Inference in Instrumental Variable Regression Analysis with Heterogeneous Treatment Effects

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Main goal of the paper

Valid inference (standard errors) in linear IV model when

1. Treatment effects (TE) heterogeneous (each pair of instrument values defines potentially different LATE)
   ▶ Need to define estimand (what are we estimating with more than one instrument?)

2. Number of instruments $K$ and covariates $L$ potentially large
   ▶ Many instrument asymptotics under which $K$ and $L$ may both increase with sample size
   ▶ Instruments may be weak: Concentration parameter $\hat{r}_n \to \infty$ as $n \to \infty$ (rules out Staiger and Stock (1997) asymptotics), maybe more slowly than $n$
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Empirical motivation

▶ Most empirical papers careful to interpret IV exercise as estimating TEs for compliers
  ▶ However, reported standard errors typically only valid under constant TE: too small under LATE model
▶ In addition, in many papers $K$ and $L$ both large, usually when both instruments and covariates are indicators:
  ▶ Angrist and Krueger (1991) interact quarter and state of birth
  ▶ Judge indicators as instruments: in Aizer and Doyle (2015) judges assigned randomly to juvenile offenders in each neighborhood; in Dobbie and Song (2015) to bankruptcy cases in each bankruptcy office
  ▶ Silver (2016): physician’s peer group in each work shift random conditional on physician fixed effects
  ▶ Autor and Houseman (2010): Random assignment of social workers to cases
In the LATE model, no single $\beta$ satisfies moment condition

$$E[(Y_i - X_i\beta - W_i'\delta)Z_i] = 0$$

what are we estimating?

Unconditional inference: estimand $\beta_U = \text{plim of estimator under standard asymptotics}$

- For TSLS (and jackknife estimators) $\beta_U = \text{weighted average of LATEs}$ (Imbens and Angrist, 1994)
- For LIML no causal interpretation of $\beta_U$ in general (Kolesár, 2013)
- Under standard asymptotics, can use delta method and normality of reduced form coefficients to derive standard errors (appendix in Imbens and Angrist (1994))
- Under many-instrument asymptotics, no results available
Conditional inference

Conditional inference: estimand $\beta_C = \text{what we'd estimate if reduced form errors set to 0}$

- For tsls and some jackknife estimators: $\beta_C$ also weighted average of LATEs, but can only show (so far) weights positive when covariates indicator variables
  - Heuristically, weights reflect strength of instruments based on their variability in sample (rather than population)
- Standard errors are smaller: s.e. for $\beta_U$ needs to also reflect variability of conditional estimand under i.i.d. sampling of instruments and covariates
  - Similar to distinction between conditional vs unconditional best linear predictor in misspecified linear regression (Abadie, Imbens, and Zheng, 2014)
Main results: Many instruments + LATEs

- Focus on jackknife estimators due to problems with LIML interpretation

- Consistency:
  - Bias of jive1, original jackknife IV estimator (Phillips and Hale, 1977; Angrist, Imbens, and Krueger, 1999) of order $L/\hat{r}_n$
  - Bias of iJIVE1, modification by Ackerberg and Devereux (2009), of order $L/\hat{r}_n \cdot K/n$ if design is “balanced”

- Asymptotic normality:
  - “many instruments term” needs to be added to (un-) conditional variance formula, similar form to constant TE case
Main results: Standard errors

- No consistently estimable “structural error”
  - Harder to estimate s.e.
  - Need to jackknife s.e. formulas
- Conditional inference: need \( \frac{(K + L)}{n} \to 0 \) + “balanced design” + bias negligible (for \( \text{iJIVE1}, \frac{KL}{n\hat{r}_n^{1/2}} \to 0 \))
  - First two conditions similar to those for consistency of EHW s.e. in linear regression with many covariates (Cattaneo, Jansson, and Newey, 2016b)
- Unconditional case: need \( \frac{(K + L) \log(K + L)}{n^{1/2}} \to 0 \)
  - Work in progress on relaxing rate condition
(When) should we run IV with many instruments?

