MODELS OF SET THEORY

1 Models

1.1 Syntax

Familiarity with notions and results pertaining to formal languages and formal theories is assumed. The theory of most concern will be ZFC, the language of most concern will be the language LST of ZFC (which has just the one non-logical symbol, the two-place relation-symbol $\in$).

Coding: In order to develop mathematical logic, like other branches of mathematics, within set theory, its objects, each formula must be identified with some set-theoretic object. The most obvious way to do this is to identify each symbol with some natural number–say $\sim = 1$, $\wedge = 2$, $\vee = 3$, and so on–since natural numbers have already been indentified with set-theoretic objects, and identify a string of symbols each of which has been identified with some set-theoretic object with the sequence of those set-theoretic objects. Also, in basic set theory it is shown how any pair (α,β) of natural numbers (or ordinals) can be coded by a single natural number (or ordinal) $\gamma = [\alpha,\beta]$, and similarly for sequences, so in fact each formula Φ of LST can be coded by some natural number $\#\Phi$. (This is just the so-called Gödel numbering of formulas used in intermediate level mathematical logic for the proof of the incompleteness theorem.) For a various extensions of LST having constants for various sets a, one can still identify a formula with a set theoretic object, say by letting the constant for a = (0,a) and otherwise proceeding as before, though for this extension of LST there will be no coding by natural numbers (at least not if it has constants 'a' for uncountably many sets s).

Complexity: The **bounded** quantifiers $(\forall x{\in}y)\Phi(x)$ and $(\exists x{\in}y)\Phi(x)$ are abbreviations for $\forall x(x{\in}y\rightarrow\Phi(x))$ and $\exists x(x{\in}y\wedge\Phi(x))$ respectively. (Then $\forall x{\in}{\in}z$ and $\exists x{\in}{\in}z$ can be introduced as further abbreviations for $\forall y{\in}z\forall x{\in}y$ and $\exists y{\in}z\exists x{\in}y$ respectively.) A Δ_0 formula is one built up from atomic formuals $u{\in}v$ and $u{=}v$ by sentential connectives $(\sim,\wedge,\vee)$ and bounded or limited quantifiers. They are also called Δ_0 or Σ_0 or Π_0 formulas. A Σ_{n+1} formula is one of the form $\exists u_1\exists u_2\ldots\exists u_p\Phi$ where Φ is a Π_n formula, and a Π_{n+1} formula is one of the form $\forall u_1\forall u_2\ldots\forall u_p\Phi$ where Φ is a Σ_n formula. (It is allowed to have $p{=}0$, and Σ_n and Π_n formulas count as Σ_{n+1} and Π_{n+1} formulas.)

Lemma:

(a) Every formula is logically equivalent to a Σ_n formula for some n.
Every formula is logically equivalent to a Π_n formula for some n.

(b) A conjunction or disjunction of Σ_n formulas is equivalent to a Σ_n formula.
A conjunction or disjunction of Π_n formulas is equivalent to a Π_n formula.

(c) The negation of a Σ_n formula is equivalent to a Π_n formula.
The negation of a Π_n formula is equivalent to a Σ_n formula.

Proof: Use the logical equivalences:

$$(\exists x\Phi(x)\wedge\exists y\Psi(y)) \leftrightarrow \exists x\exists y(\Phi(x)\wedge\Psi(y))$$
$$(\exists x\Phi(x)\vee\exists x\Psi(x)) \leftrightarrow \exists x(\Phi(x)\vee\Psi(x))$$
$$(\forall x\Phi(x)\wedge\forall x\Psi(x)) \leftrightarrow \forall x(\Phi(x)\wedge\Psi(x))$$
$$(\forall x\Phi(x)\vee\forall y\Psi(y)) \leftrightarrow \forall x\forall y(\Phi(x)\vee\Psi(y))$$
$$\sim\exists x\Phi(x) \leftrightarrow \forall x\sim\Phi(x)$$
$$\sim\forall x\Phi(x) \leftrightarrow \exists x\sim\Phi(x)$$

A formula reducible to a Σ_n formula by such equivalences will be called Σ_n in an extended sense.

Examples: Important notions expressible by $\Delta_0=\Sigma_0=\Pi_0$ formulas:

$\forall z\in x(z\in y)$	$x\subseteq y$
$\forall w\in x\exists z\in w(w\in z)\wedge\forall w\in\in x(w\in y)$	$y=\cup x$
$\forall w\in z(w\in x\wedge w\in y)\wedge\forall w\in x(w\in y\rightarrow w\in z)$	$y=x\cap y$
$x\in z\wedge y\in z\wedge\forall w\in z(w=x\vee w=y)$	$z=\{x,y\}$
$\exists z_0\in z\exists z_1\in z(z=\{z_0,z_1\}\wedge z_0=\{x,x\}\wedge z_1=\{x,y\})$	$z=(x,y)$
$\forall w\in z\exists x'\in x\exists y'\in y(w=(x',y'))\wedge$ $\forall x'\in x\forall y'\in y\exists w\in z(w=(x',y'))$	$z=x\otimes y$
$\forall w\in z\exists x\in\in w\exists y\in\in w(w=(x,y))\wedge$ $\forall w_0\in z\forall w_1\in z\forall x\in\in w_0\forall y_0\in\in w_0\forall y_1\in\in w_1$ $(w_0=(x,y_0)\wedge w_1=(x,y_1)\rightarrow y_0=y_1)$	z is a function
(z is a function)$\wedge\exists w\in z(w=(x,y))$	$z(x)=y$
$\forall u\in\in\in z\forall v\in\in\in z(z(u)=v\rightarrow u\in x)$	$\mathrm{dom}z\subseteq x$
$\forall u\in\in\in z\forall v\in\in\in z(z(u)=v\rightarrow v\in y)$	$\mathrm{ran}z\subseteq y$
$\forall u\in x\exists v\in\in\in z(z(u)=v)$	$x\subseteq\mathrm{dom}z$
$\forall v\in y\exists u\in\in\in z(z(u)=v)$	$y\subseteq\mathrm{ran}z$
$\forall y\in\in x(y\in x)$	x is transitive
(x is transitive)$\wedge\forall y\in x\forall z\in x(y\in z\vee y=z\vee z\in y)$	x is an ordinal
(x is an ordinal)$\wedge\exists y\in x\forall z\in x(z\in y\vee z=y)$	x is a successor
(x is an ordinal)$\wedge\sim$(x is a successor)$\wedge\exists y\in x$(y is a successor)	$x=\omega$
$\exists z\in x(z\subseteq y)$	$x\subseteq\wp(y)$

Important notions expressible by Π_1 formulas:

(x is an ordinal)$\wedge\forall f\forall y\in x$((f is a function)$\wedge\mathrm{dom}f\subseteq y\wedge z\subseteq\mathrm{ran}f)\rightarrow x\neq z)$	x is a cardinal
(x is a cardinal)$\wedge\forall z\forall f\forall y\in x$((f is a function) $\wedge\ \mathrm{dom}f\subseteq y\wedge z\subseteq\mathrm{ran}f)\rightarrow x\neq\cup z)$	x is regular
$\forall z(z\subseteq y\rightarrow z\in x)$	$\wp(y)\subseteq x$

1.2 Semantics

Familiarity with notions pertaining to models of formal languages and formal theories is assumed. The models of most concern will be of ZFC or some large part thereof. Note that *the existence of a model of the whole of ZFC implies the consistency of ZFC and so cannot be proved in ZFC by the incompleteness theorem*. The models of most concern will be **standard**: The universe of the model will be some transitive set of sets A, the interpretation of the relation-symbol $\in$ will be the elementhood relation on A (in other words, $\in_A=\{(a,b)|a\in b\in A\}$). For a standard model A, the usual definition of truth in A of a closed formula $\Phi(a_1,\dots,a_n)$ of the language LST(A) with a constants for each $a\in A$ is by recursion as follows:

$A\models a\in b$	iff	$a\in b$
$A\models a=b$	iff	$a=b$
$A\models\sim\Phi$	iff	not $A\models\Phi$
$A\models\Phi\wedge\Psi$	iff	$A\models\Phi$ and $A\models\Psi$
$A\models\Phi\vee\Psi$	iff	$A\models\Phi$ or $A\models\Psi$
$A\models\forall x\Phi(a_1,\dots,a_n,x)$	iff	$A\models\Phi(a_1,\dots,a_n,b)$ for all b in A
$A\models\exists x\Phi(a_1,\dots,a_n,x)$	iff	$A\models\Phi(a_1,\dots,a_n,b)$ for some b in A

Of the various further notions that can be defined in terms of truth the most important is the following: $a\in A$ is **parametrically definable** in A from the parameters $p_1,\dots,p_k$ if there is a formula $\Phi(u_1,\dots,u_k,v)$ of LST such that $A\models\Phi(p_1,\dots,p_m,a)$ but not $A\models\Phi(p_1,\dots,p_m,b)$ for any $b\neq a$. Most important is the cases where k=0, that of (outright, plain, simple) **definability**.

