

VOLATILITY AND THE YIELD CURVE

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Investors have long realized that the relative attractiveness of bonds with different maturities and coupons depends not only on expected movements in future interest rates but also on the uncertainty surrounding these moves. This linkage suggests a relationship between the level of interest rate volatility and the shape of the yield curve.

In this article we examine theoretical models that highlight this connection. Without appealing to the usual market segmentation or preferred habitat arguments, we show how such models can be used to rationalize the shape of a typical zero-coupon yield curve estimated from U.S. Treasury coupon bond prices, where yields rise at first and then decline (see Figure 1).

The models we discuss also suggest a link between interest rate volatility and the shape of the yield curve as measured by the yield spreads of certain butterfly combinations. We show that these spreads have in fact exhibited a strong correlation in the past with volatility, which suggests that they may serve as a measure of the interest rate volatility implicit in the yield curve.

Many-fixed income securities — e.g., callable bonds — contain embedded options whose prices are sensitive to the level of volatility. Modeling the additional impact of volatility on the value of the coupons allows for a better understanding of the price behavior of these securities.

THEORETICAL MODELS OF THE YIELD CURVE

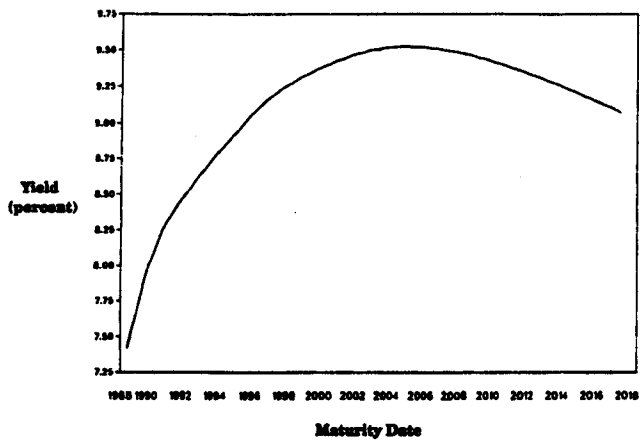
We use the term “yield curve” in this article to refer to a zero-coupon yield curve. Because a coupon bond can be thought of as a combination of zeroes, our results can easily be extended to analyze a coupon bond yield curve.

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FIGURE 1 ■ Zero-Coupon Yield Curve ■ June 1, 1988



Forming Interest Rate Expectations

Our theoretical yield curves are based on postulated statistical models for the evolution of short rates (the interest rates that apply to loans made for the shortest period considered in the analysis). Given any such model, we can derive the yield curve once we describe explicitly how investors evaluate the risk embedded in the volatility of short rates. We start by illustrating this procedure with two simple examples that — although unrealistic — will serve later to show the effect of volatility on the shape of the yield curve.

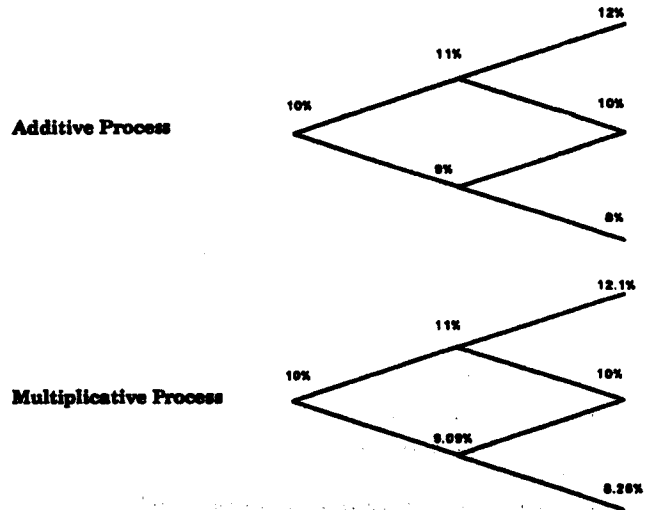
ADDITIVE PROCESS

Let us imagine that whatever the short rate is today, the short rate tomorrow will be equal to either 1) today's rate plus a fixed amount, or 2) today's rate minus the same fixed amount. We assume further that each case happens with a probability of 0.5. The upper "tree" in Figure 2 illustrates an example of the additive process where the initial rate is chosen to be 10%. Under the additive process, the prevailing short rate at any given time equals the expected future rate at any future period.