► Our paper provides inference for particular weighted average of LATEs
  ▶ Weighting may not be most policy relevant
► Alternative: use MTE framework of Heckman and Vytlacil (1999, 2005) and do inference on more policy-relevant estimands
  ▶ In general need CI for each individual LATE to be informative, or rely on parametric assumptions (e.g. Brinch, Mogstad, and Wiswall, 2015; Cattaneo, Jansson, and Ma, 2016a)
► In "judges" designs, each LATE weakly identified in general: less ambitious goal of inference for some TE may be necessary
► IV regressions still ubiquitous: useful to report valid s.e.
Some References

- Nagar '59; Phillips '89; Nelson and Starz '90; Angrist and Krueger '91; Bound, Jaeger, and Baker '95; Staiger and Stock '97; Dufour '97; Andrews, Moreira, and Stock '07; Horowitz '11; Chen and Christensen '15; Maasoumi and Phillips '82.

- Phillips and Hale '77; Angrist, Imbens, and Krueger '99; Blomquist and Dahlberg '99; Davidson and MacKinnon '04,'07; Ackerberg and Devereux '09.

- Bekker '94; Chao, Swanson, Hausman, Newey, and Woutersen '12; Andrews and Stock '05; Newey and Windmeijer '09.

- Andrews '99; Gautier and Tsybakov '11; Belloni, Chernozhukov, Fernandez-val, and Hansen '13.

- Belloni, Chernozhukov, Chetverikov, and Kato '15; Chen and Christensen '15; Tropp '12.

- Imbens and Angrist '94; Angrist, Grady, and Imbens '00; Heckman and Vytlacil '01, '05; Carneiro, Heckman and Vytlacil '11; Kolesár '13; Evdokimov and Lee '13; Kolesár, Chetty, Friedman, Glaeser, and Imbens '14.
**General setup**

- Interested in effect of treatment $X_i$ on outcome $Y_i$
- Instruments $Z_i \in \mathbb{R}^K$, valid conditional on controls/exogenous variables $W_i \in \mathbb{R}^L$
- Reduced form:

  $$Y_i = Z_i' \pi_Y + W_i' \psi_Y + \zeta_i, \quad E[\zeta_i \mid W_i, Z_i] = 0,$$

  $$X_i = Z_i' \pi + W_i' \psi + \eta_i, \quad E[\eta_i \mid W_i, Z_i] = 0.$$

- Under constant TE, $\pi_{Y,k} / \pi_k = “the causal effect”$
For matrix $A$, let $H_A = A(A'A)^{-1}A'$ denote projection (hat) matrix.

Define $\hat{Y} = (I_n - H_W)Y$, and define $\hat{X}, \hat{Z}, \hat{R}, \hat{R}_Y$ similarly, e.g. $\hat{R} = (I_n - H_W)Z\pi + W\psi = \hat{Z}\pi$.

Can think $\hat{Y}_i$ as estimator of $\tilde{Y}_i = Y_i - E[Y_i \mid W_i]$; Define $\tilde{X}, \tilde{Z}, \tilde{R}, \tilde{R}_Y$ similarly ($\tilde{R} = \tilde{Z}\pi$).

Denote predictor from the first-stage regression by $\hat{R}_{TSL} = H_Z\hat{X} = \hat{R} + H_Z\eta$, so that TSL is given by

$$\hat{\beta}_{TSL} = \frac{\sum_{i=1}^n \hat{R}_{TSL,i} \hat{Y}_i}{\sum_{i=1}^n \hat{R}_{TSL,i} \hat{X}_i}$$

Define the (sample) concentration parameter $\hat{r}_n = \sum_{i=1}^n \hat{R}_i^2$.
\[ \hat{\beta}_{\text{TLS}} \equiv \frac{\hat{X}' \hat{Z} (\hat{Z}' \hat{Z})^{-1} \hat{Z}' \hat{Y}}{\hat{X}' \hat{Z} (\hat{Z}' \hat{Z})^{-1} \hat{Z}' \hat{X}} = \frac{\sum_{i,j=1}^{n} \hat{X}_i \hat{Z}_i (\hat{Z}' \hat{Z})^{-1} \hat{Z}_j \hat{Y}_j}{\sum_{i,j=1}^{n} \hat{X}_i \hat{Z}_i (\hat{Z}' \hat{Z})^{-1} \hat{Z}_j \hat{X}_j} \]

