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Set Theory
An Introduction

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Dedicated to my wife Marilyn

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PREFACE

By its nature, set theory does not depend on any previous mathematical knowledge. Hence, an individual wanting to read this book can best find out if he is ready to do so by trying to read the first ten or twenty pages of Chapter 1. As a textbook, the book can serve for a course at the junior or senior level. If a course covers only some of the chapters, the author hopes that the student will read the rest himself in the next year or two. Set theory has always been a subject which people find pleasant to study at least partly by themselves.

Chapters 1-7, or perhaps 1-8, present the core of the subject. (Chapter 8 is a short, easy discussion of the axiom of regularity). Even a hurried course should try to cover most of this core (of which more is said below). Chapter 9 presents the logic needed for a fully axiomatic set theory and especially for independence or consistency results. Chapter 10 gives von Neumann's proof of the relative consistency of the regularity axiom and three similar related results. Von Neumann's 'inner model' proof is easy to grasp and yet it prepares one for the famous and more difficult work of Gödel and Cohen, which are the main topics of any book or course in set theory at the next level. Chapter 9 might be slightly easier for someone who has already studied logic, but it is written to be understandable by a reader with no background in logic. Actually, some of the logic given in Chapter 9 is not covered in most first year logic courses (and most of what they do cover is not needed in Chapter 9 or 10). After Chapters 1-8, the thing most required for further work in set theory (and so the thing next to be included in a longer course) is the material of Chapter 9 (and Chapter 10). The last Chapter, 11, returns to 'straight' set theory and can be read after Chapter 7. Its first part adds to the earlier cardinal arithmetic, and its second part to the earlier ordinal arithmetic. Most people find these topics attractive and easy, and will indeed read them by themselves if a course does not cover them.

For many years, the widely used introductory books on set theory all presented intuitive set theory. For the past two or three decades, the exact opposite has been true: all such books have given axiomatic set theory. But for the student, the trivial and irritating business of fooling around, as he begins to learn set theory, with axioms (saying for example that $\{x,y\}$ exists!) discourages him from grasping the main, beautiful facts about infinite unions, cardinals, etc., which should be a joy.

Therefore, we shall work in intuitive set theory in the first five of the seven main chapters. The axioms are discussed in the very short Chapter 6. By that time, many of the special features of the axiomatic business will be seen by the student to be trivial, as they should be. At the end of Chapter 6 the reader has all of Chapters 1-5 behind him *axiomatically*. In Chapter 7 (on well-orderings) we now work from the axioms, but the reader sees at once that there is practically no difference between working intuitively and working axiomatically.

Two other pedagogical devices are used to increase the reader's speed in getting the main ideas – the first (which the author learned from Azriel Levy when he was teaching in Berkeley) is this: Cardinals, order types, etc., are not defined (in some ad hoc way) until Chapter 7; but, in Chapter 2, we just 'grab' them, as Cantor did. The other device imitates the famous book (or books) of Hausdorff [F1914 and F1927 (English edition 1957)] in putting off the serious study of well-orderings as long as possible – in fact until Chapter 7. (Even in Cantor's work, some ideas are less natural and easy then others!) As a side-effect, well-ordering is studied (in Chapter 7) while working axiomatically; and it is just possible that this subject (particularly definition by induction) is one of the few more easily grasped working axiomatically than intuitively.

The author is indebted to his own teacher in set theory and logic, Alfred Tarski, for many things in this book. Some recent students have suggested various shorter proofs, which have been gratefully used. The author is very grateful to Shaughan Lavine, who prepared the index and also assisted with the proof-reading, making many corrections and improvements.

INTRODUCTION

Set theory has two overlapping aspects. In one, it is a branch of mathematics, like algebra or differential geometry, with its own special subject matter. In its other aspect, set theory is not a branch of mathematics but the very root of mathematics from which all branches of mathematics rise. (In this picture only logic lies still below set theory. Together they are often called the 'foundations of mathematics.')

This Introduction contains some remarks about the (early) history of set theory. The book proper, the mathematics, begins with Chapter 1 and does not depend on the Introduction. The remarks below should be read (rapidly for pleasure) now or perhaps after reading much of the book, or both.

