Abstract

Does stringent financial regulation always secure financial stability? No, not when shadow banking plays a role. This paper incorporates shadow banking modeled as off-balance-sheet financing in a standard continuous-time macro-finance model. In this model, regular banks pursue regulatory arbitrage by extending their businesses outside the regulatory perimeter via shadow banking. The absence of regulatory authorities in the shadow banking sector creates an enforcement problem. We show that the enforcement problem gives rise to an endogenous constraint on leverage for shadow banking. Shadow banking adds to financial instability because tightening market discipline in economic downturns forces shadow banks to sell assets at fire-sale prices to regular banks. Overall, financial instability as a function of financial regulation is U-shaped rather than monotonically decreasing as conventional wisdom predicts. This paper proposes a framework that can comprehensively evaluate the impact of different regulatory regimes on both the regulated and unregulated banking sectors.

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Introduction

The 2007-09 global financial crisis brought shadow banking—bank-like activities undertaken by unregulated financial entities and off-balance-sheet vehicles—to light in a worldwide discussion about the connection between financial instability and financial regulation. Over the course of the discussion, a new school of thought has emerged, arguing that since financial activities can always migrate to the unregulated sector, tightening financial regulation could endanger financial stability.\(^1\) This view challenges the conventional wisdom that the government can always secure financial stability by strengthening financial regulation.

Are the two schools of thought entirely incompatible for the purposes of policy-making? Or, is there a more comprehensive guideline for regulatory authorities to follow? Does financial migration always have substantial economic and financial consequences? Is it necessary for the government to arbitrarily extend its regulatory perimeter to preempt any financial migration?

\[\text{Figure 1: This stylized figure illustrates one of the paper’s main results: as regulatory authorities tighten financial regulation the shadow banking system emerges and expands (upper panel); financial instability initially declines and later rises (middle panel); social welfare initially increases and later diminishes (lower panel).}\]

To address all these questions, this paper theoretically investigates those banking activ-

\(^1\)For instance, Brunnermeier et al. (2009) mention the financial migration from the regulated to the unregulated sector and discuss the relevant regulatory perimeter issue. The IMF’s October 2014 Global Financial Stability Report shows that countries that have a large shadow banking sector typically implement tight financial regulatory rules (IMF, 2014). Richard Berner, Director of the Office of Financial Research, mentioned the risk of financial migration towards the unregulated shadow banking sector in his speech at the Money Marketeers of New York University on October 15, 2014 (Berner, 2014).
ities which chase regulatory arbitrage by migrating from the regulated traditional banking sector to the unregulated shadow banking sector. We argue that when financial regulation is sufficiently lenient, such financial migration is negligible, and in these circumstances, tightening regulation lowers financial instability and improves social welfare. However, when financial regulation becomes stringent enough, financial migration towards the shadow banking sector is sizable, and further strengthening of regulation heightens financial instability and worsens social welfare. Thus, this paper reconciles the two seemingly contradictory schools of thought about financial instability within a single framework. This framework can also evaluate the welfare implications of different regulatory regimes in light of their impacts on the shadow banking sector.

Our paper underscores the fact that the borrowing capacity of shadow banking relies on market discipline and, more importantly, on the level of financial regulation that regular banks face. This differs from the standard perception that a financial firm's internal optimal choice determines the amount of its shadow banking businesses (Plantin, 2014). Our paper highlights that if the channel connecting the borrowing capacity of shadow banking to financial regulation is turned off, tightening regulation can always lower financial instability because the contraction of the regular banking sector caused by strict regulation dominates the expansion of the shadow banking sector.

Similar to Thomas and Worrall (1988), Kehoe and Levine (1993), and Kocherlakota (1996), a standard enforcement problem in our model gives rise to the maximum borrowing capacity of shadow banking. The enforcement problem originates from the institutional details of shadow banking. To be consistent with such institutional details, our paper models regular banking as a regular bank’s on-balance-sheet financing and shadow banking as the regular bank’s off-balance-sheet financing. To regulatory authorities, a regular bank paints its shadow bank (i.e., off-balance-sheet vehicle) as a legally separate entity to circumvent any regulation of financial activities operated under the guise of the shadow bank. To creditors, however, the regular bank paints the shadow bank as part of its own business. Due to the absence of an authority, an enforcement problem arises because the creditors of a shadow bank cannot force its parent regular bank to protect them if the shadow bank is in trouble.

We next argue that tighter financial regulation causes the larger borrowing capacity of shadow banking; that is, the higher leverage that a regular bank can obtain via shadow banking. As in Thomas and Worrall (1988), Kehoe and Levine (1993), and Kocherlakota (1996), if a regular bank defaults on the obligations of its shadow bank, then the creditors of the shadow bank stop lending to the shadow bank, which deprives the regular bank of its opportunities for regulatory arbitrage. The cost to the regular bank of such default is, therefore, the present value of the future regulatory arbitrage benefits that shadow banking
offers. With more stringent regulation, comes greater opportunity for regulatory arbitrage. Thus, the cost of default is larger in economies with tighter regulation, and the leverage of shadow banking is higher in such economies.

The fact that the leverage of shadow banking is endogenously determined involves an interesting feedback loop between the cost of default and the leverage of shadow banking. A lower cost of default leads to a greater incentive to default and a narrower shadow banking channel. A narrower shadow banking channel offers less benefits to regular banks, which implies a lower cost of default. When financial regulation is sufficiently loose, the cost for regular banks to default is small. The feedback loop amplifies the effect of the small cost of default so profoundly as to rule out shadow banking completely (upper panel of Figure 1).

We embed our modeling of banking and shadow banking in a standard continuous-time macro-finance framework (Brunnermeier and Sannikov, 2014). In this framework, because of a constraint on equity financing, banks can use leverage only to finance their investments. We choose this macro-finance framework because it allows us to model financial instability as an endogenous risk generated by the financial system itself. Financial regulation that limits the use of bank leverage is essential, since the excessive use of bank leverage leads to high endogenous risk and causes a pecuniary externality for the entire economy (Lorenzoni, 2008; Stein, 2012). Shadow banking emerges as the response of regular banks to financial regulation.

The solution of a continuous-time macro-finance model characterizes the full dynamics of an economy. With this advantage, our model captures three salient dynamic features of shadow banking observed during the 2007-09 financial crisis: the pro-cyclicality of shadow banking, the reintermediation by shadow banks conducting fire sales of assets to regular banks, and the sudden collapse of the shadow banking system. By analyzing both leverage dynamics and endogenous risk dynamics, we uncover a general equilibrium channel through which shadow banking adds to financial instability.

To facilitate the illustration of the above amplification channel, we now explain why shadow banking is pro-cyclical in our model. In economic booms, both endogenous risk and the market leverage of regular banking are low. Since endogenous risk is low, a shadow bank is unlikely to be in trouble, and thus creditors are willing to lend to it. When the leverage of a regular bank is low, the regular bank must have enough incentives and resources to protect its shadow bank when a negative macroeconomic shock causes troubles for both its shadow bank and itself. Hence, regular banks have low incentives to default during economic booms.

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2The banking sector’s countercyclical leverage is a standard result in continuous-time macro-finance models such as Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). This result is consistent with findings based on market leverage, while the book leverage indicates the opposite case.
These low incentives to default, in turn, raise the leverage of shadow banking. In addition, the feedback loop between the cost of default and the leverage of shadow banking amplifies the expansion of shadow banking in economic booms.

Shadow banking increases financial instability as a general equilibrium effect in our model. The leverage of shadow banking is high in economic upturns. Since they are immune to financial regulation, shadow banks can accumulate substantial amounts of assets in upturns. When a negative macroeconomic shock hits the economy, the shrinking shadow banking channel forces shadow banks to offer assets to regular banks at fire-sale prices (i.e., reintermediation). Since regular banks are reluctant to acquire assets from shadow banks due to financial regulation, asset prices have to decline so significantly that regular banks are willing to purchase those assets. The degree of financial instability in the system naturally accelerates.

We next elaborate on the intuition behind the U-shaped relationship between financial regulation and financial instability and the hump-shaped relationship between financial regulation and social welfare (middle and lower panels of Figure 1). When regulation is loose enough, shadow banking is negligible. Tighter regulation of regular banking leads to lower financial instability and higher economic welfare. When regulation is sufficiently tight, considerable amounts of banking activities shift to the shadow banking sector thanks to the ample funding capacity of shadow banking. More stringent regulation causes a larger shadow banking system and higher financial instability, which diminishes social welfare.

**Related Literature.** The literature on shadow banking is fast growing and diverse. Different papers model shadow banking in drastically different ways. Adrian and Ashcraft (2012) provide a thorough survey of this growing literature. In this paper, we try to categorize models of shadow banking along two dimensions: the motive for shadow banking and the type of negative externalities caused by shadow banking.

The existence of shadow banking can be demand/preference driven. In Gennaioli et al. (2013), infinitely risk-averse households only value securities’ worst scenario payoffs, and shadow banking can increase such payoffs by pooling different assets together. In Moreira and Savov (2014), the preference specification of households leads directly to a demand for the liquid securities that shadow banking generates.

The second motive for shadow banking is regulatory arbitrage as discussed in this paper. Luck and Schempp (2014), Ordonez (2013), and Plantin (2014) are papers that fall into this category.

Models of shadow banking differ in the type of the externalities that shadow banking causes. The first category includes non-pecuniary externalities. In Plantin (2014),
shadow banking exposes the real sector to counter-productive uncertainty. In both Luck and Schempp (2014) and Gennaioli et al. (2013), creditors of shadow banking suffer from unexpected default caused by bank runs or crises. Generally, investments financed by shadow banking in these models have worse or more volatile fundamentals than those financed by regular banking.

Unlike the first group of papers discussed above, in Moreira and Savov (2014) and in this paper shadow banking does not involve any investments of inferior quality. Instead, we focus on the pecuniary externality that the individual choices of shadow banks for optimal leverage cause excessive endogenous risk because individual agents do not take into account the price impact of their actions in the competitive equilibrium.

Since the main result of Plantin (2014) is in the same spirit as ours, we highlights two other differences between the papers. First, financial regulation in Plantin (2014) constrains the issuance of risky securities, and regulation in this paper curbs the use of risk-free debt and leverage. Second, the size of shadow banking is a bank’s internal optimal choice in Plantin (2014), whereas the size of shadow banking relies on the credit market or is market-based in this paper.

This paper is also related to the literature on pecuniary externalities. One closely related paper is Bianchi (2011), whose quantitative examination of the non-pecuniary externality of excessive borrowing in a dynamic general equilibrium model highlights that raising the cost of borrowing can improve welfare.

In terms of methodology, this paper follows the emerging literature that consider economies with financial frictions in a continuous-time setting. This literature includes Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012b, 2013). The methodology employed in this literature has the advantage of allowing the exact characterization of full equilibrium dynamics. In our model, the tractability allows us to explicitly relate the leverage constraint to endogenous risk. Several papers, such as Adrian and Boyarchenko (2012), Danielsson et al. (2012), and Phelan (2012), have similarly related the leverage constraint to endogenous risk by assuming various forms of Value-at-Risk constraints. The leverage constraint in our paper originates from the financial friction that bankers could strategically default on securities generated by shadow banking. The leverage of shadow banking is endogenously determined by the leverage constraint in equilibrium, which is in line with the market-based concept of shadow banking. In addition, it draws the connection between shadow banking activities and financial regulation. Lastly, the microfounded leverage constraint sets the stage for the systemic event in our model, where bankers default if the constraint does not hold.
The paper is structured as follows. We lay out the baseline model in Section 1. In Section 2, we characterize the non-sunspot equilibrium of the baseline model, in which the shadow banking system never collapses on the equilibrium path, and illustrate the main results with numerical examples. Section 3 explores the welfare and policy implications of the proposed framework, and Section 4 highlights the fact that the endogenous leverage constraint for shadow banking is essential for the U-shaped result and demonstrates the robustness of the main results by varying agents’ preferences. In Section 5, we introduce the sunspot equilibrium of the baseline model, in which the shadow banking system might collapse. We also relate our model to the 2007-09 financial crisis in Section 5.

1 The Baseline Model

To analyze how shadow banking changes conventional ideas about financial regulation and financial instability, this section will model both shadow banking and regular banking and their interaction within a standard continuous-time macro-finance framework developed by Brunnermeier and Sannikov (2014). We will also specify the dynamic portfolio choice problem for each agent and list the equilibrium conditions that are characterized in Section 2.

1.1 Model Setup

The general model setup is standard in the literature on financial frictions. As in Kiyotaki and Moore (1997), He and Krishnamurthy (2012b), and others, our model has heterogeneous agents: productive bankers and less-productive households. Bankers can raise funds from households through both regulated regular banking, modeled as on-balance-sheet financing, and through unregulated shadow banking, modeled as off-balance-sheet financing.

1.1.1 Technology and Preferences

We consider a continuous-time infinite-horizon economy with two types of goods: durable physical capital goods and non-durable consumption goods.

At any time $t \in [0, \infty)$, a banker holding capital $k_t$ produces consumption goods $y_t$ according to

$$y_t = a k_t.$$

Households also have a linear production technology

$$y_t^h = a^h k_t^h,$$
although they are less productive; that is, $a^h < a$. Both bankers and households have the investment technology that an agent inverts $g(t)k_t$ units of consumption goods into $\iota_t k_t$ units of new capital, where 

$$g(t) = \iota_t + 0.5\phi (\iota_t - \delta)^2$$

with $\delta$ denoting the depreciation rate. Thus, the capital stock held by each agent grows at the rate $\iota_t - \delta$ in the absence of any shock.

The exogenous aggregate shock to the economy is driven by a Poisson process $\{N_t\}_{t=0}^{\infty}$ with intensity $\lambda$. Whenever the Poisson shock hits the economy, the capital stock held by each agent drops by a constant proportion, $\kappa$. The law of motion of the aggregate capital $K_t$ is

$$dK_t = K_{t-} (\iota_{t-} - \delta) dt - K_{t-} \kappa dN_t,$$

conditional on all agents choosing the same investment rate $\iota_{t-}$, where $K_{t-}$ denotes $\lim_{s \to t} K_s$; i.e., the left limit of the process $\{K_s, s \geq 0\}$ in period $t$. For purposes of exposition, we interpret time $t-$ as the period right before time $t$.

We assume that bankers have logarithmic utility that households are risk neutral, and that both types of agents have a time discount rate $\rho$. The expected discounted lifetime utility of a banker is

$$E_0 \left[ \int_0^\infty e^{-\rho u} \ln (c_u) \, du \right].$$

We also assume that a banker’s utility becomes negative infinity if her consumption is negative, and that households can have negative consumption. In Section 4.2, we modify the baseline model such that households have Epstein-Zin preferences.

### 1.1.2 Physical Assets

There is no friction in the market for capital goods. The market price of capital goods is in units of consumption goods, denoted by $q_t$. The law of motion of $q_t$ is denoted by

$$dq_t = q_{t-} \mu^q_{t-} dt - q_{t-} \kappa^q_d dN_t,$$
where $\mu_t$ and $\kappa_t$ are endogenously determined in equilibrium. In addition, $\kappa_t$ is allowed to be stochastic in each period $t$.

In period $t$, in the absence of a negative shock, the rate of return for a banker holding physical capital is

$$
\frac{a - g(\iota_t)}{q_t} + \iota_t - \delta + \mu_t.
$$

Other than the dividend yield $(a - g(\iota_t))/q_t$, there are two sources of gain from holding capital: the growth in the banker’s capital stock $\iota_t - \delta$ and the rise in the price of capital $\mu_t$. Similarly, there are two types of risk for holding capital: exogenous and endogenous. Exogenous risk is the possible $\kappa$ proportional decline in the banker’s capital stock, which is the direct consequence of the exogenous Poisson shock. Endogenous risk is the $\kappa_t$ proportional change in the price of capital, which is the general equilibrium effect of the Poisson shock. Endogenous risk affects the banker’s investment return through its impact on the $1 - \kappa$ proportion of physical capital left to the banker given the hit of the Poisson shock. Formally, the rate of return for bankers from holding capital is

$$\left[\frac{a - g(\iota_{t-})}{q_{t-}} + \iota_{t-} - \delta + \mu_{t-}\right] dt - \kappa_t Q_dN_t,
$$

where $\kappa_t Q \equiv \kappa + (1 - \kappa) \kappa_t$. Similarly, the rate of return for households holding capital is

$$\left[\frac{a^h - g(\iota_{t-}^h)}{q_{t-}} + \iota_{t-}^h - \delta + \mu_{t-}\right] dt - \kappa_t Q_dN_t.
$$

### 1.1.3 The Financial Market and Regulatory Authority

The financial market is incomplete. The following four assumptions detail the incompleteness of the financial market.

**Assumption 1** Households do not hold equity issued by other agents.

A banker can establish a regular bank. Via regular banking, bankers issue short-term debt and equity to finance their holdings of physical capital. The regulatory authority imposes the regulation in Assumption 2 on regular banks.

**Assumption 2** Regular banks’ debt financing is taxed at rate $\tau_t$ in period $t$; total tax revenue is instantly redistributed back to regular banks as lump-sum subsidies, whose amounts are proportional to bankers’ net worth.
Figure 2: This figure details the financial side of the model. A regular bank’s debt financing
is taxed at rate $\tau$. Households hold debt issued by regular banks and enjoy the rate of return $r$. 
Bankers earn regular banks’ residual values at rate $R - r - \tau$ as their equity. Bankers also obtain
shadow banks’ residual value at rate $R - \tilde{r}$ as their guarantors, where $\tilde{r}$ is the rate of return that
shadow banks promised to their household creditors. Bankers extend implicit guarantees to shadow
banks. The maximum size of a shadow bank is the maximum leverage of shadow banking $\bar{s}$ times
its banker’s net worth $W$.