\[ \hat{\beta}_{\text{IIJIVE2}} \equiv \frac{\sum_{i \neq j} \hat{X}_i \hat{Z}_i (\hat{Z}' \hat{Z})^{-1} \hat{Z}_j \hat{Y}_j}{\sum_{i \neq j} \hat{X}_i \hat{Z}_i (\hat{Z}' \hat{Z})^{-1} \hat{Z}_j \hat{X}_j} \]

\[ \hat{\beta}_{\text{IIJIVE1}} \equiv \frac{\sum_{i \neq j} \hat{X}_i \hat{Z}_i \left( \hat{Z}' \hat{Z} - \hat{Z}_j \hat{Z}_j \right)^{-1} \hat{Z}_j \hat{Y}_j}{\sum_{i \neq j} \hat{X}_i \hat{Z}_i \left( \hat{Z}' \hat{Z} - \hat{Z}_j \hat{Z}_j \right)^{-1} \hat{Z}_j \hat{X}_j} \]

... and more. Phillips and Hale, ’77; Angrist, Imbens, and Krueger, ’99; Blomquist and Dahlberg ’99; Ackerberg and Devereux ’09, Kolesár ’13.
Estimands

- Define $\beta_U$ as plim of estimator under standard asymptotics, and $\beta_C$ as what we would estimate if reduced-form errors were 0.

- For TSLS, JIVE1, IJIVE1:

$$
\beta_U = \frac{E[\tilde{X}_i\tilde{Z}_i]E[\tilde{Z}_i\tilde{Y}_i]^{-1}E[\tilde{Z}_i'\tilde{Y}_i]}{E[\tilde{X}_i\tilde{Z}_i]E[\tilde{Z}_i\tilde{Z}_i']^{-1}E[\tilde{Z}_i'\tilde{X}_i]} = \frac{\pi' E[\tilde{Z}_i\tilde{Z}_i'] \pi_Y}{\pi' E[\tilde{Z}_i\tilde{Z}_i'] \pi} = \frac{E[\tilde{R}_i\tilde{R}_Y]}{E[\tilde{R}_i^2]}
$$

$$
\beta_C = \frac{E_n[\hat{R}_i\hat{Z}_i]E_n[\hat{Z}_i\hat{Z}_i']^{-1}E_n[\hat{Z}_i'\hat{R}_Y,i]}{E_n[\hat{X}_i\hat{Z}_i]E_n[\hat{Z}_i\hat{Z}_i']^{-1}E_n[\hat{Z}_i'\hat{R}_i]} = \frac{\pi' E_n[\hat{Z}_i\hat{Z}_i'] \pi_Y}{\pi' E_n[\hat{Z}_i\hat{Z}_i'] \pi} = \frac{E_n[\hat{R}_i\hat{R}_Y]}{E_n[\hat{R}_i^2]}
$$

- $\beta_C$ replaces $E$ with $E_n$ (sample average) and tildes with hats

- For IJIVE2, $\beta_C$ is slightly different

- $(K + L) \log(K + L) = o(n^{1/2})$

- $\beta_U, \beta_C$ may change with $n$ under many-instrument asymptotics since distribution of data allowed to change with $n$. 
Example: interactions of a single instrument with group dummies

- Suppose $S_i \in \{1, \ldots, K\}$ with $P(S_i = k) = \lambda_k$.
- The $K$ instruments are constructed from a scalar instrument $Z_i$:  
  $Z_i = (Z_{i1}, \ldots, Z_{iK})'$, $Z_{ik} = Z_i 1 \{S_i = k\}$
- $\tilde{Z}_{ik} = (Z_i - E[Z_i|S_i = k]) 1 \{S_i = k\}$
- $\hat{Z}_{ik} = (Z_i - E_n[Z_i|S_i = k]) 1 \{S_i = k\}$
- $\tilde{R}_i = \pi' \tilde{Z}_i$, $\hat{R}_i = \pi' \hat{Z}_i$
- the estimands of TSLS, JIVE1, IJIVE1:

\[
\beta_U = \frac{\sum_{k=1}^{K} \beta_k \omega_k}{\sum_{k=1}^{K} \omega_k}, \quad \omega_k = V_k[\tilde{R}] \text{, } V_k[\hat{R}] = V[\tilde{R}_i|S_i = k] \\
\beta_C = \frac{\sum_{k=1}^{K} \beta_k \hat{\omega}_k}{\sum_{k=1}^{K} \hat{\omega}_k} \text{, where}
\]

\[
\hat{\omega}_{k}^{TSLS} = \frac{n_k}{n} \hat{V}_k[\tilde{R}], \quad \hat{\omega}_{k}^{IJIVE2} = \frac{n_k}{n} \hat{V}_k[\tilde{R}] \left(1 - \frac{1}{n_k} \hat{\kappa}_k[\tilde{R}]\right)
\]
Estimand with many instruments and covariates

Suppose \((K + L) \log(K + L) = o \left( n^{3/4} \right)\) and regularity conditions hold.

Let \(\Sigma_{WW} = E \left[ W_i W_i' \right]\). Then, for TSLS, JIVE1, IJIVE1:

\[
\beta_U \equiv \frac{E \left[ \tilde{R}_Y \tilde{R}_i \left( 1 - \frac{1}{n} W_i' \Sigma^{-1}_{WW} W_i \right) \right]}{E \left[ \tilde{R}_i^2 \left( 1 - \frac{1}{n} W_i' \Sigma^{-1}_{WW} W_i \right) \right]}
\]

So, in the previous example

\[
\omega_k = \left( \lambda_k - \frac{1}{n} \right) V_k \left[ \tilde{R} \right].
\]

[[[ TBC... ]]]
Desirable Property of the Estimand

For any estimand $\overline{\beta}$ we would like to have

$$\overline{\beta} \in \left[ \min(\overline{\beta}), \max(\overline{\beta}) \right] \quad (\ast)$$

For the unconditional estimand, Imbens and Angrist '94 show that the of IV-type estimators satisfy $(\ast)$ under weak conditions, but Kolesář '13 shows that minimum distance estimators in general do not.

Assumption 1 (LATE Model)

(Independence) $\{Y(x), X(z)\}_{x,z} \perp Z \mid W$

(Monotonicity) For all $z, z'$, $P(X(z) \geq X(z') \mid W) = 1$ a.s., or $P(X(z) \leq X(z') \mid W) = 1$ a.s.

(Linearity) $E[Z \mid W]$ is linear in $W$ and $E[(\zeta, \eta) \mid Z, W] = 0$.

Lemma 1

Suppose that the covariates contain an intercept.

- Then the estimands of TSLS, JIVE1, IJIVE1, JIVE2, IJIVE2 are weighted averages of LATEs.
- If the covariates are group dummies, then the weights of TSLS, JIVE1, IJIVE1, and IJIVE2 are non-negative.
<table>
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<th>K</th>
<th>$\hat{r}_n$</th>
<th>$\beta_{sd}$</th>
<th>OLS</th>
<th>TSLS</th>
<th>LIML</th>
<th>JIVE1</th>
<th>IJIVE1</th>
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<td>-0.007</td>
<td><strong>0.173</strong></td>
<td>0.015</td>
<td>0.004</td>
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<td>0.242</td>
<td>0.000</td>
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</tr>
</tbody>
</table>

Table: median bias; $n = 2000$, $\rho = 0.50$, Normal $X$, $Y$
Estimators

\[ \hat{\beta}_G = \frac{Y'GX}{X'GX} = \frac{\sum_{i,j} Y_i G_{ij} X_j}{\sum_{i,j} X_i G_{ij} X_j}, \]

\[ G_{TSLS} = H_Z, \]

\[ G_{IJIVE1} = (I - H_W)(I - \text{diag}(H_Z))^{-1}(H_Z - \text{diag}(H_Z))(I - H_W), \]

\[ G_{IJIVE2} = H_Z - (I - H_W) \text{diag}(H_Z)(I - H_W), \]

\[ G_{JIVE1} = (I - H_W)(I - \text{diag}(H_Q))^{-1}(H_Q - \text{diag}(H_Q)). \]
Asymptotic Theory