MULTIPLICATIVE PROCESS

The additive process has the crucial fault that short rates can become negative. Further, it fails to capture the reality that yield changes tend to be larger in absolute value at higher yield levels. A process that avoids both of these problems is one that postulates that the short rate tomorrow equals either 1) today's short rate multiplied by a constant $(1 + a)$ ($a \geq 0$) or 2) today's short rate divided by $1 + a$. Again, each case happens with a probability of 0.5. Notice that

FIGURE 2 ■ Yield Curve Generated From Short Rates



under this "multiplicative" process expected future rates exceed the prevailing short rate, and the difference increases with a .² The lower tree in Figure 2 illustrates the multiplicative process.

The Expectations Hypothesis and the Yield Curve

The expectations hypothesis has often been used to relate expectations of future interest rate evolution to the price of longer-term instruments. As used here, the hypothesis states that the one-period rates of return for all assets should be identical, and hence equal to the short rate for that period. We call it the "one-period rate of return expectations hypothesis" to distinguish it from the more commonly used "pure expectations hypothesis" that requires forward rates to equal the expected future rates.

To illustrate how this expectations hypothesis lets us compute the yield on, say, a two-period bond, we choose a particular case of the additive process considered above. The initial short rate is 10%, and each period corresponds to a year. After one year the prevailing short rate is either 8% or 12%. After one year, the two-period bond has become a one-year bond. Hence its price is either $1/1.12$ or $1/1.08$. Therefore, the expected price tomorrow is given by:

$$0.5 \times (1/1.12 + 1/1.08) = 0.90939$$

If we write y for the yield on a two-period bond, the expected rate of return from holding the two-period bond for one period is given by:

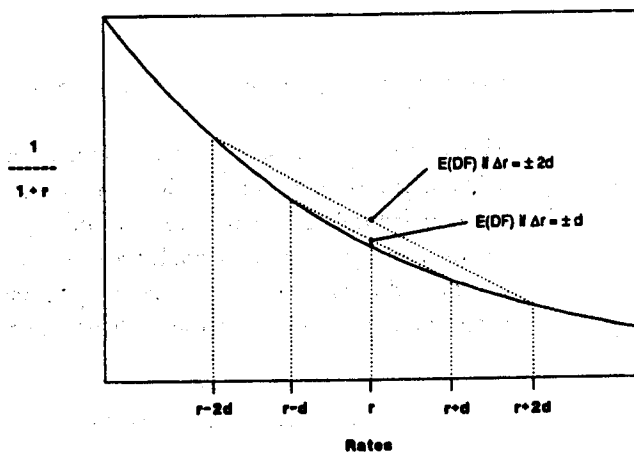
$$\text{Expected Return} = [(1 + y)^2 \times 0.90939] - 1 \quad (1)$$

For the expected rate of return on the two-period bond to be 10%, the yield on the two-period bond must be 9.98% and the yield curve must be downward-sloping. To understand why the yield curve slopes down, observe (Figure 3) that the function relating interest rates to discount factors is convex. We can express the effect of this convexity in two equivalent ways:

1. A given increase in the interest rate lowers the price of a discount bond by more when interest rates are low than when they are high.

2. Suppose the one-period interest rate tomorrow is either $r + d$ or $r - d$, each with a probability of 0.5. As of today, the expected discount factor tomorrow is $0.5 \times [1/(1 + r + d) + 1/(1 + r - d)]$, which exceeds $1/(1 + r)$ — the discount factor that would apply if the expected interest rate r were certain to prevail. Further, this difference increases as d increases. In other words, the expected price of a one-period bond issued tomorrow is larger than $1/(1 + r)$, the price that would prevail if rates were certain to be r . Figure 3 illustrates this point.

