APPENDIX

Proofs of some results in Chapter 9

Proposition 3.6. $\vdash \Phi \leftrightarrow \Phi'$ *for any variant* Φ' *of* Φ.

Proof. First a lemma:

Lemma. *Suppose* y *does not occur in* Φ. *Then: (a)* y *is free in* $\Phi\binom{x}{y}$ *at exactly the places where* x *is free in* Φ. *(b)* $\vdash \forall x\Phi \leftrightarrow \forall y\Phi\binom{x}{y}$.

(a) is obvious. We show $\leftarrow$ in (b) ($\rightarrow$ is similar but easier). Since y is not in Φ, the hypotheses of 3.5 hold, and 3.5 gives $\vdash \forall y\Phi\binom{x}{y} \rightarrow \Phi\binom{x}{y}\binom{y}{x}$. One now easily gets $\vdash \forall y\Phi\binom{x}{y} \rightarrow \forall x\Phi$, using Axioms II and III (since x is not free in $\forall y\Phi\binom{x}{y}$). So the Lemma is proved.

Let us say that Φ' is a *special* variant of Φ if it is a variant and no bound variable of Φ' occurs (at all) in Φ. It is easy to see (using 'new' variables) that *any two variants* have a common special variant. So 3.6 reduces to showing (*) *each* Φ *is equivalent to each of its special variants.*

The case where Φ is atomic is trivial. If Φ is $\Phi_1 \rightarrow \Phi_2$ clearly any special variant Ψ of Φ is of the form $\Psi_1 \rightarrow \Psi_2$ where Ψ_i is a special variant of Φ_i for $i = 1, 2$. By the induction hypothesis, $\Phi_1 \leftrightarrow \Psi_1$ and $\Phi_2 \leftrightarrow \Psi_2$ so clearly $\Phi \leftrightarrow \Psi$. The case where Φ is $\sim\Phi_1$ is similar. Now let Φ be $\forall x\Phi_1$, and let Ψ be any special variant of Φ. Ψ can be written $\forall y\Psi_1$, where x and y are different. Clearly, (**) Φ_1 and Ψ_1 are variants except that Φ has free occurrences of x exactly where Ψ has free occurrences of y. Now Ψ_1 has no bound occurrence of x (since Ψ is a special variant of Φ); and also x is not free in Ψ_1 (as then it would be free in Ψ and hence in $\Phi = \forall x\Phi_1$). So we can apply the Lemma to get $\vdash \forall y\Psi_1 \leftrightarrow \forall x\Psi_1\binom{y}{x}$. But, by (**), $\Psi_1\binom{y}{x}$ is a variant, and so clearly a special variant, of Φ_1. Thus, by the induction hypothesis, $\vdash \Phi_1 \leftrightarrow \Psi_1\binom{y}{x}$. Hence $\vdash \forall x\Phi_1 \leftrightarrow \forall y\Psi_1$, i.e., $\vdash \Phi \leftrightarrow \Psi$. Thus (*) (and 3.6) are proved.

Theorem 4.1. (a) *Assume S is safe for Φ and $\vdash \Phi$. Then $\vdash \Phi(S)$.* (*b*) *If $\vdash \Phi$ then $\vdash^{\mathcal{L}'} \Phi$ in any language $\mathcal{L}'$ containing Φ.*

Before beginning the proof, we make two remarks. It is easy to check that

(1) $\left\{\begin{array}{l}\text{If } S \text{ is safe for } \Phi \text{ and } z \text{ is free (or, respectively,}\\ \text{bound) in } \Phi(S) \text{ then either } z \text{ is free (or bound,}\\ \text{respectively) in } \Phi \text{ or else } z \text{ is a variable of } S.\end{array}\right.$

Secondly, it will be convenient (though it is not essential) to consider in the proofs of 4.1 and 5.1 permutations of the set Vbl. of all variables. (All our permutations f can have $f(x) = x$ for all but finitely many variables x, so they are 'finitary objects.')

If f is a permutation of Vbl, f induces in the obvious way a permutation $\bar{f}$ of the collection of all expressions and even sets of expression, etc. Sometimes we write $\Gamma^{(f)}$ instead of $\bar{f}(\Gamma)$. In defining 'formula', 'axiom of logic', etc., etc., we never singled out any particular variables. Hence $\bar{f}$ 'goes through everything'. For example, if $\Phi_1, \ldots, \Phi_n$ is a proof of Φ then $\Phi_1^{(f)}, \ldots, \Phi_n^{(f)}$ is a proof of $\Phi^{(f)}$. On the other hand, if Σ is any *sentence*, clearly $\Sigma^{(f)}$ is a variant of Σ, and hence $\vdash \Sigma \leftrightarrow \Pi^{(f)}$. In general, for any set Ax of sentences, $Ax^{(f)}$ is the collection of all $\Sigma^{(f)}$ such that Σ belongs to Ax; so clearly, *Ax and $Ax^{(f)}$ are 'equivalent' (have the same theorems).*

We now turn to the

Proof of Theorem 4.1. Assume S is safe for Φ and $\vdash \Phi$. Let $\Phi_1', \ldots, \Phi_p'$ be a proof of Φ. We can obviously construct a permutation f of Vbl which takes each variable in Φ to itself and takes every other variable in any Φ_i' into one not in S. Since S was safe for Φ clearly S is safe for each $\Phi_i'^{(f)}$. By the remark above, $\Phi_1'^{(f)}, \ldots, \Phi_p'^{(f)}$ is a proof of $\Phi_p'^{(f)} = \Phi$. Taking $\Phi_i = \Phi_i'^{(f)}$ we have shown:

(2) Φ has a proof $\Phi_1, \ldots, \Phi_p$ such that S is safe for each Φ_i.

We now show:

(3) $\left\{\begin{array}{l}\text{If } \Phi_1, \ldots, \Phi_p \text{ is a proof of } \Phi,\ S \text{ is safe for each } \Phi_i, \text{ and } \mathcal{L}'\\ \text{is a language containing all the formulas } \Phi_1(S), \ldots, \Phi_n(S),\\ \text{then } \vdash^{\mathcal{L}'} \Phi_k(S) \text{ for } k = 1, \ldots, p.\end{array}\right.$

(Thus, in particular, we will have $\vdash \Phi(S)$, proving (a).)

We prove that $\vdash^{\mathcal{L}'} \Phi_k(S)$ by induction on $k < p$. Suppose it is true for all $k' < k$. First suppose that Φ_k is obtained by Modus Ponens from earlier Φ_i's or is a case of Axioms I, II, or IV(a). Then the very same applies to $\Phi_k(S)$ at once since $-(S)$ 'goes through everything'. Hence, clearly, $\vdash^{\mathcal{L}'} \Phi_k(S)$. Next, suppose Φ_k is from Axiom III, say, Φ_k is $\Gamma \rightarrow \forall x\Gamma$ where x is not free in Γ. Then $\Phi_k(S)$ is $\Gamma(S) \rightarrow \forall x\Gamma(S)$. Now, S is clearly safe for Γ so by (1) if x is free in $\Gamma(S)$, then x is free in Γ (out, by hypothesis), or is a variable of S (impossible since S is safe for Φ_k). So x is not free in $\Gamma(S)$ and $\Phi_k(S)$ (as above) is a case of Axiom III. Finally, suppose Φ_k is a case of IV(b). To simplify the notation, say (as an example) that Φ_k is $x = y \rightarrow (Qxyxz \leftrightarrow Qxyyz)$ (x, y, z distinct). Then $\Phi_k(S)$ is either Φ_k (so provable in $\mathcal{L}'$) or else Q is a P_i (of S) and so $\Phi_k(S)$ is

$$x = y \rightarrow \left(\Theta_i\binom{x_{i1} \cdots x_{i4}}{x\,y\,x\,z} \longleftrightarrow \Theta_i\binom{x_{i1} \cdots x_{i4}}{x\,y\,y\,z}\right)$$

(No collisions occur here and hence this formula is a case of Proposition 3.4 since $\Theta_i(\dot{x}\dot{y}\dot{y}\dot{z})$ is obtained from $\Theta_i(\dot{x}\dot{y}\dot{x}\dot{z})$ by replacing a number of occurrences of x by *free* occurrences of y (this is so because S is safe for Φ). Hence, by 3.4, $\vdash^{\mathcal{L}'} \Phi_k(S)$, and the proof of (3) is complete.

It remains to prove (b). Let $\Phi_1, \ldots, \Phi_p$ be any proof of Φ. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be a list without repetitions of all $\mathcal{Q}$ such that $\mathcal{Q}$ is in some Φ_i but not in Φ. Let γ be a fixed sentence involving only $\approx$ and containing no variables from any Φ_i. For $j \le m$, let S take $\mathcal{P}_j$ (which is, say, l_j-ary) to γ (or, strictly speaking, to $(\gamma; w_1^j, \ldots, w_{l_j}^j)$ – where the w's are distinct variables not in any Φ_i.) Obviously S is safe for each Φ_i; and each $\Phi_i(S)$ is in the smallest language $\mathcal{L}'$ containing Φ. Hence, by (3), $\vdash^{\mathcal{L}'} \Phi(S)$. But $\Phi(S) = \Phi$, so $\vdash^{\mathcal{L}'} \Phi(S)$, as desired. The proof of 4.1 is complete.

We now turn to the proof of

Metatheorem 5.1. (a) *Suppose $Z^{(1)} \vdash \Phi$ and ψ is an instance of Φ. Then $Z^{(1)} \vdash \psi$.* (b) *$Z^{(1)}$ is a conservative extension of $Z^{(0)}$, i.e., if Φ is in $\mathcal{L}^{(0)}$ and $Z^{(1)} \vdash \Phi$, then $Z^{(0)} \vdash \Phi$.*

Proof of 5.1. Any Φ can be written as $\Phi(S)$ with S empty; so (b) will follow if we can prove (a) and

(a′) Moreover, in (a), if Ψ is in $\mathcal{L}_0$ then $Z^{(0)} \vdash \Psi$.

Assume $Z^{(1)} \vdash \Phi$ and Ψ is an instance of Φ. Then $\Psi = \Phi(S)$, where

(5)(a) Φ and S are as in (1) of Chapter 9, § 5.

By the Deduction Theorem, we also have:

(5)(b) $\left\{\begin{array}{l}\vdash \Sigma \rightarrow \Phi, \text{ where } \Sigma \text{ is a conjunction of sentences which are}\\ \text{universalizations of formulas } \Gamma_j(S_j), \text{ where (for each } j\text{) (5)(a) holds}\\ \text{for } \Gamma_j \text{ and } S_j, \text{ and } \Gamma_j \text{ belongs to } Z.\end{array}\right.$

At first, we shall suppose also that

(6) No variable of S is in Σ or any Γ_j or S_j.

(At the end we will see that the general case reduces to this one.)

