

## The Logic of Necessity

### *Senses of necessity*

Modal logic is concerned with the distinction between what merely is and what in one or another sense necessarily must be. It was pursued throughout the ancient and medieval periods, but modality is ignored by classical logic (modern textbook logic), which was developed for the analysis of mathematical arguments, where modality plays no role. The creation of modern modal logic was nonetheless a response to the development of classical logic rather than a revival of ancient and medieval tradition, which was not well understood until the advance of modern modal logic inspired historical scholars to reexamine it. This chapter will treat only modern developments.

Modal logic adds a symbol  $\Box$  for necessity to classical logic's list of symbols  $\sim$  and  $\wedge$  and  $\vee$  and  $\rightarrow$  for negation and conjunction and disjunction and conditional. Other modal notions, some with symbols of their own, may then be defined, as in TABLE 1.<sup>1</sup> (In the table,  $A$  and  $B$  appear in places grammatically appropriate for sentences except in the last two lines, where they appear in places appropriate for nominalizations of sentences. In principle,  $A \circ B$  ought to be 'its being the case that  $A$  is compossible with its being the case that  $B$ ' and similarly for  $A \Rightarrow B$ . In practice, the nominalizing phrase required by grammar is left tacit.)

INSERT TABLE 1 ABOUT HERE

In natural languages, modal distinctions are expressed through verb-inflections making grammatical moods such as indicative and subjunctive, and/or through modal auxiliaries such as the English 'must' and 'may'.

Different senses of necessity are often expressed similarly. We need to distinguish a half-dozen.

*Epistemic necessity.* With 'She must have gone, but he may have stayed', meaning 'Given what we know, she must have gone, but for all we know, he may have stayed', we have knowledge-related or *epistemic* possibility and necessity. Their logic, though in principle part of modal logic in the broad sense, has its own flavor, and is in practice treated separately as 'epistemic logic'. It has its own chapter in this book.

*Deontic necessity or obligation.* With 'He must stay, but she may go', meaning 'He is obligated to stay, but she is permitted to go', we have duty-related or *deontic* modality. Deontic logic, too, is generally treated as a separate subject. Modal logic in the narrower sense, concerned with 'vanilla' modality as opposed to the epistemic and deontic flavors, is called truth-related or *alethic* modal logic. The terminology epistemic/deontic/alethic was popularized by Von Wright (1951).

*Necessity tout court or metaphysical necessity.* The label 'alethic' conceals distinctions. Since Kripke (1972), many use 'necessity' *sans phrase* for what both is and inevitably would have been (could not have failed to be) even if the world had been otherwise, and 'possibility' *sans phrase* for what either is or isn't but potentially might have been (need not have failed to be) if only the world had been otherwise. When a distinguishing epithet is wanted, these are called *metaphysical* modalities. Within the 'alethic' category, they contrast with logical modalities, which concern not the question 'What if the world had been otherwise?' but the question 'What can without contradiction be supposed about how the world actually is?' But the distinction has often been overlooked. (Stock example: It is logically possible that water is not H<sub>2</sub>O, since there is no internal contradiction in the

traditional view that water is an element, but it is metaphysically necessary that water is H<sub>2</sub>O, since imagining a world in which some liquid with a composition other than H<sub>2</sub>O fills lakes and is called 'water' is imagining a world in which some liquid other than water fills lakes and is called 'water', not a world in which water has some composition other than H<sub>2</sub>O.)

*Linguistic necessity or analyticity.* Logical necessitation is distinctively called *implication*. But the label 'logical' itself conceals distinctions. Logicians tend to use 'logical necessity' more narrowly than philosophers, to cover 'No unmarried man is married', where all that matters is logical form, but not 'No bachelor is married', where meaning as well as form is pertinent. But there is an established expression for the broader notion, *analyticity* or *linguistic necessity*. The narrower can be called *formal necessity*.

*Validity versus demonstrability.* This last label, too, conceals a distinction, between the nonepistemological notion of what is *true* by virtue of form alone and the epistemological notion of what is *verifiable* by virtue of form alone. Logical theory analyzes the former as being true in all models, the latter as having a proof. No other labels being on offer for the contrasting pair of intuitive notions, we may use the ones for the contrasting pair of technical notions, as in TABLE 2. (In the table, consequence and deducibility are properly speaking the *converses* of implication in the model-related and proof-related senses. For classical first-order logic, the completeness theorem guarantees that validity and demonstrability coincide; in other cases, notably second-order logic, there is no completeness theorem.)

INSERT TABLE 2 ABOUT HERE

Some terminological pedantry is justifiable with modern modal logic, since it historically has been plagued by terminological misunderstandings, and indeed originated with one. Today  $\sim$  and  $\wedge$  and  $\vee$  and  $\rightarrow$  are generally pronounced 'not' and 'and' and 'or' and 'if', but about a century ago some of the founders of classical logic were erroneously pronouncing  $\rightarrow$  as 'implies'. Modern modal logic began with C. I. Lewis, correctly noting that  $\rightarrow$  does not express logical necessitation, fatefully proposed adding a symbol  $\Rightarrow$  that would. (The first major work of modern modal logic was Lewis (1918). Here the single and double arrow,  $\rightarrow$  and  $\Rightarrow$ , are being used as substitutes for older symbols, the horseshoe and fishhook,  $\supset$  and  $\prec$  used there.)

In classical logic, logical modalities are not notions for which there are symbols used in formulas, but jargon used in technical English in speaking *about* formulas. They belong not to the 'object language' but to the 'metalanguage'. Lewis differed from classical logicians by wanting to move into the object language something classical logicians wanted to leave in the metalanguage. That was the only real difference, but Lewis read  $A \rightarrow B$  as '*A materially implies B*' and  $A \Rightarrow B$  as '*A strictly implies B*', and thus created an appearance of a conflict between two theories about a single topic.

There are genuine dissenters who hold  $A \rightarrow B$  that as understood by classical logicians does not adequately represent the conditional in ordinary language outside mathematics, and who have developed non-classical *conditional* logics important enough to have their own chapter in this book. There are also genuine dissenters who object to the classical identification of *A implying B* with its being logically necessary that  $A \rightarrow B$ , since this counts in the degenerate cases where premise *A* is logically impossible and/or conclusion *B* logically necessary. They have developed so-called *relevance* or *relevant* logics. But Lewis was neither kind of dissenter. He seems never

to have evaluated or even considered the reading of  $\rightarrow$  as 'if' rather than 'implies', and he strongly defended counting in the degenerate cases of implication. Each principle that classical logic would formulate metalinguistically has an object-language counterpart in modal logic, and Lewis accepted all of them, including the counterparts of the principles admitting the degenerate cases,  $\Box \sim A \rightarrow \Box(A \rightarrow B)$  and  $\Box B \rightarrow \Box(A \rightarrow B)$ .

Since formulas involving  $\Box$  or  $\Diamond$  or  $\Rightarrow$  do not occur in the classical object language, claims about logical necessity or impossibility or implication for such formulas do not occur in the classical metalanguage. Hence the object-language counterparts of classical principles do not include formulas with *nested* modalities, boxes or diamonds or double arrows inside boxes or diamonds or double arrows. Lewis differed from classical logicians in being willing to consider such formulas, and in claiming to have intuitions about, for instance,  $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ . Lewis sought to codify his intuitions in an axiomatic system. But his intuitions (about this example among others) wavered, and the intuitions of his followers did not always agree. Soon there were five axiomatic systems, numbered **S1** through **S5**, and later works in the Lewis tradition such as Zeman (1973) list dozens.

