Inside Money and Liquidity

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Abstract

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1 Introduction

Inside money can be defined very broadly as any privately-issued long-term paper that is held by a number of agents in succession. Whenever paper circulates as a means of short-term saving (liquidity), it can properly be considered as money, or a medium of exchange, because agents hold it not for its maturity value but for its exchange value.[1] In this article, we construct a model of an economy in which agents borrow by issuing long-term paper, and use the model to ask: When and why is circulation of the paper essential to the smooth running of the economy? If there is a shortage of liquidity, what are the symptoms? How does the economy respond?

At the heart of our model lies the idea that money lubricates trade when there is a lack of double coincidence of wants over dated goods. The model is of a discrete-time production economy without uncertainty in which infinitely-lived agents each undertake a sequence of projects. Every three days, an agent starts a project that completes two days later. Given an even distribution of start times, there are in effect three equal populations, indexed by whether an agent starts a project on days 1, 4, 7, ..., or days 2, 5, 8, ..., or days 3, 6, 9, ... In such an environment, there is an apparent lack of double coincidence of wants. On the one hand, an agent starting a new project raises additional funds by selling claims to output from the project: he or she borrows by issuing "long-term" paper that matures in two days time. On the other hand, an agent completing a project wishes to lend only short-term, overnight until his or her next project starts. Goods can be stored one-for-one overnight, but this is inefficient because agents discount the utility of future consumption.

There is, however, a triangle of wants, similar to the classic example of Wicksell (1901, 1934). Let us for the moment focus on the flows of investment and output, taking these to be large relative to consumption, during a typical three-day time interval, from day t through to day t+2. Consider three typical agents, A, B and C, who start projects on days t, t+1 and t+2, respectively. Suppose

[1] Wicksell (1901, 1934) defined money as "an object which is taken in exchange, not for its own account, i.e. not to be consumed by the receiver or to be employed in technical production, but to be exchanged for something else within a longer or shorter period of time".

today is day t. A wants goods to help start his new project, and issues claims against day t+2 output. B is completing the project he started two days ago, and so has goods available to lend; but tomorrow he will need goods for his next project. C has an ongoing project that he started yesterday, so when it completes tomorrow he will have goods to lend; but on day t+2 he will need goods for his next project. Together, they have a triangle of wants: A has day t+2 goods but wants day t goods; B has day t goods but wants day t+1 goods; C has day t+1 goods but wants day t+2 goods. The difference from Wicksell's triangle is that ours is over *dated* goods rather than physically distinct commodities.

The simple market remedy is for A, B and C to trade claims to future output in an efficient Arrow-Debreu market at the initial date.[2] Thereafter, there is no need for markets to reopen or for claims to change hands.

Suppose, however, that there is *limited commitment.* Specifically, we assume that, for moral hazard reasons, an agent can mortgage no more than a fraction $\theta < 1$ of the output from his current project, and he cannot credibly pledge any of the output from future projects. In other words, all paper in the market is secured against a newly-begun or an ongoing project and matures when the project completes; and there is a limit θ on the fraction of the project's output that can be used as security. Given these constraints on the supply of paper, an Arrow-Debreu market at the initial date does not work: necessarily, markets must reopen as new projects start and new paper is issued. More interestingly, there may be a role for paper to circulate as inside money, i.e., to be resold the day after it has been issued, the day before it matures.

To understand why paper may need to circulate, consider again our trio A, B and C, today. Suppose that when C started his current project yesterday he mortgaged the maximum fraction θ of the output that he could, so today there is nothing more that he can credibly pledge. Without his participation in today's market, there are seemingly no gains from trade between A and B:

[2] Wicksell's example hinges on the absence of central marketplace where there could be a three-way exchange of commodities. The idea is that if, on account of physical trading frictions, all transactions have to be bilateral, then money can act as a lubricant to trade.

A wants to sell long-term paper maturing on day t+2, whereas B wants to buy paper maturing tomorrow. However, provided A's paper can be resold tomorrow, the three agents can do business with each other, as follows. Today, A sells his long-term paper to B in exchange for goods. Then, tomorrow, B resells A's paper to C in exchange for goods. Finally, on day t+2, A redeems his paper by giving goods to C. The key to this sequence of trades is that A's paper must be able to circulate, to act as inside money. B buys A's paper today not for its maturity value but for its exchange value. The paper provides liquidity, the means of short-term saving – for B to save tonight, and for C to save tomorrow night.

Under this trading arrangement, although A borrows from B, he actually repays C. In effect, A makes a multilateral commitment today, a promise to pay the bearer of his paper on day t+2, whoever that turns out to be. This illustrates a very general idea: the power of one agent (today, agent A) to make a multilateral commitment can substitute for the limited commitment of another agent (today, agent C). Thanks to the negotiability of A's paper, the fact that today C cannot credibly pledge any more of tomorrow's output poses no problem, because tomorrow he pays for "second-hand" paper in a spot transaction. We believe this idea underpins why the circulation of inside money is essential to the smooth running of an economy. In our symmetric model agents take turns to provide the economy with inside money, but in a different model there may be specialists, bankers.

Now, there are many reasons why paper may have *limited resaleability*, so that multilateral commitment may not be feasible. In the market today, A may be free to sell his paper to any of many potential creditors, promising to deliver goods on day t+2 (up to a fraction θ of the output from his project). But tomorrow, an initial creditor may not be able to resell the paper on to a new creditor, at least not quickly, at a fair price. For example, the initial creditor may privately learn something about the project, and given that he has an incentive to try to resell his claims against what he knows are the "lemons", adverse selection may cause the market for second-hand paper to break down. Alternatively, it may take time (longer than one day) for a new creditor to verify the authenticity of A's paper; or the initial creditor may get locked in with A, so that a new creditor is less able than the initial creditor to enforce A's promise on day t+2. Whatever the reason – adverse selection, delay in verification, or special leverage – the conclusion is broadly the same: after A's paper has been issued today, it may not be fully resaleable tomorrow.[3] In the limit, where the paper cannot be resold at all, A in effect makes only a bilateral commitment, to deliver on day t+2 to the agent who buys his paper today: the paper is nonnegotiable, the buyer has to hold on to it until it matures.

We believe that limited resaleability may be just as important as limited commitment, although only the latter has received much attention in the formal macroeconomics literature. History suggests that there are significant hurdles to overcome before private paper can freely circulate, as is evident, for example, from the slow development of negotiable bank notes. The classic work of Usher (1943) and de Roover (1948) traces modern banking back to around the thirteenth century, when trading partners used a cumbersome deposit transfer procedure that required both parties to go to their banker's premises and witness as he transferred a sum from one ledger entry to another. The demand for greater liquidity, notably in the Low Countries, eventually led, by the sixteenth century, to the circulation of financial instruments such as the bill of exchange (van der Wee, 1963). Bills were endorsed as they changed hands, but since the signatory was put under an obligation to pay if the issuer (and other signatories) defaulted, this procedure fell short of simple resaleability. It was not until the late seventeenth century in England that bank notes circulated freely.[4] A more modern example of the creation of liquidity is provided by the development of mortgage-backed securities. Traditionally, mortgages were not resold, because of their idiosyncratic nature – each has peculiarities relating to the value of the security, the risk of default, the likelihood of early redemption, etc. – plus the fact that much of this information is private to the initial lender.

[3] Akerlof (1970) showed that adverse selection can cause market failure. Costly verification was first modelled by Townsend (1979), although we have in mind that verification takes time rather than uses up resources directly. The idea that an initial creditor may have more leverage over a debtor than new creditors has been recently exploited in a series of papers by Diamond and Rajan (2001, 2002).

[4] At that time there were a number of abortive attempts to create resaleable paper, e.g. paper issued by land banks against future rents; see Horsefield (1960).

That is, an individual mortgage may hard to resell at a fair price because potential purchasers face adverse selection. Since the 1980's, however, the adverse selection problem has been partly overcome. Mortgages are now bundled together, and sliced up in clever ways, so that the resultant securities have better-defined return streams and can be more easily retraded. Case histories such as the development of negotiable bank notes and mortgage-backed securities suggest to us that, first, resaleability is by no means an automatic attribute of private paper; second, there are costs to making paper resaleable; and third, the economy-wide benefits of additional liquidity may outweigh those costs.

In the context of our model economy, we explore these issues by taking a line roughly parallel to that of mortgages. We suppose a project is made up of a number of parts, some of which will fail to produce any output. No-one can distinguish these on the day of investment, although by the following day an initial creditor (who has access to the project from the start) privately learns which are failing parts, and so has an incentive to try to resell his claims against them. The adverse selection problems is severe enough that the market for second-hand paper breaks down completely. The only way for private paper to be resaleable is if, at the time of starting the project, the investing agent bundles the parts together in such a way that an initial creditor cannot separate out the lemons later on. In effect, by bundling his project the investing agent is able to make a multilateral commitment, a promise to pay a fair return to any subsequent holder of his paper, instead of making merely a bilateral commitment to the initial creditor. Paper secured against a bundled project is thus liquid: it can circulate as inside money. Paper secured against an unbundled project is illiquid in the sense that, even though it can be freely sold on the day of issue, it cannot be resold the next day.

Bundling costs additional resources. The investing agent is willing to expend these resources if and only if resaleable (liquid) paper commands a liquidity premium over nonresaleable (illiquid) paper. We interpret bundling as a rudimentary theory of financial intermediation. In practice, a deposit in a bank can be thought of as a claim against that bank's bundled portfolio of loans to their borrowers. The counterpart to the costs of banking in our model are the cost of bundling, albeit that we have combined the roles of banker and borrower: there are no distinct intermediaries. Also in our model the three populations of agent take turns to act as borrower-cum-banker, when every three periods they invest and supply inside money to the economy.

From a social perspective, the additional resource costs of bundling are deadweight, insofar as they yield no extra output. There is no direct social payoff from banking. In a first-best equilibrium, there can be no liquidity premium, no bundling, no inside money, no multilateral commitment.

At first glance it would seem that without any bundling, so that all paper is nonresaleable and all intertemporal deals have to be bilateral, the economy must collapse to autarky because there are apparently no double coincidence of wants in dated goods. But the market is ingenious at finding gains from trade, by making use of the long horizon. As we shall explain, an agent can get around the inflexibility of nonresaleable paper by holding a long-term "overlapping" savings portfolio that meets not just his immediate investment needs but also his future investment needs.[5] Long-term saving creates a double coincidence of wants between lenders and borrowers. The downside is that these overlapping portfolios demand a lot of paper, much more than if paper were resaleable. The economy can achieve first-best, but only if the supply of paper is correspondingly large. That is, there has to be a high degree of bilateral commitment, to compensate for the absence of multilateral commitment: θ must lie above some high threshold $\theta^* < 1$.

If θ lies below this critical value θ^*, too much pressure is put on the paper market. There is a liquidity shortage, stemming from the scarcity of commitment power. The economy responds in a variety of ways. Commitment is priced high. Newly-issued illiquid paper sells for more than the corresponding two-period discount factor, i.e., it has a lower return than in first-best. And the return on liquid paper is even lower than that on illiquid paper, the difference reflecting the compensation savers must be offered for the inconvenience of having to hold illiquid paper over two nights. The liquidity premium induces investing agents to supply both kinds of paper to the market.