Under the regulation, regular banks have to pay tax $\tau_t$ for each dollar they raise. Even
though there is a tax rebate, the tax rate $\tau_t$ affects the optimal leverage of a regular bank
in period $t$ because the tax rebate is distributed as a lump-sum subsidy. The lump-sum tax
rebate setup exactly cancels the wealth effect of tax $\tau_t$.

To circumvent the above regulation, a banker can sponsor a shadow bank and earn the
residual value of the shadow bank each period as a management fee. In the real world, this
activity is often called off-balance-sheet financing.

Assumption 3 The regulatory authority treats a shadow bank as a regular bank, if bankers
hold the equity of the shadow bank.

We name debt issued by shadow banks note to distinguish it from that issued by regular
banks. Notes are also short term. Assumptions 1 and 3 imply that shadow banks are all
debt financed. Any drop in the asset value of a shadow bank causes its creditors to suffer
unless its sponsor bails it out. Assumption 4 specifies the structure of the note market and
how households manage to secure the safety of their investments in shadow banks.

Assumption 4 In the note market in any period $t$,

i. a positive measure of households can form a unit, and the unit offers a one-period note
contract $(\bar{s}_t^*, \tilde{r}_t)$;

ii. if a banker accepts the contract, she can borrow (through shadow banking) up to $\bar{s}_t^*$ times

her net worth in period $t$ and pay the principal and interest at rate $\tilde{r}_t$ in the following period.

iii. The market excludes bankers who default and allows them to come back at rate $\xi$.

### 1.2 Problems for Bankers and Households

Suppose a banker’s net worth is $W_{t-}$ in period $t-$. $S_{t-}$ denotes the value of debt that she raises via regular banking. The excess return from holding capital goods funded by regular banking is

$$S_{t-} (R_{t-} - r_{t-} - \tau_{t-}) dt - S_{t-} \kappa_t^Q dN_t,$$

where $r_t$ is the risk-free rate.

The banker also manages the shadow bank. The size of the shadow bank is $S_t^*$ in dollar terms. The banker earns the difference between the return from the capital investment $R_{t-} S_t^*$ and the interest $\tilde{r}_{t-} S_t^*$ promised to creditors. The size of the shadow bank is limited by the leverage constraint specified by the note contract $(\bar{s}_{t-}, \tilde{r}_{t-})$

$$S_t^* \leq \bar{s}_t W_t$$

In addition, the banker has a strategic choice $D_t$ to make if a Poisson shock hits in period $t$. Given that the Poisson hits the economy, if the banker does not default ($D_t = 0$), she bears the loss $S_t^* \kappa_t^Q$ for creditors of her shadow bank; otherwise, she does not bear the loss. Thus, a banker’s dynamic budget constraint is

$$dW_t = (W_{t-}R_{t-} + S_{t-} (R_{t-} - r_{t-} - \tau_{t-}) + S_t^* (R_{t-} - \tilde{r}_{t-}) + \pi_t W_{t-} - c_t) dt - (S_{t-} \kappa_t^Q + S_t^* \kappa_t^Q + (1 - D_t) S_t^* \kappa_t^Q) dN_t,$$

where $\pi_t$ is the ratio of subsidy to net worth and $c_t$ is the banker’s consumption in period $t$. Since the banker may use a mixed strategy, let $d_t$ denote the probability that $D_t = 1$.

Taking $\{q_t, r_t, \tau_t, \tilde{r}_t, \pi_t, \bar{s}_t\}_{t=0}^{\infty}$ as given, the banker chooses $\{c_t, S_t, S_t^*, \pi_t, d_t\}_{t=0}^{\infty}$ to maximize her expected lifetime utility (1) subject to the leverage constraint (2) and the dynamic budget constraint (3).

Households can invest in both capital goods and financial instruments issued by regular and shadow banks. $S_h^t$ denotes the value of capital that household $h$ holds, and $n_t$ the value of notes that it holds. The wealth $W_h^t$ of the household evolves according to

$$dW_t^h = (W_{t-}^h r_{t-} + S_{t-}^h (R_{t-}^h - r_{t-}) + n_{t-} (\tilde{r}_{t-} - r_{t-}) - c_{t-}^h) dt - (S_{t-}^h + d_t n_{t-}) \kappa_t^Q dN_t.$$
Here, we assume that all bankers follow the same rules on default decisions in a given period. Formally, a household maximizes
\[
U^h_0 = E_0 \left[ \int_0^\infty e^{-\rho t} c^h_t dt \right]
\]
by choosing \( \{c^h_t, n_t, S^h_t\}_{t=0}^\infty \).

1.3 Equilibrium

We make the following assumption to guarantee that the wealth share of bankers would not be large enough to undo all financial frictions.

Assumption 5 Each banker retires independently at rate \( \chi \). If a banker retires, she can only save her wealth and earn risk-free rate \( r_t \).

\( I = [0, 1] \) and \( J = (1, 2] \) denote sets of bankers and households, respectively. Individual bankers and households are indexed by \( i \in I \) and \( j \in J \).

Definition 1 Given the initial endowments of capital goods \( \{k^i_0, k^j_0; i \in I, j \in J\} \) to bankers and households such that
\[
\int_0^1 k^i_0 di + \int_1^2 k^j_0 dj = K_0,
\]
and a locally deterministic tax rate process \( \{\tau_t\}_{t=0}^\infty \), an equilibrium is defined by a set of stochastic processes adapted to the filtration generated by \( \{N_t\}_{t=0}^\infty \): the price of capital \( \{q_t\}_{t=0}^\infty \), risk-free rate \( \{r_t\}_{t=0}^\infty \), the maximum leverage of shadow banking \( \{\bar{s}^*_t\}_{t=0}^\infty \), the interest rate on notes \( \{\bar{r}_t\}_{t=0}^\infty \), the ratio of subsidy to net worth \( \{\pi_t\}_{t=0}^\infty \), wealth \( \{W^i_t, W^j_t\}_{t=0}^\infty \), capital holdings \( \{k^i_t, k^j_t\}_{t=0}^\infty \), investment decisions \( \{\iota^i_t\}_{t=0}^\infty \), default decisions \( \{\bar{d}^i_t\}_{t=0}^\infty \), and consumption \( \{c^i_t, c^j_t\}_{t=0}^\infty \) of banker \( i \in I \) and household \( j \in J \); such that
1. \( \{W^i_0, W^j_0\} \) satisfy \( W^i_0 = q_0 k^i_0 \) and \( W^j_0 = q_0 k^j_0 \), for \( i \in I \) and \( j \in J \);
2. bankers solve their problems given \( \{q_t, r_t, \tau_t, \bar{r}_t, \pi_t, \bar{s}^*_t\}_{t=0}^\infty \);
3. households solve their problems given \( \{q_t, r_t, \bar{r}_t, \bar{d}^i_t, i \in I\}_{t=0}^\infty \);
4. the budget of the regulatory authority is balanced;
5. markets for both consumption goods and capital goods clear
\[
\int_0^1 c^i_t dt + \int_1^2 c^j_t dj = \int_0^1 (a - g(\iota^i_t)) k^i_t dt + \int_1^2 (a^h - g(\iota^j_t)) k^j_t dj,
\]
(5)
\[
\int_0^1 k_i^i di + \int_1^2 k_j^j dj = K_t, \tag{6}
\]

where \(dK = \left(\int_0^2 (i_t^i - \delta) k_i^i di\right) dt - \kappa K_t dN_t;\)

6. the note market clears

Given this definition, the debt market automatically clears by Walras’ Law.

1.4 Financial Frictions

It is worthwhile to summarize three types of financial frictions in the baseline model. First, we do not allow for equity issuance in the model (Assumption 1). The restriction on equity financing gives rise to the standard balance sheet amplification mechanism in the financial friction literature (Krishnamurthy, 2010). Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012b) provide microfoundations for this restriction. Our model naturally inherits the amplification mechanism discussed in these two papers.

The second friction is the intervention of the regulatory authority (Assumption 2). Financial regulations are necessary given that the level of leverage chosen by individual bankers may not be the socially optimal because borrowers do not internalize the external negative impact of their private leverage choices, as discussed by Lorenzoni (2008) and Stein (2012). Section 3.1 shows that the tax on regular banking in Assumption 2 can improve bankers’ welfare by adjusting the leverage choices of regular banks and reducing the pecuniary externality.

The third financial friction is the leverage constraint on shadow banking (Assumption 4). One implication of assumptions 1 and 3 is that shadow banks issue no equity. This is realistic because the equity portion of shadow banks in the real world, such as special purpose vehicles and money market mutual funds, is typically very thin.

2 Financial Instability in the Non-Sunspot Equilibrium

In this section, we characterize the non-sunspot equilibrium of the baseline model with numerical examples. With the model characterization, we will present our main result that the relationship between financial instability and financial regulation displays a U shape.
2.1 Characterizing the Non-Sunspot Equilibrium

It is convenient to characterize the non-sunspot equilibrium because it does not involve a sudden collapse of the shadow banking sector. In addition, endogenous risk $\kappa_t^Q$ is deterministic in each period $t$. So is the loss for each dollar of investment in capital goods, $\kappa_t^Q$.

2.1.1 Households’ Optimal Choices

$\iota_t$ denotes a household’s investment in period $t$. The expression of $R_{t-}^h$ implies that the optimal level of $\iota_t$ maximizes

$$-t_t - 0.5\phi \left( \iota_t - \delta \right)^2 \overline{q_t} + \iota_t.$$ 

The first-order condition yields an expression that the optimal investment rate is a function of the price of capital, $q_t$

$$\iota_t = \delta + \frac{q_t - 1}{\phi}. \quad (7)$$ 

Bankers have the same investment function $\iota(\cdot)$ since they have the same investment technology.

Given that households are risk-neutral, the following conditions must hold in equilibrium to be consistent with the households optimal consumption and portfolio choice.

**Proposition 1** A household’s portfolio choices $\{S^h_t, n_t, t \geq 0\}$ satisfy

$$r_t = \rho \quad (8)$$

$$R_{t-}^h - r_{t-} \leq \lambda \kappa_t^Q, \quad = \ i f \ S^h_{t-} > 0, \quad (9)$$

$$\tilde{r}_{t-} - r_{t-} \leq \lambda d_t \kappa_t^Q, \quad = \ i f \ n_{t-} > 0, \quad (10)$$

for all $t \geq 0$.

This proposition assumes that all bankers follow the same default strategy $d_t$ in a given period.

2.1.2 Bankers’ Optimal Choices

For bankers’ optimal choices, we begin with the result that bankers never have negative net worth. Intuitively, bankers are risk-averse, and negative net worth leads to negative consumption and infinite negative utility.

**Lemma 1** If the initial net worth of a banker $W_0 > 0$, it is never optimal for the banker to have negative net worth.
Proof. See Appendix. ■

A corollary of Lemma 1 is that a banker’s overall leverage must be small enough that her net worth will never be wiped out. Therefore, a banker never defaults on her regular bank obligations.

Corollary 1 A banker’s overall leverage has an upper bound in any period; in particular,

\[(W_t- + S_t-) \kappa_t^Q + S_t^* \kappa_t^Q \mathbf{1}_{(d_t > 0)} < W_t-\]

always holds.

We next apply the stochastic control approach to solve for a banker’s optimal consumption and portfolio choices. Consider a banker who can access shadow banking. We first conjecture and later verify that the banker’s continuation value function take the form

\[J_t \equiv E_t \left[ \int_t^\infty e^{-\rho u} \ln(c_u) du \right] = \frac{\ln(W_t)}{\rho} + h_t,\]

where \(W_t\) is the banker’s net worth in period \(t\), and \(h_t\) is a separably additive term that depends on market conditions and evolves endogenously according to

\[dh_t = h_t- \mu h_t^- dt - h_t- \kappa h_t^- dN_t.\]

Second, we conjecture that, if the banker defaults, her continuation value is \(\hat{J}_t = \ln(W_t)/\rho + \hat{h}_t\), where \(\hat{h}_t\) follows

\[d\hat{h}_t = \hat{h}_t- \mu h_t^- dt - \hat{h}_t- \kappa h_t^- dN_t.\]

Now, we are ready to spell out the Hamilton-Jacobian-Bellman equation for the banker’s optimal control problem

\[\rho J_t^- = h_t- \mu h_t^- + \chi (J_t^r(W_t) - J_t-) \]

\[\max_{c_t-, s_t-, S_t^*, d_t, \cdot} \left\{ \frac{1}{\rho W_t-} (W_t- (R_t- + \pi_t-) + S_t- (R_t- - \tau_t- - \tau_t-) + S_t^* (R_t- - \tilde{r}_t-) - c_t-) \right. \]

\[+ \lambda (1 - d_t) \left( \frac{1}{\rho} \ln \left( W_t- - (W_t- + S_t- + S_t^*) \kappa_t^Q \right) + h_t-(1 - \kappa_t^h) \right) \]

\[+ \lambda d_t \left( \frac{1}{\rho} \ln \left( W_t- - (W_t- + S_t- \kappa_t^Q \right) + \hat{h}_t-(1 - \kappa_t^h) \right) - \lambda J_t- + \ln(c_t-) \right\}, \]

where \(J_t^r(\cdot)\) is the banker’s continuation value function if she retires in period \(t\).

Proposition 2 A banker’s choices of optimal consumption, optimal portfolio weights, and default decision \(\{c_t, s_t, s_t^*, d_t\}_{t=0}^\infty\) satisfy

\[c_t = \rho W_t, \quad (11)\]
The HJB equation for the banker is

\[
R_{t-} - r_{t-} - \tau_{t-} \leq \frac{\lambda (1 - d_t) \kappa_t^Q}{1 - (1 + s_{t-} + s^*_t) \kappa_t^Q} + \frac{\lambda d_t \kappa_t^Q}{1 - (1 + s_{t-} + s^*_t) \kappa_t^Q}, \quad \text{if } s_{t-} > 0,
\]

and

\[
R_{t-} - \tilde{r}_{t-} \geq \frac{\lambda (1 - d_t) \kappa_t^Q}{1 - (1 + s_{t-} + s^*_t) \kappa_t^Q}, \quad \text{if } s^*_t < s_{t-},
\]

and

\[
d_t = 1, \quad \text{if } \frac{1}{\rho} \ln (1 - (1 + s_{t-}) \kappa_t^Q) + \hat{h}_t > \frac{1}{\rho} \ln (1 - (1 + s_{t-} + s^*_t) \kappa_t^Q) + h_t,
\]

\[
d_t \in [0, 1], \quad \text{if } \frac{1}{\rho} \ln (1 - (1 + s_{t-}) \kappa_t^Q) + \hat{h}_t = \frac{1}{\rho} \ln (1 - (1 + s_{t-} + s^*_t) \kappa_t^Q) + h_t,
\]

\[
d_t = 0, \quad \text{otherwise},
\]

for all \( t \geq 0 \), where \( s_t = W_t / W_t \), \( s^*_t = S_t^* / W_t \), \( \hat{h}_t = \hat{h}_{t-}(1 - \kappa_t^h) \), and \( h_t = h_{t-}(1 - \kappa_t^h) \).

**Proof.** See Appendix. \( \blacksquare \)

We will interpret these expressions in order. Logarithmic agents always consume \( \rho \) fraction of their current net worth, as equation (11) shows. Inequality (12) guarantees that the banker’s net worth will be positive if a negative shock hits the economy. Equations (13) and (14) are first-order conditions with respect to the banker’s portfolio choices. Equation (15) characterizes the banker’s default decision given a negative shock.

To consider a banker’s strategic choice, we need to characterize her optimal choice if she cannot access shadow banking because of default. The law of motion of her net worth \( \hat{W}_t \) in the case of default is

\[
d\hat{W}_t = \left( \hat{W}_{t-} (R_{t-} + \pi_{t-}) + \hat{S}_{t-} (R_{t-} - r_{t-} - \tau_{t-}) - \hat{c}_{t-} \right) dt - \left( \hat{W}_{t-} \kappa_t^Q + \hat{S}_{t-} \kappa_t^Q \right) dN_t.
\]

The HJB equation for the banker is

\[
\rho \hat{J}_{t-} = \hat{h}_{t-} - \mu_{t-} \hat{J}_{t-} + \chi (J_t (\hat{W}_t) - \hat{J}_{t-}) + \xi (J_t - \hat{J}_{t-})
\]

\[
+ \max_{\hat{c}_{t-}, \hat{S}_{t-}} \left\{ \ln (\hat{c}_{t-}) + \frac{1}{\rho \hat{W}_{t-}} (\hat{W}_{t-} (R_{t-} + \pi_{t-}) + \hat{S}_{t-} (R_{t-} - r_{t-} - \tau_{t-}) - \hat{c}_{t-}) \right\}.
\]

Similar to Proposition 2, we list the banker’s first-order conditions

\[
\hat{c}_t = \rho \hat{W}_t
\]

\[
(1 + \hat{s}_{t-}) \kappa_t^Q < 1
\]

\[
R_{t-} - r_{t-} - \tau_{t-} = \frac{\lambda \kappa_t^Q}{1 - (1 + \hat{s}_{t-}) \kappa_t^Q}
\]
2.1.3 The Note Market and the Enforcement Problem

Our analysis of the note market begins with the optimal portfolio choice problem for a banker who intends to default if a negative shock hits in the subsequent period. To maximize the growth of her net worth, the banker borrows up to $\bar{s}_t^*$ times her net worth via shadow banking and adjusts her portfolio weight on regular banking $\tilde{s}_t^*$ so that

$$R_t - r_t - \tau_t = \frac{\lambda Q_t^Q}{1 - (1 + \tilde{s}_t^* Q - \bar{s}_t^*) \kappa_t^Q}. $$

In addition, she needs to make sure that her continuation value with default in the event of a negative shock is larger than that without default

$$\frac{1}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^*) \kappa_t^Q \right) + \hat{h}_t > \frac{1}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^* + \bar{s}_t^*) \kappa_t^Q \right) + h_t,$$

where $h_t = h_t (1 - \kappa_t^h)$ and $\hat{h}_t = \hat{h}_t (1 - \kappa_t^\hat{h})$.