For the sake of (a′) we extend S to S^* as follows: S^* is defined for $\mathcal{P}_1, \cdots, \mathcal{P}_m, \mathcal{P}_{m+1}, \cdots, \mathcal{P}_n$ where $\mathcal{P}_{m+1}, \cdots, \mathcal{P}_n$ is a list of those $\mathcal{Q}$ occurring in Σ but for which S was not defined. Now (as in the proof of 4.1 (b)), S^* takes each such new $\mathcal{Q}$ (i.e., for which S was not defined) to $(\gamma; \cdots)$ where γ involves only $\approx$ and all the variables of γ are new. Since (5)(a) hold for S, $\Psi = \Phi(S) = \Phi(S^*)$. By (6), S and so also S^* are clearly safe for $\Sigma \to \Phi$. Hence, by Theorem 4.1, $\vdash \Sigma(S^*) \to \Phi(S^*)$. By our assumptions and the way S^* was constructed we see that: *If* Ψ *is in* $\mathcal{L}_0$ *then so is* $\Sigma(S^*)$. Hence both (a) and (a′) reduce to showing that $\mathcal{Z}^{(1)} \vdash \Sigma(S^*)$, or that for an arbitrary, but fixed j we have:

(7) $$\mathcal{Z}^{(1)} \vdash \Gamma_j(S_j)(S^*).$$

(7) will follow at once if we can show that for some S', safe for Γ_j,

(8) $$\Gamma_j(S_j)(S^*) = \Gamma_j(S_j').$$

Just for notation, let

$$S_j = \begin{bmatrix} \mathcal{P}_1^j & & \cdots & \mathcal{P}_{m_j}^j \\ (\Theta_i^j; x_{11}^j, x_{12}^j, \ldots) & & \cdots & \cdots \end{bmatrix}.$$

By (6), S^* is safe for each Θ_i^j. Put

$$S_j' = \begin{bmatrix} \cdots & \mathcal{P}_i^j & \cdots \\ \cdots & (\Theta_1^j(S^*), x_{i1}^j, x_{i2}^j, \ldots) & \cdots \end{bmatrix} \quad i = 1, \ldots, m_j.$$

We easily see that S_j' is safe for Γ_j. (Indeed, if x is in S_j' then (by (1)) either x is in S_j – so not in Γ_j as S_j is safe for Γ_j, or else x is in S^* – so not in Γ_j by (6).) Hence, it remains only to prove (8). One can pretty much see (8) in his head; but a careful proof is nearly a page long. See Problem 1, just after the proof of 5.1.

Thus we have obtained the conjunction of (a) and (a′), assuming (6).

We now return to our position just after (5) (before we assumed (6)). Obviously, there is a permutation f which takes each variable in Φ to itself and takes every other variable in Σ or any Γ_j or S_j to one not in S. (Since S is safe for Φ, $f(x)$ is also not in S if x is in Φ.) The statements (5) and (6) are about $\Sigma, \cdots \Gamma_j \cdots, \cdots S_j \cdots, \mathcal{Z}, \Phi$ and S. We claim that

(9) $$\begin{cases} \text{(5) and (6) hold for } \Sigma_1^{(f)}, \cdots, \Sigma_j^{(f)}, \cdots, \mathcal{Z}^{(f)}, \Phi \text{ and } S \\ \text{(the last two unchanged).} \end{cases}$$

We have just chosen f so that (6) holds for $\Sigma^{(f)}$, etc., (as in (9)). We know (5) holds for $\Sigma, \ldots, \Gamma_j, \ldots, S_j, \ldots \mathcal{Z}, \Phi$, and S. Trivially, (5)(a) holds as in (9), since Φ and S are unchanged. (5)(b) passes over under f (see the remark preceding the proof of 4.1 above) to $\Sigma_1^{(f)}, \ldots, \Gamma_j^{(f)}, \ldots, S_j^{(f)}, \ldots \mathcal{Z}^{(f)}$, and $\Phi^{(f)}$. But $\Phi^{(f)} = \Phi$ so (9) holds.

We have proved above that (5) and (6) imply the conjunction of (a) and (a′). Applying this to $\Sigma^{(f)}$, etc., as in (9), we obtain: $\mathcal{Z}^{(f)(1)} \vdash \Psi$, and if Ψ is in $\mathcal{L}^{(0)}$, then $\mathcal{Z}^{(f)(0)} \vdash \Psi$. But $\mathcal{Z}^{(f)(1)} = \mathcal{Z}^{(1)(f)}$ which (see above) is equivalent to $\mathcal{Z}^{(1)}$. So $\mathcal{Z}^{(1)} \vdash \Psi$, as desired. Also $\mathcal{Z}^{(f)(0)} = \mathcal{Z}^{(0)(f)}$ which is equivalent to $\mathcal{Z}^{(0)}$. Thus, if Ψ is in $\mathcal{L}^0$, then $\mathcal{Z}^{(0)} \vdash \Psi$. The proof of 5.1 (on the putting back of substitutions) is complete.

Problem 1. Prove (8) as follows: First reduce to showing $\Delta(S_j)(S^*) = \Delta(S_j')$ for Δ an atomic subformula of Γ_j, say Δ is $\mathcal{P}_i^j z_1 z_2 \ldots$. This amounts to

(8′) $$\Theta_i^j(S^*)(s) = \Theta_i^j(s)(S^*) \text{ where } s \text{ is } \begin{pmatrix} x_{i1}^j & \cdots \\ z_1 & \cdots \end{pmatrix}.$$

(8′) reduces to the case

(8″) $\alpha(S^*)(s) = \alpha(s)(S^*)$ where α is an atomic subformula of Θ_i^j and indeed one of the form $\mathcal{P}_l u_1 u_2 \ldots$.

One can now obtain (8″) – using the fact that no u_k or x_{il}^j is in Θ.

BIBLIOGRAPHY

When possible, we depend on the bibliographies in the two books Fraenkel [1961] and Levy [1979] (see below). A reference like 'Gödel [F 1940]' directs the reader to the entry 'Gödel [1940]' in the Bibliography of Fraenkel [1961]. If 'L' is used instead of 'F', the reference is similarly to the Bibliography of Levy [1979].

Other References

1936. CHURCH, A., A note on the Entscheidungsproblem. *Journal of Symbolic Logic*, vol. 1, pp. 40-41. Correction, ibid., pp. 101-102.

1961. FRAENKEL, A., *Abstract Set Theory*. Second edition. North-Holland, Amsterdam.

1879. FREGE, G., *Begriffschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle.

1930. GÖDEL, K., Die Vollständigkeit der Axiome des logischen Funktionenkalkuls. *Monatshefte fur Math. u. Physik*, vol. 37, pp. 349-360.

1931. GÖDEL, K., Uber formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Ibid.*, vol. 38, pp. 173-198.

1960. HALMOS, P., *Naive Set Theory*. D. Van Nostrand, Princeton.

1928. HILBERT, D. and ACKERMAN, W., *Theoretische Logik*. Julius Springer, Berlin.

1978. JECH, T., *Set Theory*. Academic Press, New York.

1956. KALISH, D. and MONTAGUE, R., A simplification of Tarski's formulation of the predicate calculus. *Bulletin Amer. Math. Soc.*, vol. 62, p. 261.

1955. KELLEY, J. L., *General Topology*. Van Nostrand, Princeton.

1979. LEVY, A., *Basic Set Theory*. Springer-Verlag, Berlin, Heidelberg, New York.

1959. MONTAGUE, R. and VAUGHT, R. L., Natural models of set theories. *Fundamenta Mathematica*, vol. 47, pp. 219-242.

1982. MOORE, GREGORY, *Zermelo's Axiom of Choice – its Origin, Development and Influence*. Springer-Verlag, New York.

1929. VON NEUMANN, J., Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre. *J. reine angew. Math.*, vol. 166, pp. 227-241.

1953. SHEPHERDSON, J. C., Inner models for set theory, Part III. *Journal of Symbolic Logic*, vol. 18, pp. 145-167.

1954. SHOENFIELD, J., A relative consistency proof. *J. Symbolic Logic*, vol. 19, pp. 21-28.
1936. TARSKI, ALFRED, Der Wahrheitsbegreff in den formalisierten Sprachen. *Studia Philosophica*, vol. 1, pp. 261-405.
1951. TARSKI, A., Remarks on the formalization of the predicate calculus. *Bulletin Amer. Math. Soc.*, vol. 57, p. 81.

Recommendations for more advanced reading.

For further study in straight set theory, we recommend Levy [1979].

For reading in 'meta-set theory' (about the work of Gödel, Cohen, etc.), see Jech [1978]. Also, for an informal but readable outline, see Cohen [L1966].

INDEX

Solutions for Selected Problems

Note: Some problems are not discussed. For example, if a section contains three similar problems, only one may be solved, so the student can still solve the others for himself.

CHAPTER 1

§1. Problem 1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ so $x \in$ right side. If $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$, so $x \in$ right side. So in either case $x \in$ right side.

Then we have proved left $\subseteq$ right. Now suppose $x \in$ right side, so $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then clearly $x \in$ left side. If not, $x \in B$ and $x \in C$ so clearly again $x \in$ left side. Then $x \in$ left side.

Thus we have proved right $\subseteq$ left, so (by above) right = left.

Problem 2. Prove only the first equation.

Problem 5. Assume hypothesis and $x \in A$ (as in 'Note'). Then $x \in A \cup C = B \cup C$. So $x \in B$ (as desired) or $x \in C$ (which we assume). Hence $x \in A \cap C = B \cap C$, so $x \in B$ as desired. So we have proved $A \subseteq B$. One can 'repeat' for $B \subseteq A$. Or having proved that for *any* A, B, C, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A \subseteq B$ apply this to B, A, C (for A, B, C) getting $B \subseteq A$ at once.

Problem 6. (3), (3), (1), (2), (1), (2).

§2. Problem 1. *Additional hint:* Take $t = \{x : x \in C \text{ and } ?\}$.

§3. **Problem 1.** Let $x \in$ left side. Then $x \notin \bigcup_{i \in I} \mathcal{A}_i$, so for any $i \in I$, $x \notin \mathcal{A}_i$, i.e., $x \in \tilde{\mathcal{A}}_i$. So $x \in$ right side. Now let $x \in$ right side; the argument is similar but different.

Problem 2. *Note*, the second equation in 3.1(b) should read:

$$B \cup \bigcap_{i \in I} \mathcal{A}_i = \bigcap_{i \in I} (B \underline{\cup} \mathcal{A}_i).$$

Problem 4. ∘ means bound, □ free.

(a) for all y, $y < x$

(c) $$\sum_{i=1}^{n} (i^2 + 1)$$

§4. **Problem 1** Let $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Since $\{a\} \in$ left side, $\{a\} \in$ right side, and hence either $\{a\} = \{c\}$ (case 1) or $\{a\} = \{c, d\}$ (case 2). Continue in this style.