An extensive mathematical theory developed, of which there will be space here to present only the rudiments. The proliferation of axiomatic systems ran ahead of their intuitive interpretation — not always a bad thing, since, just as some of the many geometries that proliferated after the discovery of hyperbolic geometry turned out to have applications having nothing to do with the original conception of geometry as a theory of the space around us, so some of the technical side of modal logic has turned out to be useful in unexpected ways, beginning with the modal interpretation of intuitionistic logic in Gödel (1933).

### *Propositional modal logic: proofs*

The formulas of the language of propositional modal logic will officially comprise *atoms*  $p_0, p_1, p_2, \dots$  or  $p, q, r, \dots$  and formulas build up from these using  $\sim$  and  $\wedge$  and  $\Box$  and parentheses for punctuation. We think of  $\vee$  and  $\rightarrow$  and  $\Diamond$  and  $\Rightarrow$  as unofficial abbreviations. This language is adequate to formalize such an example as

- (1) If I could have been in Tibet, but could not have been in Tibet without having a special visa, then I could have had a special visa.

in the sense of representing its logical form by a formula, thus:

$$(2) \quad \Diamond p \wedge \sim \Diamond(p \wedge \sim q) \rightarrow \Diamond q$$

Here  $p$  and  $q$  stand for 'I am in Tibet' and 'I have a special visa'. The diamond, pronounced 'possibly', really represents the transformation of the verb from the indicative to a nonindicative mood.

In an axiomatic system, certain formulas are adopted as *axioms*, and certain forms of transition from premises to conclusion as primitive *rules of inference*. A *demonstration* or *proof* is a sequence of formulas, called *steps*, each either one of the axioms or following from earlier steps by one of the rules. A formula is *demonstrable* or a *theorem* if it is the last step of some demonstration.

Other proof-related notions are defined in terms of demonstrability or theoremhood just as with classical logic. A set  $\Gamma$  of formulas is *inconsistent* if for some  $A_1, \dots, A_n$  in  $\Gamma$  if the following is a theorem:

$$(3) \quad \sim(A_1 \wedge \dots \wedge A_n)$$

Formula  $B$  is *deducible* from  $\Gamma$  if some  $A_1, \dots, A_n$  in  $\Gamma$  if the following is a theorem:

$$(4a) \quad A_1 \wedge \dots \wedge A_n \rightarrow B$$

$$(4b) \quad \sim(A_1 \wedge \dots \wedge A_n \wedge \sim B)$$

(Here (4a) abbreviates (4b).) Inconsistency of and deducibility from a formula  $A$  are inconsistency of or deducibility from the set  $\{A\}$ .

In any axiom system, the result of making any substitution of formulas for atoms in a theorem is a theorem. In one style of system, a rule to this effect is among the primitive rules of inference. In another style, instead of the axioms being specific formulas, such as perhaps (2), what are informally called 'axioms' are properly speaking axiom *schemes*, rules to the effect that all formulas of a specified form, such as perhaps

$$(2') \quad \diamond A \wedge \sim \diamond(A \wedge \sim B) \rightarrow \diamond B$$

count as theorems. With this style, a substitution in an axiom is still an axiom, so a substitution in all the steps of a demonstration is still a demonstration, and a substitution in a theorem is still a theorem — without our having to adopt a special rule to this effect. When this style is adopted, as here, few specific formulas are seen, since what are informally called 'theorems' are properly speaking theorem *schemes*, or results to the effect that every formula of a specified form is a theorem.

By a *tautology* we mean any substitution in theorem of classical logic. We say  $B$  follows *tautologically* from  $A_1, \dots, A_n$  if (4) above is a tautology. Every modal system we consider will have every tautology as an axiom, and the rule allowing inference from any premises to any conclusion that follows

tautologically. Every such system will have as one of its axioms the following:

$$(5) \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

Every such system will have exactly one modal rule, *necessitation*, permitting inference from  $A$  to  $\Box A$ . (Necessitation expresses, not the absurd assumption that every truth is necessary, but the reasonable one that every logically demonstrable truth is.) Both axiom and rule seem intuitively correct for any of the notions of necessity considered in the preceding section, though the reader may wish to stop to think this through.

The style of axiomatization used here, replacing clumsy axiomatizations of Lewis, originated with Gödel (1933) and was developed and popularized in notes belatedly published as Lemmon et al. (1977). Alternate proof-procedures (sequent calculi or tableaux rather than axiomatic systems) are also available, as for classical logic; see Zeman (1973).

The axioms and rules we have so far are those of the system called **K** or *minimal* modal logic. A basic result is the following (wherein a *modality* is a sequence of boxes and diamonds):

(6) *Becker's rule*:

If  $A \rightarrow B$  is a theorem, then  $\Delta A \rightarrow \Delta B$  is a theorem,  
for any modality  $\Delta$ .

*Proof.* First, consider the case  $\Delta = \Box$ . Suppose we have a demonstration of  $A \rightarrow B$ . We will still have a demonstration if we add  $\Box(A \rightarrow B)$  at the end, since it follows by necessitation. Likewise if we then

add (5), since it is an axiom. Likewise if we finally add  $\Box A \rightarrow \Box B$ , since it follows tautologically. We can abbreviate the foregoing argument thus:

- |       |   |                   |
|-------|---|-------------------|
| (i)   | $A \rightarrow B$   | given             |
| (ii)  | $\Box(A \rightarrow B)$   | Nec, (i)          |
| (iii) | $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ | Ax                |
| (iv)  | $\Box A \rightarrow \Box B$                                     | Taut, (ii), (iii) |

Second, consider the case  $\Delta = \Diamond = \sim\Box\sim$ . We have:

- |       |   |                             |
|-------|---|-----------------------------|
| (i)   | $A \rightarrow B$                           | given                       |
| (ii)  | $\sim B \rightarrow \sim A$                 | Taut, (i)                   |
| (iii) | $\Box\sim B \rightarrow \Box\sim A$         | Case $\Delta = \Box$ , (ii) |
| (iv)  | $\sim\Box\sim A \rightarrow \sim\Box\sim B$ | Taut, (iii)                 |
| (v)   | $\Diamond A \rightarrow \Diamond B$         | Abbrev, (iv)                |

Finally, the general case is obtained by repeated application of the box and diamond cases.

Another basic result:

(7) *Replacement rule:*

If  $A \rightarrow B$  and  $B \rightarrow A$  are theorems,  $C(p)$  any formula, and  $C(A)$  and  $C(B)$  the results of substituting  $A$  and  $B$  for  $p$  in it, then  $C(A) \rightarrow C(B)$  and  $C(B) \rightarrow C(A)$  are theorems.

*Proof.* Since every formula is built up from atoms using  $\sim$  and  $\wedge$  and  $\Box$ , it will be enough to prove that: (i) replacement holds for  $C$  an atom; (ii) if it holds for  $C$ , it holds for  $\sim C$ ; (iii) if it holds for  $C$  and  $C'$ , it holds for  $C \wedge C'$ ; (iv) if it holds for  $C$ , it holds for  $\Box C$ . This method of proof is called *induction on complexity*. Ad (i), if the atom is  $p$ ,  $C(A)$  is  $A$ ,  $C(B)$  is  $B$ , and

$C(A) \rightarrow C(B)$  is  $A \rightarrow B$ , a theorem by hypothesis; if the atom is  $q \neq p$ ,  $C(A) \rightarrow C(B)$  is  $q \rightarrow q$ , a tautology, hence a theorem; similarly for the converse. *Ad* (ii), if  $C(A) \rightarrow C(B)$  and  $C(B) \rightarrow C(A)$  are theorems, so are  $\sim C(B) \rightarrow \sim C(A)$  and  $\sim C(A) \rightarrow \sim C(B)$ , which follow tautologically. *Ad* (iii), it resembles (ii) and left to the reader. *Ad* (iv), if  $C(A) \rightarrow C(B)$  is a theorem,  $\Box C(A) \rightarrow \Box C(B)$  is a theorem by Becker; similarly for the converse.

