DO FIVE OUT OF SIX ON EACH SET

PROBLEM SET 1. THE AXIOM OF FOUNDATION

Early on in the book (page 6) it is indicated that throughout the formal development 'set' is going to mean 'pure set', or set whose elements, elements of elements, and so on, are all sets and not items of any other kind such as chairs or tables. This convention applies also to these problems.

1. A set y is called an *epsilon-minimal element* of a set x if $y \in x$, but there is no $z \in x$ such that $z \in y$. The *axiom of foundation*, also called the *axiom of regularity*, asserts that any set that has any element at all (any nonempty set) has an epsilon-minimal element. Show that this axiom implies the following:

(a) There is no set x such that $x \in x$.

(b) There are no sets x and y such that $x \in y$ and $y \in x$.

(c) There are no sets x and y and z such that $x \in y$ and $y \in z$ and $z \in x$.

2. In the book the axiom of foundation or regularity is considered only in a late chapter, and until that point no use of it is made of it in proofs. But some results earlier in the book become significantly easier to prove if one does use it. Show, for example, how to use it to give an easy proof of the existence for any sets x and y of a set x^* such that x^* and y are disjoint (have empty intersection) and there is a bijection (one-to-one onto function) from x to x^*, a result called the *exchange principle*.

3. For any set x the *successor set* x' of x is defined to be the set $x' = x \cup \{x\}$. Show how to use the axiom of foundation to give an easy proof that if $x' = y'$, then $x = y$.

4. A set t is called *transitive* if every element of every element of t is itself an element of t, or equivalently, if every element of t is a subset of t. A set t is said to be *ordered by epsilon* if for any two elements x and y of t, either $x \in y$ or $x = y$ or $y \in x$. Give an example of each of the following:

(a) A set of exactly four elements that is transitive but not ordered by epsilon.

(b) A set of exactly four elements that is ordered by epsilon but not transitive.

(c) A set of exactly four elements that is both transitive and ordered by epsilon.

5. Let x and y be transitive sets, each ordered by epsilon.

(a) Show using foundation that if $y - x \neq \emptyset$, then $x \cap y \in y$.

[Hint: Let z be an epsilon-minimal element of $y - x$, and show $z = x \cap y$.]

(b) Show that either $x \in y$ or $x = y$ or $y \in x$.

[Hint: A proof like that of (a) shows also that if $x - y \neq \emptyset$, then $x \cap y \in x$]

Can there be two distinct sets each of exactly four elements, each both transitive and ordered by epsilon?

6. The *axiom of collection* says that for any condition C, for every set x there exists a set y such that for every element u of x, if there exists any v at all such that the condition C holds for the pair (u, v), then there exists a v that is an element of y such that the condition C holds for the pair (u, v). Later we will see, using foundation, that the axiom of collection follows from the other axioms of set-theory, including replacement. Show now conversely that, even without assuming foundation, replacement follows from collection.

PROBLEM SET 2. AN ALTERNATIVE APPROACH TO FINITENESS
The problems below show how to develop the theory of finite sets before and independently of *introducing the notion of number*. *For the space of this assignment,* forget about *the definition of finiteness given in the book, along with all definitions pertaining to number.*

1. A set X of sets is said to be *closed under deletion* if whenever $x \in X$ and $y \in x$, then $x - \{y\} \in X$. (Here $x - \{y\}$ is the result of deleting y from x.) Instead of the definition 'finite' in the book use the following alternative: x is *finite* if for every X closed under deletion, if $x \in X$ then $\varnothing \in X$. Show that under this definition

(a) $\varnothing$ is finite

(b) if x is finite and $y \notin x$, then $x \cup \{y\}$ is finite as well.

(Here $x \cup \{y\}$ is the result of inserting y into x.)

2. Call a condition C (*set*) *inductive* if (i) C holds of $\varnothing$ and (ii) whenever $y \notin x$ and C holds of x then C holds of $x \cup \{y\}$. Prove that any inductive condition holds of all finite sets. [Hint: Suppose C fails for z. Show that there is a set Z closed under deletion such that $z \in Z$ but $\varnothing \notin Z$. You will need to use the power set axiom at this point.] This result is called the *principle of* (*set*) *induction*.

3. Prove that the union $x \cup y$ of any two finite sets x and y is finite. [Hint: Use "induction on y", or in other words, use the principle of induction to show that the following condition holds of any finite set y: for all finite sets x, the union $x \cup y$ is finite.]

4. (a) Let C be any condition and u any finite set. The axiom of separation tells us that the set $v = \{x \in u: C \text{ holds of } x\}$ exists. Show that it is finite.