Problem 2. Part on 4.2(c) Both sides are clearly sets of ordered couples. So we can begin with: Let $(x, y) \in A \times (B - C)$. Then $x \in A$ and $y \in B - C$, i.e. $y \in B$ and $y \notin C$. Hence $(x, y) \in A \times B$ and $(x, y) \notin A \times C$, so $(x, y) \in$ right side. Now let $(x, y) \in$ right side...

§5. **Problem 1.** For example: $G = M/P$ where $P = M \cup F$.

Problem 2. Both sides are sets of ordered couples. Let $(x, y) \in R/S$. Then $(y, x) \in R/S$. So we can choose z such that $(y, z) \in R$ and $(z, x) \in S$. hence $(x, z) \in \breve{S}$ and $(z, y) \in \breve{R}$. But then $(x, y) \in \breve{S}/\breve{R}$. Now let $(x, y) \in$ right side, etc.

Problem 3. Let $y \in$ left side. Then for some x, xRy and $x \in \bigcup_{i \in I} \mathcal{A}_i$, i.e. we can choose i such that $x \in \mathcal{A}_i$. But then $y \in R[\mathcal{A}_i]$ so $y \in$ right side. Other direction 'similar but different'.

Problem 4. Let $x \in$ left side. Then $x \in \operatorname{dom} f$ and $f(x) \in A - B$, etc. etc.

§6. **Problem 2 (6.22.)** Let $x \in \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} \mathcal{A}_{if(i)}$. Then we can choose $f \in \prod_{i \in I} J_i$ such that $x \in \bigcap_{i \in I} \mathcal{A}_{if(i)}$, i.e., for all $i \in I$, $x \in \mathcal{A}_{if(i)}$. But $f(i) \in J_i$. So for all $i \in I$ there exists $j \in J_i$ (namely $j = f(i)$) such that $x \in \mathcal{A}_{ij}$.

§7. **Problem 1.** 5.4(a)(i) becomes $\underline{A} \underset{\mathrm{id}_A}{\cong} \underline{A}$, etc. etc.

Problem 2. Left implies right: Put $\underline{B} = \underline{A}' \mid f[A]$. Clearly $\underline{A} \underset{f}{\cong} \underline{B}$. The other direction follows at once from the fact that if $\mu, \nu \in f[A]$ then $\mu R' \nu$ iff $\mu(R' \cap (B \times B))\nu$.

Problem 3. *Hint.* For 'at most one' assume R' and R'' both work and show $R = R'$. For 'at least one', let $R' = \{(f(x), f(y)) : xRy\}$ and show R' works.

§8. **Problem 1.** Further hint: Define F on A by requiring $F(a) = \{b : b \subseteq a\}$ (for each $a \in A$).

Problem 2. First we prove 8.2: We only need to show f is one-to-one and xRy whenever $f(x)R'f(y)$. Write $\underline{A}$ for (A, R) and $\underline{A}'$ for (A', R'). Let $x, y \in A$ and $x \neq y$. Then, since $\underline{A}$ is connected, either xRy or yRx. By symmetry it is enough to deal with the case xRy: By hypothesis $f(x)R'f(y)$. Since $\underline{A}'$ is irreflexive, this implies $f(x) \neq f(y)$ as desired. Now suppose $f(x)R'f(y)$. We want xRy. If not, then $x = y$ or yRx (as $\underline{A}$ is connected). If $x = y$ then $f(x) = f(y)$, contrary to $\underline{A}'$ being irreflexive. If yRx then $f(y)R'f(x)$; but also $(f(x)R'f(y))$, contrary to $\underline{A}'$ being asymmetric. So 8.2 is proved. We have only needed: $\underline{A}$ is connected, and $\underline{A}'$ is asymmetric (as asymmetric implies irreflexive).

Problem 3. Good little problem, but no need to give answer.

Problem 4. Use the complement of the proper initial segment, etc.

Problem 6. (a) → (b) Outline: Suppose (a) holds and $X \subseteq A$, $x_0 \in X$, and b_0 is a lower bound for X. If $b_0 \in X$, then b_0 is 'clearly' the least element of X which is always a g.l.b. for X. So we can let $b_0 \notin X$. Put $X' = \{y : (\exists x \in X)(x \leq y)\}$. 'Clearly' $b_o \notin X'$. So $\widetilde{X}'$ is non-empty and bounded above. Hence by (a), $\widetilde{X}'$ has a l.u.b. u. One now shows that u is also a g.l.b. for X (as desired). The rest of Problem 6 must still be proved.

Problem 7. We give the answers, but not the proofs: False, false, true, false.

CHAPTER 2

§1. Problem 1. By 1.2.3 get t as in the Hint. Take $Z = \{t\} \times Y$.

Problem 2. We assume here the equivalence of (a)-(d). We show (b) $\to$ (e). Let $\overline{\overline{A'}} = \kappa$. Take A, B as in (b). By the Exchange Principle obtain $C)(A'$ such that $C \sim B - A$. Clearly $B' = A' \cup C$ is as desired.

Problem 3. Suppose $\kappa \leq \lambda$. Then, by 1.5, we can fix A, B with $A \subseteq B$, $\overline{\overline{A}} = \kappa$, $\overline{\overline{B}} = \lambda$. If $\kappa = 0$, we are done.d If not, take $a_0 \in A$. Let f be on B with $f(b) = a_0$ if $b \in B - A$, and $f(a) = a$ if $a \in A$. Clearly f is on B onto A, so $\kappa \leq {}^*\lambda$.

§2. Problem 1. 2.2(a) is obvious. In 2.2(b), let κ be a natural number and let X be any set such that $0 \in X$ and for every λ, if $\lambda \in X$ then $\lambda + 1 \in X$. By 2.1, $\kappa \in X$. Hence $\kappa + 1 \in X$ for any such X. So by 2.1, $\kappa + 1$ is a natural number.

Problem 2. One easily shows by induction (2.2(c)) that (*) for every n, $\mathcal{N}_n$—by (a), (b). (Note: 2.2(c) has a class variable $\mathcal{P}$ in it, and in proving * we take $\mathcal{N}$ for $\mathcal{P}$.) We now show that for any κ, if $\mathcal{N}_\kappa$ then κ is a natural number, by the 'induction' given in hypothesis (c)—taking $\mathcal{P}_\kappa$ to be 'If $\mathcal{N}_\kappa$ then κ is a natural number'. $\mathcal{P}_0$ is trivial since 0 is a natural number. Assume $\mathcal{N}_\kappa$ and assume: if $\mathcal{N}_\kappa$ then κ is a natural number. So κ is a natural number. Hence $\kappa + 1$ is a natural number (by 2.2(b)), so $\mathcal{P}(\kappa + 1)$ holds. So (*) is proved (by (c)), as desired.

Problems 3-12. The Corollaries are easy but require small arguments. For example: Prove 2.3(d)(i): Let $m \leq n$. By 1.4 there is a κ such that $m + \kappa = n$. By 1.3(a) $\kappa + m = n$, so $k \leq n$. Hence, by (b), κ is a natural number, proving 'at least one'. Suppose $m + k = n$ and $m + l = n$. Then $k + m = l + m$, so $k = l$ by (c). Thus 'at most one' is proved.

Also, we prove 2.3(d)(ii): Suppose $A \sim B$ and $B \subsetneq A$, where A is finite. Then $B \cup (A - B) = A$ and $B \cup \emptyset = B$, so $B \cup (A - B) \sim B \cup \emptyset$. Since B is finite, we can 'cancel' by (c), getting $A - B = \emptyset$, which is absurd.

The induction problems in 3-12, 13, and 14 more or less 'correct themselves'. Here is one: Problem 10 (prove 2.3(b)): Fix (A, R). We show by induction on 'n' that [for any $B \subseteq A$ of power n, if $B \neq \emptyset$ then B has a first element]. (We have just done more than half the work.) Write $\mathcal{P}(n)$ for $[\ldots n \ldots]$ above. $\mathcal{P}(0)$ holds vacuously, as $\emptyset \neq \emptyset$. Assume $\mathcal{P}(n)$. Suppose $B \subseteq A$, and $\overline{\overline{B}} = n + 1$, and $B \neq \emptyset$. Since $B \neq \emptyset$ we can choose $b \in B$. Then $B = \{b\} \cup (B - \{b\})$, a 'disjoint union'. By (b) $\overline{\overline{B - \{b\}}}$ is a natural number, say, k. Thus $k + 1 = n + 1$ so $k = n$ (by (c)). If $B - \{b\} = \emptyset$ then $B = \{b\}$, which has least element b. Suppose $B - \{b\} \neq \emptyset$. Then by the induction hypothesis (i.e., $\mathcal{P}(n)$), $B - \{b\}$ has a least element, c. It is easy to see that $\min\{b, c\}$ is a least element of B.

Problem 15. No induction is needed. $\Rightarrow$ is very easy. $\Leftarrow$ is a little bit harder. In both directions, use a result or two from 2.3(a)-(j).

§3. **Problem 1.** We want to prove that for *any* μ, ν, $(2 + \mu) + (2 + \nu) \leq (2 + \mu)(2 + \nu)$, i.e., $4 + \mu + \nu \leq 4 + 2\mu + 2\nu + \mu\nu$—which clearly holds.

Problem 2. *Hint.* First study the proof of 3.4(a) given in the book.

Problem 3. We must show that the F defined in (2) is into and onto $A^{B \times C}$ and one-to-one. We illustrate by showing onto (given into). Let $g \in A^{B \times C}$, that is, $g : B \times C \to A$. For each $c \in C$, let h_c be the function on B such that for all $b \in B$, $h_c(b) = g((b, c))$. Now let f be the function on C such that for any $c \in C$, $f(c) = h_c$. Then for any $b \in B$, and $c \in C$, $F(f)(b, c) = f(c)(b) = g((b, c))$; so $F(f) = g$.

Problem 4. Suppose $A)(B$. We will define F so that $A^{B \cup C} \underset{F}{\sim} A^B \times A^C$. Indeed, let F be the function on $A^{B \cup C}$ such that, for any $f \in A^{B \cup C}$, $F(f) = (f \restriction B, f \restriction C)$.

Problem 7. *Hint.* Take $\overline{\overline{A}} = x$ and consider $2 \times A$ and 2^A, if $\kappa \geq 3$ (the cases $\kappa = 0, 1, 2$ are trivial separately.

§4. **Problem 1.** $f(0) = g(0) = a$. Suppose $f(i) = g(i)$ and $i + 1 \leq q$. Then $f(i + 1) = \mathcal{A}_{f(i)} = \mathcal{A}_{g(i)} = g(i + 1)$. By induction, $f(i) = g(i)$ for all $i \leq q$.

Problem 2. Fix m. Take $\mathcal{A}_k = k + m$ for all k; and $a = 0$. Put $f_q =$ the unique f which works for q. f_q exists by 4.1(a) (which is applied several

times below). Then (5) becomes: $m \cdot n = f_n(n)$. Then $m \cdot 0 = f_0(0) = 0$. Finally, $m \cdot (n + 1) = f_{n+1}(n + 1) = f_{n+1}(n) + m = f_n(n) + m$ (by Lemma (i) $= m \cdot n + m$.