Set theory, in its aspect as one branch of mathematics, is devoted particularly to the theory of infinite cardinal and ordinal numbers. Perhaps not even relativity theory can be said to have sprung so completely from the mind of one man as did set theory (in this aspect)! That man was Georg Cantor. Cantor lived from 1845 to 1918, his main publications appearing between 1874 and 1897. (For reference to Cantor's papers see the bibliography of Fraenkel [1960].) Cantor was also one of the founders of point set topology which in turn had arisen in a study of trigonometric series. Cantor's set theory stands as one of the great creations of mathematics. David Hilbert is widely considered to be the leading mathematician of the last hundred years. Replying to some who thought that the paradoxes (see below) might destroy Cantor's theory of sets, Hilbert spoke of "a paradise created by Cantor from which nobody shall ever expel us" (cf. Fraenkel [1961], p. 240). Reader: Note what lies ahead for you!

The early story of the 'foundations aspect' of set theory goes back farther, and it is not concentrated in the work of one man. Actually, in the years 1800-1930 the entire mathematics went through a major change (at the very least, of *style*) into what every subject calls today its 'modern approach.' Set theory and logic played a key role in this change, but that role is nevertheless sometimes over-estimated. In fact, every branch of mathematics was going through the same convulsions. For example, in algebra the (central modern) notion of isomorphism began to appear by 1830 or earlier in the work of Galois and others, but the simple, general modern concept is not exactly present until perhaps 1910 (Steinitz on fields) or even 1920! One might think nothing special was happening here, as mathematics is always evolving. But B.L. von der Waerden's *Modern Algebra* [F1950], a graduate-level algebra book, appeared first in 1930, and is still widely used as a textbook today, fifty years later! During 1850-1930 the serious changes in style always going on would have made such a thing almost impossible. An essential feature of the modern approach is just that there is in a sense an *end of the line* in style and rigor. In most fields that was reached within ten years of 1930!

The early developments in (the foundations aspect of) set theory were closely connected with the 'convulsions' in *analysis* in the 19th century. In a reversal of the lack of rigor in the 18th century, the modern, rigorous approach to analysis began to appear in the early 1800's. By 1830, Cauchy and others were able to use *almost* the modern style in defining limits or continuity. One man ahead of his time went even further than Cauchy towards our modern analysis, namely the priest, B. Bolzano (1781-1848) (see [F1810], [F1837], and [F1851])*. Bolzano began, in particular, to use the notion of (arbitrary) *set* much more in analysis. In the same period the closely related notion of arbitrary *function* was emerging, e.g., in the work of Fourier (1810), Dirichlet (1840), and Riemann (1826-1866). (One date as in "Fourier (1810)," refers to the time of a key publication). Bolzano is indeed the only person ever proposed as a predecessor for Cantor (and was so even by Cantor himself). Bolzano had begun (but only begun) to study the notion 'A and B can be put in one-to-one correspondence' – the keystone of Cantor's theory.

By 1861, K. Weierstrass was able to give almost our idea of a 'modern' course in real variables! A decade or so later, Dedekind and Cantor created what are still the two main methods for constructing the reals from the rationals. This was a needed step in reducing all of mathematics to set theory. Richard Dedekind (1831-1916), an older mentor and longtime friend of Cantor's, contributed much to the set-theoretical study of the natural numbers. His famous book *Was sind und was sollen die Zahlen* [F1888] is full of passages which were new then but are now spoken in every beginning course in real variables or modern algebra. G. Peano (1895) is also famous for his contributions to the same subject.

Ernst Zermelo [F1904, 1908, 1908a, 1909, 1913] seems to have been the first to take each natural number to be a certain set. He also was the first to know (although in an odd way) how to get along with only ϵ and without the ordered couple† (though later Norbert Wiener in 1915 pointed out the much more elegant

*References like "Bolzano [F1851]" are explained at the beginning of the Bibliography.

†Roughly, Zermelo observed he could get along using only functions $f\colon A \to B$ where A and B are disjoint; and that such an f can be taken to be a suitable set of doubletons $\{a,b\}$. (The author learned this from Gregory Moore.)

fact that one can simply *define* a reasonable notion of ordered couple using only ϵ.) These two steps taken by Zermelo were among the last needed to show that one could do the entire mathematics in a set theory using only the notion 'belongs to.' Clearly these steps were not unrelated to Zermelo's most famous contribution, made in 1908, when he gave the first axiom system for set theory (and hence for all of mathematics). The most frequently used axioms today are nearly just his! Nevertheless, Zermelo's axiomatic work was lacking in one respect, as he did not realize that an axiomatic system cannot be fully understood until its underlying logic is fully understood.