[5] The long horizon is crucial. A three-period economy would collapse to autarky if there were no inside money: cf. Section 2 of Kiyotaki and Moore (2002).

This somewhat alleviates the need for agents to hold paper-intensive overlapping portfolios, but does not restore the economy to full efficiency.

Given the distorted returns on saving, agents no longer smooth their consumption. An agent consumes most on the days his projects complete, when his funds are abundant; and he consumes least on the days his projects start, when he needs funds for investment. On balance, too few resources are channelled from savers to investors, so that aggregate investment and output are below their efficient levels. One might say that the economy runs too slowly. The more severe is the liquidity shortage, the more pronounced are these symptoms.

For low enough θ, agents create their own liquidity. They may use storage, even though it earns zero (net) return, because output stored overnight from a completed project is liquid, ready for the start of the next project. Instead of storage, agents may use outside (fiat) money, which also earns zero return.[6] Socially, outside money has the advantage over storage that goods are not left idle.

Collecting together our findings, we can answer the question posed at the outset. The circulation of private paper as inside money is essential to the smooth running of our economy if and only if there is a shortage of commitment $\theta < \theta^*$. Notice how the two constraints, limited commitment and limited resaleability, feed into each other. On the one hand, with enough bilateral commitment ($\theta \geq \theta^*$), multilateral commitment isn't useful to the economy: just as in Arrow-Debreu, paper doesn't need to circulate. On the other hand, if there is a scarcity of commitment power ($\theta < \theta^*$), multilateral commitment is more valuable to the economy than bilateral commitment: inside money is supplied at a cost, in response to the shortage of liquidity.

The case where θ is low enough that outside money circulates alongside inside money, both earning zero return, might be considered the benchmark case to describe modern economies. It is important to recognize that the use of outside money is part of the economy's response to a liquidity shortage. No special role for outside money is imposed. Indeed, as we witness improvements in the collective monitoring of debtors, and new contractual arrangements and legal structures that

[6] We consider only steady states where the stock of outside money is fixed, and the price is constant.

shift power to creditors in the event of default, we should expect θ to rise. Our model predicts that eventually non-interest-bearing outside money will cease to be used, driven out by liquid private paper that earns higher returns.[7] Our model also suggests that whenever outside money is being used, it is less socially costly to supply than inside money because the deadweight costs of bundling – in effect, the cost of intermediation – are avoided. Of course, any welfare conclusion suggesting that government ought to be the sole supplier of liquidity must be tempered by recognizing the crucial implicit assumption, that in a world where commitment power is scarce the government is better at committing itself than are private agents. Such an assumption may not be warranted.

To end out analysis, we use the model to make predictions about long-run trends in velocities of different monetary aggregates, where we define velocity as the ratio of the value of gross output to the value of the outstanding stock of money. Suppose, as seems reasonable, that there is secular rise in θ. Then, comparing steady states, we find that for reasonable parameters the curve of long-run velocity is U-shaped. Moreover, the broader the monetary aggregate, the more delayed is the bottom of the curve. The same conclusions hold if, instead, we suppose that there is a secular fall in the costs of financial intermediation. These predictions are broadly consistent with the evidence, as presented for example in Bordo and Jonung (2004).

Section 2 lays out the model. In Section 3 we see under what conditions the first-best can be attained. When these conditions are not met, the economy suffers from a liquidity shortage. Section 4 examines the symptoms of the liquidity shortage and how the economy responds. In Section 5 we study the effects of secular trends in the model's deep parameters. Section 6 contains final remarks. All proofs are relegated to appendices.

Related literature

Before starting, we should relate our modelling approach to the literature on inside money. As we have argued, we place all the emphasis on just two constraints, limited commitment and limited

[7] There are reasons to think that outside money will not disappear altogether. For example, outside money may be a complement to inside money, not a substitute, insofar as private paper is typically a promise to pay in units of outside money. Also, cash leaves no electronic trail, and so may continue to be useful to people who want to conceal their activities. But such issues are beyond the scope of this article.

resaleability. Otherwise, our framework is perfectly competitive: there are no physical trading frictions. The modern theoretical treatments of inside money place physical trading frictions centre stage. There are two main strands to this literature, both involving spatial separation. The first, started by the beautiful papers of Townsend and Wallace (1982, 1987), that built on Townsend (1980), has agents travelling overnight between islands. During the day, a competitive market opens on each island, but agents cannot travel. Suppose that today agents A and B are together on an island, and A wishes to borrow from B by issuing paper that matures in two days' time. Unfortunately, A cannot promise to repay B if they are going to be on different islands on that day. In effect, there is limited commitment, arising from spatial separation (rather than from moral hazard, as in our model). But suppose that B is travelling tonight to another island where he will meet some third agent, C, who is in turn travelling tomorrow night to the same island that A will be on in two days' time. Then there is scope for A's paper to circulate: B buys it today, and resells it to C tomorrow. To the extent that such a connecting chain of agents may not exist, there is limited resaleability, again arising from spatial separation. This kind of framework has been used by a number of authors.[8] Related, the pioneering work by Freeman (1996a,b) (see also Green (1997, 2002)) on the payments system makes use of a model with spatial separation.

The other main strand in the recent literature on inside money adopts a matching framework. This, too, is based on spatial frictions, but goes further than Townsend-Wallace by assuming that agents are matched in pairs, so that the terms of trade are not determined competitively (at least, not through direct competition). A standard matching framework has atomless, anonymous agents: no pair meets more than once, and no-one's history is publicly known. In such a context, there is no scope for intertemporal contracting, either between a pair of agents or among a larger set, which rules out any discussion of inside money. The innovation introduced by Cavalcanti and Wallace (1999a,b) (see also Cavalcanti, Erosa and Temzelides (1999)) is to have a subgroup of non-anonymous agents, bankers, whose histories are publicly known. Bankers issue paper that is

[8] See, for example, the paper by Azariades, Bullard and Smith (2000), which embeds spatial separation into an overlapping generations framework (agents live for three periods) where inside and outside money coexist.

redeemed by other bankers. (Given an atomless set of bankers and non-bankers, there are no chains to connect agents more than once, so bankers never redeem their own paper.) In subgame perfect equilibrium, no banker dare renege, for fear that he loses his "reputation": any short-term gain from cheating is outweighed by a collectively-imposed punishment that no-one is ever willing to trade with him again.[9]

In our view, there is considerable merit in avoiding the need to model physical trading frictions, important though these may be. The model stays closer to a standard macroeconomic framework, and is therefore simpler (though simplicity is often in the eye of the beholder). By abstracting from physical trading frictions, we are able to focus on what we consider is the key to understanding the circulation of private paper: namely, the power of one agent to make a multilateral commitment substitutes for the limited commitment of another agent. And we can directly explore the separate and joint roles of the two constraints, limited commitment and limited resaleability.

2 Model

Consider a deterministic, discrete-time economy with a single homogeneous good, corn, which can be stored one-for-one. There is a continuum of infinitely-lived agents, with population size 3. At the start of period t, the utility of an agent is

$$\sum_{s=0}^{\infty} \beta^s \log c_{t+s} \tag{1}$$

where c_{t+s} denotes his or her corn consumption in period t+s, and the discount factor β lies strictly between 0 and 1. To kick-start the economy, we assume that everyone is endowed with some corn, but only at the initial period t=1.

Each agent has a technology by which corn can be invested to produce corn two periods later. An investment project is composed of a large number of parts that can be physically separated from

[9] Williamson (1999) also uses a matching environment to study the circulation of private bank notes. In his model, as in ours, there may be adverse selection.

each other. A fixed fraction of the parts will successfully produce output but the rest, a fraction α, will fail, and it is impossible to tell which at the time of investment. Overall, to produce y_{t+2} corn in period t+2 requires an investment of $G(y_{t+2})$ corn in period t, where, per unit of population, the cost function is given by

$$x = G(y) = \gamma\,(1-\lambda) \cdot y^{\frac{1}{1-\lambda}} \tag{2}$$

with $\gamma > 0$ and $0 < \lambda < 1$. The technology has increasing marginal cost because there is a fixed factor, the producer's human capital. Because output is proportional to $x^{1-\lambda}$ we can think of λ as the share of human capital. He is fully occupied throughout production, investing in period t, growing in period t+1, harvesting in period t+2; and he is unable to operate overlapping projects. So if he starts a project in period t, he cannot start a new one until period t+3. The agent has the choice to delay the start of the new project until period t+4 or later, but provided the share of human capital λ is not too small, each project generates enough profit that he wouldn't want to postpone investment; specifically, we assume[10]

$$\lambda \geq \min\left[\frac{1}{6}, \frac{4}{3}(1-\beta)\right]. \tag{Assumption 1}$$

The population is evenly divided into three groups, according to start times: one group, of size 1, invests in periods 1, 4, 7, ...; another, also of size 1, in periods 2, 5, 8, ...; the third in periods 3, 6, 9, ...

Although at the time of investment no-one can tell whether a particular part of a project is one of the fraction α that will fail to produce any harvest, by the growing period this information does become known – but only privately by those agents who have had access from the start, "insiders". Once an insider learns which parts of a project are failing, he has an incentive to sell claims against those parts to an uninformed outsider. (Later we shall be saying more about the nature of a claim,

[10] We show in Appendix D that Assumption 1 is sufficient to ensure that agents never want to postpone investment. Incidentally, none of our conclusions depends on Assumption 1. We could drop the assumption and instead simply assume that agents must start a new project every three periods because their human capital would depreciate if left idle.

and who are the insiders who have access.) In the growing period, then, a potential buyer of a claim faces the problem of adverse selection. However, we suppose this problem can be circumvented by means of additional investment beforehand. In period t an investing agent can choose to expend extra resources to bundle parts of his project together so that they cannot be individually separated in period t+1. The act of bundling at the time of investment is publicly observed, and serves as a commitment device to stop an insider from separating failures at the time of growing and selling claims against them. A bundled set of parts is guaranteed to comprise a fair mix of successes and failures, in the proportion $1-\alpha$ to α.

The cost of bundling is that investment outlays are multiplied by $1/\phi$, where $0<\phi\leq 1$. That is, to produce y_{t+2} corn in period t+2 of which $z_{t+2}\leq y_{t+2}$ is bundled requires a total investment of

$$G\left(y_{t+2}\right)+\frac{1-\phi}{\phi}G\left(z_{t+2}\right) \tag{3}$$

corn in period t.[11] The parameter ϕ reflects the ease with which output can be bundled: in the limit $\phi=1$, there are no costs of bundling. For $\phi<1$, the additional investment costs of bundling are a deadweight loss insofar as no more output is produced.

In our economy there are limitations on intertemporal contracting. We assume that, for moral hazard reasons, an agent can credibly pledge no more than a fraction θ of the output from his current investment project, and he cannot mortgage in advance any of the output from future investment. We have in mind that the agent's human capital is indispensable to obtain the full output from his investment, and he cannot write a contract that commits him to work (i.e. his human capital is inalienable). Indeed, he is free to abscond, and start a new life with a fresh identity.[12] However, a creditor buying his paper can be given keys to the current project, by way

[11] One way to think of this: The choice of which parts of output to bundle is made sequentially within an agent's investment period. Since unit cost rises ($G'(y_{t+2})$ is increasing in y_{t+2}) and since the additional cost of bundling is proportional to unit cost, it is cheapest for the agent to bundle the first z_{t+2} units in the sequence.