§5. **Problem 2.** *Hint.* Take $\mathcal{P}(n)$ to be: if $m \neq n$ then $f(m) \neq f(n)$. (Here m is fixed.) Case $n = 0$ is easy. Suppose $\mathcal{P}(n)$ and suppose $m \neq n+1$. Easys if $m = 0$, so take $m = p+1$. Then $p \neq n$. But we cannot apply the induction hypothesis which involves m not p. Apparently we chose a $\mathcal{P}$ which does not work. Start over with a different $\mathcal{P}(n)$.

Problem 3. ((c) $\to$ (a)). By recursion, put $h(0) = a$ and $h(n + 1) = f(h(n))$ for all n. We show by induction that for all n, $\mathcal{P}(n)$ holds, where

$$\mathcal{P}(n) \text{ is: for all } k \neq n,\ h(n) \neq h(k).$$

$\mathcal{P}(0)$ holds, as $a \notin \operatorname{Rng} f$ while $k \in \operatorname{Rng} f$ if $k \neq 0$. Assume $\mathcal{P}(n)$. Let $h(n + 1) = h(k)$ where $k \neq n + 1$. Then $k \neq 0$ as above. Say $k = l + 1$. Thus $f(h(n)) = f(h(l))$. Since f is $1 - 1$, $h(n) = h(l)$, where $l \neq n$ (as $l + 1 \neq n + 1$). This contradicts the induction hypothesis. So $\mathcal{P}(n + 1)$ holds. Now put $C = \operatorname{Rng} h \cup \{a\}$. We showed h is $1 - 1$ on N onto C, so $\overline{\overline{C}} = \aleph_0$. Thus $\overline{\overline{C}} \leq \overline{\overline{A}} = \kappa$, so $\aleph_0 \leq \kappa$, as desired.

CHAPTER 3

Problem 1. See any modern algebra book on the 'field of quotients' for the proof (of a more general result).

Problem 2. Similar to Problem 1.

Problem 3. *Hint.* Let $\underline{R}$ and $\underline{R}'$ be complete ordered fields. Apply the ideas in the hint in the book, Problem 3, to both $\underline{R}$ and $\underline{R}'$, getting W and W', and, by 2.2, a $(\underline{W}, \underline{W}')$-isomorphism h. Let $x \in R$. Put $D = \{w : w \in W$ and $w < x\}$. Clearly $D' = h[D]$ is non-empty and bounded above; so we can put $F(x) = \sup_{y \in D'} y$ (taken in $\underline{R}'$). It remains to show that F is the desired $(\underline{R}, \underline{R}')$-isomorphism. Be patient!

CHAPTER 4

§1. Problem 1. Recall that the phrase 'it is easy to see that' *always* means there are some details missing.

§2. Problem 1. By hypothesis, for each $i \in I$, $\{f : F(i) \underset{f}{\sim} G(i)\} \neq \emptyset$. By AC, there is a function h on I such that for each $i \in I$, $F(i) \underset{h(i)}{\sim} G(i)$. It is easy to check that $\bigcup_{i \in I} h(i)$ is a $1-1$ function on $\bigcup_{i \in I} F(i)$ onto $\bigcup_{i \in I} G(i)$, as desired.

Problem 3. As in Problem 1 above, let $F(i) \underset{h_i}{\sim} G(i)$ for all $i \in I$. let $\mathcal{T}$ be the function on $\prod_{i \in I} F(i)$ such that for all $f \in \prod_{i \in I} F(i)$, $\mathcal{T}(f)$ is the function on I such that for each $i \in I$, $\mathcal{T}(f)(i) = h_i(f(i))$. It is routine to show $\mathcal{T}$ is $1-1$ on $\prod_{i \in I} F(i)$ onto $\prod_{i \in I} G(i)$, as desired.

Problem 4. See Problems 1 and 7 of §3, Chapter 2 for special cases.

§3. Problem 1. *Hint.* Let $\overline{\overline{K_i}} = \kappa_i$ and $\overline{\overline{L_i}} = \lambda_i$ for each $i \in I$. Let $F : \bigcup_{i \in I} K_i \to \prod_{i \in I} L_i$. We show F is not onto. For each $i \in I$, let G_i be the function on K_i such that, for each $k \in K_i$, $G_i(k) = F(k)(i)$. Thus $G_i : K_i \to L_i$. But G_i is not onto L_i, since $\kappa_i < \lambda_i$ (and by 2.1.8(b)). Hence, by AC, there is an f on I such that for each $i \in I$, $f(i) \in L_i - G_i[K_i]$. Complete the proof by showing that $f \in \prod_{i \in I} L_i - F[\bigcup_{i \in I} K_i]$. (Hint^2. f has been chosen so that for each i, it differs at the i^{th} place not from *one* function but from each of a whole group of functions.)

§4. Problem 4. $\mathfrak{c} \leq \overline{\overline{A}}$, as one see by considering the set of all constant functions on R to R. If $f, g \in A$ and $f(t) = g(t)$ for all $t \in Q$, then $f = g$. Indeed, if $x \in R$ then x can be expressed as $\lim_{n \to \infty} t_n$ where each $t_n \in Q$; since f and g are continuous, $f(x) = \lim_{n \to \infty} f(t_n) = \lim_{n \to \infty} g(t_n) = g(x)$. Since each $f \in A$ is thus uniquely determined by $f \restriction Q \in R^Q$ it is easy to show $\overline{\overline{A}} \leq \overline{\overline{R^Q}} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

Problem 5. Problem 5 is harder than 4 or 6. Try it.

Problem 6. *Hint.* $U \leq R$ is open iff it is a countable union of open intervals of the form (s, t) where $s, t \in Q$ and $s < t$.

Problem 7. Denote $\mathcal{P}(N)$ by $\mathcal{S}$. Since $\mathcal{S} \sim R$ we can assume 'R' in Problem 7 has been replaced by '$\mathcal{S}$'. Suppose $A \subseteq \mathcal{S}$ and $N \underset{F_0}{\sim} A$ and put $\kappa = \overline{\overline{\mathcal{S} - A}}$. We define $W_0 = \{n \in N : n \notin F_0(n)\}$. As in Cantor's proof (cf. 3.1), $W_0 \in \mathcal{S} - A$. Define F_1 on N by: $F_1(0) = W_0$, $F_1(n+1) = F_0(n)$. Clearly $N \underset{F_1}{\sim} A \cup \{W_0\}$. Using recursion we can iterate this whole process to define W_i for each $i \in N$, where the W_i's are distinct members of $\mathcal{S} - A$. Hence $\aleph_0 \leq \kappa$. Say $\kappa = \aleph_0 + \lambda$. It follows that $\mathfrak{c} = \aleph_0 + \kappa = \aleph_0 + \aleph_0 + \lambda = \aleph_0 + \lambda = \kappa$. So $\overline{\overline{\mathcal{S} - A}} = \mathfrak{c}$, as was to be proved.

CHAPTER 5

§1. **Both problems are pretty easy, but they are important and should be done.**

§2. Problem 1. Prove 2.3(a). Let $(\sigma_i : i \in I)$ be a function on I whose values σ_i are order types. We can form $(\overline{\overline{\sigma_i}} : i \in I)$. By Assumption 4.2.1 there is a (fixed) list $(A_i : i \in I)$ such that for each $i \in I$, $\overline{\overline{A_i}} = \overline{\overline{\sigma_i}}$. For any $i \in I$, put $C_i = \{(A_i, <) : i \in I \text{ and } (A_i, <) \text{ is an order of type } \sigma_i\}$. For each $i \in I$, $C_i \neq \emptyset$ (clearly) and C_i is 'set' (easily). So, by AC, there is a function $\underline{B}$ on I such that for each $i \in I$, $\underline{B}_i \in C_i$, so $T_p \underline{B}_i = \sigma_i$. This proves 2.3.

Problem 3. Clearly $(N - \{0\}, \mathfrak{c})$ and $(N, \mathfrak{c})$ (where $\mathfrak{c}'$ is $\mathfrak{c}$ 'cut down') have type ω, and $1 + \omega$, respectively; and are isomorphic. So $1 + \omega = \omega$.

Problem 4. Clearly $1 + \omega \neq \omega + 1$. Clearly $2 \cdot \omega = \omega$ while $\omega \cdot 2 = \omega + \omega \neq \omega$.

Problem 6. Take $\alpha = \beta = 1$, $\gamma = \omega$. Then $(\alpha + \beta)\gamma = 2 - \omega = \omega$, while $\alpha\gamma + \beta\gamma = \omega + \omega \neq \omega$.

Problems 5 and 7. Problems 5 and 7 are easy but good.

Problem 8. Starting hint. Using 2.5(a), one obtains an isomorphism between B (and its induced order) and $(Q, \leq)$.

CHAPTER 6

§1 has no problems.

There are good hints in the book for the problem in §2.

§3. Problem 1. $(\mathcal{A}_x : x \in X) = \{(x, \mathcal{A}_x) : x \in X\}$, which exists as a case of $\{B_x : x \in U\}$ whose existence has been proved in item 2 in our list.

Problems 2 and 3 are the key ones.

Problem 2. There are two quite different proofs. The first depends on the Replacement Axiom but not on the Power Set Axiom: Let $b \in B$. Put $A \times \{b\} = \{(a, b) : a \in A\}$ which exists by items 2 and 8 in our list. Now, $A \times B = \bigcup\{A \times \{b\} : b \in B\}$, which exists by items 2 and 3 in our list. The second proof depends on the Power Set Axiom but not on the Replacement Axiom: Note that $\{a\}, \{a, b\} \in \mathcal{P}(A \cup B)$—if $a \in A$ and $b \in B$. So if $a \in A$ and $b \in B$ then $(a, b) = \{\{a\}, \{a, b\}\} \in \mathcal{PP}(A \cup B)$. So $A \times B = \{z : z \in \mathcal{PP}(A \cup B)$ and for some $a \in A$ and $b \in B$, $z = (a, b)\}$ which exists by the Separation Axiom.

Problem 3. Hints. Again there are two proofs. The first depends on Replacement but not on the Union Axiom. The second depends on Union but not on Replacement.

Problems 4 and 5 are routine, or soon will be.

CHAPTER 7

§1. Problem 3. Let $f, g \in H$. Put $U = \operatorname{dom} f$ and $V = \operatorname{dom} g$. Both U and V are initial segments of $\underline{A}$. Since $\underline{A}$ is connected, one can easily show (by considering $U \cap V$) that one of U, V, say U, is an initial segment of the other, V. Since $g \in H$, g is an isomorphism of $\underline{V}$ onto an initial segment of $\underline{B}$. Since 'isomorphism preserves everything', $g[U]$ is an initial segment of $g[V]$, so of $\underline{B}$. Thus both f and $g \restriction U$ are isomorphisms of $\underline{U}$ onto initial segments of $\underline{B}$. So $f = g \restriction U$, by 1.1(d), and hence $f \subseteq g$.