The notion of (outright, plain, simple) truth, that is, of truth-in-V where V is the universe of all sets, is impossible to define in ZFC. (Here what is meant by truth being definable in a theory is the existence of a formula in the language of the theory for which the basic properties of truth, namely the induction clauses above, can be proved in the theory.) Why this is impossible is most easily understood by considering the infamous König paradox: *Since there are only countably many definitions, only countably many sets are definable, and since there are uncountably many ordinals, there must be some that are undefinable, and among these there must be a least one. But that one is after all definable by the following definition:* x is the least ordinal that is not definable.

What can be done is to define in ZFC is the following: For each n, one can define truth_n, or truth for formulas_n, that is, for Σ_n formulas. But it turns out that truth_n is itself expressible by a formula_n but not by a formula_m for any m<n: To define truth for more and more complicated formulas it takes more and more complicated formulas.

Lemma: Let $\Phi(u_1,\ldots,u_m)$ be a formula, and let A,B be transitive sets with $A\subseteq B$, and let $a_1,\ldots,a_m\in A$. Then:

(a) If Φ is Δ_0, then $A\models\Phi(a_1,\ldots,a_m)$ if and only if $B\models\Phi(a_1,\ldots,a_m)$

(b) If Φ is Σ_1, then if $A\models\Phi(a_1,\ldots,a_m)$ then $B\models\Phi(a_1,\ldots,a_m)$

(c) If Φ is Π_1, then if $B\models\Phi(a_1,\ldots,a_m)$ then $A\models\Phi(a_1,\ldots,a_m)$

Proof: For (a) proceed by induction on the length of Φ, the only difficult case being the induction step for quantifiers. The cases of $\forall$ and $\exists$ being similar, only the latter will be treated here. So suppose $\Phi(u_1,\ldots,u_m) = \exists v\in u_i\Psi(u_1,\ldots,u_m,v)$, in other words = $\exists v(v\in u_i\wedge\Psi(u_1,\ldots,u_m,v))$; and suppose as induction hypothesis that claim (a) holds for Ψ. Then $A\models\Phi(a_1,\ldots,a_m)$ iff there exists $b\in A$ such that $b\in a_i$ and $A\models\Psi(a_1,\ldots,a_m,b)$. But since A is transitive, *any* $b\in a_i$ will have $b\in A$, so $A\models\Phi(a_1,\ldots,a_m)$ iff there exists $b\in a_i$ such $A\models\Psi(a_1,\ldots,a_m,b)$. Likewise $B\models\Phi(a_1,\ldots,a_m)$ iff there exists $b\in a_i$ such $B\models\Psi(a_1,\ldots,a_m,b)$. Then claim (a) for Φ follows immediately by the induction hypothesis that $A\models\Psi(a_1,\ldots,a_m,b)$ iff $B\models\Psi(a_1,\ldots,a_m,b)$. Parts (b) and (c) being similar, and only the former will be treated here. So let $\Phi=\exists v_1\ldots\exists v_n\Psi$ where Ψ is Δ_0. If $A\models\Phi(a_1,\ldots,a_m)$, then $A\models\Psi(a_1,\ldots,a_m,b_1,\ldots,b_n)$ for some $b_1,\ldots,b_n\in A\subseteq B$. By (a), $B\models\Psi(a_1,\ldots,a_m,b_1,\ldots,b_n)$ and hence $B\models\Phi(a_1,\ldots,a_m)$.

The definition of truth_0 is then this: $\models_0\Phi(a_1,\ldots,a_m)$ iff Φ is a Σ_0 formula and $A\models\Phi(a_1,\ldots,a_m)$ for some transitive A with $a_1,\ldots,a_m\in A$. The Lemma (a) is needed to prove the basic properties of truth_0, notably that if Φ is true_0 (Φ is true in *some* transitive A) then $\sim\Phi$ is not true_0 ($\sim\Phi$ is not true in *any* transitive A). The definition of truth_1 is then this: $\models_1\Phi(a_1,\ldots,a_m)$ iff Φ is of form $\exists v_1\ldots\exists v_n\Psi$ where Ψ is a Σ_0 formula and there exist $b_1,\ldots,b_n$ such that $\models_0\Psi(a_1,\ldots,a_m,b_1,\ldots,b_n)$. The definitions of truth_n for $n\geq 2$ are analogous. Φ is **absolute**$_n$ for a transitive set A if Φ is a Σ_n formula and for any $a_1,\ldots,a_m\in A$ one has $A\models\Phi(a_1,\ldots,a_m)$ iff $\models_n\Phi(a_1,\ldots,a_m)$. The Lemma (a) says any Σ_0 Φ is absolute_0 for any transitive A. Where no confusion can result, subscripts will be omitted.

1.3 Criteria

When is an axiom of ZFC is true in a transitive set A?

Proposition: Let $\Phi(u_1,\dots,u_m)$ be a formula, A transitive, and $a_1,\dots,a_m \in A$:

(a) If Φ is Δ_0, then $A \models \Phi(a_1,\dots,a_m)$ if and only if $\models \Phi(a_1,\dots,a_m)$

(b) If Φ is Σ_1, then if $A \models \Phi(a_1,\dots,a_m)$ then $\models \Phi(a_1,\dots,a_m)$

(c) If Φ is Π_1, then if $\models \Phi(a_1,\dots,a_m)$ then $A \models \Phi(a_1,\dots,a_m)$

Proof: Almost immediate from Lemma of §2.2 and definitions of truth.

The axioms of *extensionality* and *foundation* are essentially Π_1 formulas:

$\forall x \forall y(\forall z \in x(z \in y) \wedge \forall z \in y(z \in x) \rightarrow x=y)$ $\qquad$ $\forall x(\exists y \in x(y=y) \rightarrow \exists y \in x \sim \exists z \in x(z \in y))$

Hence they come out true or hold in *any* transitive set by Proposition (c).

The axiom of *pairing* is essentially a Π_2 formula:

$\forall x \forall y \exists z(z=\{x,y\})$

where $z=\{x,y\}$ occurs in the Examples in §2.1. Hence $A \models$pairing iff for all $a,b \in A$ there exists $c \in A$ such that $A \models c=\{a,b\}$. By Proposition (a), $A \models$pairing iff for all $a,b \in A$ there exists $c \in A$ such that $c=\{a,b\}$; in other words, iff whenever $a,b \in A$, then $\{a,b\} \in A$; in yet other words, iff A is closed under the pairing operation $\{,\}$.

Similarly for *union*, *infinity*, and *choice*:

$\forall x \exists y(y=\cup x)$ $\qquad$ $\exists x(x=\omega)$ $\qquad$ $\forall x$(x is a partition $\rightarrow \exists y$ (y is a selector for x))

(though the conditions x *is a partition* and y *is a selector for* x were not included in the Examples in §2.1). One has $A \models$union iff A is closed under $\cup$, $A \models$infinity iff $\omega \in A$, and $A \models$choice iff for every partition $a \in A$ there is a selector $b \in A$.

The axiom of *power* is essentially a Π_3 formula:

$\forall y \exists x(\forall z \in x(z \subseteq y) \wedge \forall z(z \subseteq y \rightarrow z \in x))$

$A \models$power iff for every $b \in A$ there is an $a \in A$ such that every c that is an element of a is a subset of b and every $c \in A$ that is a subset of b is an element of a. In other words, setting $\wp^A(b) = \wp(b) \cap A = \{c \in A | c \subseteq b\}$, one has $A \models$power iff A is closed under $\wp^A$. This is a necessary and sufficient condition. A more than sufficient condition is that A be closed under $\wp$.

As for *separation*, $A \models$separation iff for every formula $\Phi(u,w_1,\dots,w_m)$ and every $a \in A$ and $p_1,\dots,p_m \in A$ one has $\{b \in a | A \models \Phi(b,p_1,\dots,p_m)\} \in A$. $A \models \Phi((b,p_1,\dots,p_m)$ is often written $\Phi^A(b,p_1,\dots,p_m)$, and the parameters $p_1,\dots,p_m$ are often not explicitly written. In this somewhat abbreviated notation, $A \models$separation iff for every Φ and $a \in A$, $\{b \in a | \Phi^A(b)\} \in A$. This is a necessary and sufficient condition. A more than sufficient condition is that A if $a \in A$ and $a' \subseteq a$, then $a' \in A$. (Note that for supertransitive A, the relative and the absolute power operations coincide.)

As for *replacement*, $A\models$replacement iff for every formula $\Psi(u,v,w_1,\ldots,w_m)$ and $p_1,\ldots,p_m\in A$, *if* for every $a\in A$ there exists a unique $b=\psi^A(a)\in A$ with $A\models\Psi(a,b,p_1,\ldots,p_m)$, *then* for every $c\in A$, one has $\{\psi^A(a)|a\in c\}\in A$. This is a necessary and sufficient condition. A more than sufficient condition is that whenever $c\in A$, $\|c\| = \|d\|$, and $d\subseteq A$, then $d\in A$.