[12] Anonymity rules out collective arrangements in which, for example, people are disciplined not to renege on promises for fear of being subsequently shut out of the market by everyone else (as in Thomas and Worrall (1988), Kehoe and Levine (1993), Kocherlakota (1996)). On the importance of anonymity for monetary theory, see Kocherlakota (1998) and Kocherlakota and Wallace (1998).

of security. Each part of the project has its own key, and, without the key, no-one can gain access. Since the debtor has project-specific human capital and the creditor has the keys, they are both needed to extract the full output – which they end up dividing in the ratio $1-\theta : \theta$, irrespective of what contracts have been written.[13] In this environment, possession of capital (human or physical) is all that matters. Apart from the keys to his current project, the debtor has no other tangible security to offer potential creditors; his future projects do not yet have keys. On its own, his paper affords no security. In what follows, we use "paper" as a shorthand for paper secured by keys to a newly-begun or ongoing project.

As we have indicated, a fraction α of the parts of a project started in any period t will fail to produce harvest in period t+2, but no-one can distinguish them at the time of investment. By the growing period, period t+1, the failing parts are observed, both by the investing agent and by the creditor who has had access (held keys) to those parts from the start, the insiders. This information is private to them. As a result, in the absence of bundling, there is scope for adverse selection. In particular, the creditor has an incentive to resell claims to those parts that he knows will fail.[14] Indeed, if α is not too low – specifically if

$$\alpha > \frac{1-\beta^3}{1+\beta^3}, \qquad \text{(Assumption 2)}$$

– then the period t+1 market for unbundled second-hand paper breaks down completely.[15] By the same logic, if the investing agent didn't mortgage the maximum fraction θ of his project when it started in period t (i.e. if he kept the keys to some parts), and assuming he did not bundle, then he is unable to raise more funds in period t+1 (offering these keys as security), because by then he

[13] One way to rationalize this is to suppose that if the debtor were to refuse to work (or if he were to abscond, leaving the project behind) then the creditor would be capable of extracting only a fraction θ of the output. Without the debtor's skill, the remaining fraction, $1-\theta$ would be lost. In the bargain at the time of harvest, the debtor captures all the surplus from his specific human capital, but no more. For details of this kind of model, see Hart and Moore (1994).

[14] The possibility arises of using a Maskin mechanism (more specifically a Moore-Repullo mechansim) to elicit the common knowledge of investor and inital creditor, but renegotiation and/or collusion would destroy such a mechanism.

[15] This assertion is proved in Appendix C.

privately knows which of those parts will fail and no-one is willing to buy his paper.

Bundling is the only mechanism available to avoid adverse selection in our environment. Because parts of the project that were bundled in period t cannot be individually separated in period t+1, claims against those parts can be resold at fair price, as they comprise a representative mix of successes and failures.

In short, the limitations on intertemporal contracting imply that agents face two kinds of constraint in the paper market. First, a *borrowing constraint*: an agent can sell his own paper only if it is backed by his current project and issued at the time of investment, credibly promising no more than a fraction θ of the output. Second, a *resaleability constraint*: if an agent buys someone else's newly-issued paper he may not be able to resell it in the next period; that depends on whether the paper is secured against bundled parts of the project. These two constraints are at the core of our model. We want to know how they affect competitive equilibrium, individually and in tandem.

Let q_t denote the issue price of two-period paper that is secured by keys to unbundled parts of a project started in period t – the price in terms of period t corn of a claim to period t+2 corn that cannot be resold in period t+1. Call this *illiquid* paper. We must emphasize that this paper can be freely sold when newly-issued, at the time of investment when everyone is ignorant about which parts of the project will fail. It only becomes illiquid in the following period, the growing period, when there is asymmetric information between sellers (insiders) and potential buyers (outsiders).

In contrast, paper that is backed by bundled parts of a project can be resold in a second-hand market – it is *liquid* – because there is no scope for adverse selection. As we discussed in the Introduction, when liquid paper is resold, it acts as inside money. Bundling can be thought of as a proxy for financial intermediation, although in our model it is the borrowers (agents at the time of investing) who "intermediate" by bundling; we have no outside agents acting as intermediaries.

Let p_t denote the price in terms of period t corn of a claim to period t+1 corn; i.e. the second-hand price of liquid paper issued in period t-1. Without uncertainty, newly-issued and second-hand liquid paper are perfect substitutes as means of short-term saving: between periods t and t+1, both must yield a return of $1/p_t$.[16] Thus, given perfect foresight, the price of liquid paper newly-issued in period t is $p_t p_{t+1}$.

To complete our description of the model, we assume that the economy has a fixed stock of intrinsically useless, perfectly durable and divisible seashells that can circulate as outside money.

The nonresaleability of illiquid paper means that we must be somewhat careful in writing down agents' flow-of-funds constraints. Let n_t denote an agent's holding of newly-issued illiquid paper at the end of period t, that he purchased during the period and that he must hold on to until it matures two periods later in period t+2. And let m_t denote his short-term saving from period t to period t+1 (measured in terms of period t+1 corn), which could be in form of inside money (liquid paper), or, if he is willing to hold them, in the form of outside money (shells) or storage. Both n_t and m_t must be nonnegative.

Consider the flow-of-funds constraint of an agent investing in period t:

$$\begin{aligned} & G(y_{t+2}) + \frac{1-\phi}{\phi} G\left(z_{t+2}\right) + c_t + p_t m_t + q_t n_t \\ = \quad & p_t p_{t+1} \theta z_{t+2} + q_t \theta \left(y_{t+2} - z_{t+2}\right) + m_{t-1} + n_{t-2}. \end{aligned} \tag{4}$$

The left-hand side (LHS) of (4) comprises expenditures: basic investment to produce y_{t+2} corn in period t+2, plus additional investment to bundle z_{t+2} of this output, plus consumption, plus short-term saving, plus purchase of newly-issued illiquid paper. The right-hand side (RHS) comprises receipts. The first two terms are borrowing: the revenue from the sale of liquid paper, priced at $p_t p_{t+1}$, issued against the mortgageable fraction θ of the bundled output; plus the revenue from the sale of illiquid paper, priced at q_t, issued against the mortgageable fraction θ of the unbundled output. (It is convenient to arrange the accounts so that an agent always mortgages the most that he can, and then, if he chooses, he buys back his own paper as part of his holdings m_t and n_t.) The third and fourth terms are the incomes from short-term saving in period t-1 and from illiquid

[16] Throughout the article, we adopt the convention that the "return" on an asset is the gross return, and that the "rate of time preference" refers to the gross rate of time preference.

paper purchased in period t-2.

In the next period t+1, the period of growing, the agent's flow-of-funds constraint is

$$c_{t+1} + p_{t+1}m_{t+1} + q_{t+1}n_{t+1} = m_t + n_{t-1}. \tag{5}$$

Expenditures on the LHS of (5) are consumption, plus short-term saving, plus purchase of newly-issued illiquid paper. Receipts on the RHS are the incomes from short-term saving in period t and from illiquid paper purchased in period t-1. In period t+2, at the time of harvest, the agent's flow-of-funds constraint is

$$c_{t+2} + p_{t+2}m_{t+2} + q_{t+2}n_{t+2} = (1-\theta)y_{t+2} + m_{t+1} + n_t. \tag{6}$$

Expenditures on the LHS of (6) are consumption plus saving, as in the growing period. Receipts on the RHS are the portion of the harvest that was not mortgaged in period t, plus the incomes from short-term saving in period t+1 and from illiquid paper purchased in period t.

In a competitive equilibrium, the agent takes prices $p_t, q_t, p_{t+1}, q_{t+1}, p_{t+2}, q_{t+2}, ...$ as given, and chooses $\{y_{t+2}, z_{t+2}, c_t, m_t, n_t\}$ when investing in period t, $\{c_{t+1}, m_{t+1}, n_{t+1}\}$ when growing in period t+1, $\{c_{t+2}, m_{t+2}, n_{t+2}\}$ when harvesting period t+2, and so on in the production cycles starting in periods t+3, t+6, ... , to maximize his utility (1), subject to the flow-of-funds constraints $(4), (5), (6)$ and their equivalents in subsequent periods. Remember that there are two other groups of agents who make analogous choices, but who start their production cycles in periods t+1, t+4, t+7,... and t+2, t+5, t+8,... respectively. The markets for corn, illiquid paper and liquid paper all clear.

We focus on a symmetric steady-state equilibrium in which each agent's choices of quantities are in a fixed 3-period pattern, and prices are constant. Let p and q denote the prices; and let y and z denote the values of output and bundled output chosen by investing agents. Let c denote the consumption by investing agents; and let m and n denote their short-term saving and purchase of newly-issued illiquid paper. Let c', m' and n' denote the corresponding quantities chosen by growing agents. And let c”$, m$” and n” denote the quantities chosen by harvesting agents. The flow-of-funds constraints $(4), (5)$ and (6) reduce to

$$G(y) + \frac{1-\phi}{\phi} G(z) + c + pm + qn = p^2 \theta z + q_t \theta (y - z) + m\text{”} + n' \tag{7}$$

$$c' + pm' + qn' = m + n\text{”} \tag{8}$$

$$c\text{”} + pm\text{”} + qn\text{”} = (1-\theta)y + m' + n. \tag{9}$$

The market-clearing condition for corn is

$$y = c + c' + c\text{”} + G(y) + \frac{1-\phi}{\phi} G(z). \tag{10}$$

That is, output equals aggregate consumption plus investment, including the additional investment in bundling. (If there is storage, it appears on both sides of the equation.) The market-clearing condition for newly-issued illiquid paper is

$$\theta(y - z) = n + n' + n\text{”}. \tag{11}$$

That is, the supply of illiquid paper – the mortgageable part of unbundled output – equals the aggregate demand. The market-clearing condition for liquid paper is easiest to write in terms of the supply and demand for short-term saving; either

$$p < 1 \text{ and } p\theta z + \theta z = m + m' + m\text{”}$$

or

$$p = 1 \text{ and } p\theta z + \theta z \leq m + m' + m\text{”}. \tag{12}$$

The sum $p\theta z + \theta z$ represents the total supply of short-term saving provided by liquid paper (inside money). Because the paper is used twice, two vintages co-exist in each period's market: newly-issued and second-hand. The θz newly-issued paper has a market value of $p\theta z$ in the next period, whereas the θz second-hand paper yields θz corn when it matures in the next period. If $p = 1$, there may be a shortfall between $p\theta z + \theta z$ and the total demand for short-term saving, $m + m' + m$", which is made up by the circulation of shells (outside money), or by storage, or by some combination of the two.

From Walras' Law, (11) and (12) imply (10) given that the flow-of-funds constraints are satisfied.

Our goal is to characterize this equilibrium for different values of the parameters. In particular, we want to see how the equilibrium is affected by θ, the limit on bilateral commitment (which leads to the borrowing constraint), and by ϕ, the ease with which output can be bundled (which directly impinges on the resaleability constraint). To help understand the intuition, our discussion will be somewhat informal, but it contains statements of formal results (Propositions 1-4) that are proved in the Appendices.