Given this banker’s choices, we argue that no banker intends to default, if the maximum leverage of shadow banking $\bar{s}_t^*$ satisfies the enforceability constraint

$$\frac{1}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^*) \kappa_t^Q \right) + \hat{h}_t = \frac{1}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^* + \bar{s}_t^*) \kappa_t^Q \right) + h_t. \quad (18)$$

We now justify this statement. $(s_t^-, s_t^*)$ denotes the optimal portfolio choice in period $t-$ for a banker who intends to honor her shadow bank obligations in period $t$. The above no-default statement is true because

$$\frac{s_t^-(R_t - r_t - \tau_t - \tilde{s}_t^*) + s_t^*(R_t - r_t - \tilde{s}_t^*)}{\rho} + \frac{\lambda}{\rho} \ln \left( 1 - (1 + s_t^* + \tilde{s}_t^*) \kappa_t^Q \right) + \lambda \hat{h}_t > \frac{\tilde{s}_t^-(R_t - r_t - \tau_t - \tilde{s}_t^*) + \tilde{s}_t^*(R_t - r_t - \tilde{s}_t^*)}{\rho} + \frac{\lambda}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^* + \bar{s}_t^*) \kappa_t^Q \right) + \lambda \hat{h}_t \quad \text{or}

\frac{\tilde{s}_t^-(R_t - r_t - \tau_t - \tilde{s}_t^*) + \tilde{s}_t^*(R_t - r_t - \tilde{s}_t^*)}{\rho} + \frac{\lambda}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^* + \bar{s}_t^*) \kappa_t^Q \right) + \lambda \hat{h}_t = \frac{\tilde{s}_t^-(R_t - r_t - \tau_t - \tilde{s}_t^*) + \tilde{s}_t^*(R_t - r_t - \tilde{s}_t^*)}{\rho} + \frac{\lambda}{\rho} \ln \left( 1 - (1 + \tilde{s}_t^* + \bar{s}_t^*) \kappa_t^Q \right) + \lambda \hat{h}_t.$

The inequality holds since $(s_t^-, s_t^*)$ is the banker’s optimal choice given that she intends to protect her shadow bank; the equality is true because of the enforceability constraint (18). Overall, we observe that it is strictly better for the banker to bail out her shadow bank in trouble.

This enforceability constraint is “too” tight because creditors can increase $\bar{s}_t^*$ by a sufficiently small amount that bankers still strictly prefer to protect their shadow banks in period $t$. To simplify equilibrium analysis, we exclude the possibility that creditors raise the
maximum leverage of shadow banking above $\tilde{s}_{t-}$ with Assumption 6.

**Assumption 6** The creditors of a shadow bank always want to rule out its default in the worst scenario that the leverage of its sponsoring regular bank equals $\tilde{s}_{t-}$.

The rationale for Assumption 6 is as follows: since the creditors of a shadow bank cannot influence the leverage choice for its sponsoring regular bank, those creditors are very cautious, and they attempt to exclude the shadow bank’s default in the worst case for them; that is, the banker intends to default and, accordingly, maximizes the leverage of her regular bank. In Appendix A, we analyze a relatively loose enforceability constraint, with which we can characterize the market-clearing note contract ($\tilde{s}_{t-}, \tilde{r}_{t-}$) only under Assumptions 1 - 4. Appendix A also shows that Assumption 6 is not essential for our main results.

Assumption 6 avoids shadow banks’ default, which is consistent with Assumption 1. This is because if a shadow bank defaults whenever a negative shock hits, then notes issued by the shadow bank are essentially combinations of risk-free debt and equity. However, households cannot hold equity (Assumption 1). Since no shadow bank defaults given the enforceability constraint (18), the no-arbitrage condition implies that the rate of return from holding notes equals the risk-free rate, i.e., $\tilde{r}_{t} = r_{t}$.

We characterize a banker’s optimal choices given the note contract $\{\tilde{s}_{t}, \tilde{r}_{t}\}_{t=0}^{\infty}$ in Proposition 3.

**Proposition 3** Given the note contract $\{\tilde{s}_{t}, \tilde{r}_{t}\}_{t=0}^{\infty}$, a banker’s optimal portfolio weights $\{s_{t}, s_{t}^*\}_{t=0}^{\infty}$, and default decisions $\{d_{t}\}_{t=0}^{\infty}$ satisfy $d_{t} = 0$,

$$(1 + s_{t-} + s_{t-}^*)\kappa_{t-}^Q < 1,$$

$$R_{t-} - r_{t-} - \tau_{t-} \leq \frac{\lambda\kappa_{t-}^Q}{1 - (1 + s_{t-} + s_{t-}^*)\kappa_{t-}^Q}, \quad \text{if } s_{t-} > 0,$$

$$R_{t-} - r_{t-} \geq \frac{\lambda\kappa_{t-}^Q}{1 - (1 + s_{t-} + s_{t-}^*)\kappa_{t-}^Q}, \quad \text{if } s_{t-}^* < \tilde{s}_{t-},$$

for all $t \geq 0$.

To fully specify $\tilde{s}_{t-}$, we need to know the difference between $h_{t}$ and $\hat{h}_{t}$, which is denoted by $H_{t}$. The following proposition characterizes $H_{t}$, which is interpreted as the cost for a banker to default on her shadow bank obligations in period $t$.

**Proposition 4** Given that $\ln(\cdot)/\rho + h_{t}$ is the continuation value function of a banker who can access shadow banking and $\ln(\cdot)/\rho + \hat{h}_{t}$ is the continuation value function of a banker
who defaults in period $t-$,

$$H_t \equiv h_t - \hat{h}_t = E_t \left[ \int_t^\infty \exp \left( - (\rho + \xi + \chi) (u - t) \right) f_u \, du \right],$$

(19)

where $f_u$ equals

$$\frac{1}{\rho} \left( \begin{array}{c}
\text{higher leverage benefit due to cheap credit} \\
(s_{u-} + s_{u-}^* - \hat{s}_{u-}) (R_{u-} - r_{u-} - \tau_{u-}) + \hat{s}_{u-} \tau_{u-} \\
\text{tax benefit} \\
+ \lambda \left( \ln \left( 1 - \left( 1 + s_{u-} + s_{u-}^* \right) \kappa_{u} Q \right) - \ln \left( 1 - (1 + \hat{s}_{u-}) \kappa_{u} Q \right) \right)
\end{array} \right).$$

Proof. See Appendix. ■

$f_u$ is the tax benefit (i.e., regulatory arbitrage) that shadow banking offers in period $u$, and is essentially the difference between the growth rate of net worth for bankers who can use shadow banks and the growth rate of net worth for bankers who cannot. $H_t$ is the present value of future tax benefits $f_u$ that a banker will lose if she defaults in period $t$. The discount factor is the banker’s time discount factor plus the “come-back” intensity $\xi$ and the retirement rate $\chi$, because once bankers return to the shadow banking sector or retire, the advantage of accessing shadow banking effectively disappears.

We next highlight the feedback loop between the maximum leverage of shadow banking $\{s_t^*\}_{t=0}^{\infty}$ and the cost of default $\{H_t\}_{t=0}^{\infty}$. First, the enforceability constraint (18) implies that the maximum leverage of shadow banking relies on a banker’s cost of default to her shadow bank’s obligations. Second, the probabilistic representation of the cost of default (19) indicates that the maximum leverage of shadow banking directly affects how costly default is for bankers.

This feedback loop gives rise to an equilibrium where shadow banking does not exist. Conjecture that $\{s_t^* = 0\}_{t=0}^{\infty}$. The probabilistic representation (19) implies $\{H_t = 0\}_{t=0}^{\infty}$, and the enforceability constraint (18) justifies the conjecture. Therefore, the economy has an equilibrium in which shadow banking does not exist. We summarize the conclusion of these arguments in the following proposition.

**Proposition 5** There exists a non-sunspot equilibrium where shadow banking does not exist, that is, $\{s_t^* = 0, H_t = 0\}_{t=0}^{\infty}$.

This equilibrium is degenerate, and hereafter we label it the “bad” equilibrium since productive bankers are unable to leverage up via shadow banking. There might be a non-degenerate equilibrium, however, where shadow banking exists, which is labeled the “good” equilibrium.

Equilibrium selection is beyond the scope of this paper. Given the importance of shadow
banking in the real world, we assume that the “good” equilibrium prevails when both “good” and “bad” equilibria exist.

2.1.4 Miscellany

We group discussions of equilibrium conditions 4 and 5 in Definition 1 together, since all of them are straightforward. The budget of the regulatory authority is balanced if \( \pi_t = s_t \tau_t \) for \( t \geq 0 \). Since households are risk-neutral, the market for consumption goods clears automatically. The market for physical capital clears if the fractions of physical capital held by bankers and households sum to 1. Let \( \psi_t \) denote the fraction of physical capital held by bankers, which equals \( (1 + s_t + s^*_t) \omega_t \).

In our model, as in other continuous-time macro-finance papers, the wealth distribution matters for the dynamics of the economy. Later, we will capture the dynamics of an equilibrium with the bankers’ wealth share \( \omega_t \equiv \int_0^1 W_i di / q_t K_t \). Lemma 2 characterizes how \( \omega_t \) evolves.

**Lemma 2** The law of motion of \( \omega_t \) is

\[
d\omega_t = \omega_t - \mu_t\omega_t dt - \omega_t \kappa_t^Q dN_t,
\]

where

\[
\mu_t = R_t + s_t (R_t - r_t) + s^*_t (R_t - \bar{r}_t) - \mu_t^q - \mu_t^K - \rho - \chi,
\]

and

\[
\kappa_t^Q = \frac{(s_t + s^*_t) \kappa_t^Q}{1 - \kappa_t^Q}.
\]

**Proof.** See Appendix. ■

2.2 Markov Equilibrium

Our model has the property of scale-invariance with respect to \( K \). This means that, for a given equilibrium in an economy with initial capital \( \{k^i_0, k^j_0; i \in I, j \in J\} \), there exists an equivalent equilibrium with the same laws of motion of \( \omega_t, q_t, \) and \( H_t \) in any economy with initial capital \( \{\zeta k^i_0, \zeta k^j_0; i \in I, j \in J\} \), where \( \zeta \in (0, \infty) \).

Equations (2) – (19) can characterize an equilibrium specified by Definition 1. The scale-invariance property implies that we can characterize an equilibrium that is Markov in \( \omega \) with a modification of Assumption 2.\(^5\)

\(^5\)A natural tax policy is that \( \tau_t = \tau \) for \( t \geq 0 \). Given this policy, endogenous risk \( \kappa_t^Q \) jumps as shadow banks become the marginal buyer of physical capital, which complicates the computation of an equilibrium. Our setup specified in Assumption 2\(^\prime\) makes the process that shadow banks become marginal buyers smooth and simplifies the computation.
Assumption 2'. In period $t$, the tax rate $\tau_t$ equals $\min\{\tau, \tau_s\}$, where $\tau$ is a positive constant and $s_t$ denotes $\int_0^1 S_s^t \, di / \int_0^1 W_s^t \, di$; the tax rate is $\tau$ for bankers who are unable to use shadow banking due to default.

Note that the tax rate $\tau_t$ at any time $t$ depends only on the aggregate variable. Thus, individual bankers always take the tax rate as given.

In the Markov equilibrium, the dynamics of all endogenous aggregate variables $\{q_t, H_t\}_{t=0}^\infty$ can be fully described by the law of motion of the state variable and functions $q(\omega)$ and $H(\omega)$, which are defined over the domain $(0, \bar{\omega}]$. Thanks to Ito’s Lemma, we derive the law of motion of $\{q_t, H_t\}_{t=0}^\infty$.

$$
\mu^q_t = \frac{q'(\omega_t)}{q(\omega_t)} \omega_t \mu^\omega_t, \quad (23)
$$
$$
\kappa^q_t = \frac{q(\omega_t-) - q(\omega_t-(1 - \kappa_t^\omega))}{q(\omega_t-)}, \quad (24)
$$
$$
\mu^H_t = \frac{H'(\omega_t)}{H(\omega_t)} \omega_t \mu^H_t, \quad (25)
$$
$$
\kappa^H_t = \frac{H(\omega_t-) - H(\omega_t-(1 - \kappa_t^\omega))}{H(\omega_t-)} \quad (26)
$$

The following proposition describes a system of delay differential equations and their boundary conditions, which define function $q(\omega)$ and $H(\omega)$.

**Proposition 6** $q(\omega)$ and $H(\omega)$ are defined over $(0, \bar{\omega}]$. Given $(\omega, q(\omega'), H(\omega'), 0 < \omega' < \omega)$, we compute $(q'(\omega), H'(\omega))$ using the following procedure:

1. Conjecture that $\psi < 1$, find $s + s^*$ such that

$$
a - a^h - \frac{\tau}{q} = \frac{\lambda \kappa Q}{1 - (1 + s + s^*) \kappa Q} - \lambda \kappa Q, \tag{22}
$$

equations (22) and (24) hold. Derive $(\kappa^\omega, \kappa^a, \kappa^Q)$ according to Ito’s Lemma and $\mu^a$ based on equation (13). Also compute $\psi$.

2. If $\psi < 1$ does not hold, then $\psi = 1$ and $s + s^* = 1/\omega - 1$. Similarly, derive $(\kappa^\omega, \kappa^a, \kappa^Q, \kappa^H)$ according to Ito’s Lemma and $\mu^a$ based on equation (13).

3. Given $s + s^*(= \tilde{s})$ and $\kappa Q$, compute $s^*$ such that equation (18) holds and then we derive $s$. Also, derive $\mu^\omega$ according to equation (21).

4. Compute $q'(\omega)$ according to equation (23).

5. Finally, compute $f$ based on (19) and then derive $H'(\omega)$ according to

$$
(\rho + \xi + \chi) H'(\omega) = f + \omega \mu^\omega H'(\omega) + \lambda (H(\omega(1 - \kappa^\omega)) - H(\omega)). \tag{27}
$$
Boundary conditions are

\[
\begin{align*}
\mu^q(\bar{\omega}) &= \mu^H(\bar{\omega}) = \mu^\omega(\bar{\omega}) = 0, \\
\lim_{\omega \to 0} q(\omega) &= \bar{q} \text{ and } \lim_{\omega \to 0} H(\omega) = 0,
\end{align*}
\]

where \( \bar{q} \) satisfies

\[
a^h - \delta - \frac{\bar{q}^2}{2\phi} = \rho \bar{q}.
\]

(28)

For the “bad” equilibrium, we need to solve only a single differential with respect to \( q(\omega) \), since \( H(\omega) = 0 \) for all \( \omega \in (0, \bar{\omega}] \).

2.2.1 Equilibrium Uniqueness

Within the class of Markov equilibria, we can establish under what conditions the “bad” equilibrium (where shadow banking does not emerge) is unique. To achieve this result, we define mapping \( \Gamma \) which takes the cost of default function \( H(\omega) \) as input,

\[
\Gamma H(\omega) = E_t \left[ \int_t^\infty \exp(- (\rho + \xi + \chi) (u - t)) f(\omega_u) du \bigg| \omega_t = \omega \right]
\]

where

\[
f(\omega) = \frac{1}{\rho} \left( \frac{(s(\omega) + s^*(\omega) - \bar{s}(\omega))(R(\omega) - r(\omega) - \tau(\omega)) + s^*(\omega)\tau(\omega)}{+\lambda \left( \ln \left( 1 - (1 + s(\omega) + s^*(\omega)) \kappa Q(\omega) \right) - \ln \left( 1 - (1 + \bar{s}(\omega)) \kappa Q(\omega) \right) \right)} \right),
\]

and

\[
s^*(\omega) \leq \bar{s}^*(\omega),
\]

\[
\bar{s}^*(\omega) = \left( 1 - \exp \left( - \rho H(\omega) \right) \right) \left( \frac{1}{\kappa Q(\omega)} - (1 + \bar{s}(\omega)) \right).
\]

Equation (19) clearly shows that \( H(\omega) \) is a fixed point of the mapping \( \Gamma \). As we have noted, the mapping \( \Gamma \) might allow for two fixed points: one leads to the “good” non-degenerate equilibrium, and the other yields the “bad” degenerate equilibrium. The following theorem provides a sufficient condition, under which the “bad” equilibrium is unique.

**Theorem 1** If \( \tau < (\rho + \xi + \chi) \kappa \), the mapping \( \Gamma \) is a contraction mapping with the fixed point \( H(\omega) = 0 \) for all \( \omega \in (0, \bar{\omega}] \).

