

Overconfidence and Speculative Bubbles*

José Scheinkman[†] Wei Xiong[‡]

March 14, 2003

Abstract

Motivated by the behavior of asset prices, trading volume and price volatility during episodes of asset-price bubbles, we present a continuous time equilibrium model where overconfidence generates disagreements among agents regarding asset fundamentals. With short-sale constraints, an asset buyer acquires an option to sell the asset to other agents when those agents have more optimistic beliefs. As in Harrison and Kreps (1978), agents pay prices that exceed their own valuation of future dividends because they believe that in the future they will find a buyer willing to pay even more. This causes a significant bubble component in asset prices even when small differences of beliefs are sufficient to generate a trade. In equilibrium, bubbles are accompanied by large trading volume and high price volatility. Our analysis shows that while Tobin's tax can substantially reduce speculative trading when transaction costs are small, it has only a limited impact on the size of the bubble or on price volatility.

*This paper was previously circulated under the title "Overconfidence, Short-Sale Constraints and Bubbles." Scheinkman's research was supported by the Chaire Blaise Pascal and the National Science Foundation. We would like to thank Patrick Bolton, Markus Brunnermeier, John Cochrane, George Constantinides, Darrel Duffie, Ivar Ekeland, Edward Glaeser, Ravi Jagannathan, Pete Kyle, Owen Lamont, Marcelo Pinheiro, Chris Rogers, Tano Santos, Walter Schachermayer, Harald Uhlig, Dimitri Vayanos, two anonymous referees and seminar participants at various institutions and conferences for comments.

[†]Princeton University and Université Paris-Dauphine. E-mail: joses@princeton.edu; Phone: (609) 258-4020.

[‡]Princeton University. E-mail: wxiong@princeton.edu; Phone: (609) 258-0282.

1 Introduction

The behavior of market prices and trading volumes of assets during historical episodes of price bubbles presents a challenge to asset pricing theories. A common feature of these episodes, including the recent internet stock boom, is the co-existence of high prices and high trading volume.¹ In addition, high price volatility is frequently observed.²

In this paper, we propose a model of asset trading, based on heterogeneous beliefs generated by agents' overconfidence, with equilibria that broadly fit these observations. We also provide explicit links between parameter values in the model, such as trading cost and information, and the behavior of equilibrium prices and trading volume. More generally, our model provides a flexible framework to study speculative trading that can be used to analyze links between asset prices, trading volume and price volatility.

In the model, the ownership of a share of stock provides an opportunity (option) to profit from other investors' overvaluation. For this option to have value, it is necessary that some restrictions apply to short-selling. In reality, these restrictions arise from many distinct sources. First, in many markets short selling requires borrowing a security and this mechanism is costly.³ In particular the default risk if the asset price goes up is priced by lenders of the security. Second, the risk associated with short selling may deter risk-averse investors. Third, limitations to the availability of capital to potential arbitrageurs may also limit short selling.⁴ For technical reasons, we do not deal with short-sale costs or risk aversion. Instead we rule out short sales, although our qualitative results would survive the presence of limited short sales as long as the asset owners can expect to make a profit when others have higher valuations.

Our model follows the basic insight of Harrison and Kreps (1978) that, when agents agree to disagree and short selling is not possible, asset prices may exceed their fundamental value. In their model, agents disagree about the probability distributions of dividend streams - the reason for the disagreement is not made explicit. We study overconfidence, the belief of an agent that his information is more accurate than what it is, as a source of disagreement. Although overconfidence is only one of the many ways by which disagreement among investors may arise,⁵ it is suggested by some experimental studies of human behavior, and generates a mathematical framework that is relatively simple. Our model has an explicit solution, which allows us to

¹See Lamont and Thaler (2003) and Ofek and Richardson (2001) for the internet boom. Cochrane (2002) emphasizes the significant correlation between high prices and high turnover rates as a key characteristic of the 1929 boom and crash and of the internet episode. Ofek and Richardson (2002, page 1) point out that "between early 1998 and February 2000, pure internet firms represented as much as 20% of the dollar volume in the public equity market, even though their market capitalization never exceeded 6%."

²Cochrane (2002, page 6) commenting on the much discussed Palm case: "Palm stock was tremendously volatile during this period, with 15.4% standard deviation of 5 day returns, which is about the same as the volatility of the S&P 500 index over an entire year"

³Duffie, Garleanu and Pedersen [2002] provide a search model to analyze the actual short-sale process and its implication for asset prices. Jones and Lamont [2002], Geczy, Musto and Reed [2002], and D'Avolio [2002] contain empirical analysis of the relevance of short-sale costs.

⁴Shleifer and Vishny (1997) argue that agency problems limit the capital available to arbitrageurs and may cause arbitrage to fail. See also Xiong (2001), Kyle and Xiong (2001), and Gromb and Vayanos (2002) for studies linking the dynamics of arbitrageurs' capital with asset price dynamics.

⁵For example, Morris (1996) assumes non-common priors, while Biais and Bossaerts (1998) examine the role of higher order beliefs.

derive several comparative statics results and restrictions on the dynamics of observables. The model may also be regarded as a fully worked out example of the Harrison-Kreps framework in continuous time, where computations and comparison of solutions are particularly tractable.

We study a market for a single risky asset with limited supply and many risk-neutral agents in a continuous time model with infinite horizon. The current dividend of the asset is a noisy observation of a fundamental variable that will determine future dividends. In addition to the dividends, there are two other sets of signals available at each instant. The information is available to all agents, however agents are divided in two groups and they differ in the interpretation of the signals. Each group overestimates the informativeness of a different signal and as a consequence, have distinct forecasts of future dividends.⁶ Agents in our model know that their forecasts differ from the forecasts of agents in the other group, but behavioral limitations lead them to agree to disagree. As information flows, the forecasts by agents of the two groups oscillate, and the group of agents that is at one instant relatively more optimistic, may become in a future date less optimistic than the agents in the other group. These fluctuations in relative beliefs generate trade.

When evaluating an asset, agents consider their own view of fundamentals and the fact that the owner of the asset has an option to sell the asset in the future to agents in the other group. This option can be exercised at any time by the current owner, and the new owner gets, in turn, another option to sell the asset. These characteristics makes the option “American” and gives it a recursive structure. The value of the option is the value function of an optimal stopping problem. Since the buyer’s willingness to pay is a function of the value of the option that he acquires, the payoff from stopping is, in turn, related to the value of the option. This gives us a fixed point problem that the option value must satisfy. This difference between the current owner’s demand price and his fundamental valuation, which is exactly the resale option value, can be reasonably called a bubble.⁷ Fluctuations in the value of the bubble contribute an extra component to price volatility. We emphasize that the bubble in our model is a consequence of the divergence of opinions generated by the overestimation of informativeness of the distinct signals. On average our agents are neither optimists nor pessimists.

In equilibrium, an asset owner will sell the asset to agents in the other group, whenever his view of the fundamental is surpassed by the view of agents in the other group by a critical amount. Passages through this critical point determine turnover. When there are no trading costs, we show that the critical point is zero - it is optimal to sell the asset immediately after the valuation of fundamentals is “crossed” by the valuation of agents in the other group. Our agents’ beliefs satisfy simple stochastic differential equations and it is a consequence of properties of Brownian motion, that once the beliefs of agents cross, they will cross infinitely many times in any finite period of time right afterwards. This results in a trading frenzy, in which the unconditional average volume in any time interval is infinite. Since the equilibria display continuity with respect to the trading cost c , our model with small trading costs is able

⁶“We all have the same information, and we’re just making different conclusions about what the future will hold.” Henry Blodget, the former star analyst at Merrill Lynch, quoted by Michael Lewis in “In Defense of the Boom” New York Times Magazine October 27, 2002.

⁷An alternative would be to measure the bubble as the difference between the asset price and the fundamental valuation of the dividends by an agent that correctly weights the signals. We opted for our definition because it highlights the difference between beliefs about fundamentals and trading price.

to capture the excessive trading observed in bubbles.

When trading costs are small, the value of the bubble and the extra volatility component are maximized. We show that increases in some parameter values, such as the degree of overconfidence or the information content of the signals, increase these three key variables. In this way, our model provides an explanation for the cross sectional correlation between price, volume and volatility that has been observed in bubbles. However, to obtain more realistic time-series dynamics, it may be necessary to allow the parameter values to change over time. In subsection 7.5, we discuss how to accommodate fluctuations in parameter values, though we do not provide a theory that explains these oscillations.

In the model, increases in trading costs reduce the trading frequency, asset price volatility, and the option value. For small trading costs, the effect on trading frequency is very significant. The impact on price volatility and on the size of the bubble is much more modest. As the trading cost increases, the increase in the critical point also raises the profit of the asset owner from each trade, thus partially offsetting the decrease in the value of the re-sale option caused by the reduction in trading frequency. Our analysis suggests that a transaction tax, such as proposed by Tobin (1978), would, in fact, substantially reduce the amount of speculative trading in markets with small transaction costs, but would have a limited effect on the size of the bubble or on price volatility. Since a Tobin tax will no doubt also deter trading generated by fundamental reasons that are absent from our model, the limited impact of the tax on the size of the bubble and on price volatility cannot serve as an endorsement of the Tobin tax. The limited effect of transaction costs on the size of the bubble is also compatible with the observation of Shiller (2000) that bubbles have occurred in real estate market, where transaction costs are high.⁸

The existence of the option component in the asset price creates potential violations to the law of one price. Through a simple example, we illustrate that the bubble may cause the price of a subsidiary to be larger than that of its parent firm. The intuition behind the example is that if a firm has two subsidiaries with fundamentals that are perfectly negatively correlated, there will be no differences in opinion, and hence no option component on the value of the parent firm, but possibly strong differences of opinion about the value of a subsidiary. In this example, our model also predicts that trading volume on the subsidiaries would be much larger than on the parent firm. This nonlinearity of the option value may help explain the “mispricing” of carve-outs that occurred in the late 90’s such as the 3Com-Palm case.

The structure of the paper follows. In Section 2, we present a brief literature review. Section 3 describes the structure of the model. Section 4 derives the evolution of agents’ beliefs. In Section 5, we discuss the optimal stopping time problem and derive the equation for equilibrium option values. In Section 6, we solve for the equilibrium. Section 7 discusses several properties of the equilibrium when trading costs are small. In Section 8, we focus on the effect of trading costs. In Section 9, we construct an example where the price of a subsidiary is larger than its parent firm. Section 10 concludes with some discussion of implications to corporate finance.

⁸In contrast, Federal Reserve Chairman Alan Greenspan seems to believe that the low turnover induced by the high costs of transactions in the housing market are an impediment to real estate bubbles. “While stock market turnover is more than 100% annually, the turnover of home ownership is less than 10 per cent annually - scarcely tinder for speculative conflagration.” (quoted in Financial Times of April 22, 2002). The results in this paper indicate otherwise.

All proofs are in the Appendix.