It follows that, under the hypotheses of the theorem, if  $C(A)$  is a theorem, so is  $C(B)$  (which follows tautologically given the theorem  $C(A) \rightarrow C(B)$ ). Since  $A \rightarrow \sim\sim A$  and  $\sim\sim A \rightarrow A$  are tautologies, hence theorems, replacement implies that we can in any theorem switch  $A$  for  $\sim\sim A$  or vice versa — other words, put in or take out a double negation. In particular, can switch  $\sim\Box$  and  $\Diamond\sim (= \sim\Box\sim\sim)$  or  $\Box\sim$  and  $\sim\Diamond (= \sim\sim\Box\sim)$ .

Now five theorems:

- |       |  |                   |
|-------|--|-------------------|
| (i)   | $A \rightarrow (B \rightarrow A \wedge B)$   | Taut              |
| (ii)  | $\Box A \rightarrow \Box(B \rightarrow A \wedge B)$                                | Beck, (i)         |
| (iii) | $\Box(B \rightarrow A \wedge B) \rightarrow (\Box B \rightarrow \Box(A \wedge B))$ | Ax                |
| (8)   | $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$                              | Taut, (ii), (iii) |
| (i)   | $(A \wedge B) \rightarrow A$   | Taut              |
| (ii)  | $(A \wedge B) \rightarrow B$   | Taut              |
| (iii) | $\Box(A \wedge B) \rightarrow \Box A$  | Beck, (i)         |
| (iv)  | $\Box(A \wedge B) \rightarrow \Box B$  | Beck, (ii)        |
| (9)   | $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$                              | Taut, (iii), (iv) |
| (i)   | $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$                        | Taut              |
| (ii)  | $\Box(A \rightarrow B) \rightarrow \Box(\sim B \rightarrow \sim A)$                | Beck, (i)         |
| (iii) | $\Box(\sim B \rightarrow \sim A) \rightarrow (\Box\sim B \rightarrow \Box\sim A)$  | Ax                |

- (iv)  $\Box(A \rightarrow B) \rightarrow (\sim\Box\sim A \rightarrow \sim\Box\sim B)$  Taut, (ii), (iii)
- (10)  $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$  Abbrev, (iv)
- (i)  $A \rightarrow (B \rightarrow A \wedge B)$  Taut
- (ii)  $\Box A \rightarrow \Box(B \rightarrow A \wedge B)$  Beck, (i)
- (iii)  $\Box(B \rightarrow A \wedge B) \rightarrow (\Diamond B \rightarrow \Diamond(A \wedge B))$  (10)
- (11)  $(\Box A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$
- (i)  $(\Box A_1 \wedge \Box A_2) \rightarrow \Box(A_1 \wedge A_2)$  (8)
- (ii)  $\Box(A_1 \wedge A_2) \wedge \Diamond B \rightarrow \Diamond(A_1 \wedge A_2 \wedge B)$  (11)
- (12)  $(\Box A_1 \wedge \Box A_2 \wedge \Diamond B) \rightarrow \Diamond(A_1 \wedge A_2 \wedge B)$  Taut, (i), (ii)

This last generalizes:

$$(12') \quad (\Box A_1 \wedge \dots \wedge \Box A_n \wedge \Diamond B) \rightarrow \Diamond(A_1 \wedge \dots \wedge A_n \wedge B)$$

### *Propositional modal logic: models*

In an attempt to develop a notion of model to go with our notion of proof, we start from the idea that the technical notion of validity of a formula, or truth in all models, is intended to analyze the intuitive idea of truth by virtue of form alone, or truth in all instances, truth no matter what specific sentences are put in for the atoms in a formula.

In classical logic, for the truth of an instance of a formula what matters about the sentences put in for atoms is not their meaning, but only their truth value. So for a model we may simply take a *valuation* or assignment of truth values to atoms. The valuation is then extended to other formulas by the usual rules, which symbolizing 'A is true in model V' as ' $V \models A$ ', and abbreviating 'if and only if' to 'iff', read as follows:

$$(13a) \quad V \models \sim A \quad \text{iff} \quad \text{not } V \models A$$

$$(13b) \quad V \models A \wedge B \quad \text{iff} \quad V \models A \text{ and } V \models B$$

With just two atoms  $p$  and  $q$ , though there are infinitely many pairs of sentences that might be put in for them, there are only four combinations of truth values, and so only four models. In each, one of the four combinations

$$(14) \quad A_1 = p \wedge q, A_2 = p \wedge \sim q, A_3 = \sim p \wedge q, A_4 = \sim p \wedge \sim q$$

is true, the rest false. For modal logic we cannot do anything so simple-minded as add the clause

$$(13c) \quad V \models \Box A \quad \text{iff} \quad \text{necessarily, } V \models A$$

since our models are mathematical, and what is true in them is presumably necessary, so (13c) would make the truth of  $\Box A$  equivalent to that of  $A$ . We will need a more complicated notion of model.

Consider the following examples:

$$(15a) \quad \sim A_1 \wedge \Diamond A_1$$

$$(15b) \quad \sim A_1 \wedge \sim \Diamond A_1 \wedge \Diamond \Diamond A_1$$

$$(15c) \quad \Diamond(A_1 \wedge \Diamond A_2) \wedge \Diamond A_3 \wedge \sim \Diamond(A_3 \wedge \Diamond A_2)$$

$$(15d) \quad \Diamond(A_1 \wedge \Diamond A_2) \wedge \Diamond(A_1 \wedge \sim \Diamond A_2)$$

(15a) suggests that if truth in all instances is to agree with truth in all models, a model must represent not only which combination of atoms is actually true in a given instance, but also which combinations are possible.

(15b) suggests that a model indeed must represent not only actual possibilities but possible possibilities. (15c) suggests that the model must represent not only possibilities of various orders, but also which possible

possibilities are possible relative to which actual possibilities (as there is a possible possibility that  $A_2$  possible relative to the actual possibility that  $A_1$ , but not relative to the actual possibility that  $A_3$ ). (15d) suggests that the model must allow distinct possibilities at which the same combination of atoms is true (as there is one actual possibility that  $A_1$  with and another without a possible possibility that  $A_2$  possible relative to it), so possibilities must be not just valuations but objects having valuations associated with them.