**Proof.** See Appendix. \( \blacksquare \)

To show that \( \Gamma \) is a contraction mapping, we justify that \( \Gamma \) satisfies Blackwell’s sufficient conditions if \( \tau < (\rho + \xi + \chi) \kappa \).
The feedback loop illustrated earlier explains why $\Gamma$ could be a contraction mapping. Suppose the magnitude of the exogenous shock $\kappa$ increases permanently in an economy where shadow banking exists. Bankers’ incentives to default increase, and these heightened incentives lower the maximum leverage of shadow banking $\bar{s}^*$ (the enforceability constraint (18)). The decline in the leverage of shadow banking $\bar{s}^*$ reduces the cost of default $H$ (the probabilistic representation (19)), which, in turn, lowers the leverage of shadow banking. This cycle could completely eliminate any shadow banking in equilibrium. The following analyses will explore the economic implications of changes in other parameters such as the tax rate $\tau$.

![Feedback loop](image)

**Figure 3: Feedback loop**

### 2.3 Numerical Example

In this section, we present the main dynamic properties of the baseline model via a numerical example. Thanks to the global solution provided by the continuous-time approach, we are able to show the endogenous variables as functions of the state variable (bankers’ wealth share $\omega$) as well as the dynamics of the economy at any state.

#### 2.3.1 Calibration and Moments

We restrict the choice of parameter values by calibrating our model. Table 1 lists calibrated parameters, and Table 2 in Appendix D contains moments generated by the model and their targets. We set the time discount factor $\rho$ to 3% to match the real interest rate estimated by Campbell and Cochrane (1999). Bankers’ retirement rate $\chi$ is set at 16% to target the average Sharpe ratio. We set bankers’ productivity at 22.5% so that the average investment-to-capital ratio is close to 11% (He and Krishnamurthy, 2012a). The productivity of less-productive households is chosen at 10% to match the fact that the Sharpe ratio during the 2007-09 financial crisis was approximately 15 times the average level (He and Krishnamurthy, 2012a). Choices of the depreciation rate and the capital adjustment cost $\phi$ are standard in the macroeconomic literature (Christiano, Eichenbaum and Evans, 2005). We set the Poisson shock parameters to target the conditional volatility of the growth rate of bankers’ wealth in distressed periods and in non-distressed periods. The distressed periods are defined as
periods with the lowest 33% Sharpe ratios. The regulation parameter, tax rate \( \tau \), is set at 3% to target the average leverage of the entire banking sector. We set the intensity with which bankers can re-access shadow banking after default at 6% to target the ratio of securitization by non-agency issuers in the third quarter of 2006. Appendix D contains both the data and procedure used to calculate this ratio.

### 2.3.2 Capital Misallocation and the Price of Capital

The constraint on equity issuance (Assumption 1) leads to capital misallocation and to the low price of capital in economic downturns, where bankers’ wealth share is low. Panel a in Figure 4 shows that the fraction of capital goods that bankers hold \( \psi(\omega) \) is weakly increasing in bankers’ wealth share \( \omega \). When bankers are wealthy enough to hold all physical capital, there is no capital misallocation. Since bankers use leverage, they can hold all capital goods when they seize approximately 25% of the total wealth in the economy. If their wealth share declines substantially, bankers have to downsize their asset holdings; otherwise, their risk exposure would be too large because of high leverage. As the share of bankers’ wealth diminishes, bankers hold a declining fraction of capital goods, capital misallocation deepens, and aggregate productivity declines. Capital misallocation depresses the price of capital, since capital goods are less valuable in the possession of less-productive households. Panel b in Figure 4 shows that the price of capital \( q(\omega) \) is increasing in bankers’ wealth share \( \omega \). Since the high rate of return comes with the low price of capital, the excess return from holding capital is decreasing in the state variable (Panel d in Figure 4).
Besides causing capital misallocation and depressing the price of capital, the no-equity-financing friction also generates endogenous risk through the balance sheet amplification mechanism. When an aggregate negative shock hits the economy, bankers’ net worth declines disproportionately because of the leverage effect, and bankers’ overall leverage shoots up unfavorably (Panel c in Figure 4). Since all bankers are pressured to sell capital goods, the price of capital deteriorates, which further erodes bankers’ net worth and leads to another round of asset sales.

Endogenous risk caused by the balance sheet amplification varies with different states of the economy (Panel e in Figure 4). When bankers are about to conduct fire sales of physical capital to households, endogenous risk is at its most significant. When bankers’ wealth share
is either very large or very small, endogenous risk is small for the following reasons. When bankers have a small proportion of wealth, they hold only a small fraction of physical capital in the economy. A negative shock causes a small decline in the price of capital because the magnitude of bankers’ total asset sales is tiny. When bankers’ wealth share is very large, their overall leverage is mechanically low (Panel c in Figure 4). A negative shock has a small impact on bankers’ net worth and, similarly, has a small impact on the price movement of capital.

A pecuniary externality exists in the competitive equilibrium because individual bankers do not internalize the aggregate impact of their individual leverage choices on endogenous risk. The leverage decision of each banker has zero impact on endogenous risk. However, the aggregation of all bankers’ leverage choices matters considerably. Therefore, the socially optimal leverage choice is quite distinct from the private optimal choice in the competitive market. The tax on regular banking adjusts bank leverage and improves social welfare. Welfare issues are discussed in detail in Section 3.1.

2.3.4 The Dynamics of the State Variable.

We next focus on the evolution of the state variable. Equation (20) describes its motion. Upper plots in Figure 5 present values of \( \mu_\omega \) and \( \kappa_\omega \) at different states. In states where bankers’ wealth share is small, the growth rate of the state variable \( \mu_\omega \) is high owing to bankers’ high leverage \( s + s^* \) (Panel c in Figure 4) plus the high rate of return from holding physical capital (Panel d in Figure 4). The proportional decline in the state variable is also large in recessions because of bankers’ high leverage.

The lower panel of Figure 5 shows that for most of the time bankers hold about 38% of the wealth in the economy. An economy is rarely in a situation where bankers hold only a little wealth because the low price of capital and high returns from holding capital help bankers to quickly build up their wealth and pull the economy out of recessions. Bankers’ wealth share never exceeds the point \( \bar{\omega} \), where \( \mu_\omega(\bar{\omega}) = 0 \), because bankers retire randomly.

2.4 The Feedback Loop in Shadow Banking

The feedback loop between the maximum leverage of shadow banking \( \{s_t^*\}_{t=0}^{\infty} \) and the cost of default \( \{H_t\}_{t=0}^{\infty} \) is the driving force underpinning our main results: i) shadow banking

\[ \text{In He and Krishnamurthy (2012b) and He and Krishnamurthy (2013), endogenous risk does not decline as the wealth share of productive agents goes to zero because they assume that only productive agents can hold assets. Therefore, in their papers, the pressure to sell does not decline even when productive agents become very poor.} \]
This section explains each of the three main results using dynamic and comparative statics analyses.

2.4.1 Dynamic Result: Pro-cyclical Leverage of Shadow Banking

We show that the leverage of shadow banking \( \{s^*\}_{t=0}^\infty \) is pro-cyclical (Panel a in Figure 6). To demonstrate this, we rearrange the enforceability constraint (18) and find that three factors affect the maximum leverage of shadow banking in period \( t \): i) a banker’s exposure to regular banking business, \( \bar{s}_{t-} \), ii) the loss to one dollar of investment, \( \kappa_t^Q \), iii) the cost of default, \( H_t \).

\[
\bar{s}_{t-}^* = (1 - \exp(-\rho H_t))(\frac{1}{\kappa_t^Q} - (1 + \bar{s}_{t-})).
\]

Next, we illustrate how the three factors work in a dynamic setting by considering the
Figure 6: Pro-cyclical Shadow Banking

This figure presents the leverage of shadow banking $s^*$, “the leverage of regular banking” $\bar{s}$, asset volatility $\kappa Q$, and the cost of default $H$ as functions of the state variable $\omega$ (i.e., bankers’ wealth share) in the “good” equilibrium. For parameter values, see Section 2.3.1.

example of economic booms. At the same time, we will highlight the importance of the feedback loop between $\{s_t^*\}_{t=0}^\infty$ and $\{H_t\}_{t=0}^\infty$. Suppose the economy is in a boom where the bankers’ share of wealth is high, a high price of capital prevails, and a low rate of return induces bankers to take low leverage (Panel b in Figure 6), thereby lowering their incentive to default. The low asset volatility that often characterizes such a boom reduces the likelihood of bankers’ defaulting (Panel c in Figure 6). Therefore, the leverage of shadow banking is high in economic booms (Panel a in Figure 6). Given that the leverage of shadow banking is high in economic booms, equation (19) implies that the cost of default is also high (Panel d in Figure 6), which raises the maximum leverage of shadow banking further as the enforceability constraint (18) demonstrates. The shadow banking sector expands substantially during economic booms as a result of this feedback loop.
2.4.2 Comparative Statics

We next compare different economies and see how the presence of shadow banking changes the conventional understanding of financial instability and its connection to financial regulation. First, we examine economies with and without shadow banking and show that shadow banking increases financial instability. Next, we vary parameter $\tau$ and explain that there is a U-shaped relationship between financial instability and the regulation of the traditional banking sector.

![Figure 7: Reintermediation](image)

This figure presents the price of capital (blue solid line in upper left panel), the jump in the price of capital (blue solid line in upper right panel), the size of debts (lower left panel), and the size of notes (lower right panel) as functions of the state variable $\omega$ (i.e., bankers’ wealth share) in the “good” equilibrium. For comparison, this figure also shows the price of capital (red dashed line in upper left panel) and the jump in the price of capital (red dashed line in upper right panel) as functions of the state variable $\omega$ in the “bad” equilibrium. For parameter values, see Section 2.3.1

Reintermediation. The combined effect of pro-cyclical shadow banking and reintermediation increases endogenous risk in the economy. In our model, the pro-cyclicality of shadow
banking means that shadow banks purchase a large number of assets in economic upturns. The cost of funding the balance sheets of regular banks is expensive. Therefore, in economic booms, the scale of asset accumulation by shadow banks exceeds that which regular banks would pursue in absence of an accompanying shadow bank system. If an adverse shock hits the economy, shadow banks must divest large amounts of assets as the leverage constraint tightens, and regular banks are reluctant to acquire these assets because it is so expensive to expand their own balance sheets. As a result, the price of capital declines more than it would if there were no shadow banking in the economy (Panel b in Figure 7). Finally, the decline in the price of capital erodes bankers’ net worth and increases their leverage, $s + s^*$, which causes increased asset volatility and forces a further decline in the leverage of shadow banking. Overall, our calibrated model shows that shadow banking increases the price volatility of capital $q\kappa$ by 31% on average.

**Shadow Banking: Innocent or Not?** The answer is “Yes and No” in our model. The answer is “Yes” because all shadow banks hold the same type of physical capital as regular banks. The overall investment quality in the economy does not deteriorate because of the appearance of shadow banking. Thus, if we focus on the asset side of the shadow banking system, no one can blame shadow banking for increasing financial instability. Moreover, even if we move to the liability side, a single shadow bank can borrow up to the limit and still cause no harm. The answer is also “No” because when shadow banks expand in economic upturns, bankers fail take into account the negative externality of the asset fire sales that occur in economic downturns.

**Regulatory Paradox.** The conventional wisdom that tough regulation always secures financial stability may not hold when we take shadow banking into account. In the “bad” equilibrium without shadow banking, if the regulatory authority tightens regulation by raising $\tau$, banks’ leverage and the price volatility of capital decline accordingly (Panel a in Figure 8). However, in the “good” equilibrium where shadow banking plays a crucial role, the economy with tighter regulation actually experiences higher market risk (Panel b in Figure 8). The intuition is as follows: if regulation becomes tighter, bankers will face higher tax burdens if they default. Thus, tighter regulation comes with a larger cost of default, as equation (19) indicates (Panel c in Figure 8). Furthermore, a larger cost of default gives rise to a higher leverage for shadow banking. Thus, the shadow banking sector is larger in economies with more stringent regulation. Since shadow banking adds to financial instability, tough regulation imposed on regular banks can jeopardize financial stability.
Regulatory Smile. The regulatory paradox does not mean that relaxing financial regulation always reduces financial instability. It depends on the relative size of the shadow banking system. Recall the feedback loop discussed earlier. If the regulatory authority lowers the tax rate \( \tau \), both the benefit of shadow banking and the cost of default decline, which, in turn, lowers the maximum leverage of shadow banking and further reduces the cost of default. The feedback loop can be so significant that the shadow banking system simply does not emerge. In our numerical example, when \( \tau \) declines from 3% to 2.5% (lower panel in Figure 9), the shadow banking system disappears. In the regime where the level of financial regulation is lenient enough to eliminate the shadow banking system, the conventional wisdom that tightening regulation secures financial stability still holds.

Our model emphasizes the non-monotonic relationship between financial instability and
Financial Instability

Figure 9: Regulatory Smile
This figure shows the investment risk $\kappa^Q$ (upper panel) and the leverage of shadow banking (lower panel) at the stochastic steady state of economies with different tax rates $\tau$. The stochastic steady state is the state where $\omega \mu^\omega - \lambda \omega \kappa^\omega = 0$. We assume that the “good” equilibrium prevails if it exists. For other parameter values, see Section 2.3.1

financial regulation in the presence of shadow banking. The instability-regulation relationship is actually U-shaped: when regulations are relaxed enough, the size of shadow banking is small and financial instability diminishes with decreased regulatory stringency; when regulation is so tight that a large shadow banking system emerges, the reverse is true. This is how our framework reconciles and accommodates the two “contradictory” schools of thought on financial regulation.

3 Welfare and Policy Implications

In this section, we analyze the welfare and policy implications of our framework. First, we characterize the optimal tax rate that maximizes agents’ welfare in the presence of shadow banking. Second, we find that counter-cyclical regulation can generally improve financial stability and welfare because such counter-cyclical regulation alleviates the risk of the asset fire sales between shadow banks and regular banks. Third, we show that our regulatory smile result still holds when the price control analyzed in the baseline model is replaced by a
quantity control (e.g., capital requirement constraint). Lastly, we extend our baseline model to consider the impact of monetary policy on shadow banking.

3.1 Welfare Implications of the Baseline Model

To avoid the problem of welfare aggregation, we reinterpret our model as one that consists of a representative banker and a representative household. Let $K_0$ denote the total capital stock in period 0. Note that the banker’s wealth share $\omega_0$ is exactly the fraction of capital goods that she owns. Thus, her net worth in period 0 is $\omega_0 K_0 q_0$, and the net worth of the representative household is $(1 - \omega_0) K_0 q_0$. Without loss of generality, we assume that $K_0 = 1$. Hence, the welfare pair of the representative household and banker is $(1 - \omega_0) q_0, \ln(\omega_0 q_0) / \rho + h_0)$. We will focus on the welfare of the representative banker because the household’s welfare does not rely directly on two critical variables in policy discussions: economic growth and financial instability.

In the following welfare analysis, we first concentrate on the “bad” equilibrium (red dashed lines in Figure 10) without shadow banking and consider the balance between economic growth and financial instability. Second, we proceed to the “good” equilibrium (blue solid lines in Figure 10) where shadow banking plays a crucial role, and emphasize that a model can mislead welfare analysis if it does not incorporate shadow banking.

Financial regulation can improve the banker’s welfare in the “bad” equilibrium as the red dashed lines in Figure 10 show. The underlying mechanism is that as the tax rate $\tau$ increases from zero, the welfare benefit brought about by the decline in the volatility of the banker’s wealth (Panel b in Figure 11) outweighs the welfare cost caused by the drop in the growth of banker’s wealth (Panel a in Figure 11). The unregulated competitive equilibrium ($\tau = 0$) is sub-optimal for the banker because of the pecuniary externality that she does not internalize the negative impact of her leverage choice on endogenous risk $\kappa_t^q$. This result essentially justifies the legitimacy of financial regulation in our model.

We now move to the “good” equilibrium and discuss the optimal tax rate in light of shadow banking. In contrast to the “bad” equilibrium, the rise in tax rate $\tau$ increases both the growth and the volatility of the banker’s wealth because i) the shadow banking sector expands as regulation tightens, and ii) the growth of shadow banking more than offsets the negative effect of regulation. In our numerical example, the growth of shadow banking contributes to the banker’s welfare because, as the shadow banking sector begins to grow, the growth benefit dominates the cost of increased risk. However, if regulation is too stringent, further tightening will heighten financial instability, the negative effect of which actually dominates the benefit of economic growth. Therefore, our baseline model indicates
Figure 10: Welfare
This figure shows the level of the banker’s continuation value \( \ln(\omega q)/\rho + h \) in 6 different states (\( \omega = 0.1 \) in upper left panel, \( \omega = 0.2 \) in upper middle panel, \( \omega = 0.36 \) in the upper right panel, \( \omega = 0.38 \) in the lower left panel, \( \omega = 0.4 \) in the lower middle panel, and \( \omega = 0.42 \) in the lower right panel), given different tax rates. For parameter values other than \( \tau \), see Section 2.3.1.

the optimal level of the tax rate with the impact of shadow banking taken into account.