2 Related literature

There is a large literature on the effects of heterogeneous beliefs. In a static framework, Miller (1977), and Chen, Hong and Stein (2002) analyze the overvaluation generated by heterogeneous beliefs. This static framework cannot generate an option value or the dynamics of trading. Harris and Raviv (1993), Kandel and Pearson (1995) and Kyle and Lin (2002) study models where trading is generated by heterogeneous beliefs. However, in all these models there is no speculative component in prices.

Psychology studies suggest that people overestimate the precision of their knowledge in some circumstances, especially for challenging judgement tasks (see Alpert and Raiffa (1982) or Lichtenstein, Fischhoff, and Phillips (1982)). Camerer (1995) argues that even experts can display overconfidence. A similar phenomena is *illusion of knowledge* the fact that persons who do not agree become more polarized when given arguments that serve both sides (Lord, Ross and Lepper (1979)).⁹ In finance, researchers have developed theoretical models to analyze the implications of overconfidence on financial markets (see *e.g.* Kyle and Wang (1997), Odean (1998), Daniel, Hirshleifer and Subrahmanyam (1998) and Bernardo and Welch (2001).) In these papers, overconfidence is typically modelled as overestimation of the precision of one's information. We follow a similar approach, but emphasize the speculative motive generated by overconfidence.

The bubble in our model, based on the recursive expectations of traders to take advantage of mistakes by others, is quite different from the "rational bubbles".¹⁰ In contrast to our set up, rational-bubble models are incapable of connecting bubbles with turnover. In addition, in these models, assets must have (potentially) infinite maturity to generate bubbles. Although we treat, for mathematical simplicity, the infinite horizon case, the bubble in our model does not require infinite maturity. If an asset has a finite maturity the bubble will tend to diminish as maturity approaches, but it would nonetheless exist in equilibrium.

Other mechanisms have been proposed to generate asset price bubbles, *e.g.*, Allen and Gorton (1993), Allen, Morris, and Postlewaite (1993), Horst (2001) and Duffie, Garleanu, and Pedersen (2002). None of these models emphasize the joint properties of bubble and trading volume observed in historical episodes.

3 The model

There exists a single risky asset with a dividend process that is the sum of two components. The first component is a fundamental variable that determines future dividends. The second is "noise". The cumulative dividend process D_t satisfies:

$$dD_t = f_t dt + \sigma_D dZ_t^D, \tag{1}$$

⁹See Hirshleifer (2001) and Barber and Odean (2002) for reviews of this literature.

¹⁰See Blanchard and Watson (1982) or Santos and Woodford (1997).

where Z^D is a standard Brownian motion and σ_D is a constant volatility parameter. The fundamental variable f is not observable. However, it satisfies:

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f, \quad (2)$$

where $\lambda \geq 0$ is the mean reversion parameter, \bar{f} is the long-run mean of f , $\sigma_f > 0$ is a constant volatility parameter and Z^f is a standard Brownian motion. The asset is in finite supply and we normalize the total supply to unity.

There are two sets of risk-neutral agents. The assumption of risk neutrality not only simplifies many calculations, but also serves to highlight the role of information in the model. Since our agents are risk-neutral, the dividend noise in equation (1) has no direct impact in the valuation of the asset. However, the presence of dividend noise makes it impossible to infer f perfectly from observations of the cumulative dividend process. Agents use the observations of D and any other signals that are correlated with f to infer current f and to value the asset. In addition to the cumulative dividend process, all agents observe a vector of signals s^A and s^B that satisfy:

$$ds_t^A = f_t dt + \sigma_s dZ_t^A \quad (3)$$

$$ds_t^B = f_t dt + \sigma_s dZ_t^B, \quad (4)$$

where Z^A and Z^B are standard Brownian motions, and $\sigma_s > 0$ is the common volatility of the signals. We assume that all four processes Z^D , Z^f , Z^A and Z^B are mutually independent.

Agents in group A (B) think of s^A (s^B) as their own signal although they can also observe s^B (s^A). Heterogeneous beliefs arise because each agent believes that the informativeness of his own signal is larger than its true informativeness. Agents of group A (B) believe that innovations dZ^A (dZ^B) in the signal s^A (s^B) are correlated with the innovations dZ^f in the fundamental process, with ϕ ($0 < \phi < 1$) as the correlation parameter. Specifically, agents in group A believe that the process for s^A is

$$ds_t^A = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^A.$$

Although agents in group A perceive the correct unconditional volatility of the signal s^A , the correlation that they attribute to innovations causes them to over-react to signal s^A . Similarly, agents in group B believe the process for s^B is

$$ds_t^B = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^B.$$

On the other hand agents in group A (B) believe (correctly) that innovations to s^B (s^A) are uncorrelated with innovations to Z^B (Z^A). We assume that the joint dynamics of the processes D , f , s^A and s^B in the mind of agents of each group is public information.

Lemma 1 below shows that a larger ϕ , increases the precision that agents attribute to their own forecast of the current level of fundamentals. For this reason, we will refer to ϕ as the overconfidence parameter.¹¹

¹¹In an earlier draft, we assumed that agents overestimate the precision of their signal. We thank Chris Rogers for suggesting that we examine this alternative framework. The advantage of the present set-up is that the probability measure used by each group of agents is equivalent to the true probability measure.

Each group is large and there is no short selling of the risky asset. To value future cash flows, we may either assume that every agent can borrow and lend at the same rate of interest r , or equivalently that agents discount all future payoffs using rate r , and that each group has infinite total wealth. These assumptions will facilitate the calculation of equilibrium prices.

4 Evolution of beliefs

The model described in the previous section implies a particularly simple structure for the evolution of the difference in beliefs between the groups of traders. We show that this difference is a diffusion with volatility proportional to ϕ .

Since all variables are Gaussian, the filtering problem of the agents is standard. With Gaussian initial conditions, the conditional beliefs of agents in group $C \in \{A, B\}$ is Gaussian with mean \hat{f}^C and variance γ^C . We will characterize the stationary solution. Standard arguments¹² allow us to compute the variance of the stationary solution and the evolution of the conditional mean of beliefs. The variance of this stationary solution is the same for both groups of agents and equals

$$\gamma \equiv \frac{\sqrt{(\lambda + \phi\sigma_f/\sigma_s)^2 + (1 - \phi^2)(2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2)} - (\lambda + \phi\sigma_f/\sigma_s)}{\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}}.$$

The following lemma justifies associating the parameter ϕ to “overconfidence.”

Lemma 1 *The stationary variance γ decreases with ϕ .*

In addition, the conditional mean of the beliefs of agents in group A satisfies:

$$\begin{aligned} d\hat{f}^A &= -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt) \\ &\quad + \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt). \end{aligned} \quad (5)$$

Since f mean-reverts, the conditional beliefs also mean-revert. The other three terms represent the effects of “surprises.” These surprises can be represented as standard mutually independent Brownian motions for agents in group A :

$$dW_A^A = \frac{1}{\sigma_s}(ds^A - \hat{f}^A dt), \quad (6)$$

$$dW_B^A = \frac{1}{\sigma_s}(ds^B - \hat{f}^A dt), \quad (7)$$

$$dW_D^A = \frac{1}{\sigma_D}(dD - \hat{f}^A dt). \quad (8)$$

Note that these processes are only Wiener processes in the mind of group A agents. Due to overconfidence ($\phi > 0$), agents in group A over-react to surprises in s^A .

¹²*e.g.* section VI.9 in Rogers and Williams (1987) and Theorem 12.7 in Liptser and Shiryaev (1977)

Similarly, the conditional mean of the beliefs of agents in group B satisfies:

$$\begin{aligned} d\hat{f}^B &= -\lambda(\hat{f}^B - \bar{f})dt + \frac{\gamma}{\sigma_s^2}(ds^A - \hat{f}^B dt) \\ &\quad + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^B - \hat{f}^B dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^B dt), \end{aligned} \quad (9)$$

and the surprise terms can be represented as mutually independent Wiener processes: $dW_A^B = \frac{1}{\sigma_s}(ds^A - \hat{f}^B dt)$, $dW_B^B = \frac{1}{\sigma_s}(ds^B - \hat{f}^B dt)$, and $dW_D^B = \frac{1}{\sigma_D}(dD - \hat{f}^B dt)$. These processes are a standard 3-d Brownian only for agents in group B .

Since the beliefs of all agents have constant variance, we will refer to the conditional mean of the beliefs as their *beliefs*. We let g_A and g_B denote the differences in beliefs:

$$g^A = \hat{f}^B - \hat{f}^A, \quad g^B = \hat{f}^A - \hat{f}^B.$$

The next proposition describes the evolution of these differences in beliefs:

Proposition 1

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A,$$

where

$$\begin{aligned} \rho &= \sqrt{\left(\lambda + \phi\frac{\sigma_f}{\sigma_s}\right)^2 + (1 - \phi^2)\sigma_f^2 \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)}, \\ \sigma_g &= \sqrt{2}\phi\sigma_f, \end{aligned}$$

and W_g^A is a standard Wiener process for agents in group A , with innovations that are orthogonal to the innovations of \hat{f}^A .

Proposition 1 implies that the difference in beliefs g^A follows a simple mean reverting diffusion process in the mind of group A agents. In particular, the volatility of the difference in beliefs is zero in the absence of overconfidence. A larger ϕ leads to greater volatility. In addition, $\frac{\rho}{2\sigma_g^2}$ measures the pull towards the origin. A simple calculation shows that this *mean-reversion*¹³ decreases with ϕ . A higher ϕ causes an increase in fluctuations of opinions and a slower mean-reversion.

In an analogous fashion, for agents in group B , g^B satisfies:

$$dg^B = -\rho g^B dt + \sigma_g dW_g^B,$$

where W_g^B is a standard Wiener process, and it is independent of innovations to \hat{f}^B .

¹³Conley *et al.* (1997) argue that this is the correct measure of mean-reversion.

5 Trading

Fluctuations in the difference of beliefs across agents will induce trading. It is natural to expect that investors that are more optimistic about the prospects of future dividends will bid up the price of the asset and eventually hold the total (finite) supply. We will allow for costs of trading - a seller pays $c \geq 0$ per unit of the asset sold. This cost may represent an actual cost of transaction or a tax.

At each t , agents in group $C = \{A, B\}$ are willing to pay p_t^C for a unit of the asset. The presence of the short-sale constraint, a finite supply of the asset, and an infinite number of prospective buyers, guarantee that any successful bidder will pay his reservation price.¹⁴ The amount that an agent is willing to pay reflects the agent's fundamental valuation and the fact that he may be able to sell his holdings at a later date at the demand price of agents in the other group for a profit. If we let $o \in \{A, B\}$ denote the group of the current owner, \bar{o} be the other group, and E_t^o be the expectation of members of group o , conditional on the information they have at t , then:

$$p_t^o = \sup_{\tau \geq 0} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} dD_s + e^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right], \quad (10)$$

where τ is a stopping time, and $p_{t+\tau}^{\bar{o}}$ is the reservation value of the buyer at the time of transaction $t + \tau$.