AXIOM	NECESSARY & SUFFICIENT [MORE THAN SUFFICIENT]
EXTENSIONALITY	automatic
FOUNDATION	automatic
PAIRING	A closed under {,}
UNION	A closed under $\cup$
INFINITY	$\omega\in A$
CHOICE	for every partition $a\in A$ there is a selector $b\in A$
POWER	A closed under $\wp^A$ [A closed under $\wp$]
SEPARATION	for every $a\in A$, $\{b\in a|\Phi^A(b)\}\in A$ [$a'\in A$ if $a'\subseteq a\in A$]
REPLACEMENT	for every $c\in A$, $\{\psi^A(a)|a\in c\}\in A$ [$d\in A$ if $d\subseteq A$, $\|d\|=\|c\|$ for some $c\in A$]

1.4 Reflection

By transfinite recursion define:

$V(0)=\varnothing$ $\quad V(\beta+1)=\wp(V(\beta))$ $\quad V(\alpha)=\cup\{V(\beta)|\beta<\alpha\}$ at limits

Lemma A:

(i) if $\alpha'<\alpha$, then $V(\alpha')\subseteq V(\alpha)$

(ii) $V(\alpha)$ is transitive

(iii) for every x there is an α with $x\in V(\alpha)$:
the least β such that $x\in V(\beta+1)$ is called the **rank** $\rho(x)$ of x

(iv) $V(\alpha)=\{x|\rho(x)<\alpha\}$

(v) $\rho(x)=\sup\{\rho(y)|y\in x\}+1$

(vi) if $y\in x$ then $\rho(y)<\rho(x)$

(vii) $\rho(\alpha)=\alpha$

Proof: Let (o) be the proposition $V(\alpha) = \cup\{V(\beta+1) | \beta<\alpha\}$. Note that if (o) holds for all $\alpha\leq\gamma$, then (i) holds for all $\alpha<\gamma$, since then if $\alpha'<\alpha$ one has $\cup\{V(\beta+1)|\beta<\alpha'\}\subseteq\cup\{V(\beta+1)|\beta<\alpha\}$. Note that if (o) and (i) hold for all $\alpha<\gamma$, then (ii) holds for all $\alpha<\gamma$, since then if $y\in x\in V(\alpha)$, one has $x\in V(\beta+1)=\wp(V(\beta))$ for some $\beta<\alpha$ by (o), whence for that $\beta<\alpha$ one has $y\in x\subseteq V(\beta)\subseteq V(\alpha)$ by (i). To prove (i),(ii) it thus suffices to show that if (o),(i),(ii) hold for all $\alpha<\gamma$, then (o) holds for γ. There are two cases, γ and limit and γ a successor, and both are left as exercises, as are the proofs of (iii), which uses foundation, and (vii), which uses induction. (iv) is immediate from (o) and the definition of rank. Taking (v) and (vi) in reverse order, for (vi), note that if $y\in x\in V(\alpha+1) = \wp(V(\alpha))$, then $y\in x\subseteq V(\alpha) = \cup\{V(\beta+1)|\beta<\alpha\}$ by (o), and so $y\in V(\beta+1)$ for some $\beta<\alpha$. For (v) let $\sigma = \sup\{\rho(y)+1|y\in x\}$. The inequality $\sigma\leq\rho(x)$ is immediate from (vi). For the opposite inequality, for any $y\in x$, one has $y\in V(\rho(y)+1)\subseteq V(\sigma)$ by (i) and the definition of rank, so $x\subseteq V(\sigma)$ and $x\in\wp(V(\sigma))=V(\sigma+1)$ so that $\rho(x)\leq\sigma$.

	Lemma B:	**Lemma C**: if α be a limit ordinal:
(a)	$\rho(\{a,b\})\leq\max(\rho(a),\rho(b))+1$	$V(\alpha)$ is closed under $\{,\}$
(b)	$\rho(\cup a)\leq\rho(a)$	$V(\alpha)$ is closed under $\cup$
(c)	$\rho(\omega)\leq\omega+1$	if $\alpha>\omega$, then $\omega\in V(\alpha)$
(d)	if b is a selector for the partition a, then $\rho(b)\leq\rho(a)$	for any partition $a\in V(\alpha)$ there exists a selector $b\in V(\alpha)$
(e)	$\rho(\wp(a))\leq\rho(a)+1$	$V(\alpha)$ is closed under $\wp$
(f)	if $b\subseteq a$ then $\rho(b)\leq\rho(a)$	if $a\in V(\alpha)$ and $a'\subseteq a$ then $a'\in V(\alpha)$

Proof: For B, (a) and (b) follow almost immediately from Lemma A(v); (c) from Lemma A(vii); (d) from Lemma A(v) again and the fact that if b is a selector for the partition a, then $b\subseteq\cup a$; (e),(f) from Lemma A(v) again. The clauses of C follow from the corresponding clauses of B (using Lemma A(iv)).

Theorem: Let α be a limit ordinal $>\omega$. Then $V(\alpha)$ is a model of all the axioms of ZFC other than replacement.

Proof: Immediate using the criteria of §2.3 and Lemma C.

Reflection Principle: For any n the following is provable in ZFC: For every β there exists a limit ordinal $\alpha>\beta$ such that all formulas_n are absolute_n for $V(\alpha)$.

Proof: The following intuitive argument becomes formalizable in ZFC if subscripts $_n$ are added: For any existential formula $\Phi(u_1,\ldots,u_m) = \exists v\Psi(u_1,\ldots,u_m,v)$, let $f_\Phi(a_1,\ldots,a_m)$ = the least α such that (i) $\rho(a_1),\ldots,\rho(a_m)<\alpha$ and (ii) if there exists any b at all such that $\Psi(a_1,\ldots,a_m,b)$ is true, then there exists such a b with $\rho(b)<\alpha$. Next, define $F(\beta) = \sup\{f_\Phi(a_1,\ldots,a_m)|a_1,\ldots,a_m\in V(\beta),\ \Phi$ an existential formula$\}$. Note that if $\gamma<\beta$, then (since $V(\gamma)\subseteq V(\beta)$) one has $F(\gamma)\leq F(\beta)$. Also, define $G^0(\beta)=\beta$, $G^{n+1}(\beta)=F(G^n(\beta)+1)$, $G^\omega(\beta)=\sup\{G^n(\beta)|n\in\omega\}$. Note that $G^\omega(\beta)$ is a limit ordinal $\alpha>\beta$, and that if $\gamma<\alpha$, then $F(\gamma)\leq\alpha$ (since then $\gamma<G^n(\beta)$ for some n, and then $F(\gamma)\leq F(G^n(\beta))=G^{n+1}(\beta)<\alpha$). Now it can be proved by induction on the length of the formula that any formula Φ is absolute for $V(\alpha)$. The only difficult case is the induction step for quantifiers, and the cases of $\forall$ and $\exists$ being similar, only the latter will be treated here. So suppose $\Phi(u_1,\ldots,u_m) = \exists v\Psi(u_1,\ldots,u_m,v)$ and suppose as induction hypothesis that Ψ is absolute for $V(\alpha)$. Let $a_1,\ldots,a_m\in V(\alpha)$. It suffices to show that if there exists any b at all such that $\Psi(a_1,\ldots,a_m,b)$ is true, then there exists such a $b\in V(\alpha)$. But this is so since then $a_1,\ldots,a_m\in V(\gamma)=\{x|\rho(x)<\gamma\}$ for some $\gamma<\alpha$, and if there exists any such b there exists one with $b\in\{x|\rho(x)<F(\gamma)\}=V(F(\gamma))\subseteq V(\alpha)$.

Corollary: (It is provable in ZFC that:) For any finitely many replacement axioms, there exists a model of all the axioms of ZFC other than replacement and of those finitely many replacement axioms.

Proof: Immediate from the Theorem and the reflection principle applied to an n large enough that all the replacement axioms in question are (logically equivalent to) formulas$_n$.

1.4* Transitivization

Proposition: For every x there exists a (necessarily unique) $x^\dagger$ such that (i) t is transitive, (ii) $x \in t$, and (iii) if u is transitive and $x \in u$, then $u \subseteq t$.

Proof: Using foundation, it suffices to show that if $y^\dagger$ with properties (i)-(iii) exists for every $y \in x$, then letting $x^\dagger = \cup\{y^\dagger | y \in x\} \cup \{x\}$ it has properties (i)-(iii). For (i), if $v \in u \in x^\dagger$, then either u=x in which case $v \in x$ and $v \in v^\dagger \subseteq \cup\{y^\dagger | y \in x\} \subseteq x^\dagger$, or else $u \in w^\dagger$ for some $w \in x$ in which case by (i) for $w^\dagger$ one has $v \in w^\dagger \subseteq \cup\{y^\dagger | y \in x\} \subseteq x^\dagger$. (ii) is immediate. For (iii) let u be transitive with $x \in u$ and let $v \in x^\dagger$ to show $v \in u$. Well either $v = x \in u$ or else $v \in w^\dagger$ for some $w \in x$. In the latter case, note that since $w \in x \in u$ by transitivity $w \in u$, and by (iii) for $w^\dagger$ one has $v \in w^\dagger \subseteq u$.

Corollary: $x^\dagger = \cup\{y^\dagger | y \in x\} \cup \{x\}$

The **hereditary cardinality** of x is $\sigma(x) = \|x^\dagger\|$. For κ and infinite cardinal, $H(\kappa) = \{x | \sigma(x) < \kappa\}$. $H(\aleph_0)$ and $H(\aleph_1)$ are also called HF (**hereditarily finite** sets) and HC (**hereditarily countable** sets).