Problem 4. Partial solution. First we show: h is a function. Clearly h is a set of ordered couples. Suppose $(a, b), (a, b') \in h$. Then $(a, b) \in f$ and $(a, b') \in f'$, for some $f, f' \in H$. Since H is a chain, either $f \subseteq g$ or $g \subseteq f$, say the former. Then $(a, b), (a, b') \in g$, so $b = b'$.

Using several similar arguments one may show also that: $\operatorname{Dom} h$ is an initial segment of A, $\operatorname{Rng} h$ is an initial segment of $\underline{B}$, h is $1 - 1$, and for any $a_1, a_2 \in \operatorname{Dom} h$, $a_1 \underset{A}{<} a_2$ iff $h(a_1) \underset{B}{<} h(a_2)$.

Problem 7. Fix $b \in B$. Assume that for each $c < b$, there is exactly one f which works for $\underline{B}_c$ (call it f_c). We will show there is an f which works for $\underline{B}_b$. (By (6) there is always at most one such f.) By (5),

$$\text{(a) if } c < d < b \text{ then } f_c = f_d \restriction \operatorname{pred} c.$$

Consider first the case in which b is a limit element or the first element. Let f be on $\operatorname{pred} b$ with $f(c) = f_{c^+}(c)$ for all $c < b$. (c^+ is the immediate successor of c.) Then $f(c) = f_{c^+}(c) = \mathcal{A}_{f_{c^+}\restriction \operatorname{pred} c}$ (as f_{c^+} works for $\underline{B}_c$) $= \mathcal{A}_{f_c}$ (as, by (5), $f_c = f_{c^+} \restriction \operatorname{pred} c$). Thus f works for $\underline{B}_b$, as desired. The other case is where $b = c^+$. This time put $f = f_c \cup \{(c, \mathcal{A}_{f_c})\}$. Clearly f is a function on $\operatorname{pred} b$, and also (b) $f_c = f \restriction \operatorname{pred} c$. If $d < c$, then $f(d) = f_c(d) = \mathcal{A}_{f_c \restriction \operatorname{pred} d} = \mathcal{A}_{f \restriction \operatorname{pred} d}$ as desired. Finally, $f(c) = \mathcal{A}_{f_c} = \mathcal{A}_{f \restriction \operatorname{pred} c}$, by (b). Thus f works for $\underline{B}_b$.

Problem 8. Let $\underline{A}$ be any well-order. By 1.5, there is exactly one f which works for $\underline{A}$ over $\mathcal{B}$. Clearly, by our definition of $\mathcal{O}$, $\mathcal{O}(\underline{A}) = \mathcal{B}_f$. We want to prove that (7) holds for our $\mathcal{O}$ and $\underline{A}$, i.e. $\mathcal{O}(\underline{A}) = \mathcal{B}_{(\mathcal{O}(\underline{A}_a) : a \in A)}$. Thus it is enough to show $f(a) = \mathcal{O}(\underline{A}_a)$, for each $a \in A$. Since f works for $\underline{A}$ over $\mathcal{B}$, we have: $f(a) = \mathcal{B}_{f.\text{pred}\,a}$. By definition, $\mathcal{O}(\underline{A}_a) = \mathcal{B}_g$ where g works for $\underline{A}_a$ over $\mathcal{B}$. Thus it is enough to prove that $f \restriction \text{pred}\, a = g$. But this holds by (5) and (6) (and (0)).

Problem 9. Using (0) for $\underline{A}$ and for $\underline{A}'$, we see it is enough to show:

$$\text{If } \underline{A} \underset{f}{\cong} \underline{A}' \text{ then } (*) \text{ for any } a \in A,\ \mathcal{O}(\underline{A}_a) = \mathcal{O}(\underline{A}'_{f(a)}).$$

(To this it is enough to apply (0) to (the original) $\underline{A}$ and $\underline{A}'$.) Assume $\underline{A} \underset{f}{\cong} \underline{A}'$. We prove (∗) by $\underline{A}$-induction on 'a'. Suppose $\mathcal{O}(\underline{A}_b) = \mathcal{O}(A'_{f(a)})$ for all $b < a$. Then

$$\begin{aligned}\mathcal{O}(\underline{A}_a) &= \mathcal{A}_{\{\mathcal{O}(\underline{A}_b):b<a\}} \text{ (by (7), since if } b < a \text{ then } (\underline{A}_a)_b = \underline{A}_b)\\ &= \mathcal{A}_{\{\mathcal{O}(\underline{A}'_{f(a)}):b<a\}} \text{ (by the inductive hypothesis)}\\ &= \mathcal{A}_{\{\mathcal{O}(\underline{A}'_x):x<f(a)\}} \text{ (as pred } f(a) = f[\text{pred}\, a])\\ &= \mathcal{O}(\underline{A}'_{f(a)}) \text{ (by (7) with } \underline{A}'_{f(a)} \text{ for "}\underline{A}\text{"), as was to be proved.}\end{aligned}$$

§2. **Problem 1.** Let K be any set of ordinals. We will show there is a β such that for any $\alpha \in K$, $\alpha \leq \beta$. Put $\underline{B}$ = the ordered sum $\sum_{\alpha\in(K,a)} \underline{W}(\alpha)$. (Note that $\underline{W}_\alpha = \underline{W}(\alpha)$.) $\underline{B}$ is a well-order by 5.1.1(a); put $\beta = \text{Ord}\, \underline{B}$. By 1.3, for any $\alpha \in K$, $\underline{W}(\alpha)$ is isomorphic to an initial segment of $\underline{B}$, so $\alpha = \text{Ord}\, \underline{W}(\alpha) \leq \text{Ord}\, \underline{B} = \beta$, as desired.

Problem 2. (Of course, $\varepsilon_x = \{(\mu, \nu) : \mu, \nu \in x \text{ and } \mu \in \nu\}$; and well-order, here means strict well-order.) $\Rightarrow$ follows at once from results in 2.12-13 and 2.7-8.

Proof of $\Leftarrow$: For readability write $A = x$. Assume A is a transitive set and $\underline{A} = (A, \varepsilon_A)$ is a (strict) well-order. Put $\alpha = \text{Ord}\, \underline{A}$ (cf. 2.2). By 2.8(b) plus 2.12-13, $\underline{A} \underset{f}{\cong} (\alpha, \varepsilon_\alpha)$ where $f(a) = \text{Ord}\, \underline{A}_a$ for each $a \in A$. We will prove by $\underline{A}$-induction on 'a' that $f(a) = a$ for all $a \in A$. Suppose $a \in A$ and $f(b) = b$ for all $b \in_A a$. Then

$$\begin{aligned}f(a) &= \text{Ord}\, \underline{A}_a = \{\text{Ord}\, \underline{A}_b : b \in \text{pred}_{\underline{A}}\, a\} \text{ (by 2.2, as } (\underline{A}_a)_b = \underline{A}_b \text{ when } b <_{\underline{A}} a)\\ &= \{\text{Ord}\, \underline{A}_b : b \in_A a\} \text{ (at once)}\\ &= \{b : b \in_A a\} \text{ by the induction hypothesis}\\ &= a \text{ (since } a \text{ is transitive).}\end{aligned}$$

Thus $f = \text{Id}_A$ and then $A = \alpha$, so A is an ordinal, as was to be proved.

§3. **Problem 1.** We call I an ideal in A if $1 \notin I$, $0 \in I$, $x - y \in I$ whenever $x, y \in I$, and $ax \in I$ whenever $x \in I$ and $a \in A$. Assume I_0 is an ideal (in A). We want to show A_0 is included in a maximal ideal. Let K be the family of all ideals $I \supseteq I_0$. Let C be any chain of members of I. Put $I^* = \bigcup_{I\in C} I$. It is easy to check that I^* is an ideal and $I_0 \subseteq I^*$. Thus $I^* \in K$. By Zorn's Lemma (with $(K, \subseteq_K)$ for the $\underline{A}$ *there*), K has a maximal member I. If I' is any ideal and $I \subseteq I'$, then obviously $I = I'$. So I is as desired.

Problem 2. Hints given.

Problem 3. *Hint.* Show $\leq$ can be ($\subseteq$) extended to a maximal partial order $\leq'$ for A. Then show $\leq'$ must be an order.

§5. **Problem 2.** There are several proofs of 5.4. Here are hints for one. First show that for any α, $\omega_\alpha \geq \alpha$. (One can imitate the proof of 1.1(a), but over the class of all ordinals.) Thus $\aleph_\kappa \geq \kappa$, so we can let α be the least ordinal such that $\aleph_\alpha \geq \kappa$. Suppose $\aleph_\alpha > \kappa$. Each of the two cases (α limit or not) easily leads to a contradiction.

Problem 4. *Hint for the Hint*: By recursion over ω, put

$$\begin{cases}\kappa_0 = \omega\\ \kappa_{n+1} = \omega_{\kappa_n} \text{ for all } n.\end{cases}$$

Take $\sup\{\kappa_n : n \in \omega\}$. Continue.

§6. **Problem 2.** First prove easily the general facts:

(A) If Q is a set of ordinals then $\sup_{\alpha\in Q} \alpha = \bigcup_{\alpha\in Q} \alpha$.

(B) $\overline{\overline{\bigcup_{i\in I} A_i}} \leq \sum_{i\in I} \overline{\overline{A_i}}$ (which was needed for the second sentence in 6.4). Hint: If $i \in I$ and $a \in A_i$, put $F((i, a)) = a$. F maps $\bigcup_{i\in I} (\{i\} \times A_i)$ onto $\bigcup_{i\in I} A_i$. Apply 2.1.8(b). (Compare 4.4.4 and 4.4.2.)

(C) If κ is infinite, then κ is a limit ordinal. (By, for example, 6.2(b), $\kappa = \kappa + 1$. Now one easily shows that for any α, $x \neq \alpha \cup \{\alpha\}$.)

Proof of 6.5(a): Put $\lambda = \sum_{\alpha\in Q} \overline{\overline{\alpha}} \leq \sum_{\alpha\in Q} \kappa \leq \kappa \cdot \kappa = \kappa$. It remains to prove $\kappa \leq \lambda$. First, suppose κ is a limit cardinal (i.e., κ is a limit element in the class of cardinals, or, equivalently, for some limit ordinal δ, $\kappa = \aleph_\delta$). For each (cardinal) $\nu < \kappa$, there is an $\alpha \in Q$ such that $\nu \leq \alpha$ and so $\nu \leq \overline{\overline{\alpha}} \leq \lambda$. Since κ is a limit cardinal, $\kappa = \bigcup_{\nu<\kappa} \nu \leq \lambda$. Finally, suppose $\kappa = \mu^+$. We claim $\overline{\overline{Q}} = \kappa$. If not, $\overline{\overline{Q}} < \kappa$, so $\overline{\overline{Q}} \leq \mu$. Hence $\kappa = \bigcup_{\alpha \in Q} \alpha$ (by (A)) $= \overline{\overline{\bigcup_{\alpha \in Q} \alpha}} \leq \sum_{\alpha \in Q} \overline{\overline{\alpha}}$ (by (B)) $\leq \overline{\overline{Q}} \cdot \mu \leq \mu \cdot \mu = \mu$, which is absurd. Thus $\overline{\overline{Q}} = \kappa$, as claimed. Hence $\lambda = \sum_{\alpha \in Q} \overline{\overline{\alpha}} = \sum_{\alpha \in Q'} \overline{\overline{\alpha}}$ (where $Q' = Q - \{0\}$) $\geq \sum_{\alpha \in Q'} = \overline{\overline{Q'}} = \overline{\overline{Q}}$ (as Q is infinite by (C)) $= \kappa$. Thus (a) is proved.