Since, $dD = \hat{f}_t^o dt + \sigma_D dW_D^o$, we have, using the equations for the evolution of the conditional mean of beliefs (equations (5) and (9) above) that:

$$\int_t^{t+\tau} e^{-r(s-t)} dD_s = \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + M_{t+\tau},$$

where $E_t^o M_{t+\tau} = 0$. Hence, we may rewrite equation (10) as:

$$p_t^o = \max_{\tau \geq 0} E_t^o \left\{ \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + e^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right\}. \quad (11)$$

We will start by postulating a particular form for the equilibrium price function, equation (12) below. Proceeding in a heuristic fashion, we derive properties that our candidate equilibrium price function should satisfy. We then construct a function that satisfies these properties, and verify that we have produced an equilibrium.¹⁵

Since all the relevant stochastic processes are Markovian and time-homogeneous, and traders are risk-neutral, it is natural to look for an equilibrium in which the demand price of the current owner satisfies

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + q(g_t^o). \quad (12)$$

¹⁴This observation simplifies our calculations, but is not crucial for what follows. We could partially relax the short sale constraints or the division of gains from trade, provided it is still true that the asset owner expects to make speculative profits from other investors.

¹⁵The argument that follows will also imply that our equilibrium is the only one within a certain class. However, there are other equilibria. In fact, given any equilibrium price p_t^o and a process M_t that is a martingale for both groups of agents, then $\tilde{p}_t^o = p_t^o + e^{rt} M_t$ is also an equilibrium.

with $q > 0$ and $q' > 0$. This equation states that prices are the sum of two components. The first part, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda}$, is the expected present value of future dividends from the viewpoint of the current owner. The second is the value of the resale option, $q(g_t^o)$, which depends on the current difference between the beliefs of the other group's agents and the beliefs of the current owner. We call the first quantity the owner's fundamental valuation and the second the value of the resale option. Using (12) in equation (11) and collecting terms, we obtain:

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + \sup_{\tau \geq 0} \mathbb{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) e^{-r\tau} \right].$$

Equivalently, the resale option value satisfies

$$q(g_t^o) = \sup_{\tau \geq 0} \mathbb{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) e^{-r\tau} \right]. \quad (13)$$

Hence to show that an equilibrium of the form (12) exists, it is necessary and sufficient to construct an option value function q that satisfies equation (13). This equation is similar to a Bellman equation. The current asset owner chooses an optimal stopping time to exercise his re-sale option. Upon the exercise of the option, the owner gets the "strike price" $\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}})$, the amount of excess optimism that the buyer has about the asset's fundamental value and the value of the resale option to the buyer, minus the cost c . In contrast to the optimal exercise problem of American options, the "strike price" in our problem depends on the re-sale option value function itself.

It is apparent from the analysis in this section that one could, in principle, treat an asset with a finite life. Equations (10) to (11) would apply with the obvious changes to account for the finite horizon. However, the option value q will now depend on the remaining life of the asset, introducing another dimension to the optimal exercise problem. The infinite horizon problem is stationary, greatly reducing the mathematical difficulty.

6 Equilibrium

In this section, we derive the equilibrium option value, duration between trades, and contribution of the option value to price volatility.

6.1 Resale option value

The value of the option $q(x)$ should be at least as large as the gains realized from an immediate sale. The region where the value of the option equals that of an immediate sale is the stopping region. The complement is the continuation region. In the mind of the risk neutral asset holder, the discounted value of the option $e^{-rt}q(g_t^o)$ should be a martingale in the continuation region, and a supermartingale in the stopping region. Using Ito's lemma and the evolution equation for g^o , these conditions can be stated as:

$$q(x) \geq \frac{x}{r + \lambda} + q(-x) - c \quad (14)$$

$$\frac{1}{2} \sigma_g^2 q'' - \rho x q' - r q \leq 0, \text{ with equality if (14) holds strictly.} \quad (15)$$

In addition, the function q should be continuously differentiable (smooth pasting). We will derive a smooth function q that satisfies equations (14) and (15) and then use these properties and a growth condition on q to show that in fact the function q solves (13).

To construct the function q , we guess that the continuation region will be an interval $(-\infty, k^*)$, with $k^* > 0$. k^* is the minimum amount of difference in opinions that generates a trade. As usual, we begin by examining the second order ordinary differential equation that q must satisfy, albeit only in the continuation region:

$$\frac{1}{2}\sigma_g^2 u'' - \rho x u' - r u = 0 \quad (16)$$

The following proposition helps us construct an “explicit” solution to equation (16).

Proposition 2 *Let*

$$h(x) = \begin{cases} U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) & \text{if } x \leq 0 \\ \frac{2\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)} M\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) - U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) & \text{if } x > 0 \end{cases} \quad (17)$$

where $\Gamma(\cdot)$ is the Gamma function, and $M : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $U : \mathbb{R}^3 \rightarrow \mathbb{R}$ are two Kummer functions described in the appendix. $h(x)$ is positive and increasing in $(-\infty, 0)$. In addition h solves equation (16) with

$$h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}.$$

Any solution $u(x)$ to equation (16) that is strictly positive and increasing in $(-\infty, 0)$ must satisfy: $u(x) = \beta_1 h(x)$ with $\beta_1 > 0$.

We will also need properties of the function h that are summarized in the following Lemma.

Lemma 2 *For each $x \in \mathbb{R}$, $h(x) > 0$, $h'(x) > 0$, $h''(x) > 0$, $h'''(x) > 0$, $\lim_{x \rightarrow -\infty} h(x) = 0$, and $\lim_{x \rightarrow -\infty} h'(x) = 0$.*

Since q must be positive and increasing in $(-\infty, k^*)$, we know from Proposition 2 that

$$q(x) = \begin{cases} \beta_1 h(x), & \text{for } x < k^* \\ \frac{x}{r+\lambda} + \beta_1 h(-x) - c, & \text{for } x \geq k^*. \end{cases} \quad (18)$$

Since q is continuous and continuously differentiable at k^* ,

$$\begin{aligned} \beta_1 h(k^*) - \frac{k^*}{r+\lambda} - \beta_1 h(-k^*) + c &= 0, \\ \beta_1 h'(k^*) + \beta_1 h'(-k^*) - \frac{1}{r+\lambda} &= 0. \end{aligned}$$

These equations imply that

$$\beta_1 = \frac{1}{(h'(k^*) + h'(-k^*))(r+\lambda)}, \quad (19)$$

and k^* satisfies

$$[k^* - c(r + \lambda)](h'(k^*) + h'(-k^*)) - h(k^*) + h(-k^*) = 0. \quad (20)$$

The next theorem shows that for each c , there exists a unique pair (k^*, β_1) that solves equations (19) and (20). The smooth pasting conditions are sufficient to determine the function q and the “trading point” k^* .

Theorem 1 *For each trading cost $c \geq 0$, there exists a unique k^* that solves (20). If $c = 0$ then $k^* = 0$. If $c > 0$, $k^* > c(r + \lambda)$.*

The next theorem establishes that the function q described by equation (18), with β_1 and k^* given by (19) and (20), solves (13). The proof consists of two parts. First, we show that (14) and (15) hold and that q' is bounded. We then use a standard argument¹⁶ to show that in fact q solves equation (13).

Theorem 2 *The function q constructed above is an equilibrium option value function. The optimal policy consists of exercising immediately if $g^o > k^*$, otherwise wait until the first time in which $g^o \geq k^*$.*

It is a consequence of Theorem 2 that the process g^o will have values in $(-\infty, k^*)$. The value k^* acts as a barrier, and when g^o reaches k^* , a trade occurs, the owner’s group switches and the process is restarted at $-k^*$. $q(g^o)$ is the difference between the current owner’s demand price and his fundamental valuation and can be legitimately called a bubble. When a trade occurs this difference is

$$b \equiv q(-k^*) = \frac{1}{(r + \lambda)} \frac{h(-k^*)}{(h'(k^*) + h'(-k^*))}. \quad (21)$$

Using equation (21), we can write the value of the re-sale option as

$$q(x) = \begin{cases} \frac{b}{h(-k^*)} h(x), & \text{for } x < k^* \\ \frac{x}{r+\lambda} + \frac{b}{h(-k^*)} h(-x) - c, & \text{for } x \geq k^*. \end{cases} \quad (22)$$

6.2 Duration between trades

We let $w(x, k, r) = E^o[e^{-r\tau(x, k)} | x]$, with $\tau(x, k) = \inf\{s : g_{t+s}^o > k\}$, given $g_t^o = x \leq k$. $w(x, k, r)$ is the discount factor applied to cashflows received the first time that the difference in beliefs reaches the level of k given that the current difference in beliefs is x . Standard arguments¹⁷ show that w is a non-negative and strictly monotone solution to:

$$\frac{1}{2} \sigma_g^2 w_{xx} - \rho x w_x = r w, \quad w(k, k, r) = 1.$$

¹⁶See *e.g.* Kobila (1993) or Scheinkman and Zariphopoulou (2001) for similar arguments.

¹⁷*e.g.* Karlin and Taylor (1981), page 243

Therefore, Proposition 2 implies that

$$w(x, k, r) = \frac{h(x)}{h(k)}. \quad (23)$$

Note that the free parameter β_1 does not affect w .

If $c > 0$, trading occurs the first time $t > s$ when $g_t^o = k^*$ given that $g_s^o = -k^*$. The expected duration between trades provides a useful measure of trading frequency. Since w is the moment generating function of τ ,

$$\mathbb{E}[\tau(-k^*, k^*)] = - \left. \frac{\partial w(-k^*, k^*, r)}{\partial r} \right|_{r=0}.$$

When $c = 0$, the expected duration between trades is zero. This is a consequence of Brownian local time, as we discuss below.

6.3 An extra volatility component

The option component introduces an extra source of price volatility. Proposition 1 states that the innovations in the asset owner's beliefs \hat{f}^o and the innovations in the difference of beliefs g^o are orthogonal. Therefore, the total price volatility is the sum of the volatility of the fundamental value in the asset owner's mind, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda}$, and the volatility of the option component.

Proposition 3 *The volatility from the option value component is*

$$\eta(x) = \frac{\sqrt{2}\phi\sigma_f}{(r+\lambda)} \frac{h'(x)}{(h'(k^*) + h'(-k^*))}, \quad \forall x < k^*. \quad (24)$$

Since $h' > 0$, the volatility of the option value is monotone.

The variance of an agent's valuation of the discounted dividends is:

$$\begin{aligned} & \frac{1}{(r+\lambda)^2} \left\{ \left(\frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s} \right)^2 + \left(\frac{\gamma}{\sigma_s} \right)^2 + \left(\frac{\gamma}{\sigma_D} \right)^2 \right\} \\ = & (2/\sigma_s^2 + 1/\sigma_D^2)^{-1} (r+\lambda)^{-2} [2\lambda^2 + 2\lambda\phi\sigma_f/\sigma_s + 2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2 \\ & - 2\lambda(\lambda^2 + 2\lambda\phi\sigma_f/\sigma_s + (2-\phi^2)\sigma_f^2/\sigma_s^2 + (1-\phi^2)\sigma_f^2/\sigma_D^2)^{\frac{1}{2}}], \end{aligned}$$

which increases with ϕ if $\lambda > 0$, and equals $\frac{\sigma_f^2}{(r+\lambda)^2}$ if $\lambda = 0$. Therefore, an increase in overconfidence increases the volatility of the agent's valuation of discounted dividends. In the remaining of the paper, we ignore this effect, that vanishes when $\lambda = 0$, to focus on the extra volatility component caused by the option value.