All this suggests a model consisting of a set  $U$  of elements representing possibilities of all orders, a relation  $\prec$  representing relative possibility, plus a function  $V$  associating with each element  $u$  in  $U$  a valuation of atoms, telling us for each  $p_i$  whether it is true under possibility  $u$ , or as is said, true 'at'  $u$ . A little thought shows that we may define what it is for a formula  $A$  other than an atom to be true at an element  $u$  in a model  $M$ , symbolized  $M \models A[u]$ , as follows:

$$(16a) \quad M \models \sim A[u] \quad \text{iff} \quad \text{not } M \models A[u]$$

$$(16b) \quad M \models (A \wedge B)[u] \quad \text{iff} \quad M \models A[u] \text{ and } M \models B[u]$$

$$(16c) \quad M \models \Box A[u] \quad \text{iff} \quad \text{for all } v \text{ with } u \prec v, M \models A[v]$$

The definitions of  $\vee$  and  $\rightarrow$  and  $\Diamond$  and  $\Rightarrow$  in terms of  $\sim$  and  $\wedge$  and  $\Box$  then give

$$(16d) \quad M \models (A \vee B)[u] \quad \text{iff} \quad M \models A[u] \text{ or } M \models B[u]$$

$$(16e) \quad M \models (A \rightarrow B)[u] \quad \text{iff} \quad \text{if } M \models A[u] \text{ then } M \models B[u]$$

$$(16f) \quad M \models \Diamond A[u] \quad \text{iff} \quad \text{for some } v \text{ with } u \prec v, M \models A[v]$$

$$(16g) \quad M \models (A \Rightarrow B)[u] \quad \text{iff} \quad \text{for all } v \text{ with } u \prec v, \\ \text{if } M \models A[v] \text{ then } M \models B[v]$$

We could require a model to have one element distinguished as representing actuality, define truth in the model as a whole to be truth at that element, and validity as truth in all models. Instead we just define validity as truth at all elements in all models.

Other proof-related notions are defined in terms of models much as with classical logic. A set  $\Gamma$  of formulas is *satisfiable* if all its members are true at some element in some model. A formula  $B$  is a *consequence* of  $\Gamma$  if  $B$  is true at any element in any model where all members of  $\Gamma$  are true, Satisfiability of consequence for a formula  $A$  are satisfiability and consequence for the set  $\{A\}$ .

Does this notion of model fit our earlier notion of proof? It is not hard to establish the following:

(17) *Soundness theorem:*

Every theorem is valid.

*Proof.* Let  $M = (U, \prec, V)$  be any model. It is enough to show that every axiom has the property of being true in  $M$  at every  $u$  in  $U$ , and that each rule preserves this property. For then it follows that in every demonstration each step has the property, including the last. The tautology axiom and tautological-following rule will be left to the reader. For axiom (5), unpacking the definitions (16) we see that

(18)  $M \models (\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))[u]$

amounts to

(19) if for every  $v$  with  $u \prec v$ , if  $M \models A[v]$ , then  $M \models B[v]$ ,  
 then if for every  $v$  with  $u \prec v$  we have  $M \models A[v]$ ,  
 then for every  $v$  with  $u \prec v$ , we have  $M \models B[v]$

which is clear. For the necessitation rule, unpacking the definitions we see that what we need to show is

(20) If for every  $u$  we have  $M \models A[u]$ ,  
 then for every  $u$  and every  $v$  with  $u \prec v$  we have  $M \models A[v]$

which is equally clear (since  $v$  is as much an element of  $M$  as  $u$  is, so to speak).

Soundness can be used to show that formulas are *not* theorems. For  $\Box p \rightarrow \Diamond p$ , consider a model with a single element  $u$ , not having to  $u \prec u$ . Since there is no  $v$  with  $u \prec v$ , the condition that  $p$  is true at all such  $v$  holds vacuously, and  $\Box p$  is true at  $u$ , while the condition that  $p$  is true at some such  $v$  fails trivially, and  $\Diamond p$  is not true at  $u$ . Hence  $\Box p \rightarrow \Diamond p$  is not true at  $u$ , and so by soundness not a theorem.

It is harder to establish the following:

(21) *Completeness theorem:*

Every valid formula is a theorem.

This is equivalent to saying that every consistent formula is satisfiable. Given a consistent formula, rather than going off immediately to search for a model, let us consider what a model would look like if we had one. Our formula would be an element of the set  $\Gamma$  of sentences true at some element  $u$  of the model. What does such a set look like?

For one thing, such a set  $\Gamma$  must be consistent. For if (3) is a theorem, it will be true at  $u$  by soundness, and so by (16ab) not all the  $A_i$  will be true at  $u$ , or elements of  $\Gamma$ . For another thing,  $\Gamma$  must be a *maximal* consistent set in the sense that adding any formula not in  $\Gamma$  will produce inconsistency. For if  $A$  is not in  $\Gamma$ , not true at  $u$ , then  $\sim A$  is true at  $u$ , and in  $\Gamma$ , and adding  $A$  to  $\Gamma$  would produce a set that contains both  $A$  and  $\sim A$ , hence is inconsistent (since  $\sim(A \wedge \sim A)$  is a tautology, hence a theorem).

Further,  $\Gamma$  will be *deductively closed*, in the sense that any formula deducible from  $\Gamma$  will be in  $\Gamma$ . But this is not really a *further* property: it follows from maximal consistency. For if  $A$  is deducible from  $\Gamma$ , there are  $B_i$  in  $\Gamma$  such that

$$(22) \quad \sim(B_1 \wedge \dots \wedge B_m \wedge \sim A)$$

is a theorem. And if  $\Gamma$  is a maximal consistent set and  $A$  not in  $\Gamma$ , then adding  $A$  to  $\Gamma$  produces inconsistency, and there are  $C_j$  in  $\Gamma$  such that

$$(23) \quad \sim(C_1 \wedge \dots \wedge C_n \wedge A)$$

is a theorem. But then

$$(24) \quad \sim(B_1 \wedge \dots \wedge B_m \wedge C_1 \wedge \dots \wedge C_n)$$

is a theorem since it follows tautologically from (22) and (23), and since (24) is the negation of a conjunction of formulas in  $\Gamma$ ,  $\Gamma$  is inconsistent, contrary to hypothesis.

Further, we will have the following:

$$(25a) \quad \sim A \text{ is in } \Gamma \quad \text{iff } A \text{ is not in } \Gamma$$

$$(25b) \quad A \wedge B \text{ is in } \Gamma \quad \text{iff } A \text{ is in } \Gamma \text{ and } B \text{ is in } \Gamma$$

These also follow from maximal consistency. The 'only if' direction of (25a) we have already seen to follow from consistency. For the 'if' direction, if neither  $A$  nor  $\sim A$  is in a maximal consistent set  $\Gamma$ , adding either will produce inconsistency, and hence there will be  $B_i$  and  $C_j$  in  $\Gamma$  such that (22) and (23) are theorems, which we have already seen to be contrary to hypothesis.

(25b) follows easily from deductive closure, and is left to the reader.

It follows from all this that if a formula  $A$  is to be satisfiable, it must belong to a set  $\Gamma$  that has the property of maximal consistency and the other properties we have just seen to follow from that. But it turns out that any consistent  $A$  *does* belong to a maximal consistent, applying to  $\{A\}$  the following lemma (wherein 'can be extended to' just means 'is a subset of'):

(26) *Lindenbaum's lemma:*

Any consistent set can be extended to a maximal consistent set.

*Proof.* We first show

(27) The formulas of our language can be enumerated  $A_1, A_2, A_3, \dots$

For each formula is a finite sequence of symbols, each having an ASCII number of no more than four digits. Add zeros at the front to make it exactly four if it is less. Any  $n$ -symbol formula can be given a code number of  $4n + 1$  digits, with numeral consisting of a one followed by four-digit blocks representing its symbols. Formulas can then be listed in order of increasing code number.