FIGURE 3 ■ Function Relating Interest Rates to Discount Factors



This convexity implies that, under the one-period rate of return expectations hypothesis, the yield curve for the additive process is downward-sloping. Because the expected price of a one-period bond tomorrow exceeds $1/(1 + r)$, the only way to equate the expected rate of return of a two-period bond to that of a one-period bond (r) is to have a yield that is less than r . Repeating this reasoning, we conclude that the yield on a three-period bond is smaller than that on a two-period bond, and so on.

Term Premiums

Investors may in reality demand a higher one-period expected rate of return for securities with longer maturities.

The difference between the one-period expected rate of return on a zero-coupon bond that matures in T periods and the return on a one-period bond is called the holding term premium.

It is simple to rederive the yield curve when accounting for term premiums. Suppose that, as above, the initial short rate is 10%, and after one year the prevailing short rate is either 8% or 12%. Then if y again denotes the yield on a two-period bond, we know from Equation (1) that

$$\text{Expected Return} = [(1 + y)^2 \times 0.90939] - 1$$

For example, if the holding-term premium is one basis point per period, the expected rate of return on the two-period bond is 10.01%, and the yield on the two-period bond must therefore be 9.99%.

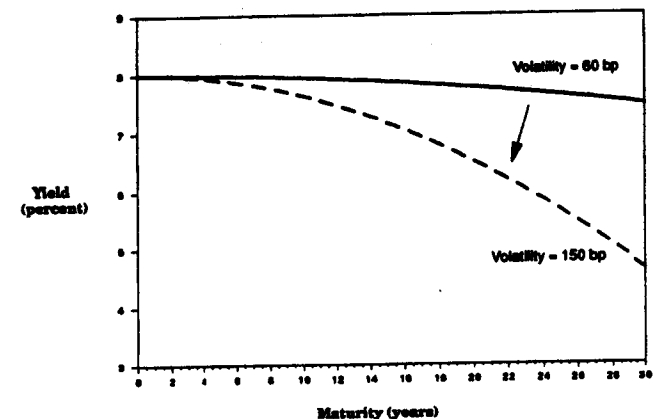
Volatility and Shape

Convexity has implications both for the shape of the yield curve and for the impact of volatility on this shape. We illustrate this with the two rate expectation processes discussed above.

ADDITIVE PROCESS

If the short rate today is r , then the rate tomorrow is either $r + d$ or $r - d$, where $d > 0$. The parameter d measures the volatility of short rates. Expected future short rates equal r , but, as shown above, the convexity of the function relating interest rates to discount factors implies that the yield curve is downward-sloping. As the impact on longer yields is proportional to d , an increase in volatility tilts the yield curve more steeply downward, as illustrated in Figure 4A.

FIGURE 4A ■ Effects of an Increase in Volatility ■ Additive Process



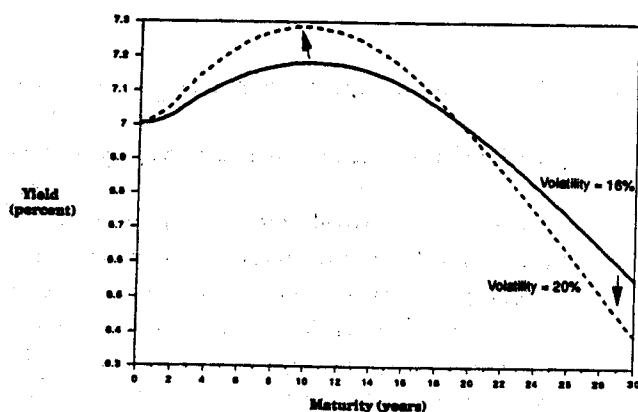
MULTIPLICATIVE PROCESS

If the short rate today is r , then the rate tomorrow is either $r \times (1 + a)$ or $r/(1 + a)$, where $a \geq 0$. The parameter "a" measures the volatility, in percentage terms, of the short rates. Unless $a = 0$, expected future rates increase over time. Offsetting this effect, however, is the impact of the convex discount factor function, which tends to make the yield curve slope downward. For longer maturities, the convexity prevails.