7 Properties of equilibria for small trading costs

In this section, we discuss several of the characteristics of the equilibrium for small trading costs, including the volume of trade and the magnitudes of the bubble and of the extra volatility component. We also provide some comparative statics and show how parameter changes co-move price, volatility, and turnover.

7.1 Trading volume

It is a property of Brownian motion that if it hits the origin at t , it will hit the origin at an infinite number of times in any non-empty interval $[t, t + \Delta t)$. In our limit case of $c = 0$, this implies an infinite amount of trade in any non-empty interval that contains a single trade. When the cost of trade $c = 0$, in any time interval, turnover is either zero or infinity, and the unconditional average volume in any time interval is infinity.¹⁸ The expected time between trades depends continuously on c , so it is possible to calibrate the model to obtain any average daily volume. However, a serious calibration would require accounting for other sources of trading, such as shocks to liquidity, and should match several moments of volume, volatility and prices.

7.2 Magnitude of the bubble

When $c = 0$, a trade occurs each time traders' fundamental beliefs "cross". Nonetheless, the bubble at this trading point is strictly positive, since

$$b = \frac{1}{2(r + \lambda)} \frac{h(0)}{h'(0)}.$$

Owners do not expect to sell the asset at a price above their own valuation, but the option has a positive value. This result may seem counterintuitive. To clarify it, it is worthwhile to examine the value of the option when trades occur whenever the absolute value of the differences in fundamental valuations equal an $\epsilon > 0$. An asset owner in group A (B) expects to sell the asset when agents in group B (A) have a fundamental valuation that exceeds the fundamental valuation of agents in group A (B) by ϵ , that is $g^A = \epsilon$ ($g^A = -\epsilon$). If we write b_0 for the value of the option for an agent in group A that buys the asset when $g^A = -\epsilon$, and b^1 for the value of the option for an agent of group B that buys the asset when $g^A = \epsilon$, then $b_0 = \left[\frac{\epsilon}{r + \lambda} + b_1 \right] \frac{h(-\epsilon)}{h(\epsilon)}$, where $\frac{h(-\epsilon)}{h(\epsilon)}$ is the discount factor from equation (23). Symmetry requires that $b_0 = b_1$ and hence $b_0 = \frac{\epsilon}{(r + \lambda)} \frac{h(-\epsilon)}{[h(\epsilon) - h(-\epsilon)]}$. As $\epsilon \rightarrow 0$,

$$b_0 \rightarrow \frac{1}{2(r + \lambda)} \frac{h(0)}{h'(0)} = b.$$

In this illustration, as $\epsilon \rightarrow 0$, trading occurs with higher frequency and the waiting time goes to zero. In the limit, traders will trade infinitely often and the small gains in each trade compound

¹⁸The unconditional probability, that it is zero, depends on the volatility and mean reversion of the process of the difference of opinions and on the length of the interval. As the length of the interval goes to infinity, the probability of no trade goes to zero.

to a significant bubble. This situation is similar to the cost from hedging an option using a stop-loss strategy studied in Carr and Jarrow (1990).

It is intuitive that when σ_g becomes larger, there is more difference of beliefs, resulting in a larger bubble. Also, when ρ becomes larger, for a given level of difference in beliefs, the re-sale option is expected to be exercised quicker, and therefore there is also a larger bubble. In fact we can show that:

Lemma 3 *If c is small, b increases with σ_g and ρ , and decreases with r and θ . For all $x < k^*$, $q(x) = b \frac{h(x)}{h(-k^*)}$ increases with σ_g and ρ , and decreases with r and θ .*

The proof of Lemma 3 actually shows that whenever c is small, the effect of a change in a parameter on the barrier is second order.

Proposition 1 allows us to write σ_g and ρ using the parameters ϕ , λ , σ_f , $i_s = \frac{\sigma_f}{\sigma_s}$, and $i_D = \frac{\sigma_f}{\sigma_D}$. i_s and i_D measure the information in each of the two signals and the dividend flow respectively. To simplify calculations, we set $\lambda = 0$, then,

$$\begin{aligned}\sigma_g &= \sqrt{2}\phi\sigma_f \\ \rho &= \sqrt{(2 - \phi^2)i_s^2 + (1 - \phi^2)i_D^2}\end{aligned}$$

Differentiating these equations, one can show the following:

As σ_f increases, σ_g increases and ρ is unchanged. Therefore, b and $q(x)$, for $x < k^*$, increase. The bubble increases with the volatility of the fundamental process.

As i_s or i_D increases, σ_g is unchanged and ρ increases, since $0 < \phi < 1$. Therefore, b and $q(x)$, for $x < k^*$, increase. The bubble increases with the amount of information in the signals and the dividend flow.

As ϕ increases, σ_g increases and ρ decreases. Thus, an increase in ϕ has offsetting effects on the size of the bubble. However, numerical exercises indicate that the size of bubble always increases with ϕ .

7.3 Magnitude of the extra volatility component

As the difference of opinions x approaches the trading point, the volatility of the option value approaches is $\frac{\sqrt{2}\phi\sigma_f}{(r+\lambda)} \frac{h'(k^*)}{h'(k^*)+h'(-k^*)}$. We have:

Lemma 4 *If c is small, $\eta(k^*)$ decreases with the interest rate r and the degree of mean reversion λ , and increases with the overconfidence parameter ϕ and the fundamental volatility σ_f .*

This Lemma implies that an increase in the volatility of fundamentals has an additional effect on price volatility at trading points, through an increase in the volatility of the option component.

7.4 Price, volatility and turnover

Our model provides a link between asset prices, price volatility and share turnover. Since these are endogenous variables, their relationship will typically depend on which exogenous variable

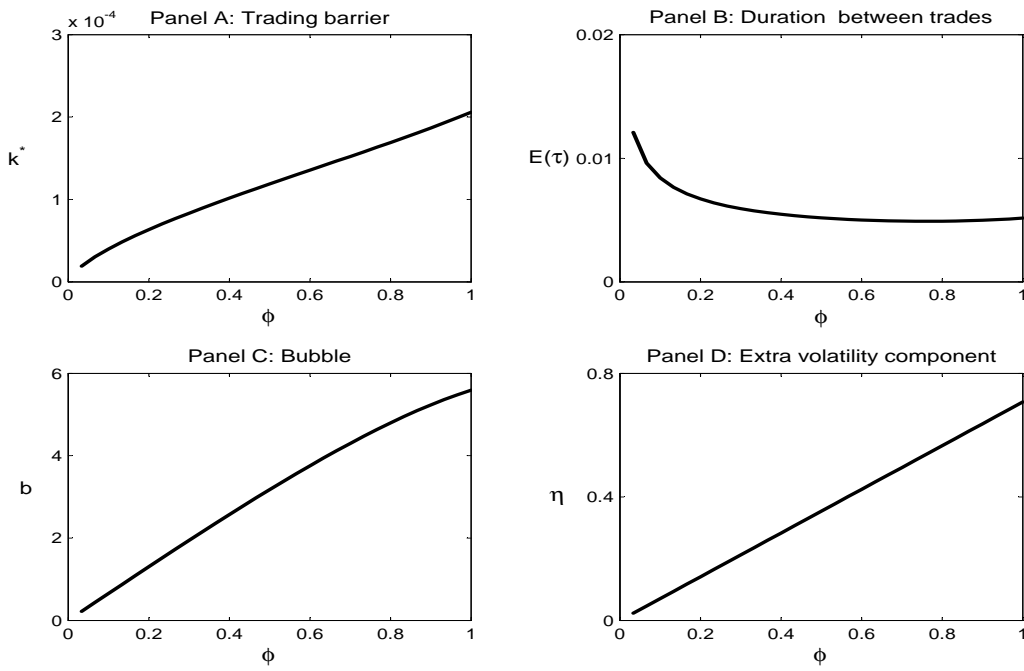


Figure 1: Effect of overconfidence level. Here, $r = 5\%$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$, and $c = 10^{-6}$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble and the extra volatility component are measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

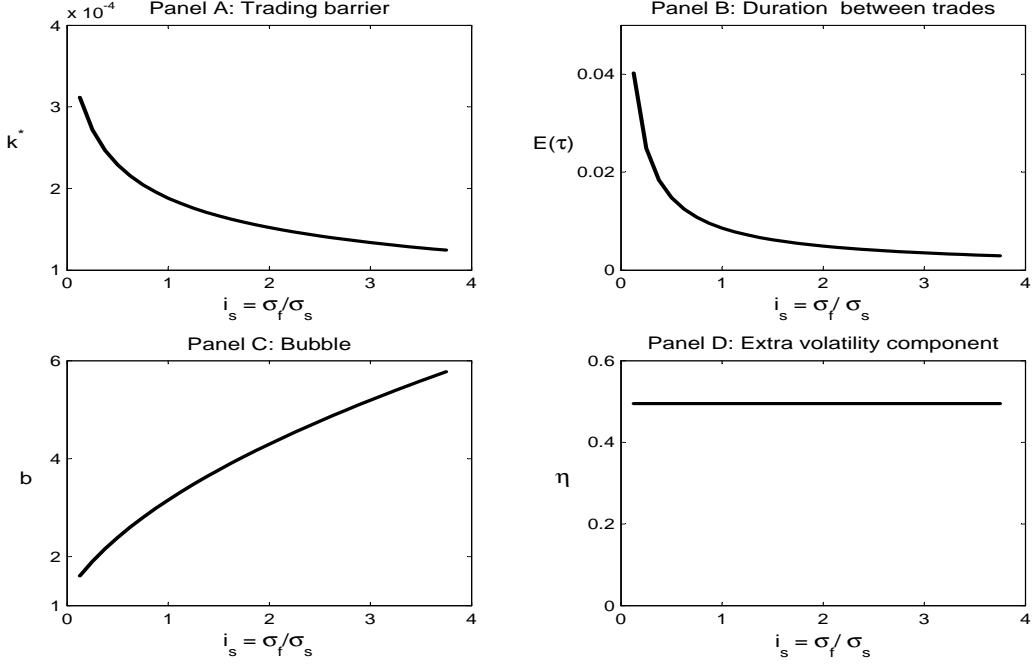


Figure 2: Effect of information in signals. Here, $r = 5\%$, $\lambda = 0$, $\theta = 0.1$, $\phi = 0.7$, $i_D = 0$, and $c = 10^{-6}$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble and the extra volatility component are measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

is shifted. In this section, we illustrate this link using numerical examples with a small trading cost.