Earlier, at exactly the same time as Cantor, there was a man, of perhaps the same brilliance as Cantor, who leaped ahead fifty years in the subject of *logic*. That man was Gottlob Frege. (His key publications were in 1879-1903). Some of his contributions will be mentioned a bit later on.

In the years 1897-1902, paradoxes (that is, contradictions) were discovered by Bertrand Russell and others using what perhaps *might* be taken as acceptable set-theoretical axioms. (Frege, almost alone, had actually assumed these axioms!) For many decades, the paradoxes were considered to be the central feature of set theory or at least of its foundations. For example, many books seemed to imply that the whole purpose of the axiomatic approach in set theory was to deal with the paradoxes (awkward for Euclid!) In recent times, the paradoxes have been assigned a lesser importance, though certainly a great one. In fact, Cantor himself knew that it was necessary to distinguish between 'ordinary' sets and very large, 'bad' collections. Obviously, Zermelo had very good reasons for studying the axiomatization of set theory (and the whole of mathematics!) – even without the paradoxes. But the paradoxes have certainly played a central role in mathematical philosophy for decades. Another controversy has been over the axiom of choice – which was made famous by Zermelo in 1904 when he derived from it the well-ordering principle. Both the paradoxes and the axiom of choice will be discussed (mathematically) in later chapters, where some brief remarks will be made on their history and, in general, on the history of set theory after 1900.

Let us return to the history of a fully correct axiomatization of set theory (which we shall follow up to 1930). Frege had provided just the understanding of logic and logistic systems needed for a fully correct axiomatization. (A logistic system is a deductive system in which the notion of what is a proof is so clear it can be decided by a machine.) Unfortunately, Frege's work was not widely known even by 1900. However, at almost the same time as Zermelo's axiomatization, another very different axiomatization of set theory was carried out by Bertrand Russell and Alfred North Whitehead in their famous three volume work, *Principia Mathematica.* From the point of view of a working mathematician, Zermelo's axioms were (and are) tremendously better than the awkward system of the *Principia.* However, Russell and Whitehead knew and appreciated Frege's work in logic. They included formal logic in their work and, although possibly with some flaws, their work was a logistic system (for set theory and the whole of mathematics). It is thus clear why this awkward work was so acclaimed and so influential.

There remained the (actually very easy) problem of getting a system both workable and fully logistic. In [F1923], Thoralf Skolem succeeded, say, 95% in simply making the working system of Zermelo into a logistic one. He proposed that Zermelo's 'schemas' (which caused the trouble) be replaced by the infinite set of their 'ϵ-instances' (exactly as is often done today). But, alas, it seems unlikely that Skolem grasped then the idea of a logistic system for logic. (In his other papers of the time, he always considers that the only meaning of 'logically valid' is 'holds in all models.') Nevertheless, it seems that in 1923, to the advanced logicians in the famous 'Hilbert group' in Germany and the famous 'Polish group,' Skolem's paper must have suggested the full logistic system, just as it does today.

In the very same paper, Skolem shared at least partly with Adolf Fraenkel [L1922] and Dmitri Mirimanoff [L1917] the (independent) discovery of the principal missing axiom in Zermelo's system, needed to carry out Cantor's original work – the axiom of replacement. (Fraenkel's influence in the matter was greatest and the axiom is often called 'Fraenkel's axiom.')

There is one other axiom usually included today, namely, the axiom of regularity. This axiom concerns 'partial universes' which are loosely related to Russell's 'types.' The partial universes and the axiom of regularity were first considered by Mirimanoff [L1917] (and also in Skolem [F1923]). They were given a definitive study by Johan (later John) von Neumann in a group of papers on set theory [F1923-1928].

In the same group of papers, the young von Neumann, who was to become perhaps the greatest mathematician of the first half of the twentieth century, published the first flawless axiomatization of set theory. In fact, it had its own awkwardness and has rarely been used! A workable modification of it was given by P. Bernays [F1937-1954]. Notice that the two systems of set theory (which do not differ very much) most used today (Zermelo-Fraenkel and von Neumann-Bernays) were achieving exactly their present form in just the years when van der Waerden's *Modern Algebra* appeared!