	Lemma A:	**Lemma B**:	**Lemma C**: if κ be an infinite cardinal:
(a)	$(\{a,b\})^\dagger \subseteq \{\{a,b\}\} \cup a^\dagger \cup b^\dagger$	$\sigma(\{a,b\}) \leq \max(\sigma(a),\sigma(b))+1$	$H(\kappa)$ is closed under {,}
(b)	$(\cup a)^\dagger \subseteq \{\cup a\} \cup a^\dagger$	$\sigma(\cup a) \leq \sigma(a)+1$	$H(\kappa)$ is closed under $\cup$
(c)	$\alpha^\dagger \subseteq \alpha+1$	$\sigma(\omega) \leq \omega$	if κ uncountable: $\omega \in H(\kappa)$
(d)		if b selector partition a, then $\sigma(b) \leq \sigma(a)+2$	for any partition $a \in H(\kappa)$, exists selector $b \in H(\kappa)$
(e)	$(\wp(a))^\dagger \subseteq a^\dagger \cup \wp(a) \cup \{\wp(a)\}$	$\sigma(\wp(a)) \leq 2^{\sigma(a)} + \sigma(a) + 1$	if κ strong limit: $H(\kappa)$ is closed under $\wp$
(f)	if $b \subseteq a$ then $b^\dagger \subseteq \{b\} \cup a^\dagger$	if $b \subseteq a$ then $\sigma(b) \leq \sigma(a)+1$	if $a' \subseteq a \in H(\kappa)$, then $a' \in H(\kappa)$
(g)		$\sigma(a) = (\Sigma\{\sigma(b) | b \in a\}) + 1$	if κ regular: if $d \subseteq H(\kappa)$, $\|d\| = \|c\|$, $c \in H(\kappa)$, then $d \in H(\kappa)$

Proof: For A, in each case, it is easily shown that the set on the right hand side is transitive. For B, (a),(b),(c),(e),(f) follow from the corresponding clauses of Lemma A; (d) follows from (b),(f) and the fact that if b is a selector for the partition a, then $b \subseteq \cup a$; (g) follows from the Corollary. For C, each clause follows from the corresponding clause of Lemma B.

Theorem:

If κ is:	then $H(\kappa)$ is a model of all of ZFC except (perhaps):
$\aleph_0$	infinity
uncountable	power, replacement
uncountable, strong limit	replacement
uncountable, regular	power
inaccessible	[no exceptions]

Proof: Immediate from Lemma C and the criteria of §2.3.

Corollary: The existence of an inaccessible cardinal cannot be proved in ZFC (if ZFC is consistent).

2 Choice

2.1 Consistency

The method of "inner models", and a particular inner model, the "constructible universe" were introduced by Kurt Gödel to prove the consistency of CH relative to ZFC=ZF+AC and of AC relative to ZF. (Here consistency of a hypothesis Φ relative to a theory Γ means: if Γ is consistent, then $\Gamma+\Phi$ is consistent.) A simpler inner model, the "hereditarily ordinal definable universe" suffices for the latter proof, which will be outlined here.

The results in intermediate level mathematical logic are not depend on AC, and the results in a basic course in set theory that are dependant on AC are identified as so dependant and come late in the course, and no such results have been used here. The only use here of AC has been in showing that AC is true in a model $V(\alpha)$ or $H(\kappa)$: After proving *without* AC that if a partition is an element of such a model, then any selector for it is an element of that model, the assumption that every parition has a selector is used to conclude that every partition in the model has a selector in the model. *No use of AC will be made in the present chapter*.

The intuitive notion of ordinal definable (OD) is: (i) a is OD if there exist a formula $\Phi(x,y)$ of LST and an ordinal β such that $\Phi(a,\beta)$ is true and $\Phi(b,\beta)$ is not true for any $b\neq a$. The official definition of **ordinal definable** (OD) is: (ii) a is OD if there exist a formula $\Phi(x,y)$ of LST, an ordinal β, and an ordinal $\alpha>\beta$ such that $\Phi(a,\beta)$ is true in $V(\alpha)$ and $\Phi(b,\beta)$ is not true in $V(\alpha)$ for any $b\neq a$. In that case a is said to be OD *by* Φ *from* β *at* α, and to be OD *through* δ where δ is the code $[[\alpha,\beta],\#\Phi]$. Notion (i) is *not* definable in ZF, since truth is not; but notion (ii) *is* definable in ZF, since truth-in-a-model is. However, intuitively the two definitions are equivalent: If (i) holds, then (ii) holds for any $\alpha>\beta$ such that Φ is absolute for $V(\alpha)$, and such exist by the Reflection Principle. If (ii) holds, then a is OD in the sense of notion (i) from the code δ by the formula expressing that x is OD in the sense of notion (ii) through y.

Metalemma. For every n the following is a theorem of ZF: If $a_1,\ldots,a_n$ are OD and b is definable$_n$ from parameters $a_1,\ldots,a_n$, the b is OD.

Proof: Let b be definable from $a_1,\ldots,a_n$ by Ψ, and let $a_1,\ldots,a_n$ be OD through $\gamma_1,\ldots,\gamma_n$ respectively. Let Φ express that there exist $z_1,\ldots,z_n$ and $w_1,\ldots,w_n$ such that $\Psi(x,z_1,\ldots,z_n)$ and y is the code $[w_1,\ldots,w_n]$ and z_1 is OD through w_1 and … and z_n is OD through w_n. Let β be the code $[\gamma_1,\ldots,\gamma_n]$ and let $\alpha>\beta$ be such that Φ is absolute for $V(\alpha)$. Then a is OD by Φ from β at α.

Also define: a is **hereditarily ordinal definable** (HOD) if b is OD for every $b\in a^\dagger$. For any formula Φ, let Φ^{HOD} be the result of replacing each quantifier $\forall x$ or $\exists x$ in Φ by $\forall x(HOD(x)\to\ldots)$ and $\exists x(HOD(x)\wedge\ldots)$.

	Lemma A	**Lemma B**	**Lemma C**:
the following are provable in ZF for all formulas Φ,Ψ,Θ:			
(a)	if OD(a) and OD(b) then OD({a,b}	if HOD(a) and HOD(b) then HOD({a,b})	(pairing axiom)^{HOD}
(b)	if OD(a) then OD($\cup$a)	if HOD(a) then HOD($\cup$a)	(union axiom)^{HOD}
(c)	OD(ω)	HOD(ω)	(infinity axiom)^{HOD}
(d)	if OD(a) then OD($\{b\in\wp(a)\mid\Theta(b)\}$)	if HOD(a) then HOD($\{b\in\wp(a)\mid\Theta(b)\}$)	(power axiom)^{HOD}
(e)	if OD(a) then OD($\{b\in a\mid\Psi(b)\}$)	if HOD(a) then HOD($a'=\{b\in a\mid\Psi(b)\}$)	(Φ-separation axiomΦ)^{HOD}
(f)	if OD(c) then OD($\{\psi^{\Theta}(a)\mid a\in c\}$)	if HOD(c) then HOD($\{\psi^{\Theta}(a)\mid a\in c\}$)	(Φ-replacement axiomΦ)^{HOD}

Proof: A is immediate from the Metalemma. B(a) follows from B(b) and the observation that $(\{a,b\})^{\dagger}=\{\{a,b\}\}\cup a^{\dagger}\cup b^{\dagger}$; and similarly for the other clauses of B. Though the HOD sets do not form a set, still the criteria of §2.3 apply, and yield C: for C(d) take $\Theta(x)$ in B(d)to be HOD(x); for C(e) and C(f), given Φ take Ψ in B(e) and B(f) to be $\Phi^{HOD.}$

Lemma D: The following is provable in ZF (*without* AC): AC^{HOD}

Proof: If OD(a) let δ_a be the least ordinal δ such that a is OD through δ. Define $a<_{OD}b$ to mean OD(a) and OD(b) and $\delta_a<\delta_b$. Let a be a partition and for $c,d\in\cup a$ write $c\equiv_a d$ to mean that c,d are elements of the same element of a. If HOD(a), then OD(c) for every $c\in\cup a$, and then $b=\{c\in\cup a\mid\delta_c<\delta_d$ for any $d\equiv c$, $d\neq c\}$ is a selector for a, while also HOD(b).

Metatheorem: If AC is consistent relative to ZF.

2.2 Independence

The method of "forcing" was introduced by Paul to prove the independence of CH relative to ZFC=ZF+AC and of AC relative to ZF. (Here independence of a hypothesis Φ relative to a theory Γ means: if Γ is consistent, then $\Gamma+\sim\Phi$ is consistent.) The latter proof also used the method of "permutation models", essentially due to Frankel, who introduced it for the simpler proof of the independence of AC relative to ZFU, a modification of ZF allowing **individuals** (in German, *Urelemente*). This last proof will be outlined here.

ZFU is like ZF with the following changes: There is an extra one-place relation-symbol, either U(x) expressing that x is an individual or V(x) expressing that x is a set (either one of these can be taken as primitive, and the other defined as its negation). There are two extra axioms, one asserting that only sets and not individuals have elements (if $x\in y$ then V(y)), the other asserting that there exists a set of all individuals. The extensionality axiom is formulated as $\forall x\forall y(V(x)\wedge V(y)\wedge\forall z(z\in x\leftrightarrow z\in y)\rightarrow x=y)$. The axiom of infinity may be replaced by an axiom asserting that the set of all individuals is infinite.