Figure 10 shows that the optimal tax rate in the “good” equilibrium differs from that in the “bad” one. Although extremely tight regulation hurts the banker’s welfare in both “good” and “bad” equilibria, the underlying reason is different for each of the “good” and “bad” cases. In the “bad” equilibrium, tightening regulation has the negative effect of slowing economic growth (Panel a in Figure 11). However, in the “good” equilibrium, raising tax rate \( \tau \) increases financial instability (Panel d in Figure 11). Therefore, a model that omits shadow banking will completely misguide the policy-making process if the economy has a large or fast-growing shadow banking sector.
Figure 11: Welfare
This figure shows how the average growth rate (left panels) and the average volatility (right panels) of the representative banker’s wealth change as the tax rate changes in economies without shadow banking (upper panels) and with shadow banking (lower panels). We use the stationary distribution to calculate moments. For parameter values other than $\tau$, see Section 2.3.1.

3.2 Counter-Cyclical Regulation

In this section, we substitute the constant tax rate regulation (Assumption 2$'$) with a counter-cyclical regulation specified by the following assumption.

**Assumption 2$''$** In period $t$, the tax rate $\tau_t$ equals $\min\{\tau(\omega_t), \tau(\bar{\omega})s_t\}$. $\tau(\omega)$ is defined by $(\tau(\bar{\omega} - \omega) + \bar{\tau}\omega)/\bar{\omega}$, where $\tau$, $\bar{\tau}$, and $\bar{\omega}$ are constants, $\tau < \bar{\tau}$, and $\bar{\omega}$ is larger than $\bar{\omega}$. The interpretation of Assumption 2$''$ is that the regulatory authority alleviates the tax burden on the regular banking sector in recessions and discourage the bankers’ use of leverage in economic booms. We use the same algorithm to solve for the equilibrium of the modified model.
Figure 12: Counter-Cyclical versus Constant Regulation.
This figure compares the “good” equilibrium of an economy with a counter-cyclical tax rate (red solid line) and that of an economy with a constant tax rate (blue dashed line) based on seven endogenous variables in the equilibria of two economies: the price of capital (upper left panel), overall leverage (upper middle panel), fraction of capital held by bankers (upper right panel), decline in the price of capital (lower left panel), leverage of shadow banking (lower middle panel), and continuation values of bankers and households (lower right panel). The counter-cyclical tax rate policy is specified by $\bar{\tau} = 0.02$, $\bar{\tau} = 0.07$, and $\bar{\omega} = 0.8$. The economy with the counter-cyclical policy has an average tax rate 0.0445, which is calculated based on the stationary distribution, and the economy with the constant tax rate policy has a tax rate 0.0445. The black dash-dot lines link the household-banker continuation value pairs $((1 - \omega)q, \ln(\omega q) + h)$ at the same states of the two economies. For other parameter values, see Section 2.3.1.
Counter-cyclical tax rate regulation outperforms constant regulation in a number of aspects, as we illustrate in Figure 12. The price of capital is higher with the counter-cyclical regulation than it is in the baseline model (Panel a in Figure 12) because bankers can assume higher leverage in economic downturns due to the low tax rate on leverage (Panel b in Figure 12). For the same reason, it is easier for bankers to seize all physical capital in the modified model (Panel c in Figure 12). In addition, the counter-cyclical regulation enhances the financial stability of an economy (Panel d in Figure 12): when shadow banks divest assets in fire sales during economic downturns, the counter-cyclical regulation implies that regular banks are more willing to purchase assets than they are when they face the constant tax rate. Therefore, the counter-cyclical regulation mitigates the impact of asset fire sales on the price of capital. Although the counter-cyclical regulation alleviates financial instability, it makes the leverage of shadow banking more volatile because bankers face a higher cost of default in economic upturns than they face in economic downturns, and thus the leverage of shadow banking is much higher in upturns than it is in downturns (Panel e in Figure 12). Given the foregoing discussion, it is readily evident that counter-cyclical tax rate regulation improves the welfare of both bankers and households (Panel f in Figure 12).

The “regulatory paradox” result still holds in the model with the counter-cyclical policy, although the financial market is more stable. The left panel of Figure 13 shows that counter-cyclical regulation can substantially enhance financial stability when the average tax is high. This is true despite of the fact that the volume of reintermediation is larger in the economy with counter-cyclical regulation (right panel in Figure 13). This is because regular banks face declining borrowing costs when they need to raise funds to acquire assets dumped by shadow banks in economic downturns. It is this property of counter-cyclical regulation that substantially lowers the financial instability of an economy.

3.3 Quantity Control

In this section, we investigate a modified model, in which the regular banking sector is subject to a quantity control rather than the price control of the baseline model. In particular, we consider the capital-requirement constraint that commercial banks in the real world often face, and this constraint essentially imposes an upper bound \( \bar{s} \) for a regular bank’s liability-to-equity ratio.

Given a quantity-oriented capital-requirement constraint, the expected lifetime utility maximization problem for a banker who can access shadow banking differs from that in the baseline model in two respects. First, the banker faces a capital-requirement constraint \( S_t \leq \bar{s}W_t \). Second, there is neither a tax nor a lump-sum subsidy in the dynamic budget.
**Figure 13: Financial Instability and Counter-Cyclical Regulation**

This figure shows how the average volatility (red solid line, left panel), the average leverage of shadow banking (red solid line, middle panel), the average overall leverage (black dashed line, middle panel), and the volatility of the securitization ratio (red solid line, right panel) change with the average tax rate in the “good” equilibrium of an economy with a counter-cyclical tax rate policy. All moments are based on the stationary distribution. For comparison, left and right panels show how the average volatility and the volatility of the securitization ratio changes, respectively, as the tax rate moves in an otherwise fixed economy. For parameter values, see both Section 2.3.1 and Figure 12.

Constraint

\[
dW_t = (W_t - R_t - (S_t - S^*_t)(R_t - r_t) - c_t) \, dt - (W_t + S_t + S^*_t) \, \kappa_t^Q \, dN_t.
\]

There are two similar changes for the maximization problem of bankers who cannot access shadow banking. We use the same numerical procedure to solve for the Markov equilibria of the modified model. The choice of parameter values follows: \(\rho = 4\%\), \(\chi = 1\%\), \(a = 0.225\), \(a^h = 0.1\), \(\delta = 10\%\), \(\phi = 3\), \(\lambda = 1\), \(\kappa = 4\%\), and \(\bar{s} = 4\).

We first focus on the dynamics of endogenous variables and then move to the regulatory smile result of this quantity-control model. A number of endogenous variables, such as the price of capital, the leverage of regular banking, the excess return of running regular banks, and the price volatility of capital, have dynamics similar to the baseline model (Panels a-e in Figure 14). The leverage dynamics of shadow banking change drastically as the price control is replaced by the quantity control. The different dynamics result from the fact...
that when bankers’ share of wealth is small, the excess return is high, and the incentives to build up leverage are strong. In these states of high excess return and leverage build-up, if a banker defaults, then she can raise funds only through regular banking, which allows for only comparatively low leverage. Therefore, when bankers’ share of wealth is small, the cost of default is extremely large, and the leverage of shadow banking is very high. This property is absent in the baseline model because bankers do not face a binding leverage constraint for their regular banking. Thus, in the baseline model, both the cost of default and the leverage of shadow banking in economic downturns are low.

Constant quantity control is less appealing than price control because it prevents productive bankers from leveraging when the entire economy needs them to do so. When bankers’ wealth share is small, the aggregate productivity could be as low as households’ productiv-
ity if the capital-requirement constraint is binding for regular banks and shadow banking is not available. We can see that micro-prudential policies such as a capital-requirement constraint may not benefit the overall economy, even though it capably contains the credit risk of individual banks.

**Financial Instability**

![Graph showing Financial Instability](image)

**Figure 15:** This figure shows the investment risk $\kappa Q$ (upper panel) and the leverage for shadow banking (lower panel) at the stochastic steady state of different economies with different capital-requirement constraints in the “good” equilibrium of the modified model. The stochastic steady state is the state where $\omega^\omega - \lambda \omega \kappa^\omega = 0$. For the choice of parameter values, see Section 3.3.

Figure 15 shows that the regulatory smile result of our baseline model continues to hold when regular banks face quantity control. If financial regulation is very lenient, the leverage of shadow banking is low and financial instability is high. As financial regulation tightens (i.e., the maximum leverage of regular banking $\bar{s}$ declines), financial instability initially diminishes. However, if the regulation is so tight that the shadow banking sector becomes sizable, tighter regulation causes higher financial instability.

### 3.4 Monetary Policy and Shadow Banking

Although this paper is primarily about financial regulation, in this section we digress and discuss how shadow banking responds to certain changes in monetary policy. We relate our model to recent work by Drechsler et al. (2014), which associates the nominal interest rate...
and banks’ reserve requirement to the use of bank leverage. We will show that their setup has the same effect as our model’s tax on regular banking.

With reference to Drechsler et al. (2014), we next interpret the tax on regular banking as the reserve-requirement constraint and associate the tax to the nominal interest rate. Consider a regular bank that issues debt worth $S_{t-}$. Due to the reserve requirement, the bank must hold reserves worth $\nu S_{t-}$. If we interpret reserves as the numéraire, the rate of return for holding reserves in real terms is the negative of the inflation rate because the real value of reserves is the inverse of the price level of consumption goods.\(^7\) Given the reserve requirement, the dynamic budget constraint of the banker becomes

\[
dW_t = (W_t - R_t - r_t) + \nu S_{t-} (-i_t - dt - r_t - dt) + S_{t-}^* (R_t - \tilde{r}_t - c_t) dt \\
- (W_{t-} + S_{t-} + S_{t-}^*) \kappa_t^Q dN_t,
\]

where $i_t$ is the inflation rate. The tax rate $\tau_t$ in the baseline model is equivalent to $\nu (i_t + r_t)$, the product of the reserve requirement parameter $\nu$ and the nominal interest rate $i_t + r_t$. By adding the specification that the central bank redistributes its revenue back to bankers in proportion to their net worth, it then becomes evident that tight monetary policy has the same effect as tight financial regulation in the baseline model.

### 4 Robustness

To identify what specification is critical for our main results, we vary the setup of the baseline model along two dimensions: the cost of default and the preference of households. We find that it is crucial to have the cost of default depend on financial regulation in the baseline model and that households’ preferences are not essential for our main results.

#### 4.1 Exogenous Leverage Constraint for Shadow Banking

For “regulatory paradox” result to obtain, the maximum leverage of shadow banking must be dependent on financial regulation. To demonstrate this, we characterize a variant of the

\(^7\)Given that each unit of consumption goods is equivalent to $p_t$ units of reserves, the inflation rate is $dp_t / p_t$. The rate of return for holding reserves in real terms is

\[
\frac{d(\frac{1}{p_t})}{\frac{1}{p_t}} = \frac{dp_t}{p_t} = -\frac{dp_t}{p_t}
\]
baseline model where the endogenous cost of default \( \{H_t\}_{t=0}^{\infty} \) is replaced by a constant \( \bar{H} \). To characterize the equilibrium of the modified model, we need to solve only a single delayed differential equation with respect to \( q(\omega) \).

\[
\bar{s}_t = (1 - \exp(-\rho \bar{H}))(\frac{1}{\kappa_Q} - (1 + \bar{s}_t -)).
\]

Although the cost of default is constant in the modified model, the leverage of regular banking is counter-cyclical and asset volatility is low in upturns. Hence, the pro-cyclicality of shadow

---

\(^8\)Huang (2014) constructs a special scheme to deter bankers from defaulting on their shadow bank obligations such that the cost of default is a constant.
banking still holds.

\[
\text{fire sales of assets by shadow banks}
\]

\[
\text{vol} \left( \frac{s^*}{1 + s + s^*} \right)
\]

\[
\tau, \text{ tax rate}
\]

\[
\text{overall leverage}
\]

\[
1 + s + s^*
\]

\[
0.01 \quad 0.02 \quad 0.03 \quad 0.04 \quad 0.05
\]

\[
0 \quad 0.05 \quad 0.1
\]

The “regulatory paradox” result does not hold in the modified model, as shown in the upper panel of Figure 16. Two facts drive this outcome. First, the size of the shadow banking sector does not change substantially with financial regulation (lower panel of Figure 16). When the regulatory authority raises the tax on regular banking, there are not many banking activities that migrate to the shadow banking sector. Second, when the level of financial regulation varies, the magnitude of the reintermediation does not change as significantly as it does in the baseline model (upper panel of Figure 17). Since the amount of shadow banking and the degree of reintermediation do not change significantly with financial regulation in the modified model, neither do we observe any significant change in financial stability caused by shadow banking. The primary reason that tighter regulation leads to reduced financial instability is simply because the overall leverage of the economy declines as financial regulation tightens (lower panel of Figure 17).
### 4.2 Households with Epstein-Zin Preference

In this section, we investigate whether our main results are robust to the preference specification of households. In particular, we modify the baseline model so that households have Epstein-Zin preferences with the time discount rate $\rho$, the relative risk-aversion coefficient $\gamma$, and the elasticity of intertemporal substitution $b^{-1}$. To simplify the characterization of the model, we modify Assumption 5 such that bankers become households at rate $\chi$.

In the modified model, a household chooses $\{c^h_t, S^h_t, n_t, t \geq 0\}$ to maximize

$$U^h_0 = E_0 \left[ \int_0^\infty f(c^h_s, U^h_s) \, ds \right],$$

where

$$f(c^h, U^h) = \frac{1}{1-b} \left\{ \frac{\rho (c^h)^{1-b}}{(1-\gamma) U^h} - \rho (1-\gamma) U^h \right\}$$

and

$$U^h_t = E_t \left[ \int_t^\infty f(c^h_s, U^h_s) \, ds \right], \text{ for } t > 0,$$

subject to the dynamic budget constraint

$$dW^h_t = (W^h_t - r_t - S^h_t (R^h_t - r_{t-}) + n_{t-} (\bar{r}_t - r_{t-}) - c^h_t) \, dt - S^h_t \kappa_t \bar{Q} \, dN_t.$$

In solving for the equilibrium of the modified model, we need to characterize the optimal consumption and portfolio choices of households. To do so, we conjecture that the continuation value of a household with net worth $W^h_t$ has the following functional form

$$V^h_t = V(\zeta_t, W^h_t) \equiv \left( \frac{\zeta_t W^h_t}{1-\gamma} \right)^{1-\gamma},$$

where $\zeta_t$ follows

$$d\zeta_t = \zeta_t - \mu_t \zeta_t \, dt - \zeta_t \kappa_t \bar{Q} \, dN_t.$$

As in Di Tella (2012), $\zeta_t$ is interpreted as the continuation value multiplier of a household’s net worth since the function $V(\zeta_t, W^h_t)$ is homogeneous of degree $1 - \gamma$ with respect to $W^h_t$. Given our conjecture, the Hamilton-Jacobi-Bellman equation of the household’s dynamic programming problem is

$$0 = \max_{c^h_t, S^h_t} \left\{ f(c^h_t, V^h_t) + D^h_t c^h_t V(\zeta_t, W^h_t) \right\},$$

(30)
where

\[
D^{\phi, S}V(\zeta_t, W^h_t) = (W^h_t - r^h_t\zeta_t - R^h_t - r^h_t - n_t - (\zeta_t - W^h_t) \zeta_t^1 - \gamma (W^h_t)^{-\gamma}
+ \mu_t^c (\zeta_t - W^h_t)^{-\gamma} + \lambda \left( \frac{((\zeta_t - \zeta_t^c)(W^h_t - S^h_t)^1 - \gamma}{1 - \gamma} \right) - \frac{(\zeta_t - W^h_t)^{1 - \gamma}}{1 - \gamma}.
\]

We summarize the key results of the problem in the following proposition.

**Proposition 7** Each household's optimal consumption weight \( \{\hat{c}^h_t, t \geq 0\} \), optimal portfolio weight \( \{s^h_t, n_t\}_{t=0}^{\infty} \), and the process \( \{\zeta_t\}_{t=0}^{\infty} \) satisfy

\[
(\hat{c}^h_t)^{b} = \frac{\rho}{\zeta_t^{1-b}}, \tag{31}
\]

\[
R^h_t - r^h_t \leq \frac{\lambda \kappa_t^Q}{(1 - s^h_t\kappa_t^Q)} (1 - \nu_t^c)^{1 - \gamma}, \quad = \text{if } s^h_t > 0, \tag{32}
\]

\[
\tilde{r}_t \leq r_t, \quad = \text{if } n_t > 0, \tag{33}
\]

\[
0 = \frac{\rho (\hat{c}^h_t)^{1-b}}{(1-b)s^h_t \zeta_t^{1-b}} - \frac{\rho}{1-b} + r_t + s^h_t (R^h_t - r^h_t) - \hat{c}^h_t + \mu_t^c
+ \lambda \left( \frac{(1 - \nu_t^c)(1 - s^h_t\kappa_t^Q)}{1 - \gamma} \right) - \frac{1}{1 - \gamma}, \tag{34}
\]

where \( \hat{c}^h_t = c^h_t / W^h_t \) and \( s^h_t = S^h_t / W^h_t \).

