA.) Since CH follows from $V = L$, we get a model of set theory where (A) and (B) fail and CH holds. It remains an open question whether there is a model where (A), (B), and CH all fail.

Let p, q, r, s range over the rationals \mathbb{Q} . As the *code* for an open subset U of the reals \mathbb{R} take the set of all $\langle p, q \rangle$ such that $p < q$ and the open interval (p, q) is contained in U . The condition " u codes an open set" amounts to " u is a set of pairs of $\langle p, q \rangle$ with $p < q$ and for any $p < r < s < q$, if $\langle p, q \rangle$ is in u , then $\langle r, s \rangle$ is in u ", which is easily seen to be Δ_1 over HC, as are the conditions " u codes an open dense set" and "the real x belongs to the open set with code u " and "the real x is linearly independent over \mathbb{Q} of the reals in the countable set y ".

Given a strong Σ_1 -well-ordering \angle of HC, the sequence $\langle u_\alpha: \alpha < \Omega \rangle$ enumerating all codes for open dense sets will be Δ_1 over HC, as will be the sequence $\langle w_\alpha: \alpha < \Omega \rangle$ defined by transfinite induction as follows: w_α is the \angle -least real w that is linearly independent over \mathbb{Q} of the w_β for $\beta < \alpha$ and belongs to the open dense set coded by u_β for all $\beta \leq \alpha$. (Since the intersection of the open dense sets coded by the u_β for $\beta \leq \alpha$ is comeager, while the span of the w_β is countable and hence meager, such a w will always exist. The condition w_α is supposed to be the \angle -least real to satisfy is Δ_1 over HC, and w is the \angle -least real to satisfy it if it is satisfied by w and by no element of $f(w) = \{y: y \angle w\}$. It is at this point that the hypothesis that this function f is Δ_1 over HC is needed.) Then $W = \{w_\alpha: \alpha < \Omega\}$ is Δ_1 over HC, consists of reals linearly independent over \mathbb{Q} , and has countable intersection with any closed nowhere dense set F (since such an intersection contains only w_β for $\beta < \alpha$, where u_α is the code for the complement of F), from which it follows that every uncountable subset of W is nonmeager.

Let now P be any set of countable ordinals that is properly Π_1 over HC, and let I (respectively J) be the set of all w_α for α of form $\omega\beta$ (respectively, $\omega\beta + 1$) with β in P . Then I and J will then also be properly Π_1 over HC. Let A, B , and G be the vector spaces over \mathbb{Q} spanned by $W - I, W - J$, and $W - (I \cup J)$. All are Σ_1 over HC, or equivalently are PCA, and all have uncountable intersection with W , hence are nonmeager. Considering G as a subgroup of the additive group of \mathbb{R} , acting on \mathbb{R} by translation, we have a special action in which A and B are invariant, and G is the orbit of 0.

Then $A \cap B = G$, while $W \cap (A - G) = J$ and $W \cap (B - G) = I$, by the linear independence of the elements of W . Since W and A and B are, while I and J are not, Σ_1 over HC, or equivalently PCA, neither $A - G$, $B - G$, nor G can be. Hence G is an invariant PCA set that is not analytic, and (A1) and (A) fail. That (A) fails if $V = L$ was actually mentioned in [1], but with the present example we can now add that (B) fails also. For since $A \cap B = G$ is a single equivalence class, the only pairs of invariant sets reducing A and B are the pair A and $B - G$ and the pair $A - G$ and B .

ON THE POSITIVE SIDE

It will be shown that (A) and (B) follow from Martin's Axiom (MA) plus the negation of CH. MA will be used only through its consequence, the Strong Baire Category Theorem (SBCT), according to which in any special topological space the union of fewer than continuum many meager sets is meager. It remains an open question whether there is a model where (A), (B), and CH all hold.

Given a continuous action of a special group G on a Polish space X , a subset A of X is called *normal* if for every x the set $A^x = \{g \in G: gx \in A\}$ has the Baire property. SBCT implies that any union of fewer than continuum many sets with the Baire property has the Baire property, and hence similarly for normal sets. Since any PCA set is a union of \aleph_1 Borel sets, and Borel sets are normal, SBCT together with the negation of CH imply that all PCA sets are normal.

Another consequence of the same set-theoretic assumptions is the following lemma: A union of \aleph_1 sets each having the Baire property will be comeager if and only if some union of countably many of them is comeager. For let the sets be B_α for $\alpha < \Omega$, let $C_\alpha = B_\alpha - \bigcup_{\beta < \alpha} B_\beta$. The C_α will have the Baire property since the B_α do, so let O_α be open sets such that each symmetric difference $N_\alpha = C_\alpha \Delta O_\alpha$ is meager. For all $\alpha < \beta < \Omega$ we will have $O_\alpha \cap O_\beta \subseteq N_\alpha \cup N_\beta$, and so by the Baire category theorem $O_\alpha \cap O_\beta = \emptyset$. Since there is a countable weak basis, it follows that $O_\beta = \emptyset$ and C_β is meager for all but

countably many β , which is to say, for all $\beta \geq \alpha$ for some α . Since the union of the C_β for $\beta \geq \alpha$ will then be meager by our set-theoretic assumptions, and the union of *all* the B_β is comeager, the union of the B_β for $\beta < \alpha$ must be comeager.