In place of the V(α) hierarchy one defines:

$U(0)=\{i\mid U(i)\}$ $\quad$ $U(\beta+1)=\wp(U(\beta))$ $\quad$ $U(\alpha)=\cup\{U(\beta)\mid\beta<\alpha\}$ at limits

A permuation of a set is a bijection from the set to itself. Given a permutation π of the set U(0) of individuals there is a unique extension of π to a permutation also denoted π of the universe of all individuals and sets, satisfying (*) $\pi(x)=\{\pi(y)|y\in x\}$. (It is easily proved by transfinite induction that for each α there is a unique extension of π to a permutation π_α of $U(\alpha)$ satisfying (*).) Individual or set a is **fixed** by permutation π if $\pi(a)=a$. Set A is **stabilized** by $I\subseteq U(0)$ if every permutation that fixes the elements of I also fixes A. A is **finitely stabilized** (FS) is some finite $I\subseteq U(0)$ stabilizes A. A is **hereditarily finitely stablized** (HFS) if every $B\in A^\dagger$ is finitely stabilized.

Metalemma X: For every Φ the following is a theorem of ZF: $\forall a_1\ldots\forall a_n(\Phi(a_1,\ldots,a_n)\leftrightarrow\Phi(\pi(a_1),\ldots,\pi(a_n)))$.

Metalemma Y: For every n the following is a theorem of ZF: If $a_1,\ldots,a_n$ are FS and b is definable$_n$ from parameters $a_1,\ldots,a_n$, the b is FS.

Proof: For X, proceed by induction on the length of Φ, (*) being the case of the simplest formula $x\in y$. For Y, let b be definable from $a_1,\ldots,a_n$ by Ψ, and let $a_1,\ldots,a_n$ be stablized by $I_1,\ldots,I_n$ respectively. Let $\Phi(z,y_1,\ldots,y_n)$ express that there exists x such that $\Psi(x,y_1,\ldots,y_n)$ and $z\in x$, so that $b=\{c|\Phi(c,a_1,\ldots,a_n)$ is true$\}$. Let $J=I_1\cup\ldots\cup I_n$, so that any π that fixes J fixes $a_1,\ldots,a_n$. For such π by (*) one has $\pi(b)=\{\pi(c)|\Phi(c,a_1,\ldots,a_n)$ is true$\}$, by X one has $\Phi(c,a_1,\ldots,a_n)$ is true iff $\Phi(\pi(c),a_1,\ldots,a_n)$ is true. It follows $\pi(b)=\{\pi(c)|\Phi(\pi(c),a_1,\ldots,a_n)\}=b$.

	Lemma A	**Lemma B**	**Lemma C**:
the following are provable in ZF for all formulas Φ,Ψ,Θ:			
(a)	if FS(a) and FS(b) then FS({a,b}	if HFS(a) and HFS(b) then HFS({a,b})	(pairing axiom)^{HFS}
(b)	if FS(a) then FS($\cup$a)	if HFS(a) then HFS($\cup$a)	(union axiom)^{HFS}
(c)	FS(ω)	HFS(ω)	(infinity axiom)^{HFS}
(d)	if FS(a) then FS($\{b\in\wp(a)\mid\Theta(b)\}$)	if HFS(a) then HFS($\{b\in\wp(a)\mid\Theta(b)\}$)	(power axiom)^{HFS}
(e)	if FS(a) and FS(b) for every $b\in a$ then FS($\{b\in a|\Psi(b)\}$)	if HFS(a) then HFS($a'=\{b\in a|\Psi(b)\}$)	(Φ-separation axiomΦ)^{HFS}
(f)	if FS(c) and FS(a) and FS(ψ(a)) for every $a\in c$, then FS($\{\psi(a)|a\in c\}$)	if HFS(c) and HFS(ψ(a)) for every $a\in c$, then HFS($\{\psi(a)|a\in c\}$)	(Φ-replacement axiomΦ)^{HFS}

Proof: Like the corresponding three lemmas in §3.1.

Lemma D: The following is provable in ZF (*without* AC): $(\sim AC)^{HFS}$

Proof: One needs to use an alternate criterion for AC, using its equivalent, the well-ordering principle WO. Using this it suffices to show that if R is a well-ordering of U(0), then ~FS(R), using the hypothesis that U(0) is infinite. Indeed, suppose $I \subseteq U(0)$ is finite, and let $i,j \in U(0)-I$ with iRj and let $\pi(i)=j$, $\pi(j)=i$, $\pi(k)=k$ for all $k \in U(0)-\{i,j\}$. Apply metalemma X to the formula expressing that $(x,y) \in z$, to conclude $j\pi(R)i$, whence $\sim i\pi(R)j$ and $\pi(R) \neq R$ and I does not stabilize R.

Metatheorem: If AC is independent relative to ZFU.

Problems

A. Let X be a nonempty set, A a nonempty set of subsets of a given set X. Then A is called a *field* of sets if we have: (i) $(X\text{-}a)\in A$ whenever $a\in A$; and (ii) $(a\cup b)\in A$ whenever $a,b\in A$. Let B be a nonempty set of subsets of X. Then there exists a field A with $B\subseteq A$ and with $A\subseteq A'$ for any other field A' with $B\subseteq A'$. This A is called the field *generated* by B.

Show that the existence of the field generated by a nonempty set of subsets of a given set can be proved without using the power axiom (but using replacement).

B. Show that the existence of the field generated by a nonempty set of subsets of a given set can be proved without using the replacement (but not powerset).

C. Show that the following notion is expressible by a $\Delta_0=\Sigma_0=\Pi_0$ formula:
u is a [total] order on x

D. Show that the following notion is expressible by a $\Delta_0=\Sigma_0=\Pi_0$ formula:
$v=\in_y$ [where $\in_y$ is by definition $\{(p,q)|p,q\in y \text{ and } p\in q\}$]

E. Show that the following notion is expressible by a Σ_1 formula:
[u is an order on x and v is an order on y and] u is isomorphic to v

F. Show that the following notion is expressible by a Π_1 formula:
w is a well-order on x

G. Show that the notion of the preceding problem [or more precisely, a condition provably an equivalent to it] is also expressible by a Σ_1 formula.

H. Show that $V(\omega)=H(\aleph_0)$.

I. Show that $H(\aleph_1)\subseteq V(\omega_1)$.

J. Show that $V(\omega_1)\neq H(\aleph_1)$

THE AXIOM OF DETERMINACY

AN ALTERNATIVE TO THE AXIOM OF CHOICE

3 Rudiments of Descriptive Set Theory

3.1 ZF + DC

We work throughout in Zermelo-Frankel (ZF) set theory with the Axiom of Dependent Choice (DC). Recall that this axiom implies the Axiom of Countable Choice (CC), which in turn implies that a union of countably many countable sets is countable, which in turn implies that the supremum of a countable set of countable ordinals is countable. When the full Axiom of Choice (AC) is used as an hypothesis, it will be noted explicitly.

3.2 Baire Space & Cantor Space

In real analysis one usually begins with the real numbers or an interval in the real numbers, but for many purposes it is almost equivalent, and much more convenient, to work with one or the other of two auxiliary spaces. Most results transfer almost automatically back and forth among these spaces, though details will not be gone into here.

The **Baire space *W*** consists of all infinite sequences of natural numbers, or in set-theoretic terms, all functions from ω to ω. If x is such a function, we call the value $x(n)$ the *nth term* of the sequence, and $x|n$, the restriction of x to $n = \{0, 1, 2, \ldots, n\text{-}1\}$ the *initial segment of length n* of the sequence. Working with the Baire space is strictly equivalent to working with the irrational numbers (to which infinite sequences of positive integers correspond by continued fraction representation, but there will be no need here to go into that). The **Cantor space *C*** consists of all infinite sequences of zeros and ones, or in set-theoretic terms, all functions from ω to 2. For any subset a of ω, the **characteristic function** of a is the function χ_a from ω to 2 with $\chi_a(n) = 1$ if $n \in a$, and $= 0$ if $n \notin a$. It is a point in the Cantor space. The following definitions are made for ***W***, the definitions for its subspace ***C*** would be exactly parallel.

A **basic** set is a subset of ***W*** of the form $U_s = \{x: x|n = s\}$ where s is a finite sequence and n is its length. Note that the intersection of two basic sets U_s and U_t is either equal to one of them (if one of s and t is an initial segment of the other) or empty (otherwise). An **open** set is a union of basic sets. Note that the complement of any basic set is an open set. (Why? Because the complement of U_s is the union of all the U_t for t a finite sequence of length n other than s.) Note also that the intersection of any two open sets is open and the union of any number of open sets is open. (Why? For the intersection claim, by the distributive law an intersection of two unions of (many) basic sets is the union of (many) intersections of two basic sets. The union claim should be obvious.) Intuitively, if an infinite sequence x is going to get into an open set U, it will effectively have got in by some finite initial segment: There will be an n such that not only x but every sequence that agrees with x on its first n terms is in U. (Why? Because x belongs to one of the basic U_s of which U is the union.)