Having (27), starting with a given consistent set  $\Gamma_0$ , go through the  $A_i$  in order, adding each when one comes to it iff this can be done without producing inconsistency, producing in the end a set  $\Gamma$ . Any finitely many

formulas in  $\Gamma$  will have gotten in by some stage along the way, and since we maintained consistency at each stage, the negation of their conjunction will not be a theorem, and  $\Gamma$  will be consistent. But if  $A_i$  is not in  $\Gamma$ , it is because to add it to what we had at that stage when its turn came would have produced inconsistency. Hence adding it to  $\Delta$  would certainly produce inconsistency, showing that  $\Delta$  is a maximal consistent set.

We have seen some properties a set  $\Gamma$  must have if it is to be the set of formulas true at some  $u$  in some model. What properties must a pair of sets  $\Gamma$  and  $\Delta$  have if they are to be the sets of formulas true at some  $u$  and  $v$  in some model, where  $u \prec v$ ? From (16c) it must be that for any formula  $A$ , if  $\Box A$  is in  $\Gamma$ , then  $A$  is in  $\Delta$ . When this condition holds, let us say  $\Delta$  is *potential* relative to  $\Gamma$ , and write  $\Gamma \triangleleft \Delta$ . With this notation, (16c) gives us further property a set  $\Gamma$  must have if it is to be the set of formulas true at some element in some model. The following lemma says that this property, too, follows from maximal consistency:

(28) *Main lemma:*

If  $\Gamma$  is a maximal consistent set, then for any formula  $B$ , if  $\sim\Box B$  is in  $\Gamma$ , then there is a maximal consistent  $\Delta$  such that  $\sim B$  is in  $\Delta$  and  $\Gamma \triangleleft \Delta$ .

*Proof.* It is enough to show that if  $\sim\Box B$  is in  $\Gamma$ , or (what is equivalent by replacement and deductive closure) if  $\Diamond\sim B$  is in  $\Gamma$ , then the set  $\Delta_0$  consisting of (i) all  $A$  such that  $\Box A$  is in  $\Gamma$ , plus (ii)  $\sim B$  is consistent. For then Lindenbaum's lemma will imply that it can be extended to a maximal consistent  $\Delta$ , and (i) will guarantee that  $\Gamma \triangleleft \Delta$  while (ii) will guarantee that  $\sim B$  is in  $\Delta$ . But if  $\Delta_0$  were inconsistent, there would be  $\Box A_i$  in  $\Gamma$  such that (4) is a theorem. But then the following would be theorems as well:

$$(29a) \quad \Box \sim (A_1 \wedge \dots \wedge A_n \wedge \sim B)$$

$$(29b) \quad \sim \Diamond (A_1 \wedge \dots \wedge A_n \wedge \sim B)$$

using necessitation to get (29a) and then replacement to get (29b). But since the  $\Box A_i$  and  $\Diamond \sim B$  are in  $\Gamma$ , and (12') is a theorem,

$$(29b') \quad \Diamond (A_1 \wedge \dots \wedge A_n \wedge \sim B)$$

is deducible from  $\Gamma$ , hence by deductive closure in  $\Gamma$ , and so by consistency (29b) cannot be in  $\Gamma$  after all.

We now have everything we need to put together the *canonical model*  $M = (U, \triangleleft, V)$ , which consists of the set  $U$  of all maximal consistent sets, the relation of relative potentiality  $\triangleleft$ , and the valuation  $V$  that makes  $p_i$  true at  $u$  (as an element of the model) iff  $p_i$  belongs to  $u$  (as a set of formulas). For this model we have, for all elements  $u$  and formulas  $A$ , the following:

$$(30) \quad M \models A[u] \text{ iff } A \text{ is in } u.$$

*Proof.* (30) holds by definition for atoms  $p_i$ . To prove by induction on complexity that it holds for all formulas, we must prove that: (i) if (30) holds for  $A$ , it holds for  $\sim A$ ; (ii) if (30) holds for  $A$  and for  $B$ , it holds for  $A \wedge B$ ; (iii) if (30) holds for  $A$ , it holds for  $\Box A$ . As for (i), by (16a),  $\sim A$  is true at  $u$  iff  $A$  is not true at  $u$ , which supposing (30) holds for  $A$  means iff  $A$  is not in  $u$ , which by (26a) means iff  $\sim A$  is in  $u$ . As for (ii), it is similar, using (16b) and (26b). As for (iii), in one direction, if  $\Box A$  is in  $u$ , then for any  $v$  with  $u \triangleleft v$ ,  $A$  is in  $v$  by definition of  $\triangleleft$ , which supposing (30) holds for  $A$  means  $A$  is true at  $v$ ; hence  $\Box A$  is true at  $u$ . In the other direction, if  $\Box A$  is not in  $u$ , then  $\sim \Box A$  is in  $u$  by (26a), so by the main lemma there is a  $v$  with  $u \triangleleft v$  and

$\sim A$  in  $v$ , so  $A$  is not in  $v$  by (26a), which supposing (30) holds for  $A$  means  $A$  is not true at  $v$ ; hence  $\Box A$  is not true at  $u$ .

Thus any consistent formula is true in the canonical model at any maximal consistent set to which it belongs, completing the proof of completeness. This theorem and the notion of model it involves are due to Saul Kripke, though the method of proof used here is that popularized in Lemmon et al. (1977). By soundness and completeness, our notions of proof and model for **K** agree with each other. The methods used are flexible, and can be applied to other systems. There will be space here to consider just three systems, with successively larger sets of theorems, and successively smaller classes of models.

Consider these axioms:

$$(33) \quad \Box A \rightarrow A$$

$$(34) \quad \Box A \rightarrow \Box \Box A$$

$$(35) \quad A \rightarrow \Box \Diamond A$$

The system **T** (respectively **S4**) (respectively **S5**) is obtained by adding to **K** the axiom (33) (respectively (33) and (34)) (respectively (33) and (34) and (35)). Consider these conditions (to hold for all elements in a model):

$$(36) \quad \textit{Reflexivity}: \quad u \prec u$$

$$(37) \quad \textit{Transitivity}: \quad \text{if } u \prec v \text{ and } v \prec w, \text{ then } u \prec w$$

$$(38) \quad \textit{Symmetry}: \quad \text{if } u \prec v, \text{ then } v \prec u$$

Axioms and conditions match, thus:

(39) *Correspondence theorem:*

(39a) A formula is a theorem of **T** iff

it is true at all elements of all reflexive models.

(39b) A formula is a theorem of **S4** iff

it is true at all elements of all reflexive, transitive models.

(39c) A formula is a theorem of **S5** iff

it is true at all elements of all reflexive, transitive, symmetric models.

*Proof.* For soundness it suffices to show that if (36) holds for a model, then (33) is true at every element of it, and similarly for (37) and (34) and for (38) and (35). Well,  $M \models (33)[u]$  and  $M \models (34)[u]$  and  $M \models (35)[u]$  respectively amount to

(40) if for every  $v$  with  $u \prec v$ , if  $M \models A[v]$ , then  $M \models A[u]$

(41) if for every  $v$  with  $u \prec v$ , if  $M \models A[v]$ , then

for every  $w$  with  $u \prec w$  and  $w'$  with  $w \prec w'$ ,  $M \models A[w']$

(42) if  $M \models A[u]$ , then for every  $v$  with  $u \prec v$

there is a  $w$  with  $v \prec w$  such that  $M \models A[w]$

(40) follows from (36) because reflexivity guarantees that  $u$  itself is among the  $v$  with  $u \prec v$ . (41) follows from (37) because transitivity guarantees that  $w'$ , too, is among the  $v$  with  $u \prec v$ . (42) follows from (38) because symmetry guarantees that  $u$  itself is among the  $w$  with  $v \prec w$ .

