PROBLEM SET 1. THE AXIOM OF FOUNDATION

Early on in the book (page 6) it is indicated that throughout the formal development 'set' is going to mean 'pure set', or set whose elements, elements of elements, and so on, are all sets and not items of any other kind such as chairs or tables. This convention applies also to these problems.

1. A set y is called an *epsilon-minimal element* of a set x if $y \in x$, but there is no $z \in x$ such that $z \in y$, or equivalently $x \cap y = \emptyset$. The *axiom of foundation*, also called the *axiom of regularity*, asserts that any set that has any element at all (any nonempty set) has an epsilon-minimal element. Show that this axiom implies the following:

(a) There is no set x such that $x \in x$.

(b) There are no sets x and y such that $x \in y$ and $y \in x$.

(c) There are no sets x and y and z such that $x \in y$ and $y \in z$ and $z \in x$.

2. In the book the axiom of foundation or regularity is considered only in a late chapter, and until that point no use of it is made of it in proofs. But some results earlier in the book become significantly easier to prove if one does use it. Show, for example, how to use it to give an easy proof of the existence for any sets x and y of a set x^* such that x^* and y are disjoint (have empty intersection) and there is a bijection (one-to-one onto function) from x to x^*, a result called the *exchange principle*.

3. For any set x the *successor set* x' of x is defined to be the set $x' = x \cup \{x\}$. Show how to use the axiom of foundation to give an easy proof that if $x' = y'$, then $x = y$.

4. A set t is called *transitive* if every element of every element of t is itself an element of t, or equivalently, if every element of t is a subset of t. A set t is said to be *ordered by epsilon* if for any two elements x and y of t, either $x \in y$ or $x = y$ or $y \in x$. Give an example of each of the following:

(a) A set of exactly four elements that is transitive but not ordered by epsilon.

(b) A set of exactly four elements that is ordered by epsilon but not transitive.

(c) A set of exactly four elements that is both transitive and ordered by epsilon.

5. Assume foundation, Let x and y be transitive sets, each ordered by epsilon. Show that if $y - x \neq \emptyset$, then $x \cap y \in y$.

[Hint: Let z be an epsilon-minimal element of $y - x$, and show $z = x \cap y$.]

6. Assume foundation, Let x and y be transitive set, each ordered by epsilon.

(a) Show that either $x \in y$ or $x = y$ or $y \in x$.

[Hint: Note the preceding problem and note that, reversing the roles of x and y that if $x - y \neq \emptyset$, then $x \cap y \in x$]

(b) Can there be two distinct sets each of exactly four elements, each both transitive and ordered by epsilon?