- Why are there stock market crashes?
- Why is France overwhelmingly Christian and Thailand overwhelmingly Buddhist?
- Why did the great depression occur?
- Why do crime rates vary so much over time and space?
- Why did the adoption of hybrid corn follow an s-shaped curve?
- Why is there racial segregation?
- What determines the success of a residential neighborhood?
- Why do mass cultural phenomena like the Hula Hoop and Harry Potter occur?

Social interactions

- Large changes in endogenous variables from small changes in exogenous variables.
  - much of economics is about smoothing.
  - if marginal utility of an agent for an action increases as other agents increase their action (*strategic complementarity*,) we obtain a *social multiplier* and possibly multiple equilibrium.

- Externality: agents utility depends on other agents choices.

- Agents choices inform other agents about utility from choice.

- Schelling [1971, 1972, 1978], Follmer [1974].

An abstract model of social interactions

- Agent indexed by $a \in \mathbb{A}$, $\mathbb{A} \subset \mathbb{Z}^d$

- Agent chooses an action x^a from a common compact set of possible actions X.

- Set of action profiles $S := \left\{x = (x^b)_{b\in\mathbb{A}} : x^b \in X\right\}$

- In S use product topology.

- $U^a(x^a, \{x^b\}_{b\neq a}, \vartheta^a)$, ϑ^a a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Local and *global* components

- $\vartheta^a = (J^a, \theta^a)$

- θ^a taste shock, assumes values in Θ.

- $J^a = (J^{a,b})_{b \neq a}$, takes values on $\Xi := \mathbb{R}^{\mathbb{A} \setminus \{0\}}$.

- The realization of $J^{a,b}$ gives effect of choice of $b \neq a$ on utility of a.

- The utility function depends also on the empirical distribution $\varrho(x)$ associated with the action profile x.

-
$$U^a(x^a, \{x^b\}_{b \neq a}, J^a, \theta^a) \equiv u^a(x^a, \{J^{a,b} x^b\}_{b \neq a}, \varrho(x), \theta^a). \qquad (1)$$

- The argument $\{J^{a,b}x^b\}_{b\neq a}$ is called the *local component*, the argument $\varrho(x)$ is called the *global component*.

- "Frozen" ϑ^a

  – $U^a(x^a, \{E(x^b)\}_{b\neq a}, \vartheta^a)$

- Interpretation $\times$ restrictions.

- Fix $(\Omega, \mathcal{F}, \mathbb{P})$, making the canonical choice $(\Omega, \mathcal{F}, \mathbb{P}) = ((\Xi \times \Theta)^{\mathbb{A}}, \mathcal{B}(\Xi \times \Theta)^{\mathbb{A}}, \mathbb{P})$ and assume that $\omega \mapsto (J, \theta)(\omega)$ is the projection mapping. Here $\mathcal{B}(\Xi \times \Theta)$ denotes the Borel σ-field on $\Xi \times \Theta$. Thus, once $\mathbb{A}$ is fixed, all variations are coded in the probability distribution $\mathbb{P}$.

Examples

- Cooper and John [1988].
  - X an interval on the line.
  - $m(\varrho) = \int x\varrho(dx)$.
  - $u^a = u(x^a, m(\varrho))$.
- Heterogeneity: $u^a = u(x^a, m(\varrho), \theta^a)$.
- Glaeser, Sacerdote and Scheinkman [1995]
  - $\mathbb{A} = \{1, 2, \ldots n\}$
  - $\Theta = \{-1, 0, 1\}$, $X = [0, 1]$
  - $N(a) = \{a - 1\}$, where $0 \equiv n$.
  - $u^a = \theta^a x^a - (1 - |\theta|)|x^a - x^{a-1}|$

- Housing segregation (Becker and Murphy [2000])
  - two groups $\{S_1\}$ and $\{S_2\}$, each with m agents.
  - neighborhood $x \in \{0, 1\}$ with m houses, p excess rent in neighborhood 1.
  - $u^i(x^a, \sum_b J^{a,b} x^b) + \theta_a x^a - p x^a$, $J^{a,b} = \frac{1}{m} I_{\{b \in S_1\}}$.
  - $u^i(0, \cdot)$ decreasing, $u^i(1, \cdot)$ increasing, that is S_1 are preferred neighbors.
  - solve for equilibrium $\bar{p}$ and $\bar{x}^a$.
  - if $u^1 = u^2$ and $\theta_a = \theta^i$ for $a \in S_i$, $\theta^1 > \theta^2$,
    * no "integrated" equilibrium
    * $u(1, 1) - u(0, 0) + \theta^1 \geq \bar{p}$ $\geq u(1, 1) - u(0, 0) + \theta^2$
  - Veblen, Van Goghs, Monets

Discrete action, "large", "mean-field" models.