For completeness, it suffices to show (i) that if (33) is a theorem,  $\prec$  is reflexive, and (ii) that if (34) is a theorem,  $\prec$  is transitive, and (iii) that if (35) is a theorem,  $\prec$  is symmetric. So let  $\Gamma$  and  $\Delta$  and  $\Theta$  be maximally consistent. Given (33), for any  $\Box A$  in  $\Gamma$ ,  $A$  is in  $\Gamma$  by deductive closure, so  $\Gamma \prec \Gamma$ . Given (34), if  $\Gamma \prec \Delta$  and  $\Delta \prec \Theta$ , then for any  $\Box A$  in  $\Gamma$ ,  $\Box \Box A$  is in  $\Gamma$

by deductive closure, so  $\Box A$  is in  $\Delta$  and  $A$  is in  $\Theta$ , and  $\Gamma \triangleleft \Theta$ . Given (35), if  $\Gamma \triangleleft \Delta$ , then for any  $\sim A$  in  $\Gamma$ ,  $\Box \Diamond \sim A$  is in  $\Gamma$  by deductive closure and  $\Diamond \sim A$  and equivalent (by replacement and deductive closure)  $\sim \Box A$  is in  $\Delta$ , and (by (26a) above)  $\Box A$  is *not* in  $\Delta$ ; so  $\Box A$  is in  $\Delta$ ,  $\sim A$  is *not* in  $\Gamma$  and (by (26a) again)  $A$  is in  $\Gamma$ , so  $\Delta \triangleleft \Gamma$ .

Alternate axiomatizations are possible. For instance, for **S5**, it is an easy exercise to show that (35) could be replaced by

$$(35^*) \quad \Box \Diamond A \rightarrow A$$

and a not-so-easy exercise to show that (34) and (35) or (35\*) could be replaced by

$$(43) \quad \sim \Box A \rightarrow \Box \sim \Box A$$

Many more examples of correspondence are treated in standard textbooks such as Hughes & Cresswell (1996), which take up questions of decidability as well as completeness, and there is now besides this basic modal logic there is now an 'advanced' modal logic, concerned not with specific instances but with a general theory of correspondence, as in Blackburn et al. (2002).

For **S5**, the model theory simplifies. By (16abc) whether a formula is true or not at an element in a model depends only on what atoms are true at that element, or elements possible relative to it, or elements possible relative to such elements, and so on; and for reflexive and transitive models this just means what axioms are true at elements possible relative to the given element. Any other elements are irrelevant, and might as well be thrown out. If they are, then if the model is also symmetric, all elements left will be

possible relative to each other, and there will be no need to mention relative possibility at all: (16c) can be replaced by

(16c')  $M \models \Box A[u]$  iff for all  $v$ ,  $M \models A[v]$

Further, if two elements agree about the truth values of all atoms — or if we are only interested in a single formula, all atoms in it — we do not need both. When duplicates are deleted — which if we are only interested in finitely many atoms will leave only finitely many elements — there is no longer any reason to distinguish an element  $u$  from the valuation of atoms associated with it. We can take a model to be simply a finite set  $\mathbf{V}$  of valuations, and define truth as follows:

(44a)  $\mathbf{V} \models \sim A[V]$       iff not  $\mathbf{V} \models A[V]$

(44b)  $\mathbf{V} \models (A \wedge B)[V]$       iff  $\mathbf{V} \models A[V]$  and  $\mathbf{V} \models B[V]$

(44c)  $\mathbf{V} \models \Box A[V]$       iff for all  $V'$  in  $\mathbf{V}$ ,  $\mathbf{V} \models A[V']$

Validity of a formula will be truth at all elements in all such models.

### *Propositional modal logic: interpretation*

But which of the many modal logics is the right one? That depends on the sense of necessity in question. The label 'semantics' is used sometimes used for model theory, sometimes used for meaning theory, but the models we have been considering tell us little about meaning, even if we adopt the common fanciful usage that calls the elements of a model 'worlds' and the relation of the model 'accessibility'. Different senses of 'necessary' do not correspond in any obvious way to different assumptions about accessibility of worlds. Technical soundness and completeness results have no obvious direct implications for the intuitive question which axioms are appropriate

for which sense of necessity. The right modal logic for a more technical sense of necessity than any on our original list of six has been identified in the specialized branch of modal logic called *provability logic*, as in Boolos (1993). For most other senses the problem is open.

With epistemic or deontic or metaphysical or linguistic necessity, all that can be legitimately asked of the logician is to lay out the options for the epistemologist or deontologist or metaphysician or linguist. For instance, with deontic necessity the logician may point out that the candidate axiom  $\Box p \rightarrow \Diamond p$  or  $\sim(\Box p \wedge \Box \sim p)$  stands or falls with the principle that there are no conflicts of obligation. For truth or verifiability by virtue of form, however, there is no other discipline but logic involved. If there is a prevailing view among logicians, it is that of Halldén (1963), which associates **S5** with truth by virtue of form, **S4** with verifiability by virtue of form. As these opinions are about intuitive notions, not notions with mathematically rigorous definitions, they cannot have mathematically rigorous proofs. The cases for the two opinions are examined in Burgess (1999), where the case for the one about **S5**, to be presented here, is found stronger than that for the one about **S4**. The basic idea goes back to Carnap (1946).

In one direction, we must argue one by one that each axiom or rule of **S5** is intuitively correct reading the box as 'it is true by virtue of form that' (from which it will follow that all theorems are). We earlier invited the reader to think about the axioms and rules of **K**, and will now discuss only the further axioms of **S5**, which we may take to be (33) and (43). Imagine putting in specific sentences for the atoms in  $A$ , to obtain a specific sentence  $\alpha$ , and writing out logical symbols in words. (33) and (43) become:

(33') If it is true by virtue of form that  $\alpha$ , then  $\alpha$ .

(43') If it is not true by virtue of form that  $\alpha$ , then

it is true by virtue of form that it is not true by virtue of form that  $\alpha$ .

The first is obvious. For the second, suppose the antecedent holds. Then there is some  $\beta$  of the same form as  $\alpha$  such that  $\beta$  is not true. To show the consequent holds, we must show that anything of the same form as 'it is not true by virtue of form that  $\alpha$ ' is true. Well, any such thing will be 'it is not true by virtue of form that  $\gamma$ ' for some  $\gamma$  that is of the same form as  $\alpha$ , hence of the same form as  $\beta$ , and the thing will be true since  $\beta$  is not.