Proof of 6.5(b): If $\mu \in Q$, put $I_\mu = \{i \in I : \overline{\overline{I_i}} = \mu\}$. Suppose $\kappa \neq \sup_{i \in I} \lambda_i =$ (say) λ. Then $\kappa = \sum_{i \in I} \lambda_i = \sum_{\mu \in Q} \sum_{i \in I_\mu} \mu = \sum_{\mu \in Q} \mu \cdot \overline{\overline{I_\mu}} \leq \sum_{\mu \in Q} \lambda \cdot \overline{\overline{I}} = \overline{\overline{Q}} \cdot \lambda \cdot \overline{\overline{I}} < \kappa$ (as κ is infinite and $\overline{\overline{Q}} \leq \overline{\overline{I}}$). This is absurd, so (b) holds.

CHAPTER 8

§1. **Problem 1.** 1.1(a),(b),(c) are excellent exercises in making proofs by transfinite induction (each is easy, but not too easy).

Problem 2. Prove 1.2(a),(b),(c). (a) and (b) can easily be done (without induction). Proof of (c):

By the class version of 7.1.1(a) (see the solution of Problem 2 of §7.5) we obtain $\text{Rk}\,\alpha \geq \alpha$. We prove $\text{Rk}\,\alpha \leq \alpha$ by induction on α. The case $\alpha = 0$ is trivial. Suppose $\alpha = \delta$, a limit ordinal, and $\gamma \subseteq V_\gamma$ for all $\gamma < \alpha$. Then $\delta = \bigcup_{\alpha < \delta} \alpha \subseteq \bigcup_{\alpha < \delta} V_\alpha = V_\delta$. Finally, suppose $\alpha = \beta + 1$ and $\beta \subseteq V_\beta$. Then $\beta \in P\beta \subseteq PV_\beta = V_{\beta+1} = V_\alpha$. Since V_α is transitive, $\beta \subseteq V_\alpha$, so $\alpha = \beta \cup \{\beta\} \subseteq V_\alpha$.

Hence, in general $\alpha \subseteq V_\alpha$, so $\alpha \in V_{\alpha+1}$. Hence V_α holds and $\text{Rk}\,\alpha \leq \alpha$. We already saw that $\text{Rk}\,\alpha \geq \alpha$, so $\text{Rk}\,\alpha = \alpha$, and 1.2(c) is proved.

Problem 3. Fix x. Define W_n by:

$$W_0 = x$$
$$W_{n+1} = \bigcup(W_n) \ (= \textstyle\bigcup_{t \in W_n} t).$$

Since ω exists, $\{W_n : n \in \omega\}$ exists by Replacement and hence $y = \bigcup_{n \in \omega} W_n$ exists (by Union). Clearly $x \subseteq y$. Let $u \in v \in y$. Then $v \in W_n$ for some n. Hence $u \in \bigcup(W_n) = W_{n+1} \subseteq y$. So y is transitive (in fact, one easily shows y *is* $Tc(x)$).

Problem 4. 1.7(a) is easily proved. Proof of 1.7(b). Suppose (2) facts. Thus there exists $A \neq \emptyset$ such that A has no ε-minimal element. For each $x \in A$, (ε-Pred x) $\cap\, x \neq \emptyset$. By Replacement $\{(\varepsilon\text{-Pred } x) \cap A : X \in A\}$ exists, so by Choice it has a choice function F. Pick $x \in A$. By ordinary recursion (and Replacement) define f by

$$\begin{cases} f(0) = x \\ f(n+1) = F(\text{Pred } f(n) \cap A). \end{cases}$$

Clearly f is an ε-descending sequence, so (3) fails.

§2. Problem 1. One can easily check that

(A) $\mathrm{Rk}\,x$ is finite if for some n, $x \in V_n$.

(B) $Tc\{x\} = \{x\} \cup Tcx$

Still fairly simple is the important

(C) $\mathrm{Rk}\,x = \sup_{y\in x}(\mathrm{Rk}\,y + 1)$

In (C), $\leq$ is nothing. For $\geq$, you might consider two cases: $\{\mathrm{Rk}\,y+1 : y \in x\}$ has a largest member, and 'otherwise'.

(2.4) ($\Leftarrow$): By a simple induction, one shows each V_n is finite. Now suppose $x \in V_n$. Then $\{x\} \subseteq V_n$ so $Tc\{x\} \subseteq V_n$. Thus if $y \in Tc\{x\}$ then $y \in V_n$, so $y \subseteq V_n$ and y is finite.

($\Rightarrow$): Let (α) be the statement: For every x of rank $< \alpha$, if every member of $Tc\{x\}$ is finite then $\mathrm{Rk}\,x$ is finite. We will prove by induction on 'α' that for every α, (α) holds. Assume (α) holds for all $\alpha < \beta$. Suppose $\mathrm{Rk}\,x = \alpha < \beta$ and (*) each member of $Tc\{x\}$ is finite. Suppose $y \in x$. Then $\mathrm{Rk}\,y < \mathrm{Rk}\,x < \beta$, so $\mathrm{Rk}\,y < \alpha$. Also, every member z of $Tc\{y\}$ ($= \{y\} \cup Tcy$ (by B)) is finite. Indeed, if $z = y$ then $y \in x \subseteq Tcx \subseteq Tc\{x\}$, so $z = y$ is finite by (*). Otherwise, $z \in Tcy \subseteq Tcx \subseteq Tc\{x\}$, so again z is finite, by (*). It now follows from the induction hypothesis that $\mathrm{Rk}\,y$ is finite. Thus $\mathrm{Rk}\,x = \sup_{y\in x}(\mathrm{Rk}\,y + 1)$ (by (C)), where x is a finite set, and each $\mathrm{Rk}\,y + 1$ (for $y \in x$) is finite. Hence $\mathrm{Rk}\,x$ is finite. By induction, (α) holds for all α, i.e., ($\Rightarrow$) holds, as desired.

§3. Problem 1. By 7.1.2 one of t and t' is isomorphic to an initial segment of the other. Say f is an isomorphism of t onto t'', an initial segment of t'. We claim that $f(U) = U$ for all $U \in t$. (Clearly the theorem follows.) The claim will be proved by t-induction on 'U'. Suppose $f(U) = U$ for all $U \subsetneq V$, where $V \in t$. If V is first in t, then by (iii) $V \neq \emptyset$, and likewise $f(V) = \emptyset = V$. Suppose there in a largest $U \in t$ such that $U \subsetneq V$. Then by (ii), $V = P(U)$. Also $f(V)$ is the immediate successor of $f(U)$ in t'' and hence, clearly, in t'. Hence by (ii), $f(V) = P(f(U)) = P(U) = V$, as desired. Now let V be a limit element in t. Note that (*) $W \in t$ and $W \subsetneq V$ iff $f(W) \in t'$ and $f(W) \subsetneq f(V)$ iff $W \in t'$ and $W \subsetneq f(V)$. Since V is a limit element, $f(V)$ is a limit element in t' (or t'') so, by (iii),

$$f(V) = \bigcup_{\substack{W\in t' \\ W\subsetneq f(V)}} W \quad = \bigcup_{\substack{W\in t \\ W\subsetneq V}} W \text{ (by (*))} = V \text{ (by (iii))},$$

completing the proof.

CHAPTER 9

§1. Problem 1. We prove by induction on Φ, that:

No proper initial segment of a formula Φ is a formula.

If Φ is an atomic formula, say Qxy (Q 2-place), then clearly no proper initial segment of Φ is a formula. Let us write $\mathcal{P}(\Phi)$ for 'no proper initial segment of Φ is a formula'. Suppose $\mathcal{P}(\Theta)$ and $\Phi = \sim\Theta$. Clearly $\sim$ is not a formula, so any proper initial segment of Φ which is a formula is of the form $\sim X$. But clearly any formula (like $\sim X$) which starts with $\sim$ is of the form $\sim\Theta'$, where $X = \Theta'$ is a proper initial segment of Θ, contrary to hypothesis. The case where $\Phi = \forall x\Theta$ is similar. Now suppose $\mathcal{P}(\Theta)$ and $\mathcal{P}(\Theta')$ and $\Phi = (\Theta \to \Theta')$ and assume Φ' is a proper initial segment of Φ. Clearly (using induction) any formula, like Φ', starting with (is of the form $(\Psi \to \Psi')$. If $l(\Psi) < l(\Theta)$ then Ψ is a proper initial segment of Θ, absurd. If $l(\Psi) = l(\Theta)$ then clearly Ψ' is a proper initial segment of Θ', absurd. So we must have $l(\Psi) > l(\Theta)$. Thus the ('main') occurrence of $\to$ in $(\Theta \to \Theta')$ must be in Ψ. But again, clearly (by induction) such an occurrence of $\to$ in a formula Ψ must have a formula, α, just to its left and another formula, β, just to its right (all inside Ψ). But then β is a proper initial segment of Θ', a contradiction, and the proof is complete.

Problem 2. First read the nine lines of hints starting at "Proof". Note: The proof that logical axioms satisfy (*) divides into inner axioms and showing that if Φ satisfies (*) so does $\forall x\Phi$.

Problem 3. The last sentence before 4.4 should end with: "is deductively closed *provided* $Ax \vdash \exists x\Psi$; that is:". Also, 4.4(a) should begin: "Suppose $Ax \vdash \exists x\Psi$. If..."

Assume all the hypotheses of (the new) 4.4(a). By 4.3 and our hypotheses $\vdash \exists xPx \to (\Sigma_1 \wedge \cdots \wedge \Sigma_n \to \Sigma)^{(P)}$. By considering the definition of $\Phi^{(P)}$, one easily sees $(\Sigma_1 \wedge \cdots \wedge \Sigma_n \to \Sigma)^{(P)}$ is identical with $\Sigma_1^{(P)} \wedge \cdots \wedge \Sigma_n^{(P)} \to \Sigma^{(P)}$, so we have $\vdash \exists xPx \to (\Sigma_1^{(P)} \wedge \cdots \wedge \Sigma_n^{(P)} \to \Sigma^{(P)})$. But then clearly $\vdash \exists xPx \wedge \forall x(Px \leftrightarrow \Psi) \to \Sigma_1^{*\Psi} \wedge \cdots \wedge \Sigma_n^{*\Psi}$. By hypothesis, $Ax \vdash \Sigma_1^{*\Psi} \wedge \cdots \wedge \Sigma_n^{*\Psi}$, hence $Ax \vdash \forall x(Px \leftrightarrow \Psi) \to \Sigma^{*\Psi}$. Hence clearly $Ax \vdash \Sigma^{*\Psi}$, as desired.