Regularity conditions

▶ For both conditional and unconditional inference need
   ▶ reduced form errors are independent, and \( \max_i E[\eta_i^4 + \zeta_i^4 | Z_i, W_i] \)
     bounded.
   ▶ Other regularity conditions (stated in paper) that hold in running example if \( \pi, \pi_Y \) are bounded

▶ For unconditional inference, assume also that sampling is i.i.d., and eigenvalues of \( E[\tilde{Z}_i \tilde{Z}_i'] \) and \( E[W_i W_i'] \) are bounded above and below (balanced design condition)

▶ Some results go through under weaker conditions
Many weak instrument asymptotics of Bekker (1994) and Chao and Swanson (2005)

Two important changes:

1. Allow TE to be heterogeneous
2. Allow $L \rightarrow \infty$

Distributional results generalize those in Newey and Windmeijer (2009) and Chao, Swanson, Hausman, Newey, and Woutersen (2012)

Regularity conditions in paper, will focus on rate conditions and substantive restrictions
Theorem 2 (Conditional asymptotic distribution)

Suppose regularity conditions hold, \( \max_i (H_{\hat{Z},W})_{ii} / n \to 0 \) and \( L \max_i (H_{\hat{Z}})_{ii} / \hat{r}_n^{1/2} \to 0 \) a.s. Then for IJIVE1 and IJIVE2, 
\[ \mathcal{V}_C^{-1/2}(\hat{\beta} - \beta_C) \xrightarrow{d} \mathcal{N}(0, 1), \]
where \( \mathcal{V}_C = (\mathcal{V}_{TEXT} + \mathcal{V}_{LATE} + \mathcal{V}_{MW}) / \hat{r}_n^2, \)

\[
\mathcal{V}_{TEXT} = \sum_{i=1}^{n} \hat{R}_i^2 \sigma_{\nu,i}^2 \\
\mathcal{V}_{LATE} = \sum_{i=1}^{n} (\hat{R}_{\Delta,i}^2 \sigma_{\eta,i}^2 + 2\hat{R}_i \hat{R}_{\Delta,i} \sigma_{\nu,i} \sigma_{\eta,i}) \\
\mathcal{V}_{MW} = \sum_{i \neq j} (H_{\hat{Z}})_{ij}^2 \left( \sigma_{\eta,j}^2 \sigma_{\nu,i} + \sigma_{\nu,i} \sigma_{\nu,j} \right)
\]

- \( \mathcal{V}_{TEXT} \) is usual “textbook” variance of TSLS, \( \mathcal{V}_{LATE} \) reflects variability of LATEs, and \( \mathcal{V}_{MW} \) is many-instruments component

- Under homogeneous TE, \( \nu_i \) is “structural error” and \( \hat{R}_{\Delta,i} = 0 \)
Theorem 3 (Unconditional asymptotic distribution)

Suppose regularity conditions hold, and that \( K + L = o(\sqrt{n/\log(n)}) \).

Then for \( \text{IJIVE}_1 \) and \( \text{IJIVE}_2 \)

\[
V_U^{-1/2}(\hat{\beta} - \beta_U) \xrightarrow{d} \mathcal{N}(0, 1),
\]

where

\[
V_U = V_C + V_E / \hat{r}_n^2
\]

\[
V_E = \sum_{i=1}^{n} \hat{R}_{\Delta,i}^2 \hat{R}_i^2
\]

and

\[
\left( \frac{V_E}{\hat{r}_n} \right)^{-1} \frac{E[\hat{R}_i^2 \hat{R}_{\Delta,i}^2]}{E[\hat{R}_i^2]} \rightarrow p 1.
\]