Two market-clearing conditions in the modified model differ from their counterparts in the baseline model. First, the risk-free rate, which is constant in the baseline model, is jointly determined by the optimal portfolio choices of both bankers and households (equations 32 and 33) and the dynamics of the continuation value multiplier \( \{\zeta_t\}_{t=0}^{\infty} \) (equation 34). Second, the market for consumption goods does not clear automatically in the modified model since households are risk-averse. Given households’ optimal consumption choice condition (31), we have the market-clearing condition for consumption goods in period \( t \),

\[
a \psi_t + a^h (1 - \psi_t) - g(\omega_t) = \left( \rho \omega_t + \rho^b \beta \zeta_t^{\frac{b-1}{2}} (1 - \omega_t) \right) q_t.
\]

As in our previous analyses, we focus on the Markov equilibrium of the modified model where shadow banking exists. To characterize the equilibrium, we need to solve a delay differential equation with respect to \( \zeta(\omega) \) in addition to differential equations for \( q(\omega) \) and \( H(\omega) \). Results found in the baseline model survive in the modified model, as the following
Figure 18: This figure shows the investment risk $\kappa^Q$ (upper panel) and the leverage for shadow banking (lower panel) at the stochastic steady state of different economies with different tax rates $\tau$ in the modified model with households of Epstein-Zin preference. The stochastic steady state is the state where $\omega\mu - \lambda\omega\kappa = 0$. We assume that the “good” equilibrium prevails if it exists. For the choice of parameter values, see Section 4.2.

The numerical example shows. The choice of parameter values is that $\rho = 4\%$, $\gamma = 2$, $b = 0.5$, $\chi = 0.1$, $a = 0.225$, $a^h = 0.1$, $\delta = 10\%$, $\phi = 3$, $\lambda = 1$, $\kappa = 4\%$, $\tau = 3\%$, and $\xi = 5\%$. Figure 22 in Appendix E shows that the main endogenous variables have dynamic properties similar to those in the baseline model. The regulatory smile result holds in the modified model, as Figure 18 shows.

5 Systemic Risk in the Sunspot Equilibrium

In this section, we proceed to the more general sunspot equilibrium of the baseline model. Using a sunspot equilibrium, we are able to consider the systemic risk that the shadow banking system may suddenly collapse. Analogous to the preview section, we also use the numerical method to solve for a sunspot equilibrium. We show that the shadow banking sector faces the risk of a sudden collapse, and that its breakdown could cause a spike in the price volatility of capital. Furthermore, we illustrate how loosening financial regulation could reduce the size of the shadow banking sector and lower the risk of its sudden collapse.
5.1 Sunspot Equilibria and Systemic Events

We name a family of sunspot equilibria defined by Definition 2 as $S$ equilibria.

**Definition 2** An $S$ equilibrium is defined by a sunspot process \( \{ l_t, t \geq 0 \} \), a deterministic but time-varying process \( \{ l^*_t, t \geq 0 \} \), and a transition rule:

i. The economy first stays in a “good” phase where shadow banking exists until
   
ii. it falls into the “bad” equilibrium when the Poisson shock hits the economy and \( l_t > l^*_t \).

The transition from the “good” phase to the “bad” equilibrium involves the systemic event that all shadow banks default at the same time. The reason is simple: at the moment when the economy falls into the “bad” equilibrium, the extra benefit of accessing shadow banking vanishes \( (H = 0) \), and all bankers find it unprofitable to support their shadow banks. Thus, all shadow banks default simultaneously.

The “good” equilibrium is a special $S$ equilibrium, in which \( \{ l_t > l^*_t, t \geq 0 \} \) is a zero probability event. The difference between the “good” equilibrium analyzed in the previous section and the “good” phase here is that endogenous risk \( \kappa^q_t \) in the “good” phase could take either of two values: one involves the systemic event \( \kappa^q,r_t \) and the other does not \( \kappa^q_t \). Thus, we need to characterize only the “good” phase to fully describe an $S$ equilibrium.

5.2 The “Good” Phase

Here, \( \{ q_t, t \geq 0 \} \) denotes the probability of the event \( \{ l_t > l^*_t \} \) in period \( t \). In the following proposition, which is the counterpart of Proposition 2, we list bankers’ optimal choices in $S$ equilibrium.

**Proposition 2’** In an $S$ equilibrium, a banker’s optimal consumption \( \{ c_t \}_{t=0}^\infty \) and optimal portfolio weights \( \{ s_t, s^*_t \}_{t=0}^\infty \) satisfy

\[
c_t = \rho W_t, \tag{35}
\]

\[
\max \left\{ \left(1 + s_{t-} + s^*_{t-}\right) \kappa^Q_t, (1 + s_{t-}) \kappa^Q,r_t \right\} < 1, \tag{36}
\]

\[
R_{t-} - r_{t-} - r_{t-} \leq \frac{\lambda (1 - q_t) \kappa^Q_t}{1 - (1 + s_{t-} + s^*_{t-}) \kappa^Q_t} + \frac{\lambda q_t \kappa^Q,r_t}{1 - (1 + s_{t-}) \kappa^Q_t}, \quad \text{if } s_{t-} > 1, \tag{37}
\]

\[
R_{t-} - \tilde{r}_{t-} \geq \frac{\lambda (1 - q_t) \kappa^Q_t}{1 - (1 + s_{t-} + s^*_{t-}) \kappa^Q_t} = \text{if } s_{t-} < \bar{s}_t^*, \tag{38}
\]

where \( \kappa^Q,r_t \equiv \kappa + (1 - \kappa) \kappa^Q,r_t \), \( s_t = S_t/W_t \), and \( s^*_t = S^*_t/W_t \).

**Proof.** Analogous to the proof of Proposition 2. ■
Given that an adverse shock hits the economy, if no systemic event occurs, bankers’ net worth drops by proportion \((1 + s_{t-} + s^*_t) \kappa_t^Q\); otherwise, it declines by proportion \((1 + s_{t-}) \kappa_t^{Q,r}\) since bankers do not bear the loss for creditors of their shadow banks. Equations (13) and (14) are the special cases of equations (37) and (38) when \(\varrho_t = 0\).

We next discuss the note market. First, if the systemic risk materializes, the default of shadow bank is inevitable because there is no benefit to shadow banking access. Assumption 1 is satisfied in equilibrium provided default can be avoided when no systemic event occurs. We can reapply the same argument to derive the maximum leverage of shadow banking

\[
\rho H_t (1 - \kappa_t^H) = \ln \left(1 + \frac{s^*_t \kappa_t^Q}{1 - (1 + s_{t-} + s^*_t) \kappa_t^Q}\right),
\]

where \(\kappa_t^H\) is the jump size of \(H\) that does not involve the systemic event. Furthermore, \(H_t\) has a probabilistic representation,

\[
H_t \equiv h_t - \hat{h}_t = E_t \left[\int_t^\infty \exp (- (\rho + \xi + \chi) u) \exp \left(-\lambda \int_u^t \varrho_s ds\right) f_u du\right]. \tag{39}
\]

Equation (39) incorporates the intensity of the systemic event \(\lambda \varrho_t\), which effectively terminates the negative effects of the bankers’ default.

We characterize a Markov \(S\) equilibrium with a single state variable \(\omega_t \equiv \int_0^1 W_i^t dW_i / q_t K_t\). \(q(\omega)\) denotes the price of capital in the “good” phase and \(q^b(\omega)\) the price of capital in the “bad” equilibrium. The following proposition draws the connection between the decline in the price of capital and the decline in the state variable, given the systemic event. Intuitively, when the systemic event occurs, all shadow banks liquidate their assets together, and the market illiquidity of physical capital decreases substantially. The significant decline in the price of capital erodes bankers’ wealth again, which, in turn, leads to a further decline in the state variable.

**Proposition 8** The realization of the systemic risk in period \(t\) would cause the price of capital to decline by factor \(\kappa_t^{q,r}\), one dollar invested in physical capital loses \(\kappa_t^{Q,r}\) dollar, and the state variable drops by factor \(\kappa_t^{\omega,r}\), where \(\kappa_t^{\omega,r}\), \(\kappa_t^{Q,r}\), and \(\kappa_t^{\omega,r}\) satisfy

\[
\kappa_t^{q,r} = 1 - \frac{q^b(\omega_{t-} (1 - \kappa_t^{\omega,r}))}{q(\omega_{t-})}, \tag{40}
\]

\[
\kappa_t^{Q,r} = \kappa + (1 - \kappa) \kappa_t^{q,r}, \tag{41}
\]

\[
\kappa_t^{\omega,r} = \frac{s_{t-} \kappa_t^{Q,r}}{1 - \kappa_t^{Q,r}}. \tag{42}
\]

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Proof. Equations (40) and (41) are the definitions of $\kappa_{t}^{q,r}$ and $\kappa_{t}^{Q,r}$ given that $\omega$ declines by fraction $\kappa_{t}^{\omega,r}$ when the systemic event occurs in period $t$. To derive $\kappa_{t}^{\omega,r}$, we first specify the loss to bankers’ total wealth, which is $W_{t-}$ before the negative shock hits in period $t$. Since bankers default and do not take on the losses to their shadow banks, they suffer losses only from their regular banking, which are $W_{t-} (1 + s_{t-}) \kappa_{t}^{Q,r}$. Since the total wealth in the economy declines by fraction $\kappa_{t}^{Q,r}$, bankers’ share of wealth after the systemic event is

$$W_{t-} \left(1 - (1 + s_{t-}) \kappa_{t}^{Q,r}\right)$$

$$q_{t-} K_{t-} \left(1 - \kappa_{t}^{Q,r}\right)$$

$$= \omega_{t-} \left(1 - s_{t-} \kappa_{t}^{Q,r} \frac{1 - \kappa_{t}^{Q,r}}{1 - \kappa_{t}^{Q,r}}\right).$$

Thus, we derive equation (42) $\blacksquare$

5.3 Numerical Example

We use a numerical example to illustrate the property of an $S$ equilibrium. To be consistent with the experience of the financial crisis of 2007-09, we choose the processes $\{l_{t}, l_{t}^{*}, t \geq 0\}$ such that a collapse of the shadow banking system becomes more likely when shadow banking plays a more important role in the economy (Panels a and b in Figure 19).

Endogenous risk $\kappa_{t}^{q}$ spikes when the shadow banking system crashes, and the economy falls from the “good” phase to the “bad” equilibrium (Panel e in Figure 19). To better understand this shift, we first look at Panel c in Figure 19, which demonstrates that the price of capital is higher in the “good” phase than it is in the “bad” equilibrium as a result of the lower cost of funding for bankers in the “good” phase. Second, when the systemic event occurs, the economy falls into the “bad” equilibrium without shadow banking. To repay their investors, all shadow banks collectively divest their assets in fire sales to regular banks. The difference between reintermediation to regular banks in this section and that discussed in Section 2.4 is that here regular banks fully anticipate that after they acquire assets in fire sales from distressed shadow banks, there is no future chance that they can sell those assets back, since the shadow banking system will close forever. Because the regular banks expect no resale, the price that they would pay for assets from shadow banks is substantially lower than in non-distressed sales (Panel c in Figure 19). It is this assessment of final sale that leads to the spike in the price volatility of physical capital.

Although the decline in the price of capital caused by the systemic event is large, the consequential loss to bankers’ wealth is not necessarily larger than a loss to bankers’ wealth
Figure 19: Panel (a) shows the estimated density of the simulated stationary distribution (solid line), and the likelihood that the systemic event would occur, conditional upon a Poisson shock (dashed line); Panel (b) shows the overall leverage of a bank $s + s^*$ (solid line) and the leverage of shadow banking $s^*$ (dashed line); Panel (c) displays the price of capital in the “good” phase (solid line) and the price of capital in the “bad” equilibrium (dashed line); Panel (d) reports the size of the decline in the state variable $\omega$ if the systemic event does not happen (solid line) and the size of the decline if it occurs (dashed line); Panel (e) reports the size of the decline in the price of capital, given the systemic event (solid line) and the size of the decline if the systemic event does not happen. For parameter values, see Section 2.3.1.

caused by a normal negative shock. Collapse of the shadow banking system makes it futile for bankers to make any effort to suffer losses in order to maintain access to shadow banking, so bankers refuse to tolerate losses for the sake of their now-useless shadow banks. Therefore, the loss to bankers comes only from their regular banks.

Regulatory Implication. All of the regulatory results that we have observed in the non-sunspot equilibrium are preserved and become more pronounced in the context of systemic risk. In a more tightly regulated economy, the decline in the price of capital due to the systemic event is more pronounced (Panel b in Figure 20), and the likelihood of the systemic event is also higher (Panel c in Figure 20). As we have seen, if the tax rate $\tau$ is small enough, the shadow banking system would be effectively eliminated, which means that the systemic
Figure 20: Regulatory Paradox.
This figure shows the decline in the price of capital without any systemic event (upper left panel); the decline in the price of capital due to the systemic event (upper right panel); the likelihood of the systemic event (lower left panel); and the leverage for shadow banking (lower right panel). The red solid line is for the economy with loose regulation ($\tau = 3.5\%$); the blue dashed line for the economy with modest regulation ($\tau = 4\%$); the black dash-dot line for the economy with tight regulation ($\tau = 4.5\%$). For other parameter values see Section 2.3.1.

risk caused by shadow banking could also be prevented completely if regulations are relaxed to a reasonable level.

5.4 The 2007-09 Financial Crisis
This section further relates our model to the 2007-09 financial crisis. To underscore the relevance of our model, we provide evidence to show that off-balance-sheet financing and implicit guarantees are the most common practice in the shadow banking system. We also show that shadow banking and asset volatility are negatively correlated in both the United States and Europe, and that reintermediation—the fire sale of assets from the shadow banking sector to the regular banking sector—contributes enormously to the collapse of the shadow banking system.
5.4.1 Off-Balance-Sheet Financing and Implicit Guarantees

Implicit guarantees are widely used by financial firms to support their off-balance-sheet financing, which expands their businesses without subjecting them to tight financial regulations. Gorton and Souleles (2007) discuss on the institutional details of securitization, SPVs, and off-balance-sheet financing extensively. They also emphasize the enforcement problem of implicit guarantees for off-balance-sheet borrowing. In particular, their empirical findings show that investors require higher yields for credit card ABS issued by riskier sponsors.

The use of implicit guarantees is also common for short-term financial instruments, such as asset-backed commercial paper (ABCP) and money market funds (MMFs). Acharya et al. (2012) investigate ABCP conduits, which receive various guarantees from their commercial bank and investment bank sponsors. Even though the most common guarantees for ABCP conduits are contractual liquidity guarantees, which nominally leave the credit risks of the underlying asset to ABCP investors, no investors in this market actually suffered losses under liquidity guarantees during the 2007-09 financial crisis according to Acharya et al. (2012). We can, then, deduce that commercial bank sponsors play a form of guarantor role beyond their agency function as the liquidity providers for their ABCP programs. For instance, we have seen that HSBC spent $35 billion in order to bring assets of its off-balance-sheet structured investment vehicles (SIVs) onto its balance sheet in late 2007.9 Around the same time, Citigroup also moved $37 billion assets in SIVs back to its balance sheet.10 In money market funds and hedge funds, we can also observe examples of implicit guarantor-like support provided by sponsor institutions. McCabe (2010) highlights reports from the Securities and Exchange Commission that at least 44 MMFs received voluntary support from their sponsors to avoid breaking the buck during the 2007-09 financial crisis. The primary purpose of such voluntary support actions is to maintain sponsors’ reputations. McCabe (2010) finds that, during the crisis, MMFs saw larger outflows if their sponsors were weaker. Even for some hedge funds, Duffie (2010) documents anecdotal evidence that broker-dealer banks, such as Bear Stearns, volunteered to bail out their internal hedge funds.

5.4.2 Growth of Shadow Banking and Low Volatility

Off-balance-sheet financing plays a crucial role in the shadow banking system which involves long and complex credit chains. The chains connecting end-borrowers and end-lenders are roughly composed of finance companies, ABS issuers, ABS underwriters, ABCP conduits/SIVs, MMFs, and different subsidiaries of broker-dealer banks and bank holding

9See Goldstein (2007).
10See Moyer (2007).
companies. Pozsar et al. (2010) survey some of the institutional details of shadow banking.

Figure 23 and Figure 24 in Appendix E show that the rapid expansion of shadow banking in both US and European markets was closely accompanied by a broad general decline in asset volatility. This phenomenon is consistent with our result that the leverage of shadow banking and the price volatility of physical capital move in opposite directions.

5.4.3 Bank Runs and Reintermediation

Our framework also captures the sudden collapse of the shadow banking system that occurred during the 2007-09 financial crisis, although it does not endogenize the trigger for this systemic event. Generally, both scholars and market participants agree that the breakdown of the shadow banking sector assumed various forms of systemic runs between financial institutions. For instance, Covitz et al. (2012) and Arteta et al. (2013) investigate the market-wide run on ABCP initiated by institutional investors such as MMFs in 2007; Gorton and Metrick (2011) document the run on the sale and repurchase market (the repo market); and McCabe (2010) considers the run on MMFs in 2008.

The consequence of runs in the shadow banking system is that a huge volume of assets was forced to find its way back from shadow banks to the balance sheets of the traditional banking system. He et al. (2010) estimate that during the 2007-09 financial crisis, securitized assets, such as mortgage-backed securities and ABS collateralized debt obligations held by hedge funds and broker-dealers, declined by around $800 billion, and commercial banks increased their holdings of these assets by $550 billion. Consistent with the results of our model, the contraction of shadow banking is accompanied by the expansion of traditional banking. This reintermediation process has an adverse impact on financial markets because of the downward pressure that it creates for asset prices, as we have discussed earlier.

Another consequence of the market-wide panic on the shadow banking sector, just as our model predicts, is the spike in market risk. At the height of the financial crisis, the VIX index climbed by 300 percent.