Let us start with some preliminary observations about any equilibrium. First, because the marginal product of investment and the marginal utility of consumption are both infinite at zero, investment and consumption will always be strictly positive. Next, it has to be the case that $p^2 \geq q$, otherwise in any period t liquid paper would dominate illiquid paper as a means of saving for period t+2 (higher return and greater liquidity). We will see that when there is a liquidity shortage, the inequality is strict: there is a liquidity premium. Also, it has to be the case that $p \leq 1$, otherwise storage would dominate liquid paper as a means of short-term saving between periods t and t+1.

Finally, whenever storage is used in equilibrium ($p = 1$), there is another equilibrium without storage in which only shells are used. Steady-state production, consumption and paper holdings are all unchanged across these equilibria, but, as in Samuelson (1958), everyone could enjoy a one-off consumption gain by switching to the equilibrium without storage since this would avoid the need to tie up corn. From now on, we assume that shells circulate as outside money in lieu of storage.

If $p < 1$, neither storage nor outside money is used, and shells do not have positive value.

3 Achieving first-best

At this point it is useful to ask: Under what circumstances do the constraints on intertemporal contracting not bind? That is, for what values of θ and ϕ and can first-best be achieved? As a benchmark, consider the symmetric, steady-state, efficient allocation. In each period, a third of the agents produces y^* (say) and, without any deadweight loss from bundling, everyone has constant consumption: $c = c' = c" = c^*$, where $c^* = \frac{1}{3}[y - G(y)]$. Marginal cost equals the discounted marginal return, using the marginal rate of substitution between consumption at the time of harvest and consumption at the time of investment as the discount factor; that is, y^* solves

$$G'(y^*) = \beta^2. \tag{13}$$

To examine if this first-best allocation can be achieved, we need to consider separately the cases of costless and costly bundling. If $\phi = 1$, there are no costs to bundling and so there need be no problem of adverse selection in the second-hand paper market. Effectively, all paper is liquid and can act as inside money. In the flow-of-funds constraints (7) (8) and (9) corresponding to an agent's three-period production cycle, we therefore have $z = y$ and $n = n' = n" = 0$. To sustain first-best as an equilibrium, there has to be adequate means of short-term saving (for example, $m > 0$ is needed for consumption at the time of growing). Outside money cannot be used because its steady-state return, 1, is strictly below the rate of time preference, $1/\beta$. The supply of inside money is dictated by θ, the fraction of output that investing agents can credibly pledge, which cannot be too low:

Proposition 1 *Suppose $\phi = 1$, so that bundling is costless and in effect all paper is liquid. Then the first-best allocation $y = y^*$, $c = c' = c" = c^*$, can be achieved as a symmetric, steady-state* *equilibrium if and only if* $\theta \geq \theta^{**}$, *where*

$$\theta^{**} = \frac{1}{3}\left[2 - \beta - \frac{\lambda\beta^2(1-\beta)}{1+\beta+\beta^2}\right]. \tag{14}$$

In such an equilibrium, the one-period return on liquid paper, $1/p$*, equals the rate of time preference,* $1/\beta$*; i.e.* $p = \beta$.

To help understand the lower bound θ^{**} in (14), consider the extreme case where consumption is negligible: $\beta \simeq 1$ and $\lambda \simeq 0$.[17] Without consumption, unmortgaged harvest $(1-\theta)y$ is saved before being reinvested, so total demand for short-term saving is at least $(1-\theta)y/p$. Supply is $p\theta y + \theta y$. But in first-best, $p = \beta \simeq 1$. Hence the critical value of θ below which there would be a supply shortage is approximately the solution to $(1-\theta)y = 2\theta y$. That is, $\theta^{**} \simeq 1/3$ – which, given $\beta \simeq 1$ and $\lambda \simeq 0$, squares with (14).

Turn now to the much richer case of costly bundling, $\phi < 1$, where some paper is illiquid and cannot circulate as inside money. As our main concern in this article is to characterize equilibrium when some paper is illiquid and to explore the nature of the distortions that may arise, it makes sense to assume that if all paper is liquid the first-best can be achieved, i.e. to assume

$$\theta \geq \theta^{**}. \tag{Assumption 3}$$

To achieve first-best when bundling is costly, there cannot be any inside money supplied to the economy, since those costs are a deadweight loss. And, with first-best consumption smoothing, outside money cannot be demanded, given that its return is strictly less than the rate of time preference. Without money of any kind, illiquid paper is the only savings instrument available. Surprisingly, it is nevertheless possible to achieve first-best, but, as we now see, only if a great deal of such paper is supplied to the market.

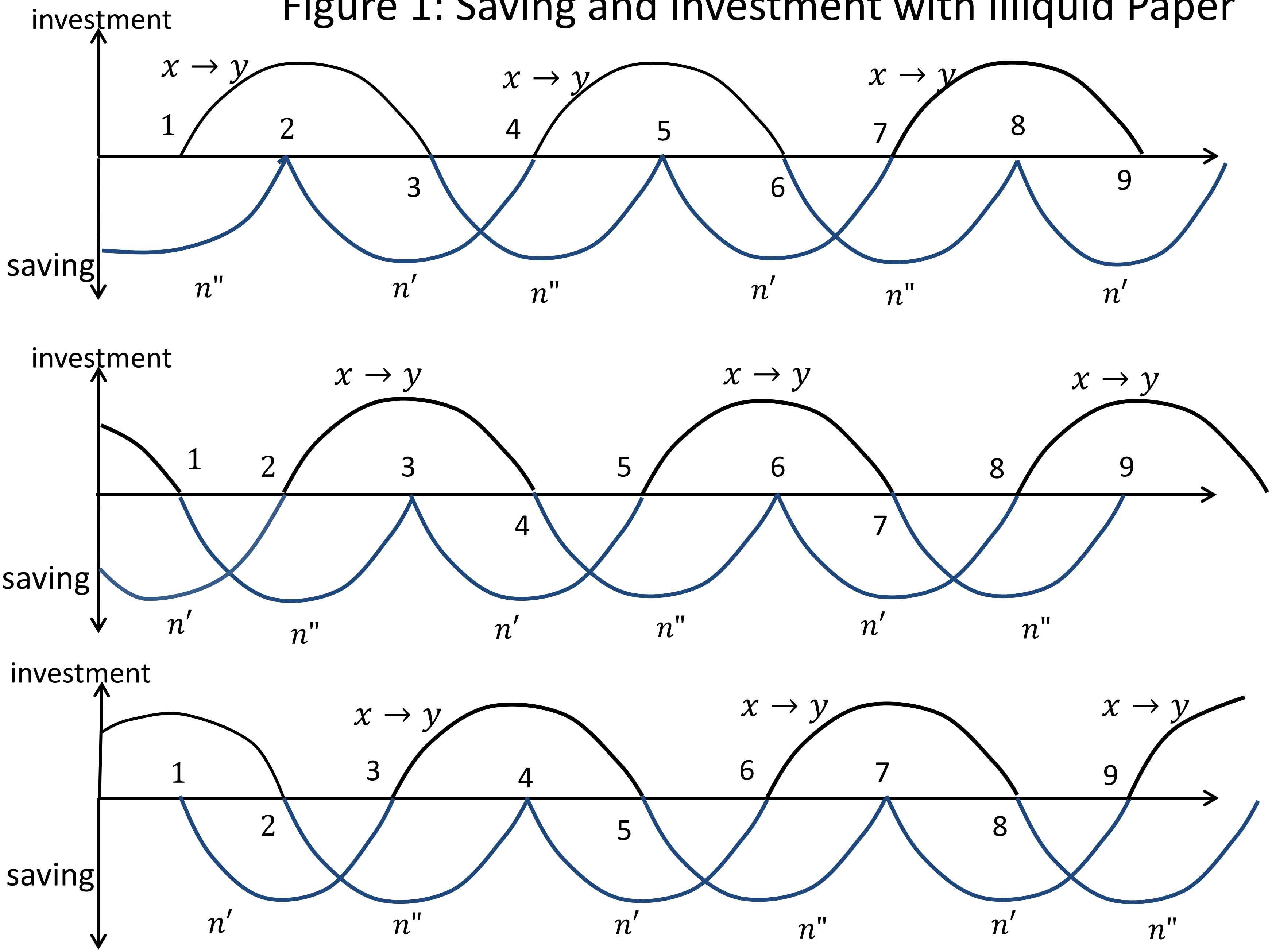
[17] We assume here that the technology has almost constant returns ($\lambda \simeq 0$) so that the consumption of profit can be ignored.

Consider a typical agent, agent A, who is investing in periods 1, 4, 7,..., growing in periods 2, 5, 8, ..., and harvesting in periods 3, 6, 9,... . See the top panel in Figure 1. The arcs above the time line denote A's production. There is a clever way for him to save income from harvest periods using illiquid paper: he holds the paper *twice* in succession: see the arcs below the time line that denote his saving. For example, (maximally levered) investment in period 1 yields him a payoff $(1-\theta)y$ in period 3, some of which he uses to purchase n" paper; the rest is used for consumption c". This n" paper matures in period 5: some of the income is used for consumption c' and the rest is used to purchase n' paper, which in turn matures in time for consumption and investment in period 7. The sequence – investment, followed by two successive purchases of illiquid paper, followed by more investment, and so on – serves to link the budgets of odd-numbered periods. A similar sequence links the even-numbered periods. But notice that, in the absence of any short-term saving, there is no cross-linkage. It is akin to two parallel turnpikes that have no cross roads which join them.

Thus, in order to implement first-best, an agent like A, who invests in periods 1, 4, 7, ..., must buy newly-issued illiquid paper in all the other periods: 2, 3, 5, 6, 8, 9,... . But from whom? Well, don't forget that when A invests he issues his own illiquid paper. So, to complete the equilibrium picture, we should consider two other typical agents, agent B who invests in periods 2, 5, 8, ..., and agent C who invests in periods 3, 6, 9,... . See the second and third panels in Figure 1. Paper flows as follows. In period 1, agent A, who is investing, issues his paper to agents B and C. Agent B, who is harvesting, buys A's paper either for consumption in period 3 or with a view to buying more paper to save for consumption and investment in period 5. And agent C, who is growing, buys A's paper to save for consumption and investment in period 3. In period 2, the same kinds of paper flows occur, except that the three agents rotate roles: B is investing, and issues paper both to C, who is harvesting, and to A, who is growing. The agents continue to rotate roles in the following periods. This arrangement emerges as the economy's ingenious response to the constraint imposed by the nonresaleability of paper, and the need to create double coincidences of wants.

Unfortunately, the arrangement places great demands on the paper market. Roughly speaking,

Figure 1: Saving and Investment with Illiquid Paper



ignoring consumption, agents are saving four times more than if they could use paper to save overnight from the harvest period, i.e. if paper were resaleable and could circulate as inside money. To obtain first-best, the supply of paper by investing agents has to be great enough, there has to be enough commitment power:

Proposition 2 *Suppose $\phi < 1$. Then the first-best allocation $y = y^*$, $z = 0$, $c = c' = c" = c^*$, can be achieved as a symmetric, steady-state equilibrium if and only if $\theta \geq \theta^*$, where*

$$\theta^* = \frac{1}{3}\left[2 - \frac{\lambda\beta^2(1 + 2\beta^2)}{1 + \beta^2 + \beta^4}\right], \tag{15}$$

which is strictly higher than the critical value θ^{**} from Proposition 1. In such an equilibrium, the implied one-period return on illiquid paper, $1/\sqrt{q}$, equals the rate of time preference, $1/\beta$; i.e. $q = \beta^2$.