Figure 1 shows the effect of changes in ϕ on the equilibrium when there is a small transaction cost on the trading barrier k^* , expected duration between trades, the bubble at the trading point b , and the volatility of the bubble at the trading point, $\eta(k^*)$. The expected duration between trades is measured in years. k^* , $\eta(k^*)$ and b are measured in multiples of the fundamental volatility $\frac{\sigma_f}{r+\lambda}$.¹⁹ Recall that, as ϕ increases, the volatility parameter σ_g in the difference of beliefs increases, while the mean reversion parameter ρ decreases. As a result, the resale option becomes more valuable to the asset owner, the bubble and the extra volatility component become larger and the optimal trading barrier becomes higher. The duration between trades is determined by two offsetting effects as ϕ increases. On the one hand, the trading barrier becomes higher making the duration between trades longer. On the other hand, the volatility σ_g of the difference in beliefs increases, causing the duration to be shorter. As we stated, the proof of lemma 3 shows that, when c is small, the change in the trading barrier k^* is second-order. Thus the duration between trades typically decreases, as illustrated in panel B.

Figure 2 shows the effect of changes in the volatility of the noise in signals σ_s on the

¹⁹Since the bubble is generated through an option value, it is natural to normalize it by the volatility of the underlying fundamental value, that is, the price volatility that would prevail if fundamentals were observable.

equilibrium, again with a small transaction cost. We measure the changes of σ_s in terms of the ratio $i_s = \frac{\sigma_f}{\sigma_s}$. As i_s increases, the mean reversion parameter ρ of the difference in beliefs increases, and the volatility parameter σ_g is unchanged. Intuitively, the increase in ρ causes the trading barrier and the duration between trades to drop. Nevertheless, the bubble at the trading point becomes larger due to the increase in trading frequency. The extra volatility component η is almost independent of i_s , since it is essentially determined by ϕ and σ_f as shown in equation 24.

In both cases, there is a monotonically increasing relationship between the size of bubble at the trading point and duration between trades. In addition, the extra price volatility either increases or it does not decrease. We have also verified that this qualitative relationship holds for many other parameter values. In our risk-neutral world, we may consider several assets and analyze the equilibrium in each market independently. In this way our comparative statics properties can be translated into results about correlations among equilibrium variables in the different markets. Thus our model is potentially capable of explaining the observed cross-sectional correlation between log market/book and log turnover for U.S. stocks in the period of 1996-2000 as documented by Cochrane (2002).

7.5 Crashes and fluctuations in parameters

There are several ways in which we can imagine a change in equilibrium that brings the bubble to zero. The fundamental of asset may become observable. The over-confident agents may correct their over-confidence. The fundamental volatility of the asset may disappear. For concreteness, imagine that agents in both groups believe that the asset fundamental will become observable at a date determined by a Poisson process that has a parameter θ , and is independent of the four Brownian motions described earlier in the model. Once the fundamental becomes observable, agents in each group believe that the beliefs of agents in the other group will collapse to their own. In this case, it is easy to see that the option value

$$q(g_t^o) = \sup_{\tau \geq 0} \mathbf{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) e^{-(r+\theta)\tau} \right].$$

Effectively, a higher discount rate $r + \theta$ is used for the profits from exercising the option.

Cochrane (2002) shows that there was a (time series) correlation between the NYSE index and NYSE volume through the 1929 boom and crash, and between the NASDAQ index and NASDAQ volume throughout the internet bubble. To reproduce such correlation in a non-trivial manner, we would have to generalize our model to account for parameter changes, in the same spirit as our discussion of crashes. For concreteness imagine that the overconfidence parameter ϕ or the informativeness of signals i_s can assume a finite number of values, and that the value of the parameter follows a Markov process with Poisson times that are independent of all the other relevant uncertainty. The model will then produce results that are qualitatively similar to the case in which these parameters are constant, except that the average size of the bubble at any time will depend on the current value of the parameter. In this way, we can admit fluctuations on the size of the bubble and turnover rates, although a more interesting discussion should account for reasons for the parameter fluctuations.

8 Effect of trading costs

Using the results established in subsection 6.1, we can show that increasing the trading cost c raises the trading barrier k^* , and reduces b , $q(x)$ and $\eta(x)$. In fact:

Proposition 4 *If c increases, the optimal trading barrier k^* increases. Furthermore, the bubble $q(x)$ and the extra volatility component $\eta(x)$ decrease for all $x < k^*(c)$. As $c \rightarrow 0$, $\frac{dk^*}{dc} \rightarrow \infty$, but the derivatives of b , $q(x)$, and $\eta(x)$ are always finite.*

In order to illustrate the effects of trading costs, we use the following parameter values from our previous numerical exercise, $r = 5\%$, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$. Figure 3 shows the effect of trading costs on the trading barrier k^* , expected duration between trades, the bubble at the trading point b , and the volatility of the bubble at the trading point, $\eta(k^*)$.

Panel A of Figure 3 shows the equilibrium trading barrier k^* . For comparison, we also graph the amount $c(r + \lambda)$, which corresponds to the difference in beliefs that would justify trade if the option value was ignored. The difference between these two quantities represents the “profits” that the asset owner thinks he is obtaining when he exercises the option to sell. When the trading cost is zero, the asset owner sells the asset immediately when it is profitable and these profits are infinitely small. As the trading cost increases, the optimal trading barrier increases, and the rate of increase near $c = 0$ is dramatic, since the derivative $\frac{dk^*}{dc}$ is infinite at the origin. As a result, the trading frequency is greatly reduced by the increasing trading cost as shown in Panel B.

Panels C and D show that trading costs also reduce the bubble and the extra volatility component, but as guaranteed by Proposition 4, at a limited rate even near $c = 0$. Although one could expect that the strong reduction in trading frequency should greatly reduce the bubble, this effect is partially offset by the increase in profits in each trade.²⁰ Similar intuition applies to the effect of the trading cost on the extra volatility component.

To estimate the impact of an increase on trading costs, measured as a proportion of price, as opposed to a multiple of fundamental volatility, we must take a stand concerning the relationship between price and volatility of fundamentals. For the parameter values used in our example, Panel C shows that the bubble at the trading point, for $c \sim 0$, is close to four times the fundamental volatility parameter $\frac{\sigma_f}{r + \lambda}$. The drop in prices of internet stocks from the late 90’s until today exceeds 80%. If we take this variation as a measure of the size of the bubble in the late 90’s, then the size of the fundamental volatility must have been approximately also 20% of trading prices. In this way, we can reinterpret the values in the figures as multiples of prices. The numerical results indicate that in this case a tax of 1% of prices would have caused a reduction of less than 20% to the magnitudes of both the bubble and the extra volatility component.

The effectiveness of a trading tax in reducing speculative trading has been hotly debated since James Tobin’s (1978) initial proposal for a transaction tax in the foreign currency markets. Shiller (2000, pages 225-228) provides an overview of the current status of this debate. Our model implies that for small trading costs, increases in trading costs have a much larger impact

²⁰Vayanos (1998) makes a similar point in a different context, when analyzing the effects of transaction cost on asset prices in a life-cycle model. Vayanos shows that an increase of transaction cost can reduce the trading frequency but may even increase asset prices.

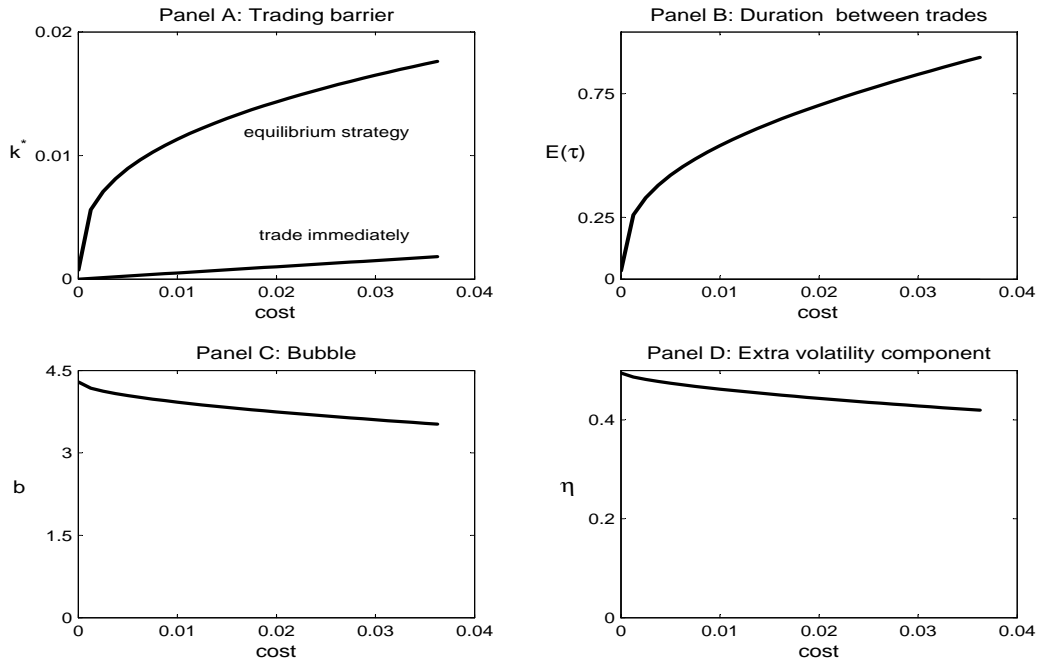


Figure 3: Effect of trading costs. Here, $r = 5\%$, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, and $i_D = 0$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble, the extra volatility component, and trading cost are measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

in trading frequency than in excess volatility or the magnitude of the price bubble. In reality, trading also occurs for other reasons, such as liquidity shocks or changes in risk bearing capacity, that are not considered in our analysis and, for this reason, the limited impact of transaction costs on volatility and price bubbles cannot serve as an endorsement of a tax on trading. Our numerical exercise can also answer a question raised by Shiller (2000) of why bubbles can exist in real estate markets, where the transaction costs are typically high.

9 Can the price of a subsidiary be larger than its parent firm?

The existence of the option value component in asset prices can potentially create violations to the law of one price and even make the price of a subsidiary exceed that of a parent company. In this section, we use a simple example to illustrate this phenomena.

There are two firms, indexed by 1 and 2. For simplicity, we assume the dividend processes of both assets follow the process in equation (1) with the same parameter σ_D , but with independent innovations, and with different fundamental variables f_1 and f_2 , respectively. The fundamental variables f_1 and f_2 are unobservable and both follow the mean-reverting process in equation (2) with the same parameters $\lambda = 0$, \bar{f} and σ_f .

There is a third firm, and the dividend flow of firm 3 is exactly the sum of the dividend flows of firms 1 and 2. In this sense, firms 1 and 2 are both subsidiaries of firm 3. In addition it is known to all participants that $(f_1)_t + (f_2)_t = f_3$, a constant. This implies that innovations to f_1 and f_2 are perfectly negatively correlated. In particular, the price of a share of firm 3 is f_3/r .

However, according to our analysis a speculative component exists in the prices of the shares of firm 1 and firm 2. Since $\hat{f}_1^C + \hat{f}_2^C = f_3$ for $C \in \{A, B\}$, when agents in group A are holding firm 1, agents in group B must be holding firm 2, and the option components in the prices of these two firms are exactly the same.