- Extra term \( V_E \) accounts for variability of conditional estimand around the unconditional estimand.
- Asymptotic normality continues to hold with

\[
(K + L) \log(K + L) = o(n^{3/4})
\]
Inference

- Naive plug-in estimators of $\mathcal{V}_C, \mathcal{V}_U$ are *upward* biased under many instrument asymptotics.
- Recall tsls estimator $\hat{R}_{TSLS,i} = \hat{Z}(\hat{Z}'\hat{Z})^{-1}\hat{Z}'\hat{X} = \sum_j (H\hat{Z})_{ij}\hat{X}_j$ of $\hat{R}_i$.
- Define jackknife estimator of $\hat{R}_i\hat{R}_{\Delta,i}$:
  \[
  \hat{J}_i(\hat{X}, \hat{Y} - \hat{X}\hat{\beta}) = \sum_{j: j \neq i} (H\hat{Z})_{ij}\hat{X}_j \sum_{k: k \notin \{i,j\}} (\hat{Y}_i - \hat{X}_i\hat{\beta})(H\hat{Z})_{ik}
  \]

$\hat{J}_i(\hat{X}, \hat{X})$ and $\hat{J}_i(\hat{Y} - \hat{X}\hat{\beta}, \hat{Y} - \hat{X}\hat{\beta})$ estimate $\hat{R}_i^2$ and $\hat{R}_{\Delta,i}^2$.
- Estimate $\nu_i$ and $\eta_i$ by $\hat{\zeta}_i - \eta_i\hat{\beta}$ and $\hat{\eta}$, where $\hat{\nu}_i$ and $\hat{\zeta}_i$ denote OLS residuals from reduced form regressions.
- Plug these estimators into asymptotic variance formulas $\mathcal{V}_C$ and $\mathcal{V}_U$. 

Leads to estimators:

\[ \hat{\nu}_{TEXT} = \sum_{i=1}^{n} \hat{j}_i(\hat{X}, \hat{X}) \hat{\nu}_i^2 \]

\[ \hat{\nu}_{LATE} = \sum_{i=1}^{n} (\hat{j}_i(\hat{Y} - \hat{X}\hat{\beta}, \hat{Y} - \hat{X}\hat{\beta}) \hat{\eta}_i^2 + 2\hat{j}_i(\hat{Y} - \hat{X}\hat{\beta}, \hat{X}) \hat{\nu}_i \hat{\eta}_i) \]

\[ \hat{\nu}_{MW} = \sum_{i \neq j} (H_{\hat{Z}})_{ij}^2 \left( \hat{\nu}_j \hat{\eta}_i^2 + \hat{\nu}_j \hat{\eta}_i \hat{\nu}_i \hat{\eta}_i \right) \]

\[ \hat{\nu}_{E} = \sum_{i} \hat{R}_{IJIVE1,i}^2 \hat{j}_i(\hat{Y} - \hat{X}\hat{\beta}, \hat{Y} - \hat{X}\hat{\beta}) \]

\[ \hat{\nu}_{C} = \frac{\hat{\nu}_{TEXT} + \hat{\nu}_{LATE} + \hat{\nu}_{MW}}{\sum_{i=1}^{n} \hat{X}_i \hat{R}_{IJIVE1,i}} \]

\[ \hat{\nu}_{U} = \hat{\nu}_{C} + \frac{\hat{\nu}_{E}}{\sum_{i=1}^{n} \hat{X}_i \hat{R}_{IJIVE1,i}} \]

For estimators other than IJIVE1, replace \( \hat{R}_{IJIVE1} \) by corresponding first-stage predictor
Theorem 4 (Standard errors)

Under same conditions leading to asymptotic normality of \( \hat{\nu}_C^{-1/2}(\hat{\beta} - \beta_C) \) and \( \hat{\nu}_U^{-1/2}(\hat{\beta} - \beta_U) \),

\[
\hat{\nu}_C^{-1/2}(\hat{\beta} - \beta_C) \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{and} \\
\hat{\nu}_U^{-1/2}(\hat{\beta} - \beta_U) \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{respectively.}
\]
Takeaways

- When TE are heterogeneous, usual standard errors *not* valid for inference
  - Conditional variance: additional term reflecting variability of LATEs
  - Unconditional variance: conditional variance + term reflecting variability of the conditional estimand
- Need to take a stance on object of interest (conditional/unconditional estimand)
- Precision trade-offs (e.g. Donald and Newey, 2001) different. Changing set of instruments
  - Changes estimand
  - Changes s.e. not only through effect $\hat{r}_n$ vs $K$ tradeoff, but also through effect on variability of LATEs