A **closed** set is the complement of an open set (and vice versa). Note that any basic set is "**clopen**", meaning both closed and open. To any closed set C there is associated a "tree" of finite sequences $T(C) = \{y|n: y \in C \text{ and } n \in \omega\}$. Note that for any t in $T(C)$, if n is the length of t and $m > n$, then there is an extension s of t having length m that is also in $T(C)$. (Why? Well, $t = y|n$ for some y in C, and we can take $s = y|m$.) Every element y of C determines a "branch" through $T(C)$, consisting of $y|0$, $y|1$, $y|2$, $y|3$, and so on. Conversely, any branch through $T(C)$ determines an element of y. (Why? Let y be the infinite sequence that is the union of the finite sequences in the branch. If it were in the complement of C rather than in C, since that complement is open there would be some finite n such that every element of $U_{y|n}$ is in the complement of C rather than C. But in that case $y|n$ wouldn't be in the tree $T(C)$ determined by C.) The **interior** $C°$ of a closed set C is the open set that is the union of all the U_s that are subsets of C (if there are any: the interior may be empty). The **boundary** of C is just the difference $C - C°$.

An $\mathbf{G_\delta}$ set is an intersection of countably many open sets. Trivially every open set is a G_δ, but so is every closed set. (Why? Because the argument given just above in effect shows that a closed set C is the intersection of the sets $C_n = \bigcup\{U_s: s \in T(C) \text{ and } n \text{ is the length of } s\}$.) An $\mathbf{F_\sigma}$ set is a union of countably many closed sets. The complement of a G_δ is an F_σ. (Why? Because the complement of an open set is a closed set, and the complement of an intersection of given sets is the union of their complements.) The reader can guess what is meant by an $F_{\sigma\delta}$ or $G_{\delta\sigma}$ set.

3.3 The Cantor-Bendixson Theorem

Let C be a closed set. A point y in C is said to be **isolated** in C if there is some basic U_s such that $C \cap U_s = \{y\}$. Such a U_s **isolates** y in C. Clearly C can have only countably many isolated points, if any (since there are only countably many U_s available to isolate them). It may have none, as is the case for the clopen sets U_s. A set is called **perfect** if it is closed and without isolated points. The result of removing the isolated points from a closed set C is called the **set-derivative** of C and denoted ∂C. It is still a closed set. (Why? Because its complement is the union of complement of C with all those U_s that isolate points of C.)

In terms of the associated tree $T(C)$, the isolation of y means that there is some n such that $s = y|n$ has, for any $m > n$, *only one* extension of length m in $T(C)$: "there is no branching above s." A little thought shows that $T(\partial C) = \{s \in T(C)\text{: there is branching above } s \text{ in } T(C)\}$, which we may call $\partial(T(C))$, by "abuse of language" using the same symbol ∂ for the derivative operation on sets and the operation on trees, which in words we may call "pruning".

We define for every ordinal α the αth iterate of ∂ applied to C, by transfinite recursion as follows:

For $\alpha = 0$, $\quad \partial^\alpha C = C$

For $\alpha = \beta + 1$, $\quad \partial^\alpha C = \partial(\partial^\beta C)$

For α a limit, $\quad \partial^\alpha C = \bigcap_{\beta<\alpha} \partial^\beta C$

By transfinite induction, each $\partial^\alpha C$ is closed and each C - $\partial^\alpha C$ for countable α is countable. For every finite sequence s, if there is any countable ordinal β such that $\partial^\beta C$ is disjoint from U_s, let β_s be the least such β; if there is no such β for s, let β_s be 0. Let α be the supremum of the β_s, which is still countable since the β_s are and there are only countably many of them. Then for any s, since $\alpha > \beta_s$, if $\partial^\alpha C$ is not disjoint from U_s, then $\partial^{\alpha'} C$ cannot be disjoint from U_s for any $\alpha' > \alpha$. In other words, we reach a fixed point with $C^* = \partial^\alpha C$, which we call the **kernel** of C. We already know C^* is closed and C - C^* is countable. In fact, C is either empty or perfect: Otherwise, the process of discarding isolated points would not have been finished at stage α. We have proved one of the very oldest results in set theory, going back to Cantor's original work on trigonometric series:

Cantor-Bendixson Theorem: Every closed set is either countable or the union of a countable and a perfect set.

Note that though countable ordinals were used in the proof, they are not mentioned in the result.

Proposition. Any perfect set has the cardinality $2^{\aleph_0}$ of continuum.

Proof. Let D be perfect. Then in the tree $T(D)$ there is "branching above every point". For any s in $T(D)$ take the least m at which there are distinct extensions of s in $T(D)$ of length m, and let s^* and $s\dagger$ be the extensions of this kind having the least and next-to-least last terms. Recursively define a mapping of the finite zero-one sequences into $T(D)$ by mapping the empty sequence to the empty sequence, and if t has been mapped to s, and t^* and $t\dagger$ are the results of adding a 0 and a 1, respectively, at the end of t, map them to s^* and $s\dagger$, respectively. Every infinite zero-one sequence determines a path through $T(D)$ — for example, a sequence that begins 0, 1, 1, 0, … determines a path that begins Ø, Ø*, Ø*†, Ø*††, Ø*††* — and thus determines an element of D. Moreover, distinct infinite zero-one sequences determining distinct elements of D. So there are as many elements of D as there are infinite zero-one sequences.

We say a set has **perfectly many** elements if it has a perfect subset. By the proposition just proved, "perfectly many" implies "continuum many". We now have some information concerning the continuum hypothesis (CH):

Corollary to Cantor-Bendixson: CH holds for closed sets: Any closed set has either countably many or (perfectly many and hence) continuum many points.

3.4 A Counterexample

It easily follows that CH holds for F_σ sets, including open sets. We cannot hope, however, to prove CH by showing that *every* subset of the Baire space has either countably or perfectly many elements.

Counterexample: Assuming AC, there is a set with continuum many but not perfectly many elements.

By AC, $2^{\aleph_0}$ is one of the alephs, and we can well-order any set of that size in the order type of that aleph, so that each element has fewer than continuum many predecessors in the well-order. There are at most continuum many open sets (each by the union of some subset of the set of all basic sets, which is countable); hence there are at most continuum many closed sets and *a fortiori* at most continuum many perfect sets. Well-order them in such a way that each has fewer than continuum many elements.

We proceed by transfinite recursion on ordinals $\alpha < 2^{\aleph_0}$ to classify progressively more and more elements of W positively or negatively, as follows. At stage 0, all elements are unclassified. Going from β to $\beta+1$, fewer than continuum many elements will have been classified so far. Consider the βth perfect set P in our well-ordering. Since it has continuum many elements, it has two that have not been classified. Classify one positively, the other negatively. At limits, we merely count as positively or negatively classified any element that has been so classified at an earlier stage.

In the end, the set Q of elements classified positively will have continuum many elements, since we added a new one at each stage, but will have no perfect subset, since we spoiled each perfect subset in turn by classifying one of its elements negatively.

3.5 Borel Sets

A **sigma-field** of subsets of a given set I is a family of subsets of I closed under taking complements and countable intersections and unions. For any nonempty I and any subset X of the power set of I, there is a *least* sigma-field of subsets of I containing X.

Why? Well, there is at least one sigma-field containing X, namely the full power set of I. The intersection of *all* sigma-fields containing X is easily verified to be a sigma-field containing X, and then will necessarily be the smallest one. This is a "top down" proof.

There is also a "bottom up" proof. For any subset Y of the power set of I let Y^* be the union of Y with the set of complements and countable intersections and countable unions of elements of Y. Define

For $\alpha = 0$, $\quad X_\alpha = X$

For $\alpha = \beta + 1$, $\quad X_\alpha = X_\beta{}^*$

For α a limit, $\quad X_\alpha = \; = \cup_{\beta<\alpha} X_\beta$

Then it is easily shown that for Ω = the smallest uncountable ordinal we have $X_\Omega = X_\Omega{}^*$ and it is the smallest sigma-field containing X.

For I = Baire space W and X = the set of all open and closed subsets of X, X_1 consists of the F_σ and G_δ sets, X_2 consists of the $G_{\delta\sigma}$ and $F_{\sigma\delta}$, and so on, and the smallest sigma-field is called the **Borel** sets.

CH is known to hold for Borel sets, and even for the larger class of sets called **analytic** sets, which are the images of Borel sets under continuous functions, a topic that will not be gone into here.

4 Infinite Games of Perfect Information

4.1 The Axiom of Determinacy (AD)

Let X be any set with at least two elements. (We will mainly be interested in the case where X is countable, and especially the case where $X = \omega$.) Let A be a subset of the set of infinite sequences of elements of X. We imagine an infinite **game** for two players as follows.

Player I picks an element x_0 of X.

Player II responds by picking an element x_1 of X.

Player I responds by picking an element x_2 of X.

Player II responds by picking an element x_3 of X.

…

Thus an infinite sequence $x = (x_0, x_1, x_2, x_3, \ldots)$ is generated. In the end — imagine each round of play happening twice as fast as the one before, so there *is* an end — Player I wins if x is in A, and Player II wins if x is not in A (is in the complement of A).