6 Final Remarks

This paper provides a framework for evaluating financial regulatory rules in the modern financial environment where the unregulated shadow banking sector plays a critical role. Our framework explicitly takes into account the unintended and indirect influence of bank regulation on the shadow banking sector. Thus, our framework offers a more comprehensive toolkit for policy evaluation.
The framework proposed in this paper could be extended in the following three directions. The first and most straightforward follow-up work is to characterize the social planner’s constrained efficient regulatory rule by, say, a process of tax rate \( \{\tau_t, t \geq 0\} \). However, this exercise requires a completely new methodology that can characterize the set of all competitive equilibria under all sorts of regulations. Recall that our paper focuses only on Markov equilibria with a single state variable.

Second, one could endogenize the collapse of the shadow banking system by exploring the stability property of the “good” equilibrium where shadow banking exists. By combining certain quantitative work, one can develop a framework that provides early warning signs of a financial crisis. We have attempted to do so by adding a few extra features into the baseline model in this paper (Huang, 2014).

Third, how creditors respond to the default of a shadow bank could also be endogenized as an equilibrium outcome. One can calibrate this extension by feeding it with the deep parameters found in credit markets.

References


Appendix

A Alternative Enforceability Constraint

Recall that the enforceability constraint (18) is so tight that creditors of a shadow bank can raise the maximum leverage for shadow banking up to the level that they are not exposed to the risk of the shadow bank’s default. In this section, we derive an enforceability constraint that is more lenient than (18). Under such an enforceability constraint, we do not have to resort to Assumption 6 so as to characterize the equilibrium of the market for notes (i.e., financial instruments issued by shadow banks).

For purposes of exposition, \((s_{t_-}, s_{t_-}^*)\) denotes the optimal leverage choice for a banker who does not intend to default if a negative shock hits the economy; \((\tilde{s}_{t_-}, \tilde{s}_{t_-}^*)\) denotes the optimal leverage choice for a banker who intends to default if a negative shock hits.

We first derive the condition under which a banker prefers \((s_{t_-}, s_{t_-}^*, d_t = 0)\) over \((\tilde{s}_{t_-}, \tilde{s}_{t_-}^*, d_t = 1)\), and second demonstrate the condition under which a banker voluntarily bails out her shadow bank in trouble given her portfolio choice \((s_{t_-}, s_{t_-}^*)\).

The HJB equation listed in Section 2.1.2 yields the first condition:

\[
\frac{s_{t_-}(R_{t_-} - r_{t_-} - \tau_{t_-}) + s_{t_-}^*(R_{t_-} - r_{t_-})}{\rho} + \frac{\lambda}{\rho} \ln \left(1 - (1 + s_{t_-} + s_{t_-}^*) \kappa_t^Q\right) + \lambda h_t \\
\geq \frac{\tilde{s}_{t_-}(R_{t_-} - r_{t_-} - \tau_{t_-}) + \tilde{s}_{t_-}^*(R_{t_-} - r_{t_-})}{\rho} + \frac{\lambda}{\rho} \ln \left(1 - (1 + \tilde{s}_{t_-} + \tilde{s}_{t_-}^*) \kappa_t^Q\right) + \lambda \hat{h}_t,
\]

where \(h_t = h_{t_-}(1 - \kappa_t^h)\) and \(\hat{h}_t = \hat{h}_{t_-}(1 - \kappa_t^{\hat{h}})\). To have a simpler condition, we focus on a special case that the leverage constraint for shadow banking is binding (i.e., \(s_{t_-}^* = \tilde{s}_{t_-}^*\)). First-order conditions discussed in Section 2.1.2 imply that \(s_{t_-}^* = \tilde{s}_{t_-}^*\) and \(s_{t_-} + \tilde{s}_{t_-}^* = \tilde{s}_{t_-}^*\). In this case, the first condition reduces to

\[
\tilde{s}_{t_-}^* \leq \frac{\rho \lambda h_t}{R_{t_-} - r_{t_-} - \tau_{t_-}}.
\]

The second condition is:

\[
\frac{1}{\rho} \ln \left(1 - (1 + s_{t_-} + \tilde{s}_{t_-}^*) \kappa_t^Q\right) + h_t \geq \frac{1}{\rho} \ln \left(1 - (1 + s_{t_-}) \kappa_t^Q\right) + \hat{h}_t,
\]

which is similar to the enforceability constraint (18). The second condition is relatively loose because the enforcement problem becomes less severe when the leverage of a banker’s regular bank is \(s_{t_-} < \tilde{s}_{t_-}\). In the special case \(s_{t_-}^* = \tilde{s}_{t_-}^*\), the second condition becomes

\[
\tilde{s}_{t_-}^* \leq (1 - \exp(-\rho H_t))(\frac{1}{\kappa_t^Q} - (1 + s_{t_-})).
\]

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Next, we argue that given Assumptions 1 - 4 the note contract that clears the credit market for shadow banking satisfies

\[
\tilde{r}_t^- = r_t^- \quad \text{and} \quad \bar{s}_t^- = \min\{(1 - \exp(-\rho H_t))\left(\frac{1}{\kappa_t} - (1 + s_t^-)\right), \frac{\rho \lambda H_t}{R_t^- - r_t^- - \tau_t^-}\}. \tag{43}
\]

First, if \(\bar{s}_t^-\) is higher than the right-hand side of condition (43), then \(d_t = 1\) in equilibrium, which means a note in period \(t^-\) is the combination of risk-free debt and equity. However, households cannot hold equity by Assumption 1. Second, if \(\bar{s}_t^-\) is lower than the right-hand side of condition (43), then the competition among creditors will push up \(\bar{s}_t^-\). The no-arbitrage condition implies that \(\tilde{r}_t^- = r_t^-\).

The enforceability constraint (18) is tighter than the alternative enforceability constraint (43) because \(\bar{s}_t^-\) that satisfies equation (18) is less than the right-hand side of equation (43).

Figure 25 - 27 in Appendix E show that main results demonstrated in Section 2 still hold if the maximum leverage of shadow banking in equilibrium is replaced by equation (43). Therefore, Assumption 6 is not essential for our results. The choice of parameter values for numerical examples shown by Figure 25 - 27 is that \(\rho = 3\%\), \(\chi = 0.15\), \(a = 0.225\), \(a^h = 0.1\), \(\delta = 10\%\), \(\phi = 3\), \(\lambda = 1\), \(\kappa = 4\%\), \(\tau = 3\%\), and \(\xi = 5\%\).

### B Stationary Distribution

We present the condition under which the stationary distribution of \(\omega\) is nondegenerate.

**Proposition 9** If \(\alpha_\pi > -1\), where \(\alpha_\pi\) is a root of the equation

\[
(1 + \alpha_\pi)\mu^\omega = \lambda \left((1 - \kappa^\omega)^{-(1+\alpha_\pi)} - 1\right),
\]

then the stationary density of \(\omega\) is nondegenerate.

**Proof.** \(\pi(\omega)\) denotes the stationary density of \(\omega\). Suppose \(\pi(\omega)\) is nondegenerate, then \(\pi(\omega)\) satisfies

\[
0 = -\frac{d(\pi(\omega) \mu^\omega)}{d\omega} - \lambda \pi(\omega) + \lambda \pi(\hat{\omega}(\omega)) \frac{d\hat{\omega}(\omega)}{d\omega},
\]

where \(\hat{\omega}\) is such that \(\omega = \hat{\omega}(1 - \kappa^\omega)\). The above equation is based on the Kolmogorov Forward Equation. To prove that \(\pi(\omega)\) is nondegenerate, we need to verify that the solution of the above equation has a finite integral over \([0, \omega]\). Since \(\omega\) is the upper bound of \(\omega\) by construction, what we need to show essentially is that the integral of \(\pi(\omega)\) between 0 and \(\varepsilon\) is finite, where \(\varepsilon\) is a small positive number. Around \(\omega = 0\), the solution of the above differential equation can be approximated by \(C_\pi \omega^{\alpha_\pi} + o(\omega^{\alpha_\pi})\), where \(C_\pi\) is a positive constant. After plugging this approximation into the above Kolmogorov Forward Equation, we derive (44). Hence, if there is a root of (44) that is larger than \(-1\), then the integral of \(\pi(\omega)\) over \((0, \varepsilon)\) is finite. ■
Algorithm. In addition, we could numerically compute the density function of the stationary distribution of \( \omega \) based on the above Kolmogorov Forward Equation. We first solve for the differential equation with respect to \( \Pi(\omega) = \pi(\omega) \mu^\omega \omega \)

\[
0 = -\Pi'(\omega) - \frac{\lambda \Pi(\omega)}{\mu^\omega \omega} + \frac{\lambda \Pi(\hat{\omega}(\omega)) d\hat{\omega}(\omega)}{d\omega},
\]

and recover \( \pi(\omega) \) from \( \pi(\omega) = \frac{\Pi(\omega)}{(\mu^\omega \omega)} \). The algorithm used to solve for the density function starts from \( \bar{\omega} \) instead of 0. This is because we only need to solve an ODE when the state variable is around \( \hat{\omega} \). To understand this, notice that \( \bar{\omega} \) is the upper bound of the state variable, which means there does not exist a state \( \hat{\omega} \) such that \( \bar{\omega} = \hat{\omega} (1 - \kappa \hat{\omega}) \). Therefore, \( \pi(\hat{\omega}(\omega)) = 0 \) when \( \omega \) is around \( \bar{\omega} \). As we proceed from \( \bar{\omega} \) towards 0, we can use our calculated results to interpolate the value of \( \Pi \) at any \( \hat{\omega} \) since \( \hat{\omega} \) has to be larger than \( \omega \).

C Proofs

Proof of Lemma 1. Now suppose in period \( t \), the banker’s net worth \( W_t \) is negative. The law of motion for the banker’s net worth is

\[
dW_t = \left( W_{t-} R_{t-} + S_{t-} (R_{t-} - r_{t-}) + S^*_t (R_{t-} - r_{t-}) - c_{t-} \right) dt - \left( W_{t-} \kappa_t^Q + S_{t-} \kappa_t^Q + S^*_t \kappa_t^Q \right) dN_t
\]

Given a fixed time \( T \), we can construct a new measure under which

\[
R_{s-} - r_{s-} = \tilde{\lambda}_{t-} \kappa^Q_s
\]

for each time \( s \) between \( t \) and \( T \). Under this new measure,

\[
\tilde{E}_t \left[ W_T \exp \left( -\int_t^T r_u du \right) + \int_t^T \left( c_s \exp \left( -\int_t^s r_u du \right) \right) ds \right] \leq W_t < 0
\]

Suppose the banker retires at the stopping time \( S \) with positive net worth \( W_S \). After the banker retires, her net worth evolves as

\[
dW_{S+u} = W_{S+u} (r_{S+u} - \rho) du.
\]

Her net worth in period \( S + s \) is \( W_S \exp \left( \int_S^{S+s} r_u du - \rho s \right) \). It is easy to see that

\[
\lim_{s \to \infty} E_S \left[ \exp \left( -\int_S^{S+s} r_u du \right) W_{S+s} \right] = 0.
\]

Thus, if \( T \) is large enough, \( E_t \left[ W_T \exp \left( -\int_t^T r_u du \right) \right] \) could be arbitrarily small. Since \( W_t < 0 \), the consumption of the banker must be negative at some point between \( t \) and \( T \) with a strictly positive
probability. Since the banker has logarithm preference, the banker’s expected lifetime discounted utility in period $t$ must be negative infinity. Therefore, we show that it is never optimal for the banker to have negative net worth. ■

**Proof of Proposition 2.** (12) holds because of Lemma 1. We proceed to show (13) and (14).

Given the conjecture that the banker’s continuation value function $\ln(W)/\rho + h$, we can establish the following Hamilton-Jacobi-Bellman (HJB) equation

$$ \rho \left( \frac{\ln(W_{t^-})}{\rho} + h_{t^-} \right) = h_{t^-} + \xi_r \left( \frac{\ln(W_t)}{\rho} + \frac{r_t - \rho}{\rho} - \frac{\ln(W_{t^-})}{\rho} + h_{t^-} \right) $$

$$ + \max_{\chi, S_{t^-}, S_{t^-}^*} \left\{ \begin{array}{l} \ln(c_{t^-}) + \frac{1}{\rho W_{t^-}} (W_{t^-} R_{t^-} + S_{t^-} (R_{t^-} - r_{t^-} - \tau_{t^-}) + S_{t^-}^* (R_{t^-} - \tilde{r}_{t^-}) + \Pi_{t^-} - c_{t^-}) \\ + \lambda (1 - d_t) \left( \frac{1}{\rho} \ln \left( W_{t^-} - (W_{t^-} + S_{t^-} + S_{t^-}^*) \kappa_t^Q \right) + h_{t^-} (1 - \kappa_t^h) \right) \\ + \lambda d_t \left( \frac{1}{\rho} \ln \left( W_{t^-} - (W_{t^-} + S_{t^-}) \kappa_t^Q \right) + \hat{h}_{t^-} (1 - \kappa_t^h) \right) - \lambda \left( \frac{\ln(W_{t^-})}{\rho} + h_{t^-} \right) \end{array} \right. $$

where $\Pi_t = S_t \tau_t$. (13) and (14) are first-order conditions with respect to $s_{t^-}$ and $s_{t^-}^*$. Furthermore, 

$$ \{h_t, t \geq 0\} $$

must satisfy

$$ \rho h_{t^-} = h_{t^-} - \mu_{t^-} + \chi \left( \frac{r_t - \rho}{\rho} - h_{t^-} \right) + \log(\rho) + \frac{1}{\rho} (R_{t^-} + s_{t^-} (R_{t^-} - r_{t^-}) + s_{t^-}^* (R_{t^-} - \tilde{r}_{t^-}) - \rho) $$

$$ + \lambda (1 - d_t) \left( \frac{1}{\rho} \ln \left( 1 + s_{t^-} + s_{t^-}^* \kappa_t^Q \right) + h_{t^-} \left( 1 - \kappa_t^h \right) \right) $$

$$ + \lambda d_t \left( \frac{1}{\rho} \ln \left( 1 + s_{t^-} \kappa_t^Q \right) + \hat{h}_{t^-} \left( 1 - \kappa_t^h \right) \right) - \lambda h_{t^-} $$

To complete the argument, the transversality condition

$$ \lim_{t \to \infty} E \left[ \exp(-\rho t) \ln(W_t)/\rho + h_t \right] = 0. $$

also must hold. ■

**Proof of Lemma 2.** $W_t$ denotes $\int_0^1 W_t^t \, dW_t$. In a Markov equilibrium, (3), the optimal choice of bankers, and the market-clearing for notes, and the balanced budget of the regulatory authority imply that

$$ dW_t = W_t \left( (R_{t^-} + s_{t^-} (R_{t^-} - r_{t^-}) + s_{t^-}^* (R_{t^-} - \kappa_t^Q) dN_t \right) $$

$$ = W_t \left( (R_{t^-} + s_{t^-} (R_{t^-} - r_{t^-}) + s_{t^-}^* (R_{t^-} - r_{t^-}) - \rho - \chi) dt - (s_{t^-} + s_{t^-}^*) \kappa_t^Q dN_t \right). $$

Note bankers retire at the intensity $\chi$. Next, consider the scaling factor $1/(q_k K_t)$.

$$ d(q_k K_t) = q_k K_t \left( \mu_{t^-}^Q + q_k^Q \right) dt - \kappa_t^Q dN_t. $$
\[d\left(\frac{1}{q_t K_t}\right) = \left(\frac{1}{q_t K_t}\right) \left(-\left(\mu_{t-}^q + \mu_{t-}^K\right) dt + \frac{\kappa_t^Q}{1 - \kappa_t^Q} dN_t\right).\]

Then,

\[d\omega_t = \omega_t \left(\mu_{t-}^\omega dt - \kappa_t^\omega dN_t\right)\]

where \(\mu_{t-}^\omega = R_{t-} + s_{t-} (R_{t-} - r_{t-}) + s_{t-}^* (R_{t-} - r_{t-}) - \mu_{t-}^q - \mu_{t-}^K - \rho - \chi\)

and \(\kappa_t^\omega = \frac{(s_{t-} + s_{t-}^*) \kappa_t^Q}{1 - \kappa_t^Q}\).

\[\textbf{Proof of Proposition 4.}\] Without loss of generality, we focus a banker with net worth \(W_t\) in period \(t\) and explicitly express her continuation value in different cases.

We start with the case that the banker is retired. Since logarithmic agents only consumer \(\rho\) fraction of their net worth, the growth rate of her net worth is \(r_{t+u} - \rho\) in period \(t+u\). Hence, the banker’s net worth in period \(t+u\) will be \(W_t \exp \left(\int_0^u r_{t+\nu} - \rho\, d\nu\right)\). The banker’s continuation value in period \(t\) is

\[\int_0^\infty \exp(-\rho u) \left(\ln(W_t) + \int_0^u r_{t+\nu} - \rho\, d\nu\right) du = \frac{\ln(W_t)}{\rho} + \ln(\rho) + \int_0^\infty \exp(-\rho u) \int_0^u r_{t+\nu} - \rho\, d\nu du = \frac{\ln(W_t)}{\rho} + \ln(\rho) + \frac{1}{\rho} \int_0^\infty \exp(-\rho v) (r_{t+\nu} - \rho)\, dv,\]

which is denoted by \(\ln(W_t)/\rho + h_t^r\).