Again, to help understand the lower bound θ^* in (15), consider the extreme case where consumption is negligible: $\beta \simeq 1$ and $\lambda \simeq 0$. Without consumption, unmortgaged harvest $(1 - \theta)y$ is saved twice before being reinvested, so $n" \geq (1 - \theta)y/q$ and $n' \geq (1 - \theta)y/q^2$, which makes a total demand for two-period paper of at least $(1 - \theta)y\left[(1/q) + \left(1/q^2\right)\right]$. Supply is the mortgaged portion of harvest, θy. But in first-best, $q = \beta^2 \simeq 1$. Hence the critical value of θ below which there would be a supply shortage is approximately the solution to $2(1 - \theta)y = \theta y$. That is, $\theta^* = 2/3$ – as per (15).

Interestingly, when consumption is negligible ($\beta \simeq 1$ and $\lambda \simeq 0$), the value $\theta^* = 2/3$ in Proposition 2 is twice the critical value $\theta^{**} = 1/3$ that we obtained in Proposition 1 for the economy with inside money. This is consistent with the observation that to sustain first-best without resaleable paper agents have to save four times more. The point is that, ceteris paribus, doubling θ serves to double the supply of paper, and halving $1 - \theta$ serves to halve the demand. Together, these just offset the quadrupling of demand stemming from the nonresaleability of paper.

The fact that $\theta^* > \theta^{**}$ reflects the substitutability between commitment power (high θ) and liquidity (high ϕ).

The remainder of the article characterizes equilibrium for values of θ below θ^* (but not lower than θ^{**}, by Assumption 3). Now, given that bundling is costly ($\phi < 1$), the demands on the paper market are such that the first-best cannot be achieved. There is a shortage of means of saving, brought about by the inflexibility of illiquid paper. At root, the economy suffers from a liquidity shortage.

4 Symptoms of liquidity shortage

In response to a liquidity shortage, the economy induces the investing agents to supply liquid paper. The issue price of liquid paper, p^2, strictly exceeds q, the price of illiquid paper – even though they have identical payoffs at maturity (both are claims to corn in two periods time). The difference in price, the *liquidity premium*, rewards the agents for their additional investment in bundling. From (7), they choose $z > 0$ satisfy

$$\frac{1-\phi}{\phi}G'(z) = \theta(p^2 - q); \tag{16}$$

i.e. the marginal cost of bundling equates to the additional revenue raised from issuing liquid paper instead of illiquid paper. It is important to recognize that if there were no liquidity shortage (as when $\theta \geq \theta^*$), then there would be no premium, no bundling and no supply of liquid paper; i.e. no inside money in the economy.

The liquid paper, both newly-issued and second-hand, is bought by agents at the time of harvest ($m" > 0$), as a means of short-term saving for consumption and investment in the following period. But at the same time these agents also buy illiquid paper ($n" > 0$), to provide funds two periods later for consumption and for the further purchase of illiquid paper ($n' > 0$) saving for consumption and investment in four periods' time.[18] That is, in addition to buying liquid paper at the time of harvest, agents follow the same long-term saving strategy that was used to achieve first-best in

[18] If θ is too low then $n' = 0$; but Assumption 3 is sufficient to rule this out. See Appendix B.

Figure 1. They have no need to hold liquid paper at other times ($m = m' = 0$). And, since agents' borrowing constraints are binding at the time of investment (see below), they have no incentive to buy paper secured against other agents' new projects ($n = 0$).

In effect, at the time of harvest an agent mixes two portfolio/investment strategies, one fast, the other slow:

fast strategy: buy liquid paper ($m" > 0$) in order to invest in the next period;

slow strategy: buy illiquid paper twice in succession ($n" > 0$ followed by $n' > 0$)

in order to invest four periods later.

See Figure 2. The drawback to the fast strategy is that the one-period return on liquid paper, $1/p$, is lower than the implied one-period return on illiquid paper, $1/\sqrt{q}$. Despite the return dominance, the agent is willing to buy liquid paper at the time of harvest because it affords him the flexibility to invest in the next period rather than having to wait four periods.

The relation between p and q must be such that the agent is indifferent between the two strategies. An indifference condition can be indirectly found by making use of the fact that in steady state the overall return from the fast strategy must equal the three-period rate of time preference $1/\beta^3$ and the overall return from the slow strategy must equal the six-period rate of time preference $1/\beta^6$. Respectively,

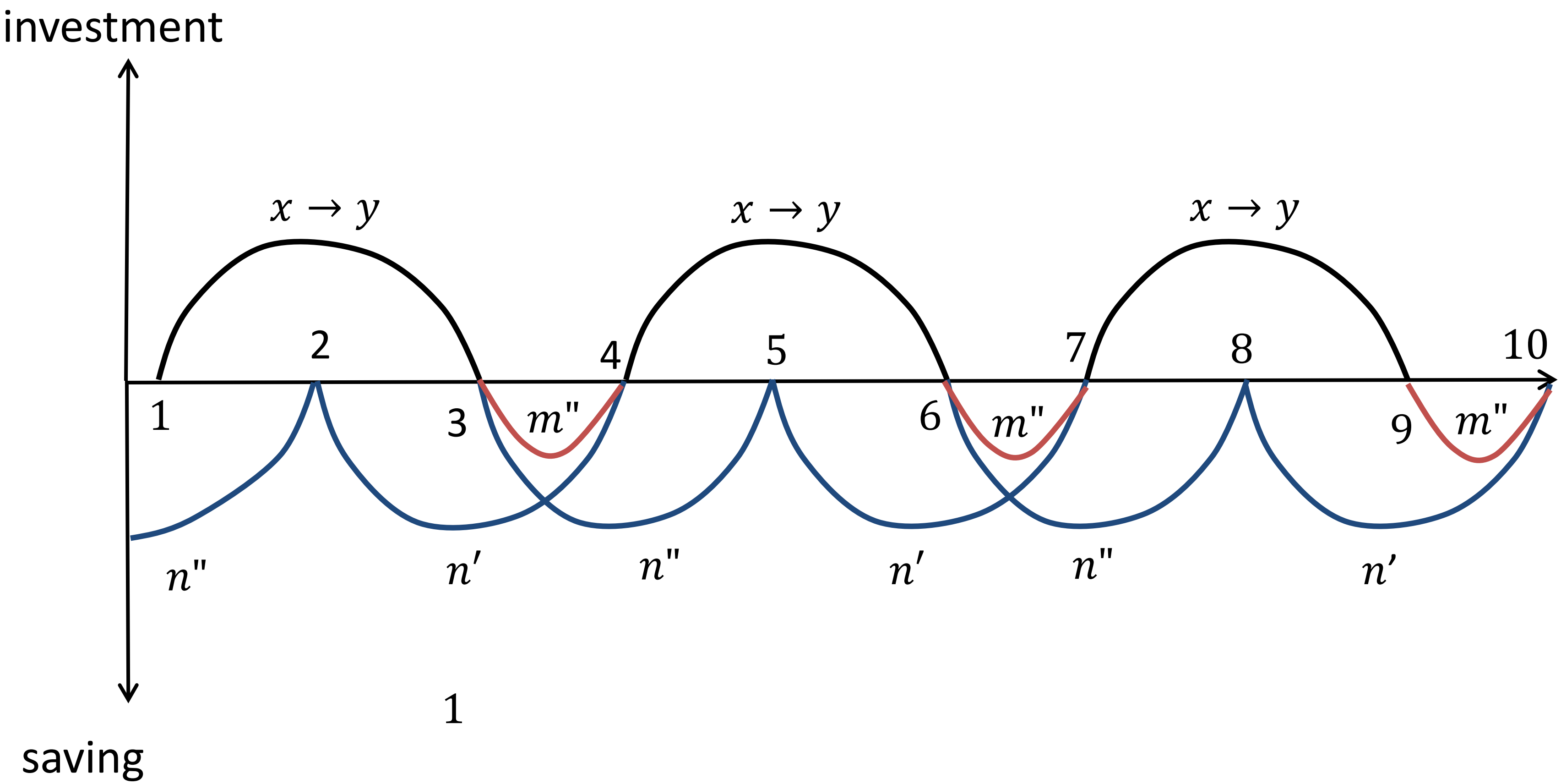
$$\frac{1}{p} \times \left[\frac{1-\theta}{G'(y) - \theta q}\right] = \frac{1}{\beta^3}, \tag{17}$$

and

$$\frac{1}{q^2} \times \left[\frac{1-\theta}{G'(y) - \theta q}\right] = \frac{1}{\beta^6}. \tag{18}$$

The term in large brackets in (17) and (18) is the marginal two-period return from investment, given maximum leverage: the fraction of harvest not mortgaged, $1 - \theta$, divided by $G'(y) - \theta q$, the

Figure 2: Investment and Saving with Money



portion of marginal cost not covered by borrowing (the "marginal downpayment"). From (17) and (18) we uncover the necessary relation between p and q:

$$q^2 = \beta^3 p. \tag{19}$$

This tells us not only that the one-period return on liquid paper is less than the rate of time preference, $1/p < 1/\beta$, but also that the return on illiquid paper, $1/q$, is less than the two-period rate of time preference, $1/\beta^2$.[19] In other words, the issue prices of liquid *and* illiquid paper, p^2 and q, are both higher than the price they would be in first-best. The high prices (low interest rates) reflect the shortage of means of saving in the economy, arising from the scarcity of commitment power and the lack of liquidity.

Low interest rates go hand in hand with borrowing constraints. To confirm this, observe from (17) and (18) that whenever $1/p < 1/\beta$ and $1/q < 1/\beta^2$, the return on investment is strictly higher than the two-period return on both kinds of paper.

The fact that agents' borrowing constraints are binding at the time of investment suggests that output might be lower than in first-best. This indeed turns out to be true. From (18), we can solve for the marginal cost of output:

$$G'(y) = \theta q + (1-\theta)\frac{\beta^6}{q^2}. \tag{20}$$

At the margin, the cost of output, $G'(y)$, is met by a combination of external finance, θq, and internal finance, $(1-\theta)\,\beta^6/q^2$, saved from harvest. Notice that if there were no liquidity shortage (θ above θ^*), so that q equalled β^2, (20) would reduce to (13) and output would be at its first-best level y^*. However, as we have just seen, when there is a liquidity shortage (θ below θ^*), q exceeds β^2. Now, on the one hand, a high q allows agents to raise more external finance - the price of their paper is high. But, on the other hand, a low return on saving, $1/q$, reduces the available internal

[19] To see why, rewrite (19) as either $p/\beta = \left(p^2/q\right)^{2/3}$ or $q/\beta^2 = \left(p^2/q\right)^{1/3}$. Given the liquidity premium $p^2 - q > 0$, it follows that $p > \beta$ and $q > \beta^2$.

finance. Also, a low θ shifts the balance from external to internal finance, which pushes investment down. Overall, we can show that the negative effects dominate: when $q > \beta^2$ output y is less than its first-best level y^*.

Aggregate consumption, $c + c' + c$", suffers a double loss relative to its first-best level, $3c^*$. From the market-clearing condition for corn (10) we see that not only is output net of basic investment $G(y)$ smaller, but also some of this corn is lost to additional investment $\frac{1-\phi}{\phi}G(z)$ in bundling that yields no output.