In the other direction, we must argue that if a given formula is not a theorem of **S5**, then some instance is intuitively incorrect when the box is read as indicated. Instances are produced by putting in specific sentences for the atoms, so we will need some assumption about what sentences are available, and we make the reasonable one that there are indefinitely many that are logically independent, with no relations holding among their truth values by virtue of logical form alone. Suppose now that  $A$  is not a theorem. Then it fails at some valuation  $V_0$  in some finite model  $\mathbf{V}$ . To produce an instance  $\alpha$  of  $A$  that is not true, we must find sentences  $\pi_i$  to put in for the atoms  $p_i$  that will have two properties: (i) the combinations of truth values that the  $\pi_i$  are not precluded by their logical form from having are precisely the combinations assigned to the corresponding  $p$  by valuations in  $\mathbf{V}$ ; and (ii) the  $\pi_i$  actually have the truth values assigned to the corresponding  $p_i$  by  $V_0$ . Towards producing such  $\pi_i$ , let  $\mathbf{V}$  contain  $n$  valuations, and let  $k$  be such that  $n \leq 2^k$ . Take  $k$  logically independent sentences  $\tau_1, \dots, \tau_k$ . There are  $2^k$  conjunctions of the form

(45)  $(\sim)\tau_1 \wedge (\sim)\tau_2 \wedge \dots \wedge (\sim)\tau_k$

where each bracketed negation may be present or absent. Independence means the truth of none is precluded by logical form, but logical form does preclude the truth of the conjunction of any two, and guarantees the truth of the disjunction of all: they are *mutually exclusive* but *jointly exhaustive*. Enumerate these conjunctions as  $\sigma_1, \sigma_2, \dots, \sigma_K$  where  $K = 2^k$ . Let  $\rho_i$  for  $i < n$  just be  $\sigma_i$ , and let  $\rho_n$  be the disjunction of the  $\sigma_j$  for  $j \geq n$ . Then the  $\rho_i$  are also mutually exclusive and jointly exhaustive, and there are exactly as many of them as of valuations in  $\mathbf{V}$ . Associate to each such valuation  $V$  a distinct  $\rho_V$ . Now for each  $p_i$  occurring in  $A$ , let  $\tau_i$  be the disjunction of the  $\rho_V$  for those  $V$  that make  $p_i$  true. Then (i) holds. As for (ii), consider the  $\rho$  corresponding to  $V_0$ . There is some assignment of truth values to the  $\tau_i$  that would make  $\rho$  true. We may suppose that the  $\tau_i$  actually have those truth values, since the only fact about them we have used is their logical independence, and that would not be affected if we replaced each that does not have the truth value we want by its negation. The actual truth of  $\rho$  guarantees (ii).

### *Quantified modal logic*

Quantifiers were introduced into modal logic in Marcus (1946, 1947) and Carnap (1946). One usual axiomatic system classical quantification theory with identity adds to classical propositional logic four axioms, thus:

- (46)  $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall xB)$        $x$  not free in  $A$
- (47)  $\forall xA \rightarrow A(y/x)$       if  $y$  free for  $x$  in  $A$
- (48)  $x = x$
- (49)  $x = y \rightarrow (A(x/z) \rightarrow A(y/z))$        $x, y$  free for  $z$  in  $A$

Also added is one rule, *universal generalization* (UG), permitting inference from  $A$  to  $\forall xA$ . Adding the axioms and rules of **K** delivers two interesting theorems:

*Converse Barcan formula:*

- |       |   |               |
|-------|---|---------------|
| (i)   | $\forall xA \rightarrow A$  | Ax            |
| (ii)  | $\Box \forall xA \rightarrow \Box A$  | Beck, i       |
| (iii) | $\forall x(\Box \forall xA \rightarrow \Box A)$   | UG, ii        |
| (iv)  | $\forall x(\Box \forall xA \rightarrow \Box A) \rightarrow (\Box \forall xA \rightarrow \forall x\Box A)$ | Ax            |
| (50)  | $\Box \forall xA \rightarrow \forall x\Box A$   | Taut, iii, iv |

*Necessity of identity:*

- |       |   |               |
|-------|---|---------------|
| (i)   | $x = x$   | Ax            |
| (ii)  | $\Box x = x$  | Nec, i        |
| (iii) | $x = y \rightarrow (\Box x = x \rightarrow \Box x = y)$ | Ax            |
| (51)  | $x = y \rightarrow \Box x = y$                          | Taut, ii, iii |

Adding the extra axioms of **S5** gives two more:

*Direct Barcan formula:*

- |        |   |               |
|--------|---|---------------|
| (i)    | $\forall x\Box A \rightarrow \Box A$  | Ax            |
| (ii)   | $\Diamond \forall x\Box A \rightarrow \Diamond \Box A$  | Beck, i       |
| (iii)  | $\Diamond \Box A \rightarrow A$   | 35*           |
| (iv)   | $\Diamond \forall x\Box A \rightarrow A$  | Taut, ii, iii |
| (v)    | $\forall x(\Diamond \forall x\Box A \rightarrow A)$   | UG, iv        |
| (vi)   | $\forall x(\Diamond \forall x\Box A \rightarrow A) \rightarrow (\Diamond \forall x\Box A \rightarrow \forall xA)$ | Ax            |
| (vii)  | $\Diamond \forall x\Box A \rightarrow \forall xA$   | Taut, v, vi   |
| (viii) | $\Box \Diamond \forall x\Box A \rightarrow \Box \forall xA$   | Beck, viii    |

- (ix)  $\forall x \Box A \rightarrow \Box \Diamond \forall x \Box A$  35  
 (52)  $\forall x \Box A \rightarrow \Box \forall x A$  Taut, viii, ix

*Necessity of distinctness:*

- (i)  $\Diamond \sim x = y \rightarrow \sim x = y$  Taut, 51  
 (ii)  $\Box \Diamond \sim x = y \rightarrow \Box \sim x = y$  Beck, i  
 (iii)  $\sim x = y \rightarrow \Box \Diamond \sim x = y$  35  
 (53)  $\sim x = y \rightarrow \Box \sim x = y$  Taut, ii, iii

Here (50) just seems wrong. (And if (51) does not seem wrong, still the value of deriving it must seem in doubt so long as the method of derivation seems in doubt.) Necessarily, everything that exists exists, but it does not follow that everything that exists necessarily exists. It appears that the logic either (i) has abandoned the leading idea of modal logic, to take seriously the distinction between 'is' and 'could have been', and is reading 'there is' as short for 'there is or could have been', or (ii) has failed to maintain metaphysical neutrality and is building in the extravagant metaphysical hypothesis that if anything that does exist hadn't existed as a concrete reality, it would still have existed as an abstract idea, or something of the sort.

Model theory tends to confirm such suspicions. As for modal propositional logic, a model for modal quantification theory consists in a set of 'possibilities' or 'worlds' and a relation of 'relative possibility' or 'accessibility', with a classical model attached to each element. At the propositional level the classical model attached is just a valuation of atoms, but at the quantificational level it consists of a universe together with an assignment to each predicate of a relation thereon. When this idea is implemented appropriately, the converse (respectively, direct) Barcan

formula is found to be valid only if it is assumed that everything in the universe attached to  $u$  is also in the universe attached to  $v$  whenever  $u \prec v$  (respectively,  $v \prec u$ ). A neutral notion of proof, sound and complete for models without such special assumptions, in which the Barcan formulas become optional extras, can be developed by replacing the above version of classical quantification theory by another in which only formulas without free variables appear in proofs; but the resulting system is not perspicuous. Generally speaking, results on quantified modal logic found in standard textbooks such as Hughes & Cresswell (1996) are less systematic than those for propositional modal logic

All this aside, the very meaningfulness of quantified modal logic was famously challenged by W. V. O. Quine. Relevant papers by and about Quine are collected in Linsky (1971), and the issue reviewed in Burgess (1998). To make sense of  $\Box\exists xPx$  one need only make sense of a sentence  $\exists xPx$  being necessarily true. But to make sense of  $\exists x\Box Px$  being true, one must make sense of an *open* sentence  $\Box Px$  being satisfied by or true of an *object*. In jargon, the former involves only *de dicto* ('of a saying') modality, while the latter involves *de re* ('of a thing') modality. The obvious way to attempt to reduce *de re* to *de dicto* would be to define  $\Box Px$  to be true of an object  $a$  iff  $\Box Pc$  is true where  $c$  is a term denoting  $a$ . The trouble is that we may have two terms  $c_1$  and  $c_2$  denoting the same object but with  $\Box Pc_1$  true and  $\Box Pc_2$  false, as in these much discussed Quinian examples:

(54a) Necessarily, eight is a perfect cube.