(b) Let D be a condition such that for every x there exists a unique y — call it d_x — such that D holds of the pair (x, y). Let u be any finite set. The axiom of replacement tells us that the set $v = \{d_x: x \in u\}$ exists. Show that it is finite.

5. Prove that the Cartesian product product $x \otimes y$ of any two finite sets x and y is finite.

6. Outline a proof that for any finite sets x and y the set x^y of all functions from y to x is finite.

PROBLEM SET 3. AN ALTERNATIVE PROOF OF CANTOR-BERNSTEIN

Cantor gave the first proof, but he used the axiom of choice. Bernstein gave the first proof not using the axiom of choice, and that is the proof given in the book. It involves the apparatus of natural numbers and recursion, and so does not come in the book until after that apparataus has been set up. Zermelo gave the proof consisting of 4-6 below, using a lemma of Dedekind contained in 1-3 below.

1. Let A be any set and $\wp(A)$ the power set of A, or set of all subsets of A. Show that for any function F from $\wp$(A) to $\wp$(A) the set of all subsets X of A such that $F(X) \subseteq X$ is nonempty, and therefore the intersection Z of all such subsets exists. A function F from $\wp(A)$ to $\wp(A)$ is called *monotone* if for any subsets X and Y of A, if $X \subseteq Y$, then $F(X) \subseteq F(Y)$. Show that if F is monotone then $F(Z) \subseteq Z$.

2. A *fixed point* for F is a subset Z of A such that $F(Z) = Z$. Show further that if F and Z are as in the preceding problem, then Z is a fixed point for F.

3. Continuing the preceding problem, show also that for any other fixed point X for F (if there are any), we have $Z \subseteq X$. Thus Z may be called the *minimum* fixed point for F.

4. Let A be any set, f any function from A to A, and D any subset of A. Define a function G from $\wp(A)$ to $\wp(A)$ by $G(X) = D \cup f[X]$, where as always $f[X] = \{f(x): x \in X\}$. Show that G is monotone and therefore has a (minimum) fixed point Z.

5. Suppose $C \subseteq B \subseteq A$, and that there is a bijection (one-to-one, onto function) f from A to C. Let $D = A - B$ and let Z be as in the preceding problem. Define a function g from A to B by letting $g(x) = f(x)$ if $x \in Z$ and $g(x) = x$ if $x \notin Z$. Show that g is a bijection from A to B.

6. Use the preceding problem to prove the Cantor-Bernstein theorem without using the axiom of choice [as in Cantor's original proof], *and* without using the apparatus of natural numbers and recursion [as in Bernstein's proof, used in the book].

PROBLEM SET 4 MORE ON FIXED POINTS

1. Let A be any nonempty set, $\wp(A)$ the power set of A, and F a function from $\wp(A)$ to $\wp(A)$ that is montone, meaning that whenever $X \subseteq Y$, then $F(X) \subseteq F(Y)$. By transfinite induction define a set $\mathrm{I}_\alpha \subseteq A$ for every ordinal α as follows:

$\mathrm{I}_\alpha = \varnothing$ if $\alpha = 0$

$\mathrm{I}_\alpha = F(\mathrm{I}_\beta)$ if $\alpha = \beta+1$

$\mathrm{I}_\alpha = \bigcup_{\beta<\alpha} \mathrm{I}_\beta$ if α is a limit

Show that $\mathrm{I}_\beta \subseteq \mathrm{I}_\alpha$ whenever $\beta \leq \alpha$.

2. Continuing the preceding problem…

(a) Show that there exists a β such that $\mathrm{I}_{\beta+1} = \mathrm{I}_\beta$. The least β for which this happens is called the *closure ordinal* for F.

(b) Can one avoid the use of the axiom of choice in proving the existence of the closure ordinal?

3. Continuing the preceding problems…

(a) Show that if β is the closure ordinal for F, then I_β is the minimum fixed point Z defined in problem #1 of problem set #3.

(b) Does there also exist a *max*imum fixed point?

4. Continuing the preceding problems…

(a) Suppose A is the set N (also known as ω) of natural numbers, and $F(X) = \{2\} \cup \{2n: n \in X\}$. What is the minimum fixed point for F?

(b) Are there any other fixed points for F?

(c) What is the closure ordinal for F?

5. Continuing the preceding problems, and keeping A = N…

(a) Give an example of a monotone F for which there are exactly three distinct fixed points.

(b) Give an example of a monotone F for which the closure ordinal is $\omega+\omega$.

6. Continuing the preceding problems, and keeping A = N, show that for every countable ordinal α there is a monotone F whose closure ordinal is α.