The numerical exercise in Subsection 7.4 shows that the magnitude of the option component can equal four or five times fundamental volatility. If fundamental volatility is large relative to the discounted value of fundamentals, the value of one of the subsidiaries will exceed the value of firm 3, even though all prices are nonnegative.²¹ Although highly stylized, this analysis may help clarify the episodes such as 3Com's equity carve-out of Palm and its subsequent spinoff.²² In early 2000, for a period of over two months, the total market capitalization of 3Com was significantly less than the market value of its holding in Palm, a subsidiary of 3Com. Other examples of this kind are discussed in Lamont and Thaler (2003), Mitchell, Pulvino and Stafford (2001), and Schill and Zhou (2001). Our model also predicts that trading in the subsidiary would be much higher than trading in the parent company, because of the higher fluctuation in beliefs about the value of the subsidiary. In fact, Lamont and Thaler (2003) show that the turnover rate of the subsidiaries' stocks was on average six times higher than that of the parent firms' stocks.

²¹Duffie, Garleanu, and Pedersen (2002) provide another mechanism to explain this phenomenon based on the lending fee that the asset owner can expect to collect.

²²The missing link is to demonstrate that the divergence of beliefs on the combined entity was smaller than the divergence of beliefs on the Palm spinoff.

This example also indicates that the diversification of a firm reduces the bubble component in the firm's stock price because diversification reduces the fundamental uncertainty of the firm, therefore reducing the potential disagreements among investors. This result is consistent with the diversification discount "puzzle" - the fact that the stocks of diversified firms appear to trade at a discount compared to the stocks of undiversified firms.²³

10 Conclusion and further discussions

In this paper, we provide a simple model to study bubbles and trading volume that result from speculative trading among agents with heterogeneous beliefs. Heterogeneous beliefs arise from the presence of overconfident agents. With a short-sale constraint, an asset owner has an option to sell the asset to other agents with more optimistic beliefs. Agents value this option, and consequently pay prices that exceed their own valuation of future dividends, because they believe that in the future they will find a buyer willing to pay even more. We solve the optimal exercise problem of an asset owner and derive, in an almost analytic form, many of the equilibrium variables of interest. This allows us to characterize properties of the magnitude of the bubble, trading frequency, and asset price volatility and to show that the model is consistent with the observation that in actual historical bubbles, volatility and turnover are also inordinate. Theoretical results and numerical exercises suggest that a small trading tax may be effective in reducing speculative trading, but it may not be very effective in reducing price volatility or the size of the bubble. Through a simple example, we also illustrate that the bubble can cause the price of a subsidiary to be larger than its parent firm, a violation of the law of one price.

It is natural to conjecture that the existence of a speculative component in asset prices has implications for corporate strategies. Firm managers may be able to profit by adopting strategies that boost the speculative component.

The underpricing of a firm's initial public offering (IPO) has been puzzling. Rajan and Servaes (1997), and Aggarwal, Krigman, and Womack (2001) show that higher initial returns on an IPO lead to more analysts and media coverage. Since investors may disagree about the precision of information provided by the media, the increase in this coverage could increase the option component of the stock. Therefore, IPO underpricing could be a strategy used by firm managers to boost the price of their stocks. If this mechanism is operative, underpricing is more likely to occur when managers hold a larger share of the firm. This agrees with the empirical results in Aggarwal, Krigman, and Womack (2002). According to our model, a bigger underpricing should be associated with a larger trading volume. In fact, Reese (2000) finds that the higher initial IPO returns is associated with larger trading volume for more than three years after issuance. In a similar fashion, our framework may also be useful for understanding returns and volume on name changes (adding "dotcom".)

In addition, if prices contain a large non-fundamental component, many standard views in both corporate finance and asset pricing that use stock prices as a measure of fundamental value will be substantially altered. For example, Bolton, Scheinkman, and Xiong (2002) analyze managerial contracts in such a model. The paper shows that the presence of overconfidence on

²³See Lang and Stultz (1994), Burger and Ofek (1995), and others.

the part of potential stock buyers could induce incumbent shareholders to use short-term stock compensation to motivate managerial behavior that increases short term prices at the expense of long term performance. This provides an alternative to the common view that the recent corporate scandals were caused by a lack of adequate board supervision.

Appendix: Proofs

Proof of Lemma 1: Let $\vartheta(\phi) = \lambda + \phi \frac{\sigma_f}{\sigma_s}$ and $\iota(\phi) = (1 - \phi^2)(2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2)$. Then

$$\frac{d\gamma}{d\phi} \sim \frac{1}{2} \frac{2\vartheta \frac{d\vartheta}{d\phi} + \frac{d\iota}{d\phi}}{\sqrt{(\vartheta^2 + \iota)}} - \frac{d\vartheta}{d\phi} = \left(\frac{\vartheta}{\sqrt{\vartheta^2 + \iota}} - 1 \right) \frac{d\vartheta}{d\phi} + \frac{1}{2\sqrt{\vartheta^2 + \iota}} \frac{d\iota}{d\phi} \leq 0.$$

Proof of Proposition 1: The process for g^A can be derived from equations (5) and (9):

$$dg^A = d\hat{f}^B - d\hat{f}^A = - \left[\lambda + \frac{2\gamma + \phi\sigma_s\sigma_f}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2} \right] g^A dt + \frac{\phi\sigma_f}{\sigma_s} (ds^B - ds^A).$$

Using the formula for γ , we may write the mean-reversion parameter as

$$\rho = \sqrt{\left(\lambda + \phi \frac{\sigma_f}{\sigma_s} \right)^2 + (1 - \phi^2) \sigma_f^2 \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right)}.$$

Using equations (6) and (7),

$$dg^A = -\rho g^A dt + \frac{\phi\sigma_f}{\sigma_s} (\sigma_s dW_B^A - \sigma_s dW_A^A).$$

The result follows by writing

$$W_g^A = \frac{1}{\sqrt{2}} (W_B^A - W_A^A).$$

It is easy to verify that innovations to W_g^A are orthogonal to innovations to \hat{f}^A in the mind of agents in group A.

Proof of Proposition 2: Let $v(y)$ be a solution to the differential equation

$$yv''(y) + (1/2 - y)v'(y) - \frac{r}{2\rho}v(y) = 0. \quad (\text{A1})$$

It is straightforward to verify that $u(x) = v\left(\frac{\rho}{\sigma_g^2}x^2\right)$ satisfies equation (16).

The general solution of equation (A1) is²⁴

$$v(y) = \alpha M\left(\frac{r}{2\rho}, \frac{1}{2}, y\right) + \beta U\left(\frac{r}{2\rho}, \frac{1}{2}, y\right).$$

²⁴See Abramowitz and Stegun (1964), chapter 13.

$M(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot, \cdot)$ are Kummer functions defined as

$$M(a, b, y) = 1 + \frac{ay}{b} + \frac{(a)_2 y^2}{(b)_2 2!} + \cdots + \frac{(a)_n y^n}{(b)_n n!} + \cdots$$

where $(a)_n = a(a+1)(a+2)\dots(a+n-1)$, $(a)_0 = 1$, and

$$U(a, b, y) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, y)}{\Gamma(1+a-b)\Gamma(b)} - y^{1-b} \frac{M(1+a-b, 2-b, y)}{\Gamma(a)\Gamma(2-b)} \right\}.$$

Furthermore, $M_y(a, b, y) > 0$, $\forall y > 0$, $M(a, b, y) \rightarrow +\infty$, $U(a, b, y) \rightarrow 0$, as $y \rightarrow +\infty$.

Given a solution u to equation (16) we can construct two solutions v to equation (A1), by using the values of the function for $x < 0$ and for $x > 0$. We will denote the corresponding linear combinations of M and U by $\alpha_1 M + \beta_1 U$ and $\alpha_2 M + \beta_2 U$. If these combinations are constructed from the same u their values and first derivatives must have the same limit as $x \rightarrow 0$. To guarantee that $u(x)$ is positive and increasing for $x < 0$, α_1 must be zero. Therefore,

$$u(x) = \beta_1 U \left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2 \right) \quad \text{if } x \leq 0.$$

The solution must be continuously differentiable at $x = 0$. From the definition of the two Kummer functions, we can show that

$$\begin{aligned} x \rightarrow 0-, \quad u(x) &\rightarrow \frac{\beta_1 \pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}, & u'(x) &\rightarrow \frac{\beta_1 \pi \sqrt{\rho}}{\sigma_g \Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)} \\ x \rightarrow 0+, \quad u(x) &\rightarrow \alpha_2 + \frac{\beta_2 \pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}, & u'(x) &\rightarrow -\frac{\beta_2 \pi \sqrt{\rho}}{\sigma_g \Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)} \end{aligned}$$

By matching the values and first order derivatives of $u(x)$ from the two sides of $x = 0$, we have

$$\beta_2 = -\beta_1, \quad \alpha_2 = \frac{2\beta_1 \pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}.$$

The function h is a solution to equation (16) that satisfies

$$h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)} > 0,$$

and $h(-\infty) = 0$. Equation (16) guarantees that at any critical point where $h < 0$, h has a maximum, and at any critical point where $h > 0$ it has a minimum. Hence h is strictly positive and increasing in $(-\infty, 0)$.

Proof of Lemma 2: Write $\alpha = \frac{\sigma_g^2}{2\rho} > 0$ and $\beta = \frac{r}{\rho} > 0$. $h(x)$ is a positive, and increasing in $(-\infty, 0)$, solution to $\alpha h'' - xh' - \beta h = 0$.

If $x^* \in R$ with $h(x^*) > 0$ and $h'(x^*) = 0$ then $h''(x^*) = \beta h(x^*)/\alpha > 0$. Hence h has no local maximum while it is positive and as a consequence it is always positive and has no local maxima. In particular h is monotonically increasing. Since $h' > 0$ for $x \leq 0$ and $h'' \geq 0$ for $x \geq 0$, $h'(x) > 0$ for all x . From the solution constructed in Proposition 2, $\lim_{x \rightarrow -\infty} h(x) = 0$.

Note that any solution to the differential equation is infinitely differentiable. Next, we show that h is convex. For $x > 0$, $h''(x) = xh'(x)/\alpha + \beta h(x)/\alpha > 0$. To prove that h is also convex for $x < 0$, let us assume that there exists $x^* < 0$ such that $h''(x^*) \leq 0$. Then

$$h'''(x^*) = x^*h''(x^*)/\alpha + (\beta + 1)h'(x^*)/\alpha > 0.$$

This directly implies that $h''(x) < 0$ for $x < x^*$. Then $\lim_{x \rightarrow -\infty} h'(x) = \infty$. In this situation the boundary condition $h(-\infty) = 0$ can not be satisfied. In this way, we get a contradiction.

Let $v(x) = h'(x)$. $v(x)$ is positive and increasing. v also satisfies $\alpha v''(x) - xv'(x) - (\beta + 1)v(x) = 0$. By repeating the proof that we use for h , we can show that $v(x)$ is convex and $\lim_{x \rightarrow -\infty} v(x) = 0$. In fact, one can show that any higher order derivative of $h(x)$ is positive, increasing and convex.

Proof of Theorem 1: Let $l(k) = [k - c(r + \lambda)](h'(k) + h'(-k)) - h(k) + h(-k)$. We first show that there exists a unique k^* that solves $l(k) = 0$.