CHAPTER 10

§1. Problem 1. Preliminary remarks needed: We say Φ is $Z_0^{(0)}$-bounded if Φ is equivalent in $Z_0^{(0)}$ to a bounded formula. (Note: The first formula in (1)(a), $u \subseteq v$, is trivially $Z_0^{(0)}$-bounded. In fact, since $Z_0^{(0)} \vdash x \subseteq y \leftrightarrow (\forall t \in x)$ $t \in y$, $x \subseteq y$ is $Z_0^{(0)}$-bounded in the stronger sense, where only ϵ-bounded quantifications occur. But we do not need to discuss this sense.) If Φ is $Z_0^{(0)}$-bounded, then so are $(\forall u \in v)\Phi$ and $(\forall u \subseteq v)\Phi$ and clearly also $(\exists u \in v)\Phi$ and $(\exists u \subseteq v)\Phi$. We often consider an iota-formula Φ (e.g. "$z = (x, y)$" or "$y = Px$"), which is not strictly speaking a formula of $Z_0^{(0)}$ since (way back in Chapter 1, for example) their definition used the ι-symbol. We understand that the ι-symbol can be removed as on line 2, page 100, to get a $Z_0^{(0)}$-formula Φ' and in say Φ is equivalent in $Z_0^{(0)}$ to bounded formula if and only if Φ' is. This usage is understood on lines 5-9 of §1. Another example is this: The formula "$B = \{x : x \in A \wedge \Phi\}$" is $Z_0^{(0)}$-bounded if Φ is. In fact, it (or its $Z_0^{(0)}$-translate) is $Z_0^{(0)}$-equivalent to $(\forall x \in B)(x \in A \wedge \Phi) \wedge (\forall x \in A)(\Phi \rightarrow x \in B)$.

Now we deal with $z = (x, y)$ in 1(a). First note that

$$Z_0^{(0)} \vdash v = \{x, y\} \leftrightarrow x \in v \wedge y \in v \wedge (\forall t \in v)(t = x \vee t = y)$$

of course $v = \{x\}$ is also $Z_0^{(0)}$-bounded. Thus

$$Z_0^{(0)} \vdash z = (x, y) \leftrightarrow (\exists u \in z)(\exists v \in z)(u = \{x\} \wedge v = \{x, y\}),$$

so $z = (x, y)$ is $Z_0^{(0)}$-bounded.

The reader should do 1(b). This will be rather long, but the main requirement is patience. (In 1(b) you will require for the first time the use of $\subseteq$-bounded quantification.)

Problem 2 will now be straightforward.

Problem 3. The last three lines of (2) should be headed by '(b)'. For 2(a) we first check that if u is a partial universe, so that for some tower t, $u \in t$, then for some tower t', $t' \in PPu$ and $u \in t'$. Indeed, take $t' = \{v \in t : v \subseteq u\}$. Clearly $t' \subseteq Pu$, so $t' \in PPu$.

Now we want to show that 'u is a partial universe' is $Z_0^{(0)}$-absolute. There are two cases in the definition of absolute. We do the one about w, and leave the one about V to the reader. We must show (working in $Z_0^{(0)}$) that: if (i) [w is a limit partial universe and $u \in w$], then (ii) {[there exists t such that t is a tower and $t \in PPu$ and $u \in t$] iff [there exists t such that $t \in w$ and $(t$ is a tower$)^{(w)}$ and $(t \in PPu)^{(w)}$ and $(u \in t)^{(w)}$]}. The three formulas relativized to w are all bounded. Hence (by 1.1) the $^{(w)}$'s can be deleted. Clearly it is enough to show that if (i) holds, t is a tower, $t \in PPu$, and $u \in t$ then $t \in w$. But, by 8.3.2 and our hypotheses, it is clear that the partial universes u, PPu, and w must lie in that ($\subseteq$)-order, so $t \in w$ (as $t \in PPu$).

Problem 6. Additional hints. By definition, u is a partial universe iff for some t, t is a tower and $u \in t$. The trouble is thatthis 'for some t' is not a bounded quantification. Try instead (perhaps differently in several cases) things *like* $(\exists t' \in u)(t'$ is a tower $\wedge (\exists v \in t)(u = PPt'))$.

§2. Problem 1. Assume $ZFC^{(0)}$ is consistent. We will prove that $ZFC^{(0)}+$ 'there is no inaccessible cardinal $> \aleph_0$' is consistent. Write 'u is a big universe' or $\Delta(u)$ for 'u is a limit universe and not the first, and u has class replacement'. There exists an inaccessible cardinal $> \aleph_0'$ and $\exists u \Delta(u)$ are equivalent in $ZFC^{(0)}$ (by 8.2.6 and 8.3.3). It remains to show that $ZFC^{(0)}+ \sim \exists u \Delta(u)$ is consistent. This is certainly so if $ZFC^{(0)} \vdash\sim \exists u \Delta(u)$. So we may assume that contrary, i.e., the theory $T = ZFC^{(0)} + \exists u \Delta(u)$ is consistent.

Let $\Psi(x)$ be '$x \in$ the first big universe'. Clearly $T \vdash \exists x \Psi(x)$. By 1.2–1.4 (always ignoring the case 'P is V'), we see that if Σ is any axiom of $ZFC^{(0)}$ then T (or less) $\vdash \Sigma^{(\Psi)}$. We finish by showing that $T \vdash (\sim \exists u \Delta(u))^{(\Psi)}$. (If so, then Ψ interprets $ZFC^{(0)}+ \sim \exists u \Delta(u)$ in T, so $ZFC^{(0)}+ \sim \exists u \Delta(u)$ is consistent, completing the proof of 2.2.) Indeed, it follows from (2) and (3) that $\Delta(u)$ is T-absolute. Hence $T \vdash \Psi(u) \wedge (\Delta(u))^{(\Psi)} \rightarrow \Delta(u)$. But $\Psi(u)$ is 'u is the first big universe', while $\Delta(u)$ is '$u \in$ the first big universe'. Thus $T \vdash\sim (\Psi(u) \wedge (\Delta(u))^{\Psi})$, so $T \vdash\sim \exists u(\Psi(u) \wedge (\Delta(u))^{(\Psi)})$, and $T \vdash (\sim \exists u \Delta(u))^{(\Psi)}$, as was to be shown.

CHAPTER 11

§1. Problem 1. *should* say: Prove 1.4(a),(b),(c). Hints for (a): Let $\lambda < \operatorname{cf}\kappa$. As in the book, if $f : \lambda < \kappa$ then $f : \lambda \to \alpha$, for some $\alpha < \kappa$. Hence $\kappa^\lambda \le \sum_{\alpha<\kappa} \overline{\overline{\alpha}}^\lambda$. Also note that by G.C.H., if $\mu, \mu' < \kappa$ and $\nu = \max(\mu, \mu')$ is finite, then $\mu^{\mu'} \le \nu^\nu = \nu^+ \le \kappa$. Now complete (a); and do (b) and (c), which are easy.

Problem 2. 1.5(a) should begin: Any *infinite* order . . . .

Proof of 1.5(a). First we prove: *any order $\underline{A}$ has a cofinal subset which is a well-order (under $<^{\underline{A}}$)*. Pick $b \notin A$ and well-order A by $\prec$. Let $\overline{\overline{A}} = \kappa$. We define a_η for $\eta < \kappa^+$ by recursion: If $(a_\eta : \eta < \xi)$ is $<^{\underline{A}}$-increasing and (its range) is not cofinal in $\underline{A}$, then let a_ξ be the $\prec$-first member of A such that $a_\eta <^{\underline{A}} a_\xi$ for all $\eta < \xi$. Otherwise (for 'purely technical reasons'), put $a_\xi = b$. Clearly $a_\xi = b$ for some $\xi < \kappa^+$ (else $\kappa = \kappa^+$); let γ be the first such ξ. Thus $\{a_\eta : \eta < \gamma\}$ is a cofinal subset of $\underline{A}$ which is well-ordered by $<^{\underline{A}}$ (indeed, in order type γ).

Now we complete (a) by proving: *Any well-order $\underline{B}$, of power $\kappa \ge \aleph_0$, has a cofinal subset $\underline{C}$ of ordinal $\le \kappa$.* Clearly we can assume $\underline{B}$ is (ζ, ϵ_ζ) where $\kappa \le \zeta < \kappa^+$. Write $\zeta = \{\alpha_0, \ldots, \alpha_\xi, \ldots\}_{\xi<\kappa}$. We now define by recursion ordinals β_ξ for $\xi < \kappa$. If there is an ordinal γ in ζ which is greater than β_η for all $\eta < \xi$, then let γ' be the first such γ and put $\beta_\xi = \max(\gamma', \alpha_\xi)$; if not put $\beta_\xi =$ (say) ζ. If $\beta_\xi < \zeta$ for all $\xi < \kappa$, then $\{\beta_\xi : \xi < \kappa\}$ is cofinal in ζ (since $\beta_\xi \ge \alpha_\xi$) and has order type κ. Otherwise, let ξ be the first ordinal such that $\beta_\xi = \zeta$. Then $\{\beta_\eta : \eta < \xi\}$ is cofinal in ζ and has order type $\xi < \kappa$.

Problem 3. The proof of 1.5(b) consists of several arguments. 1.5(a) is applied twice near the beginning.

No induction or recursion is needed.

Problem 4. Prove 1.5(c). The author cannot find a 'proof' as simple as the book thinks it knows (though one may exist). Here is an outline (to be completed) of a (well-known) proof: First the reader should give the easy proof of

Proposition (I). *The following are equivalent:*

(i) *α is a regular cardinal*

(ii) $\alpha = \operatorname{cf}\alpha$

(iii) *for some β, $\alpha = \operatorname{cf}\beta$.*

Theorem (II.) *Let $\underline{A} = (A, <)$ be any order. Suppose $\underline{B}$ and $\underline{C}$ are both cofinal in A, and* Ord B *and* Ord C *are both regular cardinals. Then* Ord $\underline{B}$ = Ord C. *((II) will be proved below), after* III *and its proof.*

Proposition (III.) (II) $\Rightarrow$ 1.5(c).

The proof of (III) is harder than that of (I) but is straightforward in, say, 5 or 6 'well-chosen' lines.