We use the same idea to express the continuation value of a banker who can access shadow banking. Given the banker’s optimal portfolio choices \((s_{t+u}, s_{t+u}^*)\), if she does not retire in period \(t+u\), her net worth is

\[W_t \exp \left(\int_0^u \left(R_{t+\nu} + s_{t+\nu} (R_{t+\nu} - r_{t+\nu}) + s_{t+\nu}^* (R_{t+\nu} - r_{t+\nu}) - \rho\right) d\nu + \int_0^u \ln(1 + s_{t+\nu} + s_{t+\nu}^*) \kappa_{t+\nu}^Q dN_{t+\nu}\right)\]
Let $t + T$ denote the stopping time that the banker retires. Her continuation value in period $t$ is

$$E_t \left[ \int_0^T \exp(-\rho u) \left( \ln(\rho W_t) + \int_0^u R_{t+u} + s_{t+u} (R_{t+u} - r_{t+u}) + s^*_{t+u} (R_{t+u} - r_{t+u}) - \rho \, dv \right) \, du \
+ \int_0^T \exp(-\rho u) \int_0^u \ln \left( 1 + s_{t+u} + s^*_{t+u} \right) \kappa^{Q}_{t+u} \, dN_{t+u} \, du + \exp(-T\rho) \left( \frac{\ln(W_{t+T})}{\rho} + h_{t+T}^r \right) \right].$$

$$= \ln(W_t) + \rho \ln(\rho) + \rho + \chi$$

$$+ E_t \left[ \int_0^\infty \exp(-(\rho + \chi) v) \left( R_{t+u} + s_{t+u} (R_{t+u} - r_{t+u}) + s^*_{t+u} (R_{t+u} - r_{t+u}) - \rho \right) \, dv \right]$$

$$+ E_t \left[ \int_0^\infty \chi \exp(-(\rho + \chi) v) \, dv \right],$$

which we denote as $\ln(W_t)/\rho + h_t$.

Finally, we consider the case that the banker who cannot use shadow banking but obtain such opportunity at intensity $\xi$. Let $\hat{s}_{t+u}$ denote her optimal portfolio choices and $T_\xi$ the stopping when the banker obtain the access to shadow banking. Her continuation value in period $t$ is

$$E_t \left[ \int_0^{\min(T,T_\xi)} \exp(-\rho u) \left( \ln(\rho W_t) + \int_0^u R_{t+u} + \hat{s}_{t+u} (R_{t+u} - r_{t+u}) + s_{t+u} (R_{t+u} - r_{t+u}) - \rho \, dv \right) \, du \
+ \int_0^{\min(T,T_\xi)} \exp(-\rho u) \int_0^u \ln \left( 1 + \hat{s}_{t+u} \right) \kappa^{Q}_{t+u} \, dN_{t+u} \, du + \exp(-T\rho) \left( \frac{\ln(W_{t+T})}{\rho} + h_{t+T}^r \right) 1_{\{\min(T, T_\xi) = T_\xi\}} \right]$$

$$= E_t \left[ \int_0^T \exp(-\rho u) \left( \ln(\rho W_t) + \int_0^u R_{t+u} + s_{t+u} (R_{t+u} - r_{t+u}) + s^*_{t+u} (R_{t+u} - r_{t+u}) - \rho \, dv \right) \, du \
+ \int_0^T \exp(-\rho u) \int_0^u \ln \left( 1 + s_{t+u} + s^*_{t+u} \right) \kappa^{Q}_{t+u} \, dN_{t+u} \, du + \exp(-T\rho) \left( \frac{\ln(W_{t+T})}{\rho} + h_{t+T}^r \right) \right]$$

$$- E_t \left[ \int_0^{\min(T,T_\xi)} \exp(-\rho u) \left( \ln(\rho W_t) + \int_0^u R_{t+u} + \hat{s}_{t+u} (R_{t+u} - r_{t+u}) + s_{t+u} (R_{t+u} - r_{t+u}) - \rho \, dv \right) \, du \
+ \int_0^{\min(T,T_\xi)} \exp(-\rho u) \int_0^u \ln \left( 1 + \hat{s}_{t+u} \right) \kappa^{Q}_{t+u} \, dN_{t+u} \, du \right]$$

$$= \ln(W_t)/\rho + h_t - H_t,$$

where

$$H_t = E_t \left[ \int_0^\infty \exp(-(\rho + \chi + \xi) v) \left( (s_{t+u} + s^*_{t+u} - \hat{s}_{t+u}) (R_{t+u} - r_{t+u} - \tau_{t+u}) + s^*_{t+u} \tau_{t+u} \right) \, dv \right]$$

$$+ \lambda \left( \ln \left( 1 + (s_{t+u} + s^*_{t+u} \right) \kappa^{Q}_{t+u} \right) - \ln \left( 1 + \hat{s}_{t+u} \right) \kappa^{Q}_{t+u}) \right) \right] dv.$$

**Lemma 3.** $H$ is bounded.

**Proof of Lemma 3.** It is easy to see that $H$ is bounded from below by 0. Suppose there are two bankers with the same net worth in period $t$, banker 1 only accesses shadow banking after a random period since $t$, and banker 2 can use shadow banking right after $t$. Since banker 1 is
unable to choice certain portfolio choices that are available to banker 2, the continuation of banker 1 cannot be larger than banker 2. Given the continuation value function of bankers with logarithm preference and the fact that banker 1 and 2 have the same net worth, we show that $H \geq 0$, for all $\omega \in (0, \bar{\omega}]$.

Next, we show that $H$ also has an upper bound. Notice that in period $u$ the banker who can access shadow banking can save the tax payment $S^\ast_u \tau_u$. Since bankers’ leverage has an upper bound $\bar{s} = 1/\kappa$, the maximum benefit a banker can get should be $W_u \bar{s} \tau$. We next calculate by how much a banker’s continuation could increase if she can get the maximum benefit every period compared to the situation that she cannot access shadow banking. Let $g_u$ and $\kappa^W_u$ denote the growth rate of the banker who cannot access shadow banking and the drop rate of her net worth after a bad shock. Then, the banker’s net worth would grow at rate $g_u + \bar{s} \tau$ if she gets the maximum benefit. The law of motion for her net worth is

$$dW_u = W_u ((g_u + \bar{s} \tau) du - \kappa^W_u dN_u).$$

In period $t$, her net worth would be

$$W_0 \exp \left( \int_0^t (g_u + \bar{s} \tau) du - \kappa^W_u dN_u \right).$$

Since logarithmic agents would only consume $\rho$ fraction of their wealth, the banker’s continuation would increase by

$$\int_0^\infty \exp (\rho t) \int_0^t \bar{s} \tau dudt = \bar{s} \tau / \rho^2 < \infty.$$

Hence, $H(\omega) \leq (\bar{s} \tau) / \rho^2$, for $\omega \in (0, \bar{\omega}]$. \hfill \blacksquare

**Alternative Proof of Proposition 4.** We first establish the differential equation that $H_t$ satisfies

$$(\rho + \xi + \chi) H(\omega) = f(\omega) + \omega \mu^\omega H'(\omega) + \lambda (H(\omega (1 - \kappa^\omega)) - H(\omega)),$$

where

$$f(\omega) = \frac{1}{\rho} \left( \left( s(\omega) + s^\ast(\omega) - \bar{s} (\omega) \right) (R(\omega) - r(\omega) - \tau(\omega)) + s^\ast(\omega) \tau(\omega) \right. \left.+ \lambda \ln \left( 1 - (1 + s^\ast(\omega) \kappa^Q(\omega)) \right) \right).$$

Suppose $H(\omega)$ solves the above differential equation. We look at

$$H(\omega_t) \exp (- (\rho + \xi + \chi) t).$$
Ito’s Lemma implies
\[
   d \left( H(\omega_t) \exp \left( - (\rho + \xi + \chi) t \right) \right) = \exp \left( - (\rho + \xi + \chi) t \right) \left( -H(\omega_t) (\rho + \xi + \chi) + H'(\omega_t) \omega_t \mu_t^\omega \right) dt + (H(\omega_t (1 - \kappa_t^\omega)) - H(\omega_t)) dN_t
\]

Given a fixed time \( T \),
\[
   H(\omega_t) \exp (- (\rho + \xi + \chi) t) = E_t \left( \int_t^T \exp \left( - (\rho + \xi + \chi) u \right) f(\omega_u) du \right) + E_t \left( H(\omega_T) \exp \left( - (\rho + \xi + \chi) T \right) \right).
\]

Since \( f(\cdot) \) is bounded, by the dominated convergence theorem,
\[
   \lim_{T \to \infty} E_t \left( \int_t^T \exp \left( - (\rho + \xi + \chi) u \right) f(\omega_u) du \right)
\]
is well defined. And, since we focus on the bounded \( H(\cdot) \), \( E_t \left( H(\omega_T) \exp \left( - (\rho + \xi + \chi) T \right) \right) \) converges to zero as \( T \to \infty \). Therefore,
\[
   H(\omega_t) = E_t \left[ \int_t^\infty \exp \left( - (\rho + \xi + \chi) (u - t) \right) f(\omega_u) du \right]
\]

\[\blacksquare\]

**Proof of Theorem 1.** We will apply the contraction mapping theorem to show the uniqueness of the solution \( H(\omega) = 0 \). First, we define a complete metric space. Since the state variable \( \omega \) is between 0 and \( \bar{\omega} \), we focus on the space \( B([0, \bar{\omega}]) \) of bounded continuous functions \( h : [0, \bar{\omega}] \to R \) under sup norm. Theorem 3.1 in Stokey et al. (1989) implies that \( B([0, \bar{\omega}]) \) is a complete metric space.

We will use Blackwell’s sufficient conditions to show \( \Gamma \) is a contraction mapping. Suppose both \( h, \tilde{h} \in B([0, \bar{\omega}]) \) and \( h(\omega) \geq \tilde{h}(\omega) \), for all \( \omega \in (0, \bar{\omega}] \), since
\[
   \tilde{s}^* = (1 - \exp(-\rho H)) \left( \frac{1}{\kappa q} - (1 + \tilde{s}) \right)
\]
all portfolio choices permitted under \( \tilde{h}(\omega) \) are feasible under \( h(\omega) \). Hence, \( \Gamma h \geq \Gamma \tilde{h} \), for all \( \omega \in (0, \bar{\omega}] \). Next, we need to show that there exists a positive constant \( \beta < 1 \) such that \( \Gamma (h + v) \leq \Gamma h + \beta v \), for all \( h \in B([0, \bar{\omega}]), \ v \geq 0, \ \omega \in (0, \bar{\omega}] \). Consider
\[
   \Gamma (h + v) [\omega] = E_0 \left[ \int_0^\infty \exp \left( - (\rho + \xi + \chi) u \right) f(\omega_u) du \right| \omega_0 = \omega,
\]
where
\[
   f(\omega) = \frac{1}{\rho} \left( \frac{(s(\omega) + s^*(\omega) - \tilde{s}(\omega)) (R(\omega) - r(\omega) - \tau(\omega)) + s^*(\omega) \tau(\omega)}{\lambda \left( \ln \left( 1 + \ln(1 + s(\omega) + s^*(\omega)) \kappa q(\omega) \right) \right) - \ln \left( 1 + \tilde{s}(\omega) \kappa q(\omega) \right)} \right),
\]

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\[ s^* \leq \bar{s}^*, \]

and
\[ \bar{s}^* = (1 - \exp (-\rho (h + v))) \left( \frac{1}{\kappa Q} - (1 + \bar{s}) \right). \]

Since the lower bound of \( h \) is zero, then
\[ \bar{s}^* = (1 - \exp (-\rho h)) \left( \frac{1}{\kappa Q} - (1 + \bar{s}) \right) \]
\[ \leq (1 - \exp (-\rho h) + \rho v) \left( \frac{1}{\kappa Q} - (1 + \bar{s}) \right) \]
\[ \leq (1 - \exp (-\rho h)) \left( \frac{1}{\kappa Q} - (1 + \bar{s}) \right) + \frac{\rho v}{\kappa}. \]

With the assistance of above inequality, we derive that
\[ \Gamma (h + v)[\omega] \leq \Gamma h + E_0 \left[ \int_0^\infty \exp (- (\rho + \xi + \chi) u) \frac{v}{\kappa} du \bigg| \omega_0 = \omega \right] \]
\[ \leq \Gamma h + \frac{\tau}{(\rho + \xi + \chi) \kappa} v. \]

If \( \tau < (\rho + \xi + \chi) \kappa \), \( \Gamma \) is a contraction mapping.

## D Data and Tables

### Table 2: Moments

<table>
<thead>
<tr>
<th>Moment</th>
<th>Model</th>
<th>Target</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>average Sharpe ratio</td>
<td>34.4%</td>
<td>40%</td>
<td>Wachter (2013)</td>
</tr>
<tr>
<td>highest Sharpe ratio</td>
<td>15</td>
<td>15</td>
<td>He and Krishnamurthy (2012a)</td>
</tr>
<tr>
<td>average Sharpe ratio</td>
<td>11.2%</td>
<td>11%</td>
<td>He and Krishnamurthy (2012a)</td>
</tr>
<tr>
<td>average investment capital ratio</td>
<td>25.5%</td>
<td>25.1%</td>
<td>ratio of securitization in third quarter of 2006</td>
</tr>
<tr>
<td>ratio of securitization at steady state (^2)</td>
<td>2.9</td>
<td>3</td>
<td>He and Krishnamurthy (2012a)</td>
</tr>
<tr>
<td>bankers' overall leverage volatility of bankers' wealth growth rate in distress periods (^3)</td>
<td>35.3%</td>
<td>31.5%</td>
<td>He and Krishnamurthy (2012a)</td>
</tr>
<tr>
<td>in non-distress periods</td>
<td>19.1%</td>
<td>17.5%</td>
<td>He and Krishnamurthy (2012a)</td>
</tr>
</tbody>
</table>

\(^1\) We use the density of the stationary distribution to calculate all moments.
\(^2\) The steady state is where \( \mu^\omega - \lambda \kappa^\omega = 0 \).
\(^3\) The distress periods are those with highest 33% Sharpe ratio.

**Securitization.** We follow Loutskina (2011) to compute the ratio of securitization. The difference
Table 3: Details of Securitization Data

<table>
<thead>
<tr>
<th>Loan Category</th>
<th>Outstanding</th>
<th>Securitized</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home Mortgages</td>
<td>FL383165105</td>
<td>FL673065105</td>
</tr>
<tr>
<td>Multifamily Residential Mortgages</td>
<td>FL143165405</td>
<td>FL673065405</td>
</tr>
<tr>
<td>Commercial Mortgages</td>
<td>FL383165505</td>
<td>FL673065505</td>
</tr>
<tr>
<td>Commercial and Industrial Loans¹</td>
<td>FL253169255</td>
<td>FL673069505</td>
</tr>
<tr>
<td>Loans</td>
<td>FL263168005</td>
<td>FL673069255</td>
</tr>
<tr>
<td>Consumer Credit</td>
<td>FL153166000</td>
<td>FL673066600</td>
</tr>
</tbody>
</table>

¹ Because item FL253169255 is not available now, we use the ratio of securitization calculated by Loutsksina (2011) to estimate the outstanding commercial and industrial loans.

is that we focus on securitization done by non-agency security issuers. All data are drawn from the “Flow of Funds Accounts of the United States”. There are 5 loan categories. The details of items for each category are listed in Table 3.

E  Figures

Figure 21: $\psi$, $q$, $qr^2$, $s$, and $s^*$ as functions of the state variable $\omega$ in the modified model with a constant cost of default $\bar{H} = 2.7709$, which equals the average cost of default in the calibrated model in Subsection 2.3.1. For other parameter values see Section 2.3.1.
Figure 22: $\psi, q, (s + s^*)$, $R - r - \tau - \lambda\kappa Q$, $q\kappa^q$, and $(1 + s + s^*)\kappa^Q$ as functions of the state variable $\omega$, i.e., bankers' wealth share, in the “good” equilibrium of the modified model in which households have Epstein-Zin preferences. For the choice of parameter values, see Section 4.2.
Figure 23: This figure shows the daily EURO STOXX 50 Volatility Index (solid line) and yearly Europe securitization (dashed line) outstanding from 1999 to 2012.  

Figure 24: This figure shows the daily Chicago Board Options Exchange Volatility Index (VIX, solid line) and yearly United States Asset-Backed Securities Outstanding (dashed line) from 1995 to 2012.  
*Source:* the Chicago Board Options Exchange and Securities Industry and Financial Markets Association
Figure 25: $\psi, q, (s + s^*), R - r - \tau - \lambda \kappa Q, q\kappa q$, and $(1 + s + s^*)\kappa Q$ as functions of the state variable $\omega$, i.e., bankers’ wealth share, in the “good” equilibrium of the modified model in which the maximum leverage of shadow banking is given by (43). For the choice of parameter values, see Section A.
Figure 26: Pro-cyclical Shadow Banking
This figure presents the leverage of shadow banking $s^*$, “the leverage of regular banking” $\tilde{s}$, asset volatility $\kappa^Q$, and the cost of default $H$ as functions of the state variable $\omega$ (i.e., bankers’ wealth share) in the “good” equilibrium of the modified model in which the maximum leverage of shadow banking is given by (43). For parameter values, see Section A.
Financial Instability

Leverage of Shadow Banking

**Figure 27:** This figure shows the investment risk $\kappa^Q$ (upper panel) and the leverage for shadow banking (lower panel) at the stochastic steady state of different economies with different tax rates $\tau$ in the modified model in which the maximum leverage of shadow banking is given by (43). The stochastic steady state is the state where $\omega\mu - \lambda\omega\kappa = 0$. We assume that the "good" equilibrium prevails if it exists. For the choice of parameter values, see Section A.