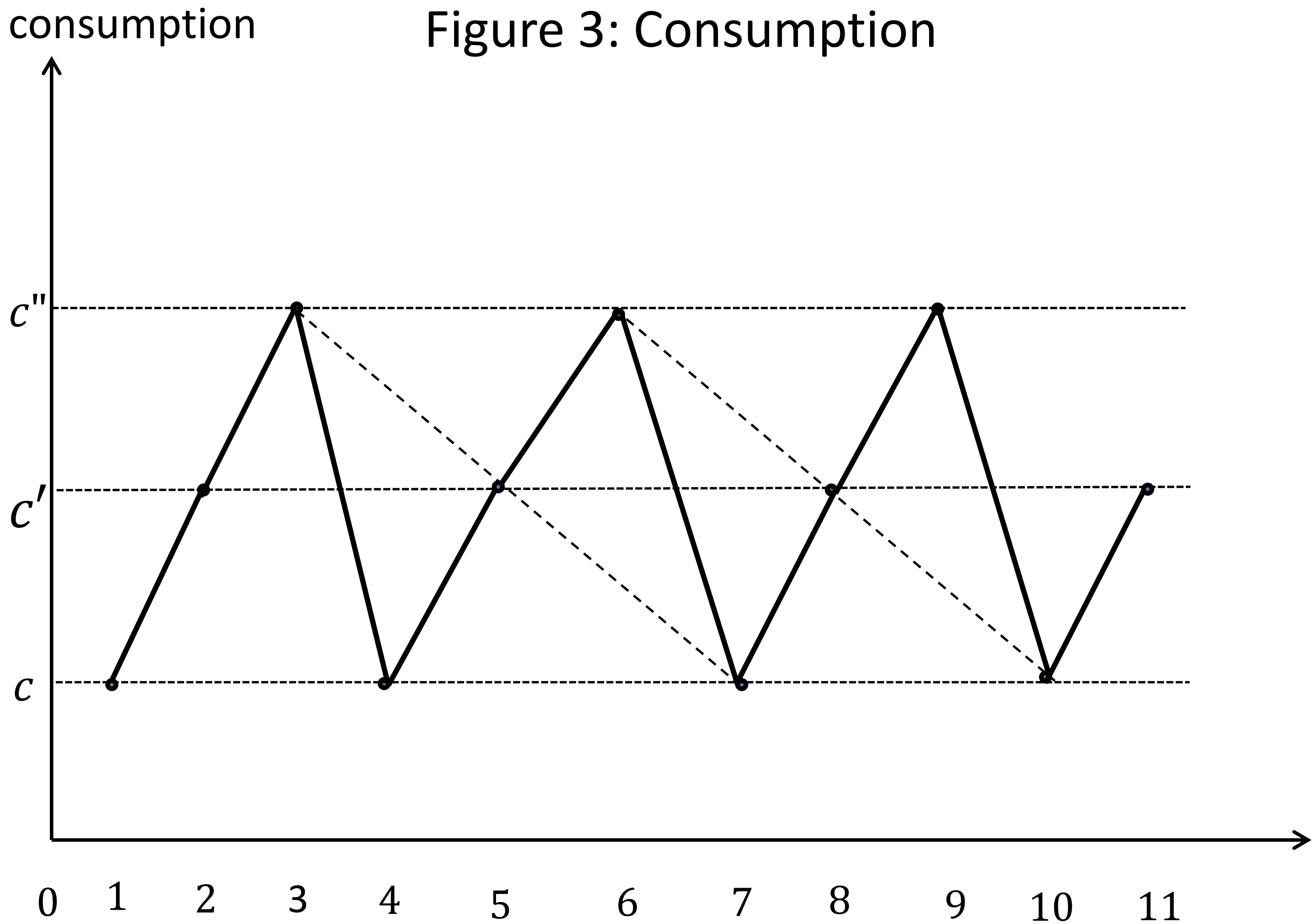
Roughly put, one might say that because agents have to hold illiquid paper, they take longer to reinvest the income from their earlier production, and the economy "runs too slowly". A better, though still incomplete, intuition for the underproduction is that, with a liquidity shortage in the economy, not enough funds are shifted from the harvesting agents to the investing agents. Discouraged by the low interest rates (even on illiquid paper), the harvesting agents tend to overconsume. Consumption is no longer smoothed. Consider the standard first-order conditions that determine an agent's choice between consumption and saving using illiquid paper, n" > 0 and $n' > 0$. Both the marginal rate of substitution between consumption at the time of harvesting and consumption two periods later (the next growing period), and the marginal rate of substitution between consumption at the time of growing and consumption two periods later (the next investment period), are equal to the two-period return on illiquid paper:

$$\frac{1/c''}{\beta^2(1/c')} = \frac{1}{q} \tag{21}$$

$$\frac{1/c'}{\beta^2(1/c)} = \frac{1}{q}. \tag{22}$$

Given that $q > \beta^2$, we see from (21) that the path of consumption every other period, starting from the time of harvest, is downward-sloping: c" $> c' > c$. Consumption is highest at the time of harvest and lowest at the time of investment, because the return on illiquid paper is less than the rate of time preference. See the dotted line in Figure 3 joining the odd-numbered periods 3, 5

Figure 3: Consumption



and 7. Repeating the same exercise for the even-numbered periods, and then connecting adjacent periods, we obtain the jagged solid line.

It should be added that since $m" > 0$, the marginal rate of substitution between consumption at the time of harvesting and consumption in the next period equals the return on liquid paper:

$$\frac{1/c"}{\beta(1/c)} = \frac{1}{p} \tag{23}$$

From (21), (22) and (23), we rediscover the relation between p and q that we found earlier in (19).[20]

If the liquidity shortage is severe enough that shells (outside money) are used as a means of short-term saving in addition to liquid paper (inside money), then $p = 1$ and, by (19), $q = \beta^{3/2}$. This happens if ϕ and θ are low enough. Our findings are summarized in Proposition 3.

Proposition 3 *Suppose $\phi < 1$ and $\theta < \theta^*$, defined in (15). Then there is a unique symmetric, steady-state equilibrium in which:*

(i) the price of two-period liquid paper p^2 strictly exceeds β^2 (but is no greater than 1);

(ii) the price of two-period illiquid paper q equals $\sqrt{\beta^3 p}$ – implying that it strictly exceeds β^2 (but is no greater than $\beta^{3/2}$), and that there is a liquidity premium $p^2 - q > 0$;

(iii) bundling is used to supply liquid paper: $z > 0$;

(iv) borrowing constraints bind at the time of investment: $m = n = 0$;

(v) output is lower than in first-best: $y < y^*$;

(vi) consumption is highest at the time of harvesting and lowest at the time of investment: $c" > c' > c$;

(vii) there is a continuous and strictly decreasing function $\Theta(\phi)$ such that outside money has value if and only if $\theta < \Theta(\phi)$. The function is such that $\theta^{**} < \Theta(\phi) < \theta^*$, implying that whenever $\theta \geq \Theta(0)$ or $\phi \geq \Theta^{-1}(\theta^{**})$, outside money is not used.[21]

[20] The equlibrium values of the ten unknowns $p, q, y, z, c, c', c", m", n'$ and $n"$ can be solved from ten equations of (7),(8),(9),(11),(12),(16),(20), (21), (22) and (23) with m = m' = 0 and n = 0.

[21] An explicit definition of $\Theta(\phi)$ is given in Appendix B.

Figure 4 collects together Propositions 1 to 3, partitioning the parameter space $\{(\theta, \phi) \mid \theta^{**} \leq \theta \leq 1,\ 0 < \phi \leq 1\}$ into essentially three regions. In region III, strictly south-west of the boundary $\theta = \Theta(\phi)$, outside money is used as a supplement to inside money. The boundary slopes down – once again a reflection of the substitutability between commitment power (high θ) and liquidity (high ϕ). In region II, north-east of the boundary but with $\theta < \theta^*$ and $\phi < 1$, only inside money is used a means of short-term saving. East of $\theta = \theta^*$ (region Ia), the first-best is achieved with illiquid paper alone: there is no money. And along the line $\phi = 1$ (region Ib), where bundling is costless, all paper is effectively liquid and the first-best is also achieved. Note that at the boundaries there are no discontinuities in equilibrium prices or quantities.

5 Financial deepening

It is tempting to take a long view and interpret the parameters θ and ϕ as indices of financial development. A higher value of θ may reflect improved contractual arrangements or legal structures that shift bargaining power from debtors to creditors in the event of default and renegotiation. A higher value of ϕ may reflect the development of negotiability or securitization. More generally, better communication among creditors makes it feasible to keep track of individual borrowers. And once there is a common record of someone's repayment history, reputation starts to matter: in effect, a borrower can make a greater commitment to all creditors, old and new, thus raising both θ and ϕ, at least indirectly.

The danger of interpreting the parameters in such a general way is that we move away from the safe ground of the particular adverse-selection-cum-moral-hazard model laid out at the start of Section 2. But, undaunted, let us conclude our analysis by exploring the effects of financial development, captured in the model by higher values of θ and ϕ. In what follows, bear in mind that we are merely comparing steady-state equilibria, not conducting a full dynamic analysis of transitions.

Proposition 4 reports on how θ and ϕ affect not only equilibrium prices and quantities, but also certain measures of "financial depth”, ratios of financial aggregates to GDP. Start with the narrowest monetary aggregate, outside money: By the same logic we used to write down the market-clearing condition for short-term saving (12), and knowing that $m = m' = 0$, we have that in aggregate

$$\frac{\text{outside money}}{\text{output}} = \frac{pm" - p^2\theta z - p\theta z}{y} \tag{24}$$

where both numerator and denominator are valued in terms of current-period corn (our numeraire). Equation (24) derives from the fact that

$$\frac{\text{inside money}}{\text{output}} = \frac{p^2\theta z + p\theta z}{y} \tag{25}$$

and, for the broader measure total money (inside plus outside money),

$$\frac{\text{total money}}{\text{output}} = \frac{pm"}{y} \tag{26}$$

The broadest financial aggregate includes illiquid paper. It is not clear how such paper should be valued in the period before maturity; but let us simply price it at $\sqrt{q}$, given that its two-period issue price is q. With this pricing rule,

$$\frac{\text{total paper}}{\text{output}} = \frac{pm" + q\theta y + \sqrt{q}\theta y}{y} \tag{27}$$

Because in our model we have combined the roles of borrower and banker, and we do not have financial intermediaries as distinct agents, in (27) we measure the stock of illiquid paper as θy, not as $\theta(y - z)$. This reflects the fact that intermediaries buy illiquid paper as an input to their production of liquid paper, and in practice this input is included in measures of broad financial aggregates.

Notice that the ratios in (24) $-$ (27) are inverse velocities of circulation.

Proposition 4 *Under the conditions of Proposition 3, within the region $\theta < \Theta(\phi)$ where outside money has value:*

(i) output increases with θ but is unaffected by ϕ;

(ii) the outside money/output ratio decreases with θ and ϕ;

(iii) the inside money/output ratio increases with θ and ϕ;

(iv) the total money/output ratio increases with ϕ and, if $\lambda \geq 1/3$, decreases with θ.