(54b) Necessarily, the number of planets is a perfect cube.

(55a) Necessarily, Hesperus is identical with Hesperus.

(55b) Necessarily, Hesperus is identical with Phosphorus.

In order for the reduction to work, we would need a class of *privileged* terms such that, whatever may happen with other terms, for any privileged  $c_1$  and  $c_2$ ,  $\Box Pc_1$  and  $\Box Pc_2$  would have the same truth value. Whether such a class can be identified depends on the nature of the objects and sense of necessity involved.

For linguistic necessity, numbers are among the few objects with a natural choice of canonical terms: numerals. With this choice, it is the truth of (54a), where the number is denoted by a numeral, and not the falsehood of (54b), where the number is denoted in another way, that matters when trying to decide whether 'Necessarily,  $x$  is a perfect cube' is true of the number in question. There are different systems of numerals, but intuitively it seems that

(54a') Necessarily, VIII is a perfect cube.

is just as true as (54a).

For metaphysical necessity, since Kripke (1972) it has been widely accepted proper names may be chosen as canonical terms for any sorts of object that have them. Or at least, it has been widely accepted that though (55b) is not though not *a priori* or analytic like (55a), it is metaphysically necessary. No matter what, the planet Venus, alias Hesperus, a.k.a. Phosphorus, would have been identical to itself, the planet Venus, alias Phosphorus, a.k.a. Hesperus. The leading idea behind the metaphysical necessity of identities connecting proper names is that when using a name to discuss what might have been, it denotes the same thing it denotes when discussing what is; therefore, if two names denote the same thing when discussing what is (as 'Hesperus' and 'Phosphorus' both denote the planet

Venus), they continue to do so when discussing what might have been. This property of names is called 'rigidity'.

Most descriptions do not share it, but rather are 'flexible', and become ambiguous in modal contexts. Thus

(56) If Bill Gates had given all his wealth to Ivana Trump, the richest person in the world would have been female.

is ambiguous between a true and a false reading:

(56a) If Bill Gates had given all his wealth to Ivana Trump, the person who then would have had more money than anyone else (namely, Ms Trump) would have been female.

(56b) If Bill Gates had given all his wealth to Ivana Trump, the person who now does have more money than anyone else (namely, Mr. Gates) would have been female.

This example brings out a deficiency in the formalism of quantified modal logic. Its operators apply to predicates, whereas the natural language operation of changing from indicative to nonindicative applies to verbs, and a single predicate may have more than one verb. The formalism does not have the resources to distinguish the following:

(57a) If Bill Gates had given half his wealth to Ivana Trump, she would have had more money than he had.

(57b) If Bill Gates had given half his wealth to Ivana Trump, she would have had more money than he has.

For attempts to deal with the limitations of expressive power of box-diamond modal logic by adding further operators (for 'actually' and the like), see Cresswell (1990).

The fact that in mathematics nothing could have been other than as it is, which accounts for the neglect of 'could have' by classical logic, does not prevent the application of mathematics to possibility in mathematical statistics. It is just that, to apply mathematics, we have to depart from our usual ways of speaking and thinking. We have to conceive of a 'space' whose 'points' are 'possibilities' or 'states'. Instead of saying that 'I am not in Tibet' or 'I could have been in Tibet' we can say 'In the actual state, I am not in Tibet' and 'In some possible state, I am Tibet'. In so speaking we are in effect shifting from an indicative-mood 'am' as contrasted with the non-indicative mood 'could have been' to a moodless 'am' as short for 'am or could have been'. This shift once made, (1) can be formalized in classical quantification theory, thus:

$$(58) \quad \exists tPt \wedge \sim\exists t(Pt \wedge Qt) \rightarrow \exists tQt$$

The variable  $t$  ranges over states, one of which we may suppose to be the actual state, and the rest merely possible states. The predicates  $Pt$  and  $Qt$  stand for 'in state  $t$ , I am in Tibet' and 'in state  $t$ , I have a special visa'.

Thus formalizing in classical logic what does not naturally invite such formalization is called *regimentation*. On this approach there are modal applications of classical logic, but no *autonomous* modal logic.

Mathematical statistics provides no obvious precedent for distinguishing possible possibilities from actual possibilities, but the model theory of modal logic suggests adding to the classical language, besides the variables  $t_0, t_1, t_2, \dots$  for 'possibilities' or 'states', and predicates  $P_0, P_1, P_2, \dots$  or

$P, Q, R, \dots$  corresponding to atoms  $p, q, r, \dots$ , a symbol  $<$  for relative possibility. We can then translate every formula  $A$  of the autonomous modal propositional language into a formula  $A^*(t_0)$  of the regimented classical language thus:

$$(59a) \quad p_i^* \quad = P_i t_0$$

$$(59b) \quad (\sim A)^* \quad = \sim A^*$$

$$(59c) \quad (A \wedge B)^* \quad = A^* \wedge B^*$$

$$(59d) \quad (\Box A)^* \quad = \forall t_1(t_0 < t_1 \rightarrow A^{*+})$$

where  $+$  indicates increasing the subscript on each variable by one. For instance,  $(\Diamond \Box \Diamond p)^*$  will amount to the following:

$$(60) \quad \exists t_1(t_0 < t_1 \wedge \forall t_2(t_1 < t_2 \rightarrow \exists t_3(t_2 < t_3 \wedge P t_3)))$$

$A$  will be true at every element in every modal model iff  $\forall t_0 A^*$  is true in every classical model.

Everything that can be said in the autonomous language can be said in the regimented language — and more also, since there are many classical formulas that are not  $*$ -translations of modal formulas. Even the kind of distinction seen in (57) can be expressed, by introducing, besides or instead of the predicate 'in state  $u$ ,  $x$  has more money than  $y$  has', a predicate ' $x$  has more money in state  $u$  than  $y$  has in state  $v$ '. Special problems of interpretation arise if one adopts the regimented rather than the autonomous approach, but these are beyond the scope of this chapter.

### *Books and papers of particular note*

In addition to other references cited above, mention should be made of: Lewis & Langford (1932), which gives the mature views of the founder

of modern modal logic; McKinsey (1941), whose decidability results show the highest level reached prior to Kripke (1963a, 1963b), which revolutionized the subject; and Goldblatt (2006), which gives an authoritative history of the field on its mathematical side, something lacking for the philosophical side.

TABLES

TABLE 1

Modal Notion	Definition	Symbol
it is necessary that $A$	$\Box A$	
it is impossible that $A$	$\Box \sim A$	
it is contingent whether $A$	$\sim \Box \sim A \wedge \sim \Box A$	
it is possible that $A$	$\sim \Box \sim A$	$\Diamond A$
$A$ is compossible with $B$	$\sim \Box \sim (A \wedge B)$	$A \circ B$
$A$ necessitates $B$	$\Box (A \rightarrow B)$	$A \Rightarrow B$

TABLE 2

Modal Notion	Model-Related	Proof-Related
necessity	validity	demonstrability
impossibility	unsatisfiability	inconsistency
possibility	satisfiability	consistency
compossibility	(joint) satisfiability	(joint) consistency
implication	consequence	deducibility

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