A **strategy** for Player I tells that player what to pick as x_0, what to pick as x_2 as a function of the opponent's previous move x_1, what to pick as x_4 as a function of the opponent's previous moves x_1 and x_3, and so on. The notion of strategy for Player II is defined similarly. Formally, a strategy for Player I would be a function from finite sequences of elements of X (with sequences of length n representing the first n moves of a hypothetical opponent) to elements of X (representing Player I's next move). A strategy for Player II would be almost the same, a function from *nonempty* finite sequences of elements of X to elements of X. (Because Player I goes first and Player II second, a strategy for the former must specify what to do when the opponent has not yet made any moves, while a strategy for the latter need not.)

A **winning** strategy for a given player is a strategy such that, if that player follows it, that player will always win, no matter how the opponent plays. It cannot be the case that both players have winning strategies. (Else what would happen when each player followed that player's supposed winning strategy?) The game is said to be **determined** and the set A **determinate** if there exists a winning strategy for one of the players. The **Axiom of Determinacy** (AD) asserts the determinateness of all games, or determinacy of all sets, in the case where X is countable. But let us first see what can be proved about the existence of winning strategies *without* assuming the axiom.

4.2 The Gale-Stewart Theorem

Let us stick to the case $X = \omega$, though the proof to follow does generalize.

Gale-Stewart Theorem: Suppose A is an open or closed subset of W. Then the game is determinate.

Proof: Since the situation is almost completely symmetrical, we just treat the open case. Suppose A is open and I does not have a winning strategy, to show that II does have a winning strategy. Call a finite sequence s of odd length a **position**. It represents a possible situation early in the game, where II is to move next. If I has a winning strategy for the continuation of the game from position s, then call s a **losing** position for II.

Claim: If s is not a losing position for II, then there is an i such at, if II plays i next, then no matter what j I plays next after that, the resulting position, which we write s^i^j, meaning "s followed by i followed by j", will still not be a losing position for II.

Proof of Claim: Suppose that whatever i II plays, there is a j_i such that if I plays, then I has a winning strategy S_i for continuing the game from s^i^j_i. Then, contrary to the assumption that s is not a losing position, I *already* has a winning strategy S for continuing the game from s: Whatever i II plays, play j_i in response, and thereafter follow S_i.

Proof of Theorem modulo Claim: Let II follow the following strategy T: At any given s where I does not have a winning strategy for continuing the game, play an i such that whatever j I plays, I will still not have a winning strategy for continuing the game from s^i^j. Suppose that II plays according to this strategy, and the result is some x in W. It only remains to prove:

Claim: x is not in A.

Proof of Claim: If x is in A then there is some n, which may be taken to be even, such that setting $s = x|n$, we have that U_s is a subset of A. But this means that I has a *trivial* strategy for winning the game continuing from s: *Whatever* I does, I has in effect *already* won. But II's strategy insures that there is no finite stage s along the way to x at which I has a winning strategy for continuing. This contradiction completes the proof.

4.3 Ultrafilters & Games

Suppose there is a nonprincipal ultrafilter F on ω. Consider the following game: I and II alternately pick finitely many natural numbers not already picked. In the end, I wins if the set of all numbers eventually picked by I is in F. (This game could be reduced to the form of a game where players alternately pick single natural numbers using a coding of finite sets by single numbers.)

Proposition. Neither player has a winning strategy for the ultrafilter game.

Proof. Suppose I has a winning strategy S (the proof being essentially the same if we suppose II has the winning strategy). When I plays a_0 as opening move according to S, II pretends that *she* is playing I and her opponent is playing II, and that *she* picked a_0 and *he* then picked the empty set of numbers. II then plays what would be the response by S if that had happened, call it b_0. The players then continue this way, with I playing the real game

$a_0, b_0, a_1, b_1, \ldots$

and II playing the virtual game

$a_0, \varnothing, b_0, a_1, b_1, \ldots$

both using S. In the end, since S is a winning strategy, I wins the real game, and the set A that is the union of all the a_n is in F. But II wins the virtual game, and the union of a_0 with the set B that is the union of all the b_n is also in F. But then the intersection of these two sets must be in F. But that intersection is just the finite set a_0, and no finite set belongs to a nonprincipal ultrafilter.

So we get an undetermined game *if* there exists an nonprincipal ultrafilter. But remember that the proof of the existence of such an ultrafilter used AC (in the guise of Zorn's Lemma).

4.4 Martin & Friedman

Various game-theorists slowly and painfully extended the Gale-Stewart from open and closed first to F_σ and G_δ, then to $G_{\delta\sigma}$ and $F_{\sigma\delta}$, then to $F_{\sigma\delta\sigma}$ and $G_{\delta\sigma\delta}$, after which there was no further progress. D. A. Martin then proved that assuming large cardinals — much larger ones than inaccessibles —one could prove determinacy for all analytic sets, a class that, as already mentioned, properly includes the class of Borel sets. Later he proved, just in ZF+DC, determinacy for all Borel sets.

The method of proof involves trading an F_σ game on the set ***W*** of infinite sequences from the set of natural numbers, a set of cardinal $\beth_0 = \aleph_0$, by an open and therefore determinate game on infinite sequences from a set of size $\beth_1$. Similarly a $G_{\delta\sigma}$ can be traded for an open and therefore determinate game on infinite sequences from a set of size $\beth_2$. To get Borel determinacy one eventually has to consider sets of size up to $\beth_\Omega$. The proof of the existence of sets this large definitely requires the use of Replacement, whereas most of "down-to-earth" mathematics can be done just in Zermelo set theory, without Replacement.

Even before Martin found this result, Harvey Friedman had proved that one could not prove analytic determinacy without using large cardinals, nor prove Borel determinacy without using Replacement.

4.5 AD & CH

There are various well known ways to code finite sequences of natural numbers by natural numbers. Given a subset A of the set ***C*** of all infinite 0,1-sequences, consider the following game:

I picks a natural number,
    and if it is not a code for a finite 0,1-sequence s_0 at once forfeits
II in response picks a natural number i_0
    and if it is ≥2 at once forfeits
I picks a natural number,
    and if it is not a code for a finite 0,1-sequence s_1 at once forfeits
II in response picks a natural number i_1
    and if it is ≥2 at once forfeits
…

If neither player forfeits, in the end I wins iff the infinite 0,1-sequence

$$s_0{}^\wedge i_0{}^\wedge s_1{}^\wedge i_1{}^\wedge \ldots$$

is in A. Of course, AD implies that either I or II has a winning strategy for this game.

Proposition: If I has a winning strategy, the set A has a perfect subset.

Proof: The point is that if S is a winning strategy for I, we have a tree of possible positions arrived at in playing the game with I following strategy S, and it is the kind of tree associated with a perfect set P: A tree such that "there is branching above any point", because every time it is II's turn, II can choose either a 0 or a 1. Since every one of the (continuum many) branches through the tree represents a play of the game in which I follows strategy S, and since S is a *winning* strategy for I, each branch of the tree, which is to say, each element of the perfect set P, is in A, and A has "perfectly many" elements.

Proposition: If II has a winning strategy, the set A is countable.

Proof: Let S be a winning strategy for II and x and element of A. It is not possible to break the whole of x up into segments

$x = s_0$^i_0^s_1^i_1^…

in such a way that we would have the following (*)

i_0 would be II's response by S to I's playing s_0

i_1 would be II's response by S to I's playing s_0, s_1

…

(Why not? Well, then x would represent a play of the game in which II follows strategy S, and since S is supposed to be a winning strategy for II, the result of such play would never be an element of A.) It follows that there is some k so that we have a representation

$x = s_0$^i_0^s_1^i_1^… ^s_k^i_k^y

where y is the rest of x, and (*) holds out to k, but thereafter for any n, II's response by strategy S to I's playing s_0, s_1, … , s_k, (y_0, y_1, … , y_{n-1}) would *not* be y_n, but rather the opposite, 1 - y_n. But this means that the whole of x can be generated recursively using the strategy S and the finite data consisting of the sequence σ of plays s_0, i_0, s_1, i_1,… , s_k, i_k. Since each element x of A is thus generated from a finite data σ, and there are only countably many possibilities for σ, there can be only countably many elements of A.

Putting the two propositions together we have:

Theorem of Morton Davis: AD implies every uncountable set has a perfect subset.

4.6 AD, AC, CH, GCH, GAD

Since we saw in §3.4 that AC implies the opposite, it follows (apart from details about switching back and forth between Baire and Cantor space) that AC and AD are inconsistent with each other. We already knew this from §4.3, but now we have another proof. We could give yet another proof involving the a construction resembling that of §3.4, given a well-ordering. (Well-order the strategies, and so on, and gradually classify elements of the space, at each stage classifying some new element positively, but frustrating some strategy.) Looking at such a proof we would see that AD implies there is no well-ordering of any set of size continuum, and even that no set of size continuum has any uncountable well-ordered subset. Adapting the material in §4.3, we could also give an ultrafilter proof of the existence of a set of size continuum with no perfect subset.

The theorem of Morton Davis can also be quoted as saying that AD implies CH. Note, however, that we must here understand CH to be the proposition that there is no cardinal number λ with $\aleph_0 < \lambda < 2^{\aleph_0}$ and *not* as the proposition that $2^{\aleph_0} = \aleph_1$. But if we ask, not what alephs can be mapped one-to-one *into* the Baire space, or the real numbers, or whatever, but rather what alephs one of these spaces can be mapped *onto*, the answer is that the least aleph that it cannot be mapped onto, its so-called *Hartog's number*, is very large, and in fact a fixed point of the alephs. So looked at one way the continuum seems small, but looked at another way it seems large, in the world of AD.