And within the region $\theta > \Theta(\phi)$ where outside money has no value:

(i) the issue prices of liquid and illiquid paper (p^2 and q), and the liquidity premium ($p^2 - q$), all decrease with θ and ϕ;

(ii) output increases with θ and ϕ if $\lambda \geq \left(1-\beta^{1/2}\right) \frac{\left(1+2\beta^{1/2}\right)\left(1+\beta^2+\beta^4\right)}{3\beta^{5/2}\left(1+\beta^{3/2}+\beta^2\right)}$.

(iii) consumption becomes smoother ($c"/c'$ and c'/c both decrease) as θ and ϕ increase;

(iv) the inside money/output ratio increases with ϕ and decreases with θ.

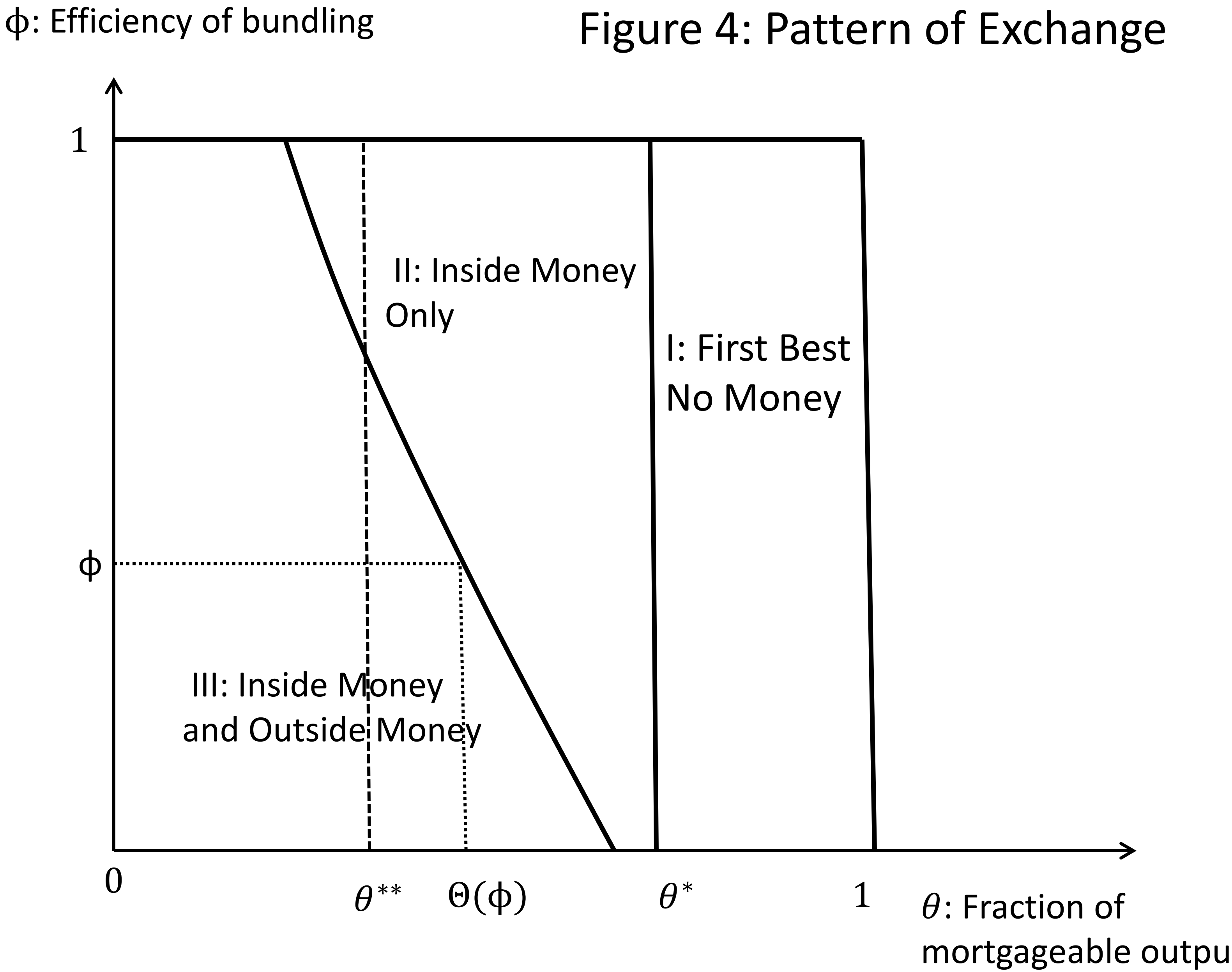
From Proposition 4 we can say more about Figure 4. As we travel north-east in region II, prices and the liquidity premium fall; and, provided λ is not too small, output rises. On reaching region Ia or Ib, prices and output stay constant at their first-best levels, $p = \beta$, $q = \beta^2$ and $y = y^*$.

In region III, prices are also constant, but here $p = 1$ and $q = \beta^{3/2}$. From (20) it immediately follows that output is increasing with θ, although independent of ϕ, as per Proposition 4.

The fact that output is (weakly) increasing as we travel north-east in region III does not imply that agents are necessarily better off. We must take into account that the higher is θ or ϕ, the greater is the additional investment $\frac{1-\phi}{\phi}G(z)$ in bundling.[22] Consider the effect on welfare of increasing in ϕ alone. From the market-clearing condition for corn (10), the higher is ϕ, the *lower* is aggregate consumption $c + c' + c"$, because there is more aggregate investment but no more output. To put it starkly: the agents in the economy would benefit from shutting down the supply of inside money and relying entirely on outside money for their short-term saving! The point is that if bundling

[22] From (16), the higher is θ or ϕ, the higher is z. Hence, combining (2) and (16), we have that $\frac{1-\phi}{\phi}G(z) = \frac{1-\phi}{\phi}G'(z)(1-\lambda)z = \theta\left(p^2 - q\right)(1-\lambda)z$ is higher too.

Figure 4: Pattern of Exchange



doesn't serve to relieve the liquidity shortage – and it doesn't in region III where prices and the liquidity premium are constant – then the costs of bundling are a pure deadweight loss.

Of course, this somewhat surprising welfare conclusion relies heavily on the fixed supply of shells that serve as outside money. If outside money were under the control of government then, since one of the root causes of the liquidity shortage in our economy is a scarcity of commitment power, we would need to ask if the government is any better at committing itself (e.g. not to raise the supply of outside money) than are private agents.

In broad terms, we may conclude that financial development not only raises output but also shifts the economy from outside to inside money. Moreover, it deepens the extent to which paper of all kinds is used.

Perhaps the model's sharpest prediction is that eventually outside money will cease to be used, driven out by liquid private paper that earns higher returns. The thesis that outside money may disappear is controversial. There are several arguments against it. First, outside money in the form of cash will always be useful to people like drug dealers who want to conceal their nefarious activities, because cash leaves no electronic trail. Next, our analysis presupposes that outside money and other assets are substitute means of saving. It can be argued that in fact outside money is complementary: after all, assets such as bonds are promises to pay in outside money. Finally, we may be focussing here too much on the supply of liquidity. As the pace of the modern world quickens, the problem of finding double coincidences of wants in dated goods becomes more severe. Our demand for liquidity may be rising in line with the supply.

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7 Appendix

7.1 Proof of Proposition 1:

This economy is the special case of the model in which there is no cost of bundling so that all the paper is liquid, i.e., $\phi = 1$, and $z = y$. In a symmetric steady state equilibrium, the flow of funds constraints for the agent in investing, growing, and harvesting periods, equations (7, 8, 9), are simplified to

$$G(y) + c + pm = p^2\theta y + m" \tag{A1}$$

$$c' + pm' = m \tag{A2}$$

$$c" + pm" = (1-\theta)y + m'. \tag{A3}$$

The market clearing condition for liquid paper are:

$$(1+p)\theta y \leq m + m' + m", \text{ where equality holds if } p < 1. \tag{A4}$$

The first order conditions for consumption and investment are

$$\frac{1/c}{\beta/c'} \geq \frac{1}{p}, \text{ and } \left(\frac{1/c}{\beta/c'} - \frac{1}{p}\right) m = 0 \tag{A5}$$

$$\frac{1/c'}{\beta/c"} \geq \frac{1}{p}, \text{ and } \left(\frac{1/c'}{\beta/c"} - \frac{1}{p}\right) m' = 0 \tag{A6}$$

$$\frac{1/c"}{\beta/c} \geq \frac{1}{p}, \text{ and } \left(\frac{1/c"}{\beta/c} - \frac{1}{p}\right) m" = 0 \tag{A7}$$

$$\frac{1/c}{\beta^2/c"} = \frac{1-\theta}{G'(y) - \theta p^2}. \tag{A8}$$

The symmetric steady state equilibrium is characterized by the eight variables $(p, y, c, c', c", m, m', m")$ which satisfies (A1-A8).

From (A5-A8), we learn that the competitive equilibrium with only liquid paper achieves the first best allocation ($G'(y) = \beta^2$ and $c = c' = c"$), if and only if $p = \beta$. In such an equilibrium, by adding (A3), β times (A1) and β^2 times (A2), we have:

$$c = c' = c" = (1-\beta)\left[(1-\theta)\, y^* + m'\right] + \frac{\lambda\beta^3}{1+\beta+\beta^2} y^*,$$

where y^* is output of the first best allocation. Then from (A3) and (A2), we get

$$\begin{aligned} m" &= (1-\theta)y^* + m' - \frac{\lambda\beta^2}{1+\beta+\beta^2} y^* \\ m &= \beta m' + (1-\beta)[(1-\theta)y^* + m'] + \frac{\lambda\beta^3}{1+\beta+\beta^2} y^*. \end{aligned}$$

Then from (A4), we have:

$$(1+\beta)\theta y^* = 3m' + (2-\beta)(1-\theta)y^* - \frac{\lambda\beta^2(1-\beta)}{1+\beta+\beta^2} y^*.$$

Thus we learn $m' \geq 0$, if and only if $\theta \geq \theta^{**}$, where θ^{**} is defined at (14). We can also check $m" > 0$ and $m > 0$, if $\theta \geq \theta^{**}$. Therefore, $\theta \geq \theta^{**}$ is the necessary and sufficient condition for the competitive equilibrium with only liquid paper to be the first best. $Q.E.D.$

7.2 Appendix B: Proof of Propositions 2, 3 and 4:

Since consumption is always positive due to infinite marginal utility at zero consumption, the first order conditions for choice between consumption and purchasing two-period illiquid papers n, n' and $n"$ are

$$\frac{1/c}{\beta^2/c"} \geq \frac{1}{q}, \text{ and } \left(\frac{1/c}{\beta^2/c"} - \frac{1}{q}\right) n = 0, \tag{A9}$$

$$\frac{1/c'}{\beta^2/c} \geq \frac{1}{q}, \text{ and } \left(\frac{1/c'}{\beta^2/c} - \frac{1}{q}\right) n' = 0, \tag{A10}$$

$$\frac{1/c"}{\beta^2/c'} \geq \frac{1}{q}, \text{ and } \left(\frac{1/c"}{\beta^2/c'} - \frac{1}{q}\right) n" = 0. \tag{A11}$$

Similarly, the first order conditions for choice between consumption and liquid saving (either liquid paper, fiat money or storage) m, m' and m" are (A5),(A6), and (A7). The first order conditions for production and bundling are given by:

$$\frac{1/c}{\beta^2/c"} = \frac{1-\theta}{G'(y) - \theta q}. \tag{A12}$$

$$\frac{1-\phi}{\phi} G'(z) \leq \theta(p^2 - q), \text{ and } z = y \text{ if } \frac{1-\phi}{\phi} G'(y) < \theta(p^2 - q). \tag{A13}$$

The symmetric steady state equilibrium is characterized by 13 variables $(p, q, y, z, c, c', c", n, n', n",$ $m, m', m")$ which satisfy 13 independent equilibrium conditions $(7), (8), (9), (11), (12),$ (A5-A7) and (A9-A13).