- A model is mean-field if

$$U^a(x^a, \{x^b\}_{b \neq a}, \varrho, J^a, \theta^a) = U^a(x^a, \varrho, \theta^a).$$

- Here will choose $U^a = U(x^a, y, \theta^a)$ where y average action of population.

- $X = \{0, 1\}$

- U smooth with respect (y, θ^a).

- $v(y, \gamma) = U(0, y, \gamma) - U(1, y, \gamma)$.

- $v_\gamma > 0$.

- There exist m and M such that if $\gamma > M (\gamma < m)$, $v(y,\gamma) > 0, (resp.\ v(y,\gamma) < 0)$, for every $0 \leq y \leq 1$.

- $\theta^a = \alpha + \sigma\mu^a$.

  - μ^a are i.i.d, with mean zero, distribution (density) F (f).

  - $\sigma > 0$ and α is independent of every μ^a and has distribution (density) G (g).

- f and g are bounded, and with positive densities on the whole real line.

Agents observe common shock

- Equilibrium: λ_σ such that, if $\mu^a < \lambda_\sigma$, $x^a = 1$, otherwise $x^a = 0$.

- $y = F(\lambda_\sigma)$.

- $$v(F(\lambda_\sigma), \alpha + \sigma\lambda_\sigma) = 0. \tag{2}$$

- $v(F(\frac{m-\alpha}{\sigma}), m) < 0$ and $v(F(\frac{M-\alpha}{\sigma}), M) > 0$,

- Equilibrium exists

- Multiple equilibrium $\Rightarrow$ there exists λ with $\alpha + \sigma\lambda \in (m, M)$ such that:

  $v_y(F(\lambda), \alpha+\sigma\lambda) f(\lambda) + \sigma v_\gamma(F(\lambda), \alpha+\sigma\lambda) < 0.$

- Uniqueness if σ large.

- Multiple equilibria requires SC ($v_y < 0.$)

- If $v_y f + \sigma v_\gamma$, *social multiplier*: $\frac{\sigma v_\gamma}{v_y f + \sigma v_\gamma}$

- Under SC, there exist $\mu_0 < \mu_1$, such that for $\mu_0 < \alpha < \mu_1$, three equilibria for deterministic model ($\sigma = 0.$)

  - choose $m \leq \mu_0 < \mu_1 \leq M$, such that $v(0, \mu_0) = v(1, \mu_1) = 0.$

- As $\sigma \to 0$ extreme equilibria are approximated.

Agents do not observe common shock

- Global games.

- Equilibrium: z such that $x^a = 1$ if and only if $\theta^a < z$.

- $h_\sigma(\mu|z)$, the conditional density of μ.

- In z equilibrium, agent that obtains z forecasts that conditional on μ, b chooses 1 iff $z - \sigma\mu + \sigma\mu^b < z$. From independence, $F(\mu)$ agents choose 1.

-
$$\int v(F(\mu), z) h_\sigma(\mu|z) d\mu = 0. \tag{3}$$

- if $\sigma^i \to 0$, corresponding equilibria z^i bounded, and hence may assume $z^i \to z^*$.

- $h_\sigma(\mu|z) \to f(\mu)$ as $\sigma \to 0$.

- $$\int v(F(\mu), z^*) f(\mu) d\mu = 0. \quad (4)$$
  $$\int_0^1 v(y, z^*) dy = 0 \quad (5)$$

- Unique z^* and equilibria for σ small $\sim z^*$.

- $\alpha > z^* (\alpha < z^*)$ in equilibrium if σ small almost everyone plays 0 (*resp.* 1).

- A model is symmetric if $v(A, 0) = -v(1 - A, 0)$. In this case $z^* = 0$.

Generalization

- Types $t = 1, \ldots, T$, $\theta^a = \eta^t + \alpha + \sigma\mu^a$.
- μ^a i.i.d.
- Each agents knows his type (η^t) and observes θ^a.
- Equilibrium is z^t such that $x^a = 1$ if and only if $\theta^a - \eta^t < z^t$

Systems with local and global interactions

- Decoupling local and global component:

$$U^a(x^a, \{x^b\}_{b\neq a}, \varrho, J^a, \theta^a) \equiv u^a(x^a, \{J^{a,b}x^b\}_{b\neq a}, \varrho, \theta^a). \qquad (6)$$

- **Definition 1** *A* system of random social interactions *is an* $\mathcal{E} = (\mathbb{A}, \mathbb{P}, X, (U^a)_{a\in\mathbb{A}})$ *such that:*

  (i) $\mathbb{A} \subset \mathbb{Z}^d$ *is a set of agents.*

  (ii) $\mathbb{P}$ *is a probability measure on* $(\Omega, \mathcal{F})$*.*

  (iii) $X \subset \mathbb{R}^l$ *is a compact convex action space.*

  (iv) $U^a : S \times \mathcal{M}(X) \times \Xi \times \Theta \to \mathbb{R}$ *is a measurable mapping satisfying equation (6) such that, for* $\mathbb{P}$*-a.e. pair* (J, θ)*, the map* $(x, \varrho) \mapsto U^a(x^a, \{x^b\}_{b\neq a}, \varrho, J^a, \theta^a)$ *is continuous, and concave in* x^a*; the utility function of agent* $a \in \mathbb{A}$*.*

Equilibrium

- From now on, to facilitate notation we fix the interaction pattern J, and only write it if necessary.