If $c = 0$, $l(0) = 0$, and $l'(k) = k[h''(k) - h''(-k)] > 0$, for all $k \neq 0$. Therefore $k^* = 0$ is the only root of $l(k) = 0$.

If $c > 0$, then $l(k) < 0$, for all $k \in [0, c(r + \lambda)]$. Since h'' and h''' are increasing (Lemma 2), for all $k > c(r + \lambda)$

$$\begin{aligned} l'(k) &= [k - c(r + \lambda)][h''(k) - h''(-k)] > 0, \\ l''(k) &= h''(k) - h''(-k) + [k - c(r + \lambda)][h'''(k) - h'''(-k)] > 0. \end{aligned}$$

Therefore $l(k) = 0$ has a unique solution $k^* > c(r + \lambda)$.

Proof of Theorem 2: First we show that q satisfies equation (14). Using the notation introduced in equation (21), and equation (22), we have

$$q(-x) = \begin{cases} \frac{b}{h(-k^*)}h(-x) & \text{for } x > -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}h(x) - c & \text{for } x \leq -k^*. \end{cases}$$

We must establish that $U(x) = q(x) - \frac{x}{r+\lambda} - q(-x) + c \geq 0$. A simple calculation shows that

$$U(x) = \begin{cases} 2c & \text{for } x < -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}[h(x) - h(-x)] + c & \text{for } -k^* \leq x \leq k^* \\ 0 & \text{for } x > k^* \end{cases}$$

Thus, $U''(x) = \frac{b}{h(-k^*)}[h''(x) - h''(-x)]$, $-k^* \leq x \leq k^*$. From Lemma 2 we know for $U''(x) > 0$ for $0 < x < k^*$, and $U''(x) < 0$ for $-k^* < x < 0$. Since $U'(k^*) = 0$, $U'(x) < 0$ for $0 < x < k^*$. On the other hand, $U'(-k^*) = 0$, so $U'(x) < 0$ for $-k^* < x < 0$. Therefore $U(x)$ is monotonically decreasing for $-k^* < x < k^*$. Since $U(-k^*) = 2c > 0$ and $U(k^*) = 0$, $U(x) > 0$ for $-k^* < x < k^*$.

We now show that equation (15) holds. By construction, it holds in the region $x \leq k^*$. Therefore we only need to show that equation (15) is valid for $x \geq k^*$. In this region, $q(x) = \frac{x}{r+\lambda} + \frac{b}{h(-k^*)}h(-x) - c$, thus $q'(x) = \frac{1}{r+\lambda} - \frac{b}{h(-k^*)}h'(-x)$ and $q''(x) = \frac{b}{h(-k^*)}h''(-x)$. Hence,

$$\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - r q(x)$$

$$\begin{aligned}
&= \frac{b}{h(-k^*)} \left[\frac{1}{2} \sigma_g^2 h''(-x) + \rho x h'(-x) - r h(-x) \right] - \frac{r + \rho}{r + \lambda} x + r c \\
&= -\frac{r + \rho}{r + \lambda} x + r c \leq -(r + \rho)c + r c = -\rho c < 0
\end{aligned}$$

where the inequality comes from the fact that $x \geq k^* > (r + \lambda)c$ from Theorem 1.

Also q has an increasing derivative in $(-\infty, k^*)$ and has a derivative bounded in absolute value by $\frac{1}{r + \lambda}$ in (k^*, ∞) . Hence q' is bounded.

If τ is any stopping time, the version of Ito's lemma for twice differentiable functions with absolutely continuous first derivatives (*e.g.* Revuz and Yor (1999), Chapter VI) implies that

$$e^{-r\tau} q(g_\tau^o) = q(g_0^o) + \int_0^\tau \left[\frac{1}{2} \sigma_g^2 q''(g_s^o) - \rho g_s^o q'(g_s^o) - r q(g_s^o) \right] ds + \int_0^\tau \sigma_g q'(g_s^o) dW_s$$

Equation (15) states that the first integral is non positive, while the bound on q' guarantees that the second integral is a Martingale. Using equation (14) we obtain,

$$\mathbb{E}^o \left\{ e^{-r\tau} \left[\frac{g_\tau^o}{r + \lambda} + q(-g_\tau^o) - c \right] \right\} \leq \mathbb{E}^o [e^{-r\tau} q(g_\tau^o)] \leq q(g_0^o).$$

This shows that no policy can yield more than $q(x)$.

Now consider the stopping time $\tau = \inf\{t : g_t^o \geq k^*\}$. Such τ is finite with probability one, and g_s^o is in the continuation region for $s < \tau$. Using exactly the same reasoning as above, but recalling that in the continuation region (15) holds with equality we obtain

$$q(g^o) = \mathbb{E}^o \left\{ e^{-r\tau} \left[\frac{g_\tau^o}{r + \lambda} + q(-g_\tau^o) - c \right] \right\}.$$

Proof of Proposition 3: Since $q(x) = \frac{1}{(r + \lambda)} \frac{h(x)}{h'(k) + h'(-k)}$, the volatility of $q(g_t^o)$ is given by $\frac{1}{(r + \lambda)} \frac{h'(g_t^o)}{h'(k) + h'(-k)}$ multiplied by the volatility of g_t^o . From the Proof of proposition 1,

$$dg_t^o = -\rho g_t^o dt + \frac{\phi \sigma_f}{\sigma_s} (ds^{\bar{o}} - ds^o).$$

From equations (3) and (4) the volatility of $s^{\bar{o}} - s^o$ is $\sqrt{2}\sigma_s$ in an objective measure. Hence the volatility of g^o is $\sqrt{2}\phi\sigma_f$.

Proof of Lemma 3: When $c = 0$, the magnitude of the bubble at the trading point is

$$b_0 = \frac{\sigma_g}{2\sqrt{2}\rho(r + \lambda)} \frac{\Gamma\left(\frac{r + \theta}{2\rho}\right)}{\Gamma\left(\frac{1}{2} + \frac{r + \theta}{2\rho}\right)}.$$

It is obvious that b_0 increases with σ_g . We can directly show that b_0 increases with ρ and decreases with r and θ by plotting it.

When $c = 0$, the bubble is $q_0(x) = b_0 \frac{h(x)}{h(0)}$ where $h(x)$ is a positive and increasing solution to

$$\frac{1}{2} \sigma_g^2 h''(x) - \rho x h'(x) - (r + \theta) h(x) = 0, \quad h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r + \theta}{2\rho}\right) \Gamma\left(\frac{1}{2}\right)}.$$

Note that $q_0(x)$ is not effected by letting $h(0) = 1$.

Assume $\tilde{\sigma}_g > \sigma_g$, let $\tilde{h}(x)$ solve $\frac{1}{2}\tilde{\sigma}_g^2\tilde{h}''(x) - \rho x\tilde{h}'(x) - (r + \theta)\tilde{h}(x) = 0$, $\tilde{h}(-\infty) = 0$, $\tilde{h}(0) = \frac{\pi}{\Gamma(\frac{1}{2} + \frac{r+\theta}{2\rho})\Gamma(\frac{1}{2})}$. We show that $\tilde{h}(x) > h(x)$ for all $x < 0$. Let $f(x) = \tilde{h}(x) - h(x)$. Then from Lemma 2, $f(-\infty) = f(0) = 0$. Suppose f has a local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, from the equations satisfied by $\tilde{h}(x)$ and $h(x)$, we have

$$\frac{1}{2}[\tilde{\sigma}_g^2\tilde{h}''(x) - \sigma_g^2h''(x)] - \rho x[\tilde{h}'(x) - h'(x)] - (r + \theta)[\tilde{h}(x) - h(x)] = 0.$$

This implies that $\tilde{\sigma}_g^2\tilde{h}''(x^*) < \sigma_g^2h''(x^*)$. Since $\tilde{\sigma}_g^2 > \sigma_g^2$, this in turn implies that $\tilde{h}''(x^*) < h''(x^*)$. This is equivalent to $f''(x^*) < 0$, which contradicts with x^* being a local minimum. Therefore, $f(x)$ cannot have a local minimum with its value less than zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x \in (-\infty, 0)$. Therefore, $\tilde{h}(x) > h(x)$ for all $x < 0$. This implies that the bubble $q_0(x)$ increases with σ_g for all $x < 0$.

Assume $\bar{\rho} > \rho$, and let $\bar{h}(x)$ solve $\frac{1}{2}\bar{\sigma}_g^2\bar{h}''(x) - \bar{\rho}x\bar{h}'(x) - (r + \theta)\bar{h}(x) = 0$, $\bar{h}(-\infty) = 0$ and $\bar{h}(0) = \frac{\pi}{\Gamma(\frac{1}{2} + \frac{r+\theta}{2\bar{\rho}})\Gamma(\frac{1}{2})}$. We can show that $\bar{h}(x) < h(x)$ for all $x < 0$. Again let $f(x) = \bar{h}(x) - h(x)$. We first establish that $f(x)$ has no local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, taking differences, we obtain: $\frac{1}{2}\bar{\sigma}_g^2[\bar{h}''(x) - h''(x)] - \rho x[\bar{h}'(x) - h'(x)] - (r + \theta)[\bar{h}(x) - h(x)] = (\bar{\rho} - \rho)x\bar{h}'(x)$. This last equation implies that $\bar{h}'(x^*) < 0$, what contradicts the fact that $\bar{h}(x)$ is an increasing function. Therefore, $f(x)$ cannot have a local minimum below zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x < 0$. This directly implies that $\bar{h}(x) > h(x)$ for all $x < 0$, and $q_0(x)$ increases with ρ for all $x < 0$. Similarly, we can prove that $q_0(x)$ decreases with r and θ for all $x < 0$.

One can extend the comparative statics we established for $c = 0$ for the case of c small. Let $\zeta \in \{\sigma_g, \rho, \theta\}$. From equation (21) it follows that if $\frac{\partial k^*(\zeta, c)}{\partial \zeta} = o(k^*)$ then the comparative statics of b with respect to ζ is preserved for small c .