PROBLEM SET 5. TOPICS IN COMBINATORIAL SET THEORY
In this problem set you may use AC and results depending on AC freely as needed.

1. Let k be a positive integer and A an infinite set of k-tuples of natural numbers. Show that there is an infinite subset B of A such that for any two k-tuples $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ in B, we either have $a_i \leq b_i$ for all i or else have $b_i \leq a_i$ for all i. [Hint: Use induction on k.]

2. Let Ω (also called ω_1, also called $\aleph_1$) be the smallest uncountable ordinal, or what is the same thing, the set of all countable ordinals. Let A be an uncountable set of finite subsets of Ω. Show that there exists an uncountable subset A^* of A and a finite subset a^* of Ω such that $b \cap c = a^*$ for all distinct b and c in A^*. [Hint: First show there is a finite n such that uncountably many a in A have cardinality n; then argue somewhat as in the preceding problem.]

3. In a partial order $(P, \leq)$ two elements q and r are called *incompatible* if there is no p such that both $q \leq p$ and $r \leq p$. A subset A of P is called an *antichain* if any two distinct elements of A are incompatible. Consider the case where P consists of all functions f such that dom f is a finite subset of Ω and ran f is a subset of $\{0, 1\}$, and where the partial order relation $\leq$ is just inclusion $\subseteq$. (Remember that officially a function is just a set of ordered pairs where no two distinct pairs have the same first element.) Show that in this case any antichain is countable. [Hint: Use the preceding problem.]

4. Let R be a symmetric relation on natural numbers. A set I of natural numbers is called *homogeneous* for R if EITHER aRb holds for ALL pairs of distinct elements a and b in I, OR ELSE aRb holds for NO pair of distinct elements a and b in I. Show that there exists an infinite homogeneous set I. [Hint: Consider the hypothesis (H) that for every infinite set A of natural numbers there exists an a in A such that aRb holds for infinitely many b in A. First prove the existence of a suitable I assuming H, then prove it assuming not-H.]

5. Let Z be a countable set. For natural numbers n, let F_n be the set of functions from n to Z, and let F be the union of the F_n. Let T be a subset of F such that for each $T_n = T \cap F_n$ the following hold:

(i) If f is in T_n and $m < n$, then the restriction $f \mid m$ is in T_m.

(ii) T_n is finite but nonempty.

Show it follows that

(iii) There is a function g from the natural numbers to Z such that for all n, the restriction $g \mid n$ is in T_n.

[Hint: First show that if f is in T_n and there are infinitely many extensions of f in T, then there is some extension of f in T_{n+1} such that there are infinitely many extensions of it in T.]

6. Let Z be a countable set. For countable ordinals α, let F_α be the set of functions from α to Z, and let F be the union of the F_α. Show that there is a subset T of F such that for each $T_\alpha = T \cap F_\alpha$ the following hold:

(i) If f is in T_α and $\beta < \alpha$, then the restriction $f \mid \beta$ is in T_β.

(ii) T_α is countable but nonempty.

But [in contrast to problem 5]

(iii) There is *no* function g from the countable ordinals to Z such that for all α, the restriction $g \mid \alpha$ is in T_α.

[Hint: To insure (iii), take as Z the set of rational numbers and let the functions f in T all be such that whenever $\beta < \alpha$ are in dom f, then $f(\beta) < f(\alpha)$, thus guaranteeing that the function is one-to-one. The hard part will be to define the T_α by recursion in such a way as to insure (ii). Futher hints may be forthcoming if too many students find this too tricky.]

PROBLEM SET 6. TOPICS IN COMBINATORIAL SET THEORY
In this problem set you may use AC and results depending on AC freely as needed.

1. A subset C of Ω is called *unbounded in* γ, where γ is a limit ordinal $\leq \Omega$ if for every $\alpha < \gamma$ there is a β in C having $\alpha < \beta < \gamma$. (When $\gamma = \Omega$ we simply say "unbounded".) A subset C of Ω is called *closed* if whenever γ is a limit ordinal $< \Omega$ and C is unbounded in γ, then γ is in C. A function f from Ω to Ω is *increasing* if $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$, and *continuous* if $f(\beta) = \sup\{f(\alpha): \alpha < \beta\}$ at limit ordinals β. An ordinal γ is a *fixed point* for f if $f(\gamma) = \gamma$. Show that the set of fixed points of any increasing continuous function is closed unbounded.

2. **(a)** Show that the intersection of any two, or indeed any countably many, closed unbounded sets is closed unbounded.

(b) Show that if $(C_\alpha: \alpha < \Omega)$ is a sequence of closed unbounded sets, then the set $\Delta = \{\gamma: \gamma \in C_\alpha \text{ for all } \alpha < \gamma\}$ is closed unbounded as well.