An increase in volatility now has two distinct, opposing effects on the yield curve. First, through the simple multiplicative process, it increases expected future short rates. Second, by increasing the convexity of the discount factor function, it reduces yields.

The first effect dominates at the short end of the yield curve, while the second one prevails at the long end. As a result, an increase in volatility moves the peak of the yield curve toward earlier maturities and increases the curvature of the yield curve. Figure 4B illustrates these effects.

FIGURE 4B ■ Effects of an Increase in Volatility ■ Multiplicative Process



VOLATILITY OF VOLATILITY

Neither of the two processes discussed above leads to yield curves that fit the data very well. One of the reasons is that volatility itself is quite volatile. The implied yield volatility of options on Treasury bond futures has ranged from a low of 7.35% to a high of over 30% in the last few years. What is more important, volatility tends to revert to a mean level — i.e., volatility has a tendency to fall when it is high and rise when it is low.

One possible way to estimate the time series properties of volatility is first to infer the bond yield volatility implied in the price of options on Treasury bond futures. Using weekly data from January 1, 1984, through June 23,

1988, a regression of V (the implied bond yield volatility) on V lagged one period gives:

$$V(t) = 0.0072 + 0.9587V(t-1) + e(t)$$

where $e(t)$ is the unexpected shock to volatility.

This regression explains 93% of the variation in V . The regression implies a mean value of volatility of 17.5% and has a standard error of 1.4%. The half-life of a volatility shock is about 16.4 weeks — i.e., after this time the effect of a shock falls below one-half of the initial level.

If volatility increases permanently, then the reduction in yields caused by the convexity of the discount factor function would be greater as maturity increased. The temporary nature of volatility increases makes the yield reduction substantially less pronounced at the end of the maturity spectrum. This has implications for the measurement of yield curve volatility.

A THREE-VARIABLE MODEL OF THE YIELD CURVE

In addition to volatile volatility, there are at least two other reasons why the simple model described above is inadequate. First, the level of the long rate helps explain changes in the short rate. Second, the data indicate that investors demand a positive term premium, suggesting that the expectations hypothesis underestimates yields at longer maturities.

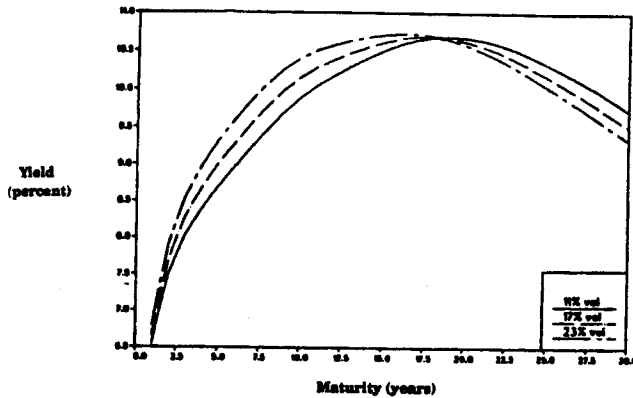
To deal with these complexities, we estimate a model where future short rates depend on 1) today's short rate, 2) the level toward which the short rate is expected, as of today, to converge — which we call the "long" rate — and 3) the volatility of this "long" rate. We allow for the possibility of term premiums, and estimate the model using zero yield curves derived from coupon bond prices over the last five years.

We confirmed that, historically, variations in these three variables satisfactorily explain the past movements in the yield curve, and that, in particular, increases in volatility accentuate the convexity of the yield curve. Figure 5 shows distinct yield curves generated by the model for a fixed short rate at different levels of volatility.

A MEASURE OF VOLATILITY IN THE YIELD CURVE

This model suggests that we can obtain a measure of volatility from the yield curve by considering the yield spread on a butterfly — i.e., a portfolio that is long two bonds of different maturities and short a bond of intermediate

FIGURE 5 ■ Model Yield Curves Under Various Volatilities ■ Short Rate Fixed



maturity. Moreover, as we discuss above, the temporary nature of volatility changes implies that this spread should involve bonds in the earlier part of the curve.