Though AD implies CH, it does not imply its generalization GCH. Without AD, GCH must be formulated as saying that for no cardinal κ is there any cardinal λ such that $\kappa < \lambda < 2^{\kappa}$. And it was shown by Sierpinski that GCH implies AC. Hartog's theorem, that for every κ there is an $\aleph$ that κ cannot be mapped onto, is the first step in Sierpinski's proof.

AD pertains to games involving infinite sequences of natural numbers, or elements of some other countable set. A stronger assertion, the *generalized axiom of determinacy* (GAD) would be that there exists a winning strategy for one or another player in games involving infinite sequences from *any* nonempty set. But GAD is inconsistent, since it not only includes AD as a special case, but also implies AC as another special case:

Just imagine, given a set X, a game in which player I picks a nonempty subset A of X and then player II picks an element a of X, and II wins if $a \in A$. Clearly I cannot have a winning strategy, since whatever set I plays, II can pick an element of it. However, a *strategy* for II would be a function telling II in advance what element to pick as a function of what set I picks. And that is just what a choice function for nonempty subsets of X amounts to, se we have AC.

5 Category (& Measure)

5.1 Baire Category

The theory of Baire category (something much older than and entirely unrelated to the category theory of Mac Lane) is a basic tool in parts of real analysis. Let us go back to the Baire space, though most of what we will have to say will apply *mutatis mutandis* to Cantor space. A subset A is called (globally) **dense** (in W) if for every basic U_s, the intersection $A \cap U_s$ is nonempty. The set A is called (locally) **dense in** U_s if for every basic $U_t \subseteq U_s$, the intersection $A \cap U_t$ is nonempty. By contrast, the set A is called **nowhere dense** or **rare** if it is not dense in any U_s: in other words, for any U_s there is a $U_t \subseteq U_s$ such that $A \cap U_t = \emptyset$. Note that any subset of a rare set is rare.

Example: For any k the set Lack(k) of infinite sequences in which k does not appear as a term, is rare. (Why ? Because for any s, taking $t = s$^k we get a $U_t \subseteq U_s$ disjoint from Lack(k).)

Example: For any closed set C, the boundary C - C° is rare. (Why? For any U_s, it cannot be contained in the boundary, else it would be contained in C and in the interior C°. Since both the complement and the interior of C are open, any point in U_s that lies in either the complement or the interior will lie in some $U_t \subseteq U_s$ that is wholly contained in either the complement or the interior, and hence disjoint from the boundary.)

A set is **first category** or **meager** if it is a union of countably many rare sets. Note that any subset of a meager set is meager, and any union of countably many meager sets is meager. An example of meager set would be the set of all x such that the not every natural number appears as a term of x. (This is just the union of the various Lack(k).)

A set is **second category** or **nonmeager** if it is not meager, and is **residual** or **comeager** if its complement is meager, and **residual in** U_s or **comeager in** U_s if its relative complement in U_s is meager.

5.2 The Baire Category Theorem & Baire Property

Intuitively, being meager is to be thought of as a way of being "small". This way of thinking is justified by the next theorem, a famous result often used in existence proofs in mathematical analysis.

Baire Category Theorem: The whole space W is nonmeager.

Proof. Let M be a meager set, the union of rare sets $R_0, R_1, R_2, \ldots$, to show that there is an element of W not belonging to any R_i and therefore not belonging to M. Since R_0 is rare, starting from $W = U_{\text{empty\_sequence}}$ there is a t such that U_t is disjoint from R_0. Pick such a t (for definiteness, the first in some fixed well-ordering of the finite sequences) and call it s_0. Then since R_1 is rare, starting from U_s for $s = s_0$, there is a t extending s_0 such that U_t is disjoint from R_1. Pick such a t and write it as $t = s_0\text{^}s_1$. Continue in this way, next picking s_2 so that U_t for $t = s_0\text{^}s_1\text{^}s_2$ will avoid R_2, and so on. Stringing together all the s_i gives an infinite sequence $x = s_0\text{^}s_1\text{^}s_2\text{^}\ldots$ avoiding all the R_k and hence avoiding M, as required.

A set A has the **Baire property** or is **almost open**, if there is an open set O such that the symmetric difference $A\Delta O = (A - O) \cup (O - A)$ is meager. Intuitively, being almost open is to be thought of as a way of being "regular" or well-behaved. The following characterization may clarify the notion.

Proposition:

(a) The complement of an almost open set is almost open.

(b) The union of countably many almost open sets is almost open.

(c) The intersection of countably many almost open sets is almost open.

(d) Every Borel set is almost open.

Proof: For (a), if O is open and $M = A\Delta O$ is meager, let C be the complement of O and $N = C - C°$ the boundary, which is rare. Then $M' = M \cup N$ is meager, and (complement A)$\Delta C°$ $\subseteq M'$.

For (b), if for each A_i we have open O_i and meager M_i with $A_i\Delta O_i \subseteq M_i$, then the union O of the O_i is open, the union M of the M_i is meager, and for the union A of the A_i we have $A\Delta O \subseteq M$.

(c) follows from (a) and (b) and the DeMorgan laws.

(d) follows from (a)-(c) and the definition of Borel sets.

It is known that analytic sets are also almost open.

Proposition: A set A is almost open if for every basic U_s there is a basic $U_t \subseteq U_s$ such that A either is meager in U_t or is comeager in U_t.

Proof. Here "A is meager in U_t" just means "$U_t \cap A$ is meager".

First suppose the condition of the theorem is met. Let U^+ be the open set that is the union of the U_t in which A is comeager, and U^- the open set that is the union of the U_t in which A is meager, and let U be the union of U^+ and U^-. By the condition, for every basic U_s there is a basic $U_t \subseteq U_s$ such that $U_t \subseteq U$, hence the complement N of U is rare. Let M be the union of N with the meager sets U_t - A for $U_t \subseteq U^+$ and the meager sets $U_t \cap A$ for $U_t \subseteq U^-$. It can be checked that $A \Delta U^+ \subseteq M$.

Conversely, suppose A is almost open, with open O and meager M such that $A \Delta O = M$. Let C be the complement of O and $N = C$ - $C°$ the boundary of C, which we proved earlier to be a rare set, making $M' = M \cup N$ meager. As we saw in that proof, any U_s either has a $U_t \subseteq U_s$ contained in O or has a $U_t \subseteq U_s$ contained in $C°$ and hence disjoint from O. In the former case, U_t - A is a subset of the meager set M', and A is comeager in U_t, while in the latter case $A \cap U_t$ is a subset of the meager set M', and A is meager in U_t. So the condition of the proposition is met.

5.3 The Banach-Mazur Game

Given a subset A of the Cantor space $\mathbf{C}$, consider the following game $G(A)$: I and II alternately pick finite 0,1-sequences:

I	s_0		s_1		s_2		…	
II		t_0		t_1		t_2		…

These are strung together to get a point x in A

$$x = s_0\hat{}t_0\hat{}s_1\hat{}t_1\hat{}\ldots$$

and I wins if $x \in A$. For any finite sequence r there is a related game $G_r(A)$, where we start with r as given and end with $x = r\hat{}s_0\hat{}t_0\hat{}s_1\hat{}t_1\hat{}\ldots$, an element of U_r.

Proposition: If II has a winning strategy in $G(A)$, then A is meager.

Proof. Let S be a winning strategy for II. The idea is to use S to develop a certain tree of possible plays (s_0, t_0, s_1, t_1, …) of the game. The sequences of x that result from branches through the tree will, since S is a winning strategy for II, all belong to the complement of A. We need to arrange the tree so that all but a meager set of elements of $\mathbf{C}$ correspond to branches through it — to be done by arranging the tree so that all but a rare set of elements of $\mathbf{C}$ correspond to branches up through the nth level of the tree for each n. The rather messy details will be given in class, time permitting (and perhaps included in a future edition of these notes).

Analogously, if II has a winning strategy in $G_r(A)$, then A is meager in U_r. If I has a winning strategy for $G(A)$ then there is an s (namely, the sequence the strategy would tell I to play on the first move), such that I would have a winning strategy if playing the role of II in the game G_s(complement of A) — for that is what the rest of the game $G(A)$ after I's first move looks like. But that means that the complement of A is meager in G_s, and A is comeager in U_s. Similarly, I's having a winning strategy in $G_r(A)$ implies A being comeager in $U_{r\hat{}s}$ for some s, which is to say, in some $U_t \subseteq U_s$. Comparing with the second proposition of the previous section we see that we have the following:

Proposition: If all the games $G_r(A)$ are determinate, then A is almost open.

Theorem: AD implies that all sets are almost open.

There is neither space nor time to develop the basics of measure theory here, but it may just be said that there is a close analogy between Lebesgue measure and Baire category, explored in some depth in the little book *Measure & Category* by John Oxtoby (Springer, 1971). It includes, in particular, in chapter 6, details about the Banach-Mazur game (in a slightly different context from the one considered here). Analogous to the theorem of the last section we have the following:

Theorem: AD implies that all sets are Lebesgue measurable.