In order to prove Propositions, we first prove the following series of lemma.

Lemma 1: In the competitive equilibrium, $\beta^2 \leq q \leq p^2$ and $\beta \leq p \leq 1$.

(Proof) Multiplying three inequalities of (A9) (A10) and (A11) side by side, we learn $q \geq \beta^2$ in the steady state. Also, multiplying (A5) (A6) and (A7) side by side, we learn $p \geq \beta$. Since everyone can use storage as liquid saving, the rate of return on liquidity cannot be strictly less than one-for-one (the rate of return on storage), and thus we have $p \leq 1$. If there are active trades of illiquid paper so that one of n, n' and n is strictly positive positive, then at least one of (A9),(A10) and (A11) holds with equality. If there is no active trade of illiquid paper, q is defined as the smallest price at which nobody wants to buy it, so that at least one of (A9),(A10) and (A11) holds with equality. Then from (A5-A7), we get $q \leq p^2$. (Intuitively, the return on two period illiquid paper cannot be dominated by return on liquidity for two periods in succession.) *Q.E.D.*

Lemma 2: The competitive equilibrium achieves the first best allocation, if and only if $p = \beta$. Also, the competitive equilibrium achieves the first best, if and only if $q = \beta^2$.

(Proof) If $p = \beta$, we learn $q = \beta^2$ from Lemma 1. Then from (A13), we know $z = 0$. Also, if $p =$

β, we learn from (A5-A7) that $c \le c' \le c" \le c$, or $c = c' = c"$. Then, from the marginal condition for production (A12), we learn $G'(y) = \beta^2$. Therefore, if $p = \beta$, the competitive equilibrium achieves the first best allocation: $c = c' = c"$, $z = 0$, and $G'(y) = \beta^2$. Conversely, if the competitive equilibrium achieves the first best allocation, then at least one of n, n' and n must be positive because the market for illiquid paper market to clear with $z = 0$. Then from (A9-A11), we learn $q = \beta^2$. Then, because $z = 0$ in the the first best, we learn from (A13) and (2) that $p^2 = q$, or $p = \beta$. *Q.E.D.*

Lemma 3: In the competitive equilibrium, $z < y$.

(Proof) Suppose that $z = y$. Then $p^2 - q \ge \frac{1-\phi}{\phi} G'(y) > 0$. Also, $n = n' = n" = 0$. Then, because $c' > 0$, we learn m must be strictly positive. If either m' or $m"$ is strictly positive, then from (A5-A7), $p^{-2} = (1/c_t) / \left(\beta^2 / c_{t+2}\right)$. Then from (A9-A11), we get $q^{-1} \le p^{-2}$. But, this contradicts $p^2 - q > 0$. If, instead, $m' = m" = 0$, then, the flow-of-funds constraint of the investing agent becomes:

$$c + G(y) + \frac{1-\phi}{\phi} G(z) + pm = \theta p^2 y.$$

But this contradicts the market clearing condition of the liquidity: $\theta p(1+p)y \le pm$. *Q.E.D.*

Lemma 4: In the competitive equilibrium, $m \cdot n' = m' \cdot n" = m" \cdot n = 0$.

(Proof) Suppose that $m \cdot n' > 0$. Then from (A10) and (A5), we learn,

$$\frac{c}{c'} = \frac{\beta^2}{q} = \frac{p}{\beta}.$$

Thus

$$q = \frac{\beta^3}{p} \le \beta^2, \text{ by Lemma 1.}$$

Then, by Lemma 1, $q = \beta^2$ and $p = \beta$. This implies $z = 0$ and there is no inside money nor outside money. This contradicts $m > 0$ in the market for liquidity. The same proof by contradiction applies to $m' \cdot n" = m" \cdot n = 0$. *Q.E.D.*

Lemma 5: In the competitive equilibrium, $n" > 0$ and $m' = 0$.

(Proof) Suppose that $n" = 0$. Since $c' > 0$, we have $m > 0$. Then from Lemma 4, we know $n' = 0$. Then, the market clearing condition of illiquid paper becomes:

$$\theta(y - z) = n.$$

Because of Lemma 3, we learn $n > 0$, which implies $m" = 0$ by Lemma 4. Then, the flow-of-funds constraint of the investing agent becomes:

$$c + G(y) + \frac{1-\phi}{\phi} G(z) + pm = p^2 \theta z \tag{A14}$$

If $m' = 0$, then the market for liquidity implies,

$$p\theta(1+p)z \leq pm.$$

which contradicts (A14). If instead m' is strictly positive, then we have

$$p^{-2} = \frac{1/c}{\beta^2 / c"} = q^{-1},$$

because m, m' and n are all positive. Then, $q = p^2$. But this contradicts (A14) with strictly positive consumption and investment. *Q.E.D.*

Lemma 6: In the competitive equilibrium, $n' + m" > 0$.

(Proof) Suppose that $n' = m" = 0$. Then, using Lemma 5, the flow-of-funds constraint and

market equilibrium conditions imply:

$$\begin{aligned} c + G(y) + \frac{1-\phi}{\phi} G(z) + pm + qn &= q\theta(y-z) + p^2\theta z \\ c' &= m + n" \\ (1+p)\theta z &\leq m \\ \theta(y-z) &= n + n". \end{aligned}$$

From these equations, we have:

$$c + G(y) + \frac{1-\phi}{\phi} G(z) \leq qn" - p\theta z = qc' - qm - p\theta z \tag{A15}$$

From the first order conditions (A5,A7,A11), we have, either

$$m > 0, \text{ and } \frac{c'}{c} = \frac{\beta}{p}, \text{ or}$$

$$m = z = 0, \text{ and } p^2 = q, \text{ and}$$

$$\frac{c'}{c} = \frac{c'/c"}{c/c"} = \frac{\beta^2/q}{\beta/p} = \frac{\beta}{q^{1/2}}.$$

Then from Lemma 1, we learn $c \geq qc'$. Then, LHS of equation (A15)is strictly larger than RHS. This is contradiction. $Q.E.D.$

Now, we prove Proposition 2.

Proof of Proposition 2:

From Lemma 2, the competitive equilibrium achieves the first best allocation if and only if $q = \beta^2$ and $p = \beta$. In such an equilibrium, all z, m, m' and $m"$ are zero. By adding (9), q times (8) and q^2 times (7) with $q = \beta^2$, we have:

$$c = c' = c" = (1-\beta^2)[(1-\theta)y^* + n] + \frac{\lambda\beta^6}{1+\beta^2+\beta^4}y^*.$$

Also, from (8) and (9), we have:

$$\beta^2 n" = \beta^2[(1-\theta)y^* + n] - \frac{\lambda\beta^6}{1+\beta^2+\beta^4}y^*$$

$$\beta^2 n' = \beta^2[(1-\theta)y^* + n] - \frac{\lambda\beta^4(1+\beta^2)}{1+\beta^2+\beta^4}y^*.$$

Then from (11), we have

$$\theta y^* = 3n + 2(1-\theta)y^* - \frac{\lambda\beta^2(1+2\beta^2)}{1+\beta^2+\beta^4}y^*.$$

Thus we learn $n \geq 0$, if and only if $\theta \geq \theta^*$, θ^* is defined in (15). We can also check $n' > 0$ and $n" > 0$, if $\theta \geq \theta^*$. Thus, the competitive equilibrium can achieve the first best allocation if $\theta \geq \theta^*$. Also, the competitive equilibrium cannot achieve the first best allocation if $\theta < \theta^*$. *Q.E.D.*

Lemma 7: If $\theta < \theta^*$, then $n = 0$ in the competitive equilibrium.

(Proof) Suppose that $n > 0$. This implies $m" = 0$ by Lemma 4. Then from Lemma 6, we learn $n' > 0$. But, then all n, n' and $n"$ would be strictly positive by Lemma 5, and $q = \beta^2$ by (A9-A11). This implies the first best allocation by Lemma 2. This contradicts Proposition 1 with assumption $\theta < \theta^*$. *Q.E.D.*

Lemma 8: If $\theta^{**} \leq \theta < \theta^*$, then $n' > 0$ and $m = 0$.

(Proof) Suppose that $n' = 0$. This implies $m" > 0$ by Lemma 6. Then we have two cases (Case i: $m > 0$) and (Case ii: $m = 0$).

(Case i: $m > 0$) From Lemma 5 and $m" \cdot m > 0$, the first order conditions (A5,A7,A11) implies $q = p^2$. This implies $z = 0$. Then in order for the market for liquidity to clear, we need $p = q = 1$. Therefore, the flow-of-funds constrains, market clearing conditions and the first order conditions imply:

$$\begin{aligned}
c + (1-\lambda)G'(y) + m &= \theta y + m" \\
c' &= m + n" \\
c" + n" + m" &= (1-\theta)y \\
n" &= \theta y \\
G'(y) &= \theta + (1-\theta)\beta^3 \\
c &= \beta c", \text{ and } c' = \beta c.
\end{aligned}$$

From these, we learn

$$c' = \beta^2 c" = \beta^2(1-\beta)\left[1 - \theta + \frac{\lambda G'(y)}{1-\beta^3}\right] y.$$

Then we learn

$$\frac{m}{y} = \beta^2(1-\beta)\left(1 + \frac{\lambda\beta^3}{1-\beta^3}\right) - \theta\left[1 + \beta^2(1-\beta)(1-\lambda)\right].$$

But, RHS of this equation is negative by $\theta \geq \theta^{**}$. This is contradiction.

(Case ii: $m = 0$) Since $n" > 0$ and $n = m' = 0$ by Lemma 5 and 7, the flow-of-fund constraints, market clearing conditions and the first order conditions imply:

$$c + \left[G'(y) - \theta q\right] y = \lambda G'(y) y + \lambda\theta\left(p^2 - q\right) z + m" \tag{A16}$$

$$c' = n" \tag{A17}$$

$$c" + pm" + qn" = (1-\theta)y \tag{A18}$$

$$n" = \theta(y - z) \tag{A19}$$

$$m" \geq (1+p)\theta z, \ (= \text{ holds if } p < 1) \tag{A20}$$

$$\frac{c}{c"} = \frac{\beta}{p}, \text{ and } \frac{c'}{c"} = \frac{\beta^2}{q}, \tag{A21}$$

and (17) and (18). From (A17-A21), for $l = z/y$, we have

$$p(1+p)\theta l \leq \frac{pm"}{y} = 1 - \theta - \left(1 + \frac{1}{\beta^2}\right) q\theta(1-l), \text{ or}$$

$$\frac{1-\theta}{\theta} \geq g_0(q) + g_1(p,q)l, \tag{A22}$$

where $g_0(q) \equiv (1+\beta^2)q/\beta^2$, and $g_1(p,q) \equiv p(1+p) - (1+\beta^2)q/\beta^2$. Also adding (A18), p times (A16) and q times (A17) with using (A21) and (17), (A19) implies:