[Remark: The proofs can be arranged so that the proof of (b) incorporates a proof of (a).]

(c) A subset S of Ω is called *stationary* if $S \cap C \neq \emptyset$ for any closed unbounded C. Show that if C is closed unbounded and C is the union of S_n for $n = 0, 1, 2, \ldots$, then at least one of the S_n is stationary.

(d) A function f from Ω to Ω *presses down* on a set S if $f(\alpha) < \alpha$ for all α in S, and is *constant* on a set T if $f(\alpha)$ is the same for all α in T. Show that if S is stationary and f presses down on S, then f is constant on some stationary subset T of S.

3. Let $(f_\alpha: \omega < \alpha < \Omega)$ be a sequence of functions, where for each α the function f_α is a bijection from $\omega = \{0, 1, 2, \ldots\}$ to α. Let $S_{n\,\alpha} = \{\beta < \Omega: \alpha < \beta \text{ and } f_\beta(n) = \alpha$.

(a) Show that two of the $S_{n\,\alpha}$ have empty intersection if the ns are the same and the αs different.

(b) Show that two of the $S_{n\,\alpha}$ have empty intersection if the ns are different and the αs the same.

(c) Show for any α, one of the $S_{n\,\alpha}$ is stationary. [Hint: Use problem 2(c).]

(d) Show that there exists an uncountable set of disjoint stationary sets. [Hint: Show that for some n, the sets $S_{n\,\alpha}$ are stationary for uncountably many α.]

4. Two sets of natural numbers are said to be *almost disjoint* if their intersection is finite. Show that there exists as set of continuum many sets of natural numbers, any two of which are almost disjoint. [Hint: Combine the facts that there exist continuum many functions from the natural numbers to $\{0,1\}$ and the fact that there is a bijection between the natural numbers and the set of finite $\{0, 1\}$-sequences.]

5. Let I be any infinite set. A subset F of the power set $\wp(\mathrm{I})$ is called a *filter* if the following hold:

(i) the intersection of any finitely many members of F is nonempty

(ii) any set having an member of F as a subset is itself a member of F.

A filter F is called an *ultrafilter* if further

(iii) for any subset A of I, either A is in F or the complement I - A is in F.

An ultrafilter U on a set I is called *principal* if there is some i in I such that $U = \{A \subseteq \mathrm{I}: i \in A\}$. Show that for any infinite I the following hold:

- **(a)** Any maximal filter is an ultrafilter.
- **(b)** The union of a chain of filters (under inclusion) is a filter.
- **(c)** Any filter can be extended to an ultrafilter.
- **(d)** There exists a nonprincipal ultrafilter.

6. An ultrafilter on an infinite set I is called *countably additive* if the intersection of any countable set of elements of the ultrafilter is in the ultrafilter. For any uncountable cardinal κ an ultrafilter is called $<\kappa$-*additive* if the intersection of any set of fewer than κ elements of the ultrafilter is in the ultrafilter.

(a) Show that there is no countably additive non-principal ultrafilter on any set of the cardinality of the continuum. [Hint: As a representative set of the cardinality of the continuum, use the power set of the set of natural numbers, and consider for each n the set of all sets of natural numbers that have n as an element.]

(b) Show that if there exists a countably additive non-principal ultrafilter on a set of uncountable cardinal κ, then for some uncountable $\lambda \leq \kappa$ there exists a $<\lambda$-additive non-principal ultrafilter on a set of cardinal λ. [Hint: Let λ be least such that there is a set of λ elements of the ultrafilter whose intersection is not in the ultrafilter.]

(c) Show that if there is a $<\lambda$-additive non-principal ultrafilter on some set of uncountable cardinal λ, then for any $\mu < \lambda$ we have $2^{\mu} < \lambda$. [Hint: Adapt the proof of (a).]

(d) Show that if there is a $<\lambda$-additive non-principle ultrafilter on some set of uncountable cardinal λ, and $\mu < \lambda$ and for each $\iota < \mu$ we have $\nu_\iota < \lambda$, then $\sum_{\iota<\mu} \nu_\iota < \lambda$. [Hint: Can a set of cardinal $< \lambda$ belong to the ultrafilter?]

(e) Show that there is no countably additive non-principal ultrafilter on $\aleph_\alpha$ for any $0 < \alpha \leq \Omega$. [Hint: By (b) it is enough to show by induction on α that there is no $<\aleph_\alpha$-additive non-principal ultrafilter on $\aleph_\alpha$ for any $\alpha \leq \Omega$.]