- A system is purely local if there exists a representation such that each U^a is independent of ϱ.

- Every finite system ($\mathbb{A}$ finite) is purely local.

- A system is purely global if there exists a representation with $U^a = w^a(x^a, \theta^a, \varrho)$ for every $a \in \mathbb{A}$.

- $\mathbb{A}_n \equiv \mathbb{A} \bigcap [-n, n]^d$.

- **Definition 2** *Let $\mathcal{E}$ be a system of random social interactions. A random variable $g(\theta) = \{g_a(\theta)\}_{a\in\mathbb{A}}$ is called an* equilibrium *for $\mathcal{E}$ if:*

  (i) If the systems is not purely local, then the empirical distribution associated with the action profile $g(\theta)$ exists and is almost surely independent of $\omega \in \Omega$. That is

  $$\lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}} \delta_{g_a(\theta(\omega))}(\cdot) = \varrho \qquad \mathbb{P}\text{-a.s.} \tag{7}$$

  for some $\varrho \in \mathcal{M}(X)$.

  (ii) No agent has an incentive to deviate from the proposed strategy. That is, for each $(a \in \mathbb{A})$

  $$g_a(\theta) \in \text{argmax}_{x^a} U^a(x^a, \{x^b\}_{b\neq a}, \varrho, \theta^a) \tag{8}$$

Microscopic equilibrium

- For a given ϱ, $g_a(\varrho, \theta)$ is a *microscopic equilibrium* if it satisfies (8) above.

- If the system is purely local, every microscopic equilibrium is an equilibrium.

- **Proposition 1** *Let $\mathcal{E}$ be a system of random social interactions. For all $\varrho \in \mathcal{M}(X)$ the system has a microscopic equilibrium $g(\varrho, \cdot)$ with respect to ϱ. In particular, every purely local system has an equilibrium.*

Uniqueness in purely local systems

- u^a strictly concave in x^a, optimal action given $\{x^b\}_{b\neq a}$:

$$g_a(\{x^b\}_{b\neq a}, \varrho, \theta^a) = \text{argmax}_{x^a\in X} U^a(x^a, \{x^b\}_{b\neq a}, \varrho, \theta^a)$$

- Assume there exists a family $(L^{a,b})_{a,b\in\mathbb{A}}$:

$$\begin{aligned} |g_a(\{x^c\}_{c\neq a}, \varrho, \theta(\omega)) &- g_a(\{y^c\}_{c\neq a}, \varrho), \theta(\omega)| \\ &\leq L^{a,b}(\theta^a)|x^b - y^b| \end{aligned}$$

where $x^c = y^c$ if $c \neq a$ or b.

- If $X \subset \mathbb{R}$, u^a is C^2, $u^a_{aa} < 0$, and optimal x^a are interior then $\frac{\partial g_a}{\partial x^b} = -\frac{u^a_{a,b}}{u^a_{a,a}}$. Choose

$$L^{a,b}(\theta^a) = \sup_x \frac{|u^a_{a,b}(x^a, J^{a,b}x^b, \theta^a)|}{|u^a_{a,a}(x^a, J^{a,b}x^b, \theta^a)|}$$

- *Moderate social influence* (MSI) hold if there exists a family $L^{a,b}$ such that

$$\sup_a \sum_{b \neq a} L^{a,b}(\theta^a) \leq \alpha < 1 (a.s.)$$

  – suffices

$$\frac{|u^a_{a,b}(x^a, J^{a,b}x^b, \theta^a)|}{|u^a_{a,a}(x^a, J^{a,b}x^b, \theta^a)|} \leq \alpha < 1.$$

- **Proposition 2** *If MSI holds, $\mathcal{E}$ has a unique microscopic equilibrium with respect to each ϱ.*

- If MSI holds then a purely local system has a unique equilibrium.

Social multiplier

- Suppose $X \subset \mathbb{R}$, u^a is C^2,$u^a_{aa} < 0$, for each θ, x^a is uniformly interior (in a), and u^a depends on a parameter p. Set

$$g(x, \theta, p) \equiv \{g_a(\{x^b\}_{b \neq a}, \theta^a, p)\}_{a \in \mathbb{A}}.$$

- Equilib. x^* solves $x^*(\theta) - g(x^*(\theta), \theta, p) = 0$.