Using the definition of function h in equation (17), we write h as $h(x, \zeta)$. From equation (20),

$$\frac{\partial k^*(\zeta, c)}{\partial \zeta} = \frac{\left(\frac{\partial h(k^*, \zeta)}{\partial \zeta} - \frac{\partial h(-k^*, \zeta)}{\partial \zeta}\right) - [k^* - c(r + \lambda)]\left(\frac{\partial^2 h(k^*, \zeta)}{\partial x \partial \zeta} + \frac{\partial^2 h(-k^*, \zeta)}{\partial x \partial \zeta}\right)}{[k^* - c(r + \lambda)]\left[\frac{\partial^2 h(k^*, \zeta)}{\partial x^2} - \frac{\partial^2 h(-k^*, \zeta)}{\partial x^2}\right]}. \quad (\text{A2})$$

As $c \rightarrow 0$, $k^* \rightarrow 0$ and the numerator and denominator go to zero. To find the limit behavior, we use the explicit form of h given in the proof of Proposition 2, and write

$$h(x, \zeta) = C_0 + C_1x + C_2x^2 + C_3x^3 + o(x^4)$$

with

$$C_0 = \frac{\pi}{\Gamma\left(\frac{r}{2\rho} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}, \quad C_1 = \frac{\pi\sqrt{\rho}}{\Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)\sigma_g},$$

$$C_2 = \frac{\pi r}{4\Gamma\left(\frac{r}{2\rho} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\sigma_g^2}, \quad C_3 = \frac{\pi\sqrt{\rho}(r + \rho)}{3\Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)\sigma_g^3}.$$

We use equation (20) to replace the term $k^* - c(r + \lambda)$ in the right hand side of equation (A2) by $\frac{h(k^*, \zeta) - h(-k^*, \zeta)}{\frac{\partial h(k^*, \zeta)}{\partial x} + \frac{\partial h(-k^*, \zeta)}{\partial x}}$. Taking limits as $k^* \rightarrow 0$, we obtain

$$\frac{\partial k^*(\zeta, c)}{\partial \zeta} \sim o(k^*),$$

$\zeta \in \{\sigma_g, \rho, \theta\}$. A small variation establishes the same result for $\frac{\partial k^*(r, c)}{\partial r}$. Hence, for small c , b increases with σ_g and ρ , and decrease with r and θ . In addition we can show that $q(x)$ for $x < k^*$ increases with σ_g and ρ , and decreases with r and θ .

Proof of Lemma 4: This is analogous to the proof of Lemma 3.

Proof of Proposition 4: Let $l(k, c) = [k - c(r + \lambda)](h'(k) + h'(-k)) - h(k) + h(-k)$. $k^*(c)$ is the root of $l(k, c) = 0$. If $c > 0$

$$\frac{dk^*}{dc} = \frac{(r + \lambda)}{[k^* - c(r + \lambda)]} \frac{[h'(k^*) + h'(-k^*)]}{[h''(k^*) - h''(-k^*)]} > 0.$$

Hence $k^*(c)$ is differentiable in $(0, \infty)$. Now suppose $c_n \rightarrow 0$. The sequence $k^*(c_n)$ is bounded and every limit point \bar{k}^* must solve $l(\bar{k}^*, 0) = 0$. Hence, as we argued in the proof in the proof of Theorem 1, $\bar{k}^* = 0$ and the function $k^*(c)$ is continuous. Hence $\frac{dk^*}{dc}$ approaches ∞ as $c \rightarrow 0$. The claims on b and $q(x)$ follow from equations equations (21) and (22), and Lemma 2. The derivative of $\eta(x)$ with respect to c is

$$\begin{aligned} \frac{d\eta(x)}{dc} &= \frac{\sqrt{2}\phi\sigma_f h'(x)(h''(k^*) - h''(-k^*))}{(r + \lambda) (h'(k^*) + h'(-k^*))^2} \left(-\frac{dk^*}{dc}\right) \\ &= -\frac{\sqrt{2}\phi\sigma_f h'(x)}{[k^* - c(r + \lambda)](h'(k^*) + h'(-k^*))} < 0. \end{aligned}$$

Therefore, $\eta(x)$ decreases with c . However, note that $\frac{d\eta(x)}{dc}$ is finite as $c \rightarrow 0$ although $\frac{dk^*}{dc} \rightarrow \infty$ as $c \rightarrow 0$.

References:

- Abramowitz, Milton and Irene Stegun (1964), *Handbook of Mathematical Functions*, Dover Publications, New York.
- Aggarwal, Rajesh, Laurie Krigman, and Kent Womack (2002), Strategic IPO underpricing, information momentum, and lockup expiration selling, *J. Finan. Econ.* 66, 105-137.
- Allen, Franklin and Gary Gorton (1993), Churning bubbles, *Rev. Econ. Stud.* 60, 813-836.
- Allen, Franklin, Stephen Morris, and Andrew Postlewaite (1993), Finite bubbles with short sale constraints and asymmetric information, *J. Econ. Theory* 61, 206-229.
- Alpert, Marc and Howard Raiffa (1982), A progress report on the training of probability assessors, in Daniel Kahneman, Paul Slovic, and Amos Tversky, ed.: *Judgement under Uncertainty: Heuristics and Biases*, Cambridge University Press, Cambridge.
- Barber, Brad and Terrance Odean (2001), The internet and the investor, *J. Econ. Perspectives* 15, 41-54.
- Bernardo, Antonio, and Ivo Welch (2001), On the evolution of overconfidence and entrepreneurs, *Journal of Economics and Management Strategy* 10, 301-330.

Biais, Bruno and Peter Bossaerts (1998), Asset prices and trading volume in a beauty contest, *Rev. Econ. Stud.* 65, 307-340.

Blanchard, Olivier and Mark Watson (1982), Bubbles, rational expectations and financial markets, in P. Wachtel (ed.) *Crisis in the Economic and Financial Structure: Bubbles, Bursts, and Shocks*, Lexington Press, Lexington, MA.

Bolton, Patrick, José Scheinkman, and Wei Xiong (2002), Executive compensation and short-termist behavior in speculative markets, Working paper, Princeton University.

Burger, Philip and Eli Ofek (1995), Diversification's effect on firm value, *J. Finan. Econ.* 37, 39-65.

Camerer, C. (1995), Individual decision making, in John Kagel and Alvin Roth (ed.) *The Handbook of Experimental Economics*, Princeton University Press, Princeton, NJ.

Carr, Peter and Robert Jarrow (1990), The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value, *Rev. Finan. Stud.* 3, 469-492.

Chen, Joseph, Harrison Hong and Jeremy Stein (2002), Breadth of ownership and stock returns, *J. Finan. Econ.* 66, 171-205.

Cochrane, John (2002), Stocks as money: convenience yield and the tech-stock bubble, NBER Working Paper 8987.

Conley, Timothy, Lars Peter Hansen, Erzo Luttmer, and Jose Scheinkman (1997), Short-term interest rates as subordinated diffusions, *Rev. Finan. Stud.* 10, 525-577.

Daniel, Kent, David Hirshleifer, and A. Subrahmanyam (1998), Investor psychology and security market under- and overreaction, *J. Finance* 53, 1839-1885.

D'Avolio, Gene (2002), The market for borrowing stock, *J. Finan. Econ.* 66, 271-306.

Duffie, Darrell, Nicolae Garleanu and Lasse Pedersen (2001), Securities lending, shorting, and pricing, *J. Finan. Econ.* 66, 307-339.

Geczy, Christopher, David Musto and Adam Reed (2002), Stocks are special too: an analysis of the equity lending market, *J. Finan. Econ.* 66, 241-269.

Gromb, Dennis and Dimitri Vayanos (2002), Equilibrium and welfare in markets with financially constrained arbitrageurs, *J. Finan. Econ.* 66, 361-407.

Harris, Milton and Artur Raviv (1993), Differences of opinion make a horse race, *Rev. Finan. Stud.* 6, 473-506.

Harrison, Michael and David Kreps (1978), Speculative investor behavior in a stock market with heterogeneous expectations, *Q.J.E.* 92, 323-336.

Hirshleifer, David (2001), Investor psychology and asset pricing, *J. Finance* 56, 1533-1597.

Horst, Ulrich (2001), Financial price fluctuations in a stock market model with many interacting agents, *ISE Finance Award Series Volume 2*, 1-28, ISE, Istanbul.

Jones, Charles and Owen Lamont (2002), Short-sale constraints and stock returns, *J. Finan. Econ.* 66, 207-239.

Kandel, Eugene and Neil Pearson (1995), Differential interpretation of public signals and trade in speculative markets, *J.P.E.* 103, 831-872.

Karlin, Samuel and Howard Taylor (1981), *A Second Course in Stochastic Processes*, Academic Press, New York.

Kobila, T. O. (1993), A class of solvable stochastic investment problems involving singular controls, *Stochastics Stochastics Rep.* 43, 29-63.

Kyle, Albert and Tao Lin (2002), Continuous speculation with overconfident competitors, Working paper, Duke University.

Kyle, Albert and Albert Wang (1997), Speculation duopoly with agreement to disagree: Can overconfidence survive the market test? *J. Finance* 52, 2073-2090.

Kyle, Albert and Wei Xiong (2001), Contagion as a wealth effect, *J. Finance* 56, 1401-1440.

Lamont, Owen and Richard Thaler (2003), Can the market add and subtract? Mispricing in tech stock carve-outs, *J.P.E.* 111, 227-268.

Lang, Larry and Rene Stultz (1994), Tobin's q, corporate diversification and firm performance, *J.P.E.* 102, 1248-1280.

Lichtenstein, S., B. Fischhoff, and L. Phillips (1982), Calibration of probabilities: The state of the art to 1980, in Daniel Kahneman, Paul Slovic, and Amos Tversky, ed.: *Judgement under Uncertainty: Heuristics and Biases*, Cambridge University Press, Cambridge.

Lord, C. G., L. Ross, and M. R. Lepper (1979), Biased assimilation and attitude polarization: The effects of prior theories on subsequently considered evidence, *Journal of Personality and Social Psychology* 37, 2098-2109.

Miller, Edward (1977), Risk, uncertainty and divergence of opinion, *J. Finance* 32, 1151-1168.

Mitchell, Mark, Todd Pulvino, and Erik Stafford (2002), Limited arbitrage in equity markets, *J. Finance*, 551-584.

Morris, Stephen (1996), Speculative investor behavior and learning, *Q.J.E.* 110, 1111-1133.

Odean, Terrance (1998), Volume, volatility, price, and profit when all traders are above average, *J. Finance* 53, 1887-1934.

Ofek, Eli and Matthew Richardson (2001), Dotcom mania: The rise and fall of internet stock prices, *J. Finance*, forthcoming.

Rajan, Raghuram and Henri Servaes (1997), Analyst following of initial public offerings, *J. Finance* 52, 507-529.

Reese, William (2000), IPO underpricing, trading volume, and investor interest, Working paper, Tulane University.

Revuz, Daniel and Marc Yor (1999), *Continuous Martingales and Brownian Motion*, Springer, New York.

Rogers, L. C. G. and David Williams (1987), *Diffusions, Markov Processes, and Martingales, Volume 2: Ito Calculus*, John Wiley & Sons, New York.

Santos, Manuel, and Michael Woodford (1997), Rational asset pricing bubbles, *Econometrica* 65, 19-57.

Scheinkman, Jose and Thaleia Zariphopoulou (2001), Optimal environmental management in the presence of irreversibilities, *J. Econ. Theory* 96, 180-207.

Schill, Michael and Chungsheng Zhou (2001), Pricing an emerging industry: Evidence from internet subsidiary carve-outs, Working paper, *Financial Management* 30, 5-33.

Shiller, Robert (2000), *Irrational Exuberance*, Princeton University Press, NJ.

Shleifer, Andrei and Robert Vishny (1997), Limits of arbitrage, *J. Finance* 52, 35-55.

Tobin, James (1978), A proposal for international monetary reform, *Eastern Econ. J.* 4, 153-159.

Vayanos, Dimitri (1998), Transaction costs and asset prices: a dynamic equilibrium model, *Rev. Finan. Stud.* 11, 1-58.

Xiong, Wei (2001), Convergence trading with wealth effect, *J. Finan. Econ.* 62, 247-292.