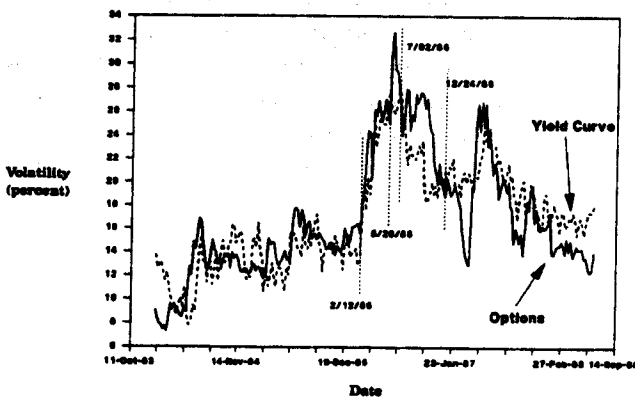
A regression of implied volatility on the level of the one-month, three-year, and ten-year zero yields confirms this implication. In fact, estimating this regression for weekly data from January 1, 1984, through June 23, 1988, gives us:

$$V(t) = 0.497 - [0.065 \times z1mo(t)] + [0.157 \times z3yr(t)] - [0.126 \times z10yr(t)]$$

The regression explains 70% of the variation in V , with a standard error of estimate equal to 0.03.

The fitted value of this regression can be interpreted as a measure of implied volatility in the yield curve. Figure

FIGURE 6 ■ History of Implied Volatilities ■ From Option on T-Bond Futures and the Yield Curve



ENDNOTES

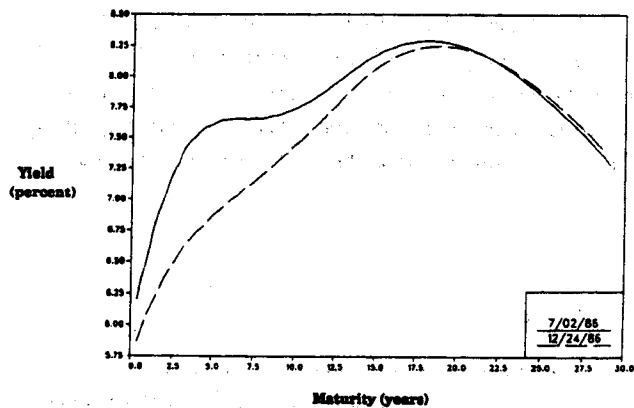
¹In practice, the zero-coupon yield curve of interest is estimated from coupon bonds.

²These problems can also be corrected by using a process in which, if the

6 compares the history of implied volatility obtained from options on Treasury bond futures with the implied volatility estimated from the yield curve.

In Figure 6 we portray several episodes where volatility levels changed markedly between two dates. In 1986, for example, volatility increased between February 12 and May 28, and decreased between July 2 and December 24. Figure 7 compares the initial and final yield curves in the latter period. The changes were similar in all the other episodes. Notice that the curvature of the yield curve for maturities up to ten years was much greater on July 2 than on December 24.

FIGURE 7 ■ Fitted Yield Curve Comparison



SUMMARY

This article reports on our examination of the role of interest rate volatility in determining the shape of the yield curve. We have developed theoretical models that portray the effect of volatility on the "hump" that most yield curves exhibit. These models suggest that a correlation should exist between the level of interest rate volatility and the yield spreads on certain butterfly combinations. Examination of past yield curves confirms the existence of this relationship.

Understanding the effect of the interest rate volatility implicit in the yield curve is essential to comprehending the behavior of fixed-income securities. This is particularly true in the case of callable bonds, where changes in volatility have an impact on both the value of the embedded option and the value of the future payments.

short rate today is r , then tomorrow's rate is $r(1 + a)$ with a probability of 0.5 or $r(1 - a)$ with a probability of 0.5, where $0 < a < 1$.