- If MSI holds, then $\frac{\partial x^*}{\partial p} = -(I - \frac{\partial g}{\partial x})^{-1} \frac{\partial g}{\partial p}$ where,

$$(I - \frac{\partial g}{\partial x})^{-1} = I + \frac{\partial g}{\partial x} + (\frac{\partial g}{\partial x})^2 + \cdots \equiv I + H$$

- If strategic complementarity the matrix $H \geq 0$ (social multiplier.)

- If unobservable heterogeneity "impossible" to distinguish multiple equilibria from large multiplier.

Homogeneous and ergodic systems

- Canonical shift transformation on Ω:

$$T^a[\{\omega_k\}_{k\in\mathbb{A}})] := \{\omega_{k-a}\}_{k\in\mathbb{A}} \in \Omega.$$

- $\mathcal{E}$ is homogeneous if $\mathbb{A} = \mathbb{Z}^d$, $J^{a,b} = J^{a+k,b+k}$ for $k \in \mathbb{Z}^d$, and:

  (i) There exists a measurable u such that

$$u^a(x^a, \{J^{a,b}x^b\}_{b\neq a}, \theta^a, \varrho) = u(x^a, \{J^{0,b-a}x^b\}_{b\neq a}, (T^a\theta)^0, \varrho) \qquad (a \in \mathbb{A}).$$

  (ii) The random sequence $\theta = \{\theta^a\}_{a\in\mathbb{A}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is stationary in the sense that $\mathbb{P}[\theta \circ T^a \in B] = \mathbb{P}[\theta \in B]$ for all $a \in \mathbb{A}$ and for each $B \in \mathcal{F}$.

- **Proposition 3** *If a homogeneous system has a unique microscopic equilibrium with respect to a* ϱ *then the equilibrium must be homogeneous:* $g_a(\varrho, \theta) = g_0(\varrho, T^a\theta)$

- If MSI holds and $\mathcal{E}$ is homogeneous, every (microscopic) equilibrium is homogeneous.

- Homogeneous $\mathcal{E}$ is ergodic if $\mathbb{P}[A] \in \{0, 1\}$ for every A invariant with respect to T^a.

- **Proposition 4** *If* $\mathcal{E}$ *is ergodic and* $g(\varrho, \cdot)$ *a homogeneous microscopic equilibrium, then*

$$\lim_{n \to \infty} \frac{1}{\mathbb{A}_n} \sum_{a \in \mathbb{A}} \delta_{g_a(\varrho, \cdot)}(\cdot) = \mu[\varrho] \ (a.s.),$$

$\mu[\varrho]$ *the distribution of* $g_0(\varrho, \cdot)$.

  - If $\mathcal{E}$ is purely local any homogeneous equilibrium has empirical distribution independent of realization of θ.

Existence and uniqueness for homogeneous systems

- Vasserstein distance $d_V(\varrho, \tilde{\varrho})$:

  $$\sup \left\{ \frac{|\int f d\varrho - \int f d\tilde{\varrho}|}{L(f)} : f : S \to R \text{ Lipschitz} \right\},$$

  $L(f)$ Lipschitz constant of f.

- Assume there exists a family $(L^{a,b})_{a,b\in\mathbb{A}}$:

  $$\begin{aligned} |g_a(\{x^c\}_{c\neq a}, \varrho, \theta^a) &- g_a(\{y^c\}_{c\neq a}, \varrho), \theta^a| \\ &\leq L^{a,b}(\theta^a)|x^b - y^b| \end{aligned}$$

  where $x^c = y^c$ if $c \neq a$ or b, and bounded random variables $L^{a,\varrho}(\theta^a)$ such that:

  $$\begin{aligned} |g_a(\{x^c\}_{c\neq a}, \widehat{\varrho}, \theta^a) &- g_a(\{x^c\}_{c\neq a}, \tilde{\varrho}), \theta^a| \\ &\leq L^{a,\varrho}(\theta^a) d_V(\widehat{\varrho}, \tilde{\varrho}) \end{aligned}$$

- If $\mathcal{E}$ homogeneous, *Average Moderate Social Influence* (AMSI) holds in its weak (strong) form if we can choose $L^{0,a}$ and $L^{0,\varrho}$ such that

$$\sum_{a \neq 0} \mathbb{E} L^{0,a} < \infty$$

$$\left(E L^{0,\varrho} \left(1 + \sum_{a \neq 0} \mathbb{E} L^{0,a} \right) < 1. \right)$$

- **Proposition 5** *If there exists homogeneous microscopic equilibrium for each ϱ and AMSI holds in its weak form then $\mathcal{E}$ has an equilibrium.*

- **Proposition 6** *If $\mathcal{E}$ has a unique microscopic equilibrium for each ϱ and AMSI holds in its strong form then $\mathcal{E}$ has a unique equilibrium.*