We now invoke Vaught's result that the strong and weak transforms $A^* = \{x \in X: A^x \text{ is comeager in } G\}$ and $A^\dagger = \{x \in X: A^x \text{ is nonmeager in } G\}$ are Borel (respectively, analytic) for any Borel (respectively, analytic) subset A of X . Given a PCA subset A of X , we can write it as a union of Borel sets B_α for $\alpha < \Omega$, where $B_\alpha \subseteq B_\beta$ when $\alpha < \beta$. By the lemma, for any x , A^x will be comeager if and only if some B_α^x is comeager. It follows that A^* is the union of the invariant Borel sets B_α^* . If A is invariant, then $A = A^*$, and so we have written A as a union of \aleph_1 invariant Borel sets, proving (A).

Under our set-theoretic assumptions, Vaught's result can be strengthened to say that the transforms A^* and A^\dagger are PCA for any PCA subset A of X . The proofs are similar in the two cases, so we consider only the strong transform A^* . Represent A as the project to X of a coanalytic subset C of the product $X \otimes Y$ for some Polish space Y . Represent C as a union of constituent Borel sets C_α for $\alpha < \Omega$, where $C_\alpha \subseteq C_\beta$ when $\alpha < \beta$. This may be so done that, setting $\text{rank}(x, y) =$ the least α such that $\langle x, y \rangle \in C_\alpha$ if such exists and Ω otherwise, and letting $Q(x, y, u, v)$ hold if and only if $\text{rank}(x, y) \leq \text{rank}(u, v)$, the relation Q is analytic as a subset of $X^2 \otimes Y^2$.

Then for any x , the condition $x \in A^*$ is equivalent to each of the following, using the lemma above to get from (1) to (2):

- (1) $\{g: \exists y (gx, y) \in C\}$ is comeager
- (2) $\exists \alpha < \Omega$ ($\{g: \exists y (gx, y) \in C_\alpha\}$ is comeager)
- (3) $\exists u \exists v (\langle u, v \rangle \in C \ \& \ \{g: \exists y Q(gx, y, u, v)\})$ is comeager
- (4) $\exists u \exists v (\langle u, v \rangle \in C \ \& \ x \in R^*)$

where R is the analytic set $\{\langle x, u, v \rangle: \exists y Q(x, y, u, v)\}$ and the transform R^* is taken in the sense of the action of G on $X^2 \otimes Y$ given by $g\langle x, u, v \rangle = \langle gx, u, v \rangle$. By Vaught's result cited earlier, R^* is analytic, so the representation (4) shows that A^* is PCA, as asserted.

The invariant PCA reduction principle (B) now follows from the ordinary PCA reduction principle using D. E. Miller's observation in [1] that if C and D reduce the invariant sets A and B , then so do the invariant sets C^* and D^\dagger .

Before closing it may be in order to mention an old negative observation of Miller's. The outstanding open problem in invariant descriptive set theory is Miller's topological generalization of Vaught's conjecture, the dichotomy according to which in any continuous action of a Polish group on a Polish space, the number of orbits is either countable or continuum. (So formulated the conjecture is trivial if the continuum is \aleph_1 ; there is an ostensibly stronger, more technical version that is nontrivial even then, but it need not detain us here.) The conjecture definitely fails for some analytic equivalence relations, for which we can prove only the trichotomy that the number of equivalence classes is either countable, or exactly \aleph_1 , or continuum. (So formulated the conjecture is trivial if the continuum is \aleph_1 or \aleph_2 ; again there is a stronger, more technical version.) Miller's observation is that even the trichotomy fails for special actions, assuming MA and that the continuum is $> \aleph_2$.

For let W be a basis for \mathbb{R} over \mathbb{Q} , let I be a subset thereof of cardinality \aleph_2 , and let G be the space spanned by $W - I$. Under our set-theoretic assumptions, fewer than continuum translates of G cover \mathbb{R} , whereas the union of fewer than continuum meager sets is meager, so G is nonmeager in \mathbb{R} and special when considered as a topological space in its own right. Letting G act on \mathbb{R} by translation, the number of orbits is exactly \aleph_2 , hence more than \aleph_1 but less than continuum.

REFERENCES

- [1] Burgess, John P. and Douglas E. Miller, 1975, "Remarks on Invariant Descriptive Set Theory," *Fundamenta Mathematicæ* 90: 53-75.
- [2] Su Gao, 2009, *Invariant Descriptive Set Theory*, Chapman & Hall/CRC Press.
- [3] Vaught, Robert L., 1974, "Invariant Sets in Topology and Logic," *Fundamenta Mathematicæ*, 82: 269-293.
- [4] Wagner, Daniel H., 1980, "Survey of Measurable Selection Theorems: An Update," in A. Dold and B. Eckmann (eds.), *Measure Theory Olberwolfach 1979*, Springer Lecture Notes in Mathematics 794, 176-219.