Proof of (II). We know that Ord $\underline{B} \le$ Ord $\underline{C}$ or Ord $\underline{C} \le$ Ord B. Clearly we can, without loss of generality, assume Ord $\underline{B} \le$ Ord C. We will obtain (using $\underline{B}$-recursion) an isomorphism f of $\underline{B}$ onto a cofinal subset of $\underline{C}$. We justify this (somewhat informally) as follows: Let $b \in B$, and suppose we have $(f(b') : b' < b)$, an isomorphism of $\{b' : b' < b\}$ onto $\{f(b') : b' < b\} \subseteq C$. Since Ord $\underline{B}$ is a cardinal, $\{b' : b' < b\}$ and hence $\{f(b') : b' < b\}$ have power $< \overline{\overline{B}} \le \overline{\overline{C}}$, so $\{f(b') : b' < b\}$ is not cofinal in $\underline{C}$, as $\underline{C}$ is regular. Hence we can take $c' =$ the $\underline{C}$-first element $> f(b')$ for all $b' \in B$, and take $c'' =$ the $\underline{C}$-first element $>_{\underline{A}} b$, and $f(b) = \max(c', c'')$. It is easy to see that f is well-defined (by $\underline{B}$-recursion). The reader should prove that $D = f[B]$ is cofinal in C (in 2 lines), so $\overline{\overline{D}} = \overline{\overline{C}}$. Also, he should check that if $\underline{U}$ and $\underline{V}$ any orders such that Ord $\underline{U}$ and Ord $\underline{V}$ are regular cardinals and $\overline{\overline{U}} = \overline{\overline{V}}$ then Ord U = Ord V. So Ord C = Ord B' = Ord B.

§2. Problems 1–4. Problems 1–4 already have hints.

Problem 5. Here is a sketch of a proof of 2.5^{AC} with much to be filled in. Theorem 2.5^{AC}: If $\alpha \ge 2$, $\beta \ne 0$, and α is infinite or β is infinite, then $\overline{\overline{\alpha^\beta}} = \max(\overline{\overline{\alpha}}, \overline{\overline{\beta}})$.

Proof. Claim (0) If κ is infinite and $\lambda \ne 0$ then $\max(\kappa, \lambda) \cdot \lambda = \max(\kappa, \lambda)$. (Just take cases $\kappa \le \lambda$ and $\kappa > \lambda$.) Henceforth assume $\alpha \ge 2$ and $\beta \ne 0$.

Claim (1) $\alpha^\beta \ge \beta$. The proof is by induction on $\beta \ge 1$ (with α fixed).

Claim (2) $\alpha^\beta \ge \alpha$. Proof in 2 lines with no induction.

By (1) and (2), $\alpha^\beta \ge \max(\alpha, \beta)$. Hence (3) $\overline{\overline{\alpha^\beta}} \ge \max(\overline{\overline{\alpha}}, \overline{\overline{\beta}})$.

Claim (4) If $\alpha \ge \omega$ then $\overline{\overline{\alpha^\beta}} \le \max(\overline{\overline{\alpha}}, \overline{\overline{\beta}})$.

Do the easy case $0 \ne \beta < \omega$, then prove (4) holds for $\beta \ge \omega$ by induction on $\beta \ge \omega$.

(Note: If $\beta = \delta$, a limit ordinal, then $\overline{\overline{\sup_{\beta<\delta} \alpha^\beta}} \leq \sum_{\beta<\delta} \overline{\overline{\alpha^\beta}}$. In the step from β to $\beta + 1$, use (0).)

Claim (5) Take $\alpha = n \geq 2$ and hence β infinite. Then $\overline{\overline{n^\beta}} \leq \overline{\overline{\beta}}$ $(= \max(\overline{\overline{n}}, \overline{\overline{\beta}}))$.

Proof is by induction on $\beta \geq \omega$. Do $\beta = \omega$ first. Next, assuming that $\beta \geq \omega$ and (5) holds for β, show (5) holds for $\beta + 1$. These two steps are easy. Finally, assume $\beta = \delta$ is a limit ordinal $> \omega$ and $\overline{\overline{n^\gamma}} \leq \overline{\overline{\gamma}}$ if $\omega \leq \gamma < \delta$. Then $\overline{\overline{n^\delta}} = \overline{\overline{\sup_{\gamma<\delta} n^\gamma}} = \overline{\overline{\sup_{\omega\leq\gamma<\delta} n^\gamma}} \leq \sum_{\omega\leq\gamma<\delta} \overline{\overline{n^\gamma}} = \sum_{\omega\leq\gamma\leq\delta} \overline{\overline{\gamma}} \leq \overline{\overline{\delta}} \cdot \overline{\overline{\delta}} = \overline{\overline{\delta}}$. Thus (5) is proved and 2.5 follows from (3), (4) and (5).

Problem 6. Theorem 2.6(c) *should* read as follows:

> (Representation to the base $\beta \geq 2$.) For any ζ there exists n, $\alpha_0, \ldots, \alpha_{n-1}$, and $\eta_0, \ldots, \eta_{n-1}$ such that $\alpha_0 > \alpha_1 > \cdots > \alpha_{n-1}$ and, for all $i < n$, $0 < \eta_i < \beta$, and $\zeta = \beta^{\alpha_0} \cdot \eta_0 + \beta^{\alpha_1} \cdot \eta_1 + \cdots + \beta^{\alpha_{n-1}} \cdot \eta_{n-1}$; moreover, the representation is unique.

Now the proofs of both existence and uniquenes (in 2.6(c)), using 2.6(a) and (b), are fairly easy.

Problem 7. We shall do (b) $\Rightarrow$ (a) (leaving (a) $\Rightarrow$ (b) as an easy but good problem for the reader).

Note: If $\gamma < \omega^\delta$, then $\gamma < \omega^{\delta'} \cdot p$ for some $\delta' < \delta$, some $p \in \omega$ (by 2.6(b)). Hence, if $\gamma, \gamma' < \omega^\delta$ then for some $\delta' < \omega$, some p, we have $\gamma, \gamma' < a^{\delta'} \cdot p$.

Now assume (b) and suppose (a) fails, *i.e.*, ω^δ can be written as $\gamma + \rho$ where $0 < \rho < \omega^\delta$ (and hence $\gamma < \omega^\delta$). By the Note above we have $\gamma, \rho < \omega^{\delta'} \cdot p$ for some $\delta' < \delta$ and some p. Hence $\omega^\delta = \gamma + \rho \leq \omega^{\delta'}(p + p) < \omega^\delta$, which is absurd.

Problem 8. Let $K_{\alpha\beta} = \{f : \beta \to \alpha : f(\gamma) = 0$ for all but finitely many $\gamma < \beta\}$. For *distinct* $f, g \in K_{\alpha\beta}$, put $f \prec g$ iff among the finitely many γ for which $f(\gamma) \neq g(\gamma)$ the largest such γ has $f(\gamma) < g(\gamma)$. Write $K^*_{\alpha\beta}$ for the *structure* $(K_{\alpha,\beta}, \prec)$.

It is clear that $\prec$ is irreflexive and connected. We now show that $\prec$ is transitive. Suppose $f \prec g$ and $g \prec h$. Let $\delta_1 = \max\{\gamma : f(\gamma) \neq g(\gamma)\}$ and $\delta_2 = \max\{\gamma : g(\gamma) \neq h(\gamma)\}$. Consider the three cases: $\delta_1 = \delta_2$, $\delta_1 < \delta_2$, $\delta_2 < \delta_1$. In each case a straightforward argument shows that $f \prec h$. We illustrate with the case $\delta_1 < \delta_2$ and leave the other two cases to the reader. Assume $\delta_1 < \delta_2$. Using the definitions of δ_1 and δ_2, we see the following: Clearly $f(\delta_2) = g(\delta_2)$; but $g(\delta_2) < h(\delta_2)$ so $(*)$ $f(\delta_2) < h(\delta_2)$. Clearly $f(\xi) = g(\xi) = h(\xi)$ for all ξ such that $\delta_2 \leq \xi < \beta$. Hence, by $(*)$, $f \prec h$, as desired. Thus $K^*_{\alpha\beta}$ is an order.

If $\gamma < \beta$, put $K^\beta_{\alpha\gamma} = \{f \in K_{\alpha\beta} :$ for all ξ, if $\gamma \leq \xi < \beta$ then $f(\xi) = 0\}$. Also, let $\Gamma_{\alpha\gamma\beta}$ be the function on $K_{\alpha\gamma}$ such that if $g \in K_{\alpha\gamma}$ then $\Gamma_{\alpha\gamma\beta}(g)$ is the f on β with $f \restriction \gamma = g$ and $f(\xi) = 0$ if $\gamma \leq \xi < \beta$. It is straightforward to check that $K^\beta_{\alpha\gamma}$ is a proper initial segment of $K^*_{\alpha\beta}$ and $\Gamma_{\alpha\gamma\beta}$ is an isomorphism of $K^*_{\alpha\gamma}$ onto $K^{\beta*}_{\alpha\gamma}$ (which is $K^\beta_{\alpha\gamma}$ as a substructure of $K^*_{\alpha\beta}$).

Now we prove by induction on 'β' that, for all β, (!) $K^*_{\alpha\beta}$ has order type α^β (and hence is a well-order). If $\beta = 0$, (!) is clear. Suppose (!) holds for β. If $f \in K_{\alpha,\beta+1}$ then clearly $f \restriction \beta \in K_{\alpha\beta}$ and $f(\beta) < \alpha$. Put $F(f) = (f \restriction \beta, f(\beta))$. We claim $K^*_{\alpha,\beta+1} \underset{F}{\cong} K^*_{\alpha\beta} \cdot \alpha^*$ (where $\cdot$ is the ordinal product of orders and α^* is $(\alpha, \epsilon_\alpha)$). Clearly F is 1-1, on and onto. But we also have $f \prec g$ iff $f(\beta) < g(\beta)$ or $(f(\beta) = g(\beta)$ and $f \restriction \beta \prec g \restriction \beta)$, as one easily checks. Thus the claim is proved. Hence $K^*_{\alpha,\beta+1} \cong K^*_{\alpha\beta} \cdot \alpha^* \cong$ (by induction hypothesis) $\alpha^\beta \cdot \alpha = \alpha^{\beta+1}$.

Finally, suppose (!) holds for all $\beta < \delta$ (a limit ordinal). Then for each $\beta < \delta$, ($K^*_{\alpha\beta}$ and hence also) $K^{\delta*}_{\alpha\beta}$ is isomorphic to (α^β, ϵ); let this uniquely determined map be $G_\beta : K^{\delta*}_{\alpha\beta} \to (\alpha^\beta, \epsilon)$. We now have $K_{\alpha,\delta} = \bigcup_{\beta<\delta} K^\delta_{\alpha\beta}$ (where $\beta \mapsto K^\delta_{\alpha\beta}$ is a chain under "is a proper initial segment of"). Also $\alpha^\delta = \bigcup_{\beta<\delta} \alpha^\beta$ (again a chain under "is a proper initial segment of"). Moreover the G_β's are a chain under $\subseteq$. It follows that $G = \bigcup_{\beta<\delta} G_\beta$ is an isomorphism of $K^*_{\alpha\delta}$ onto α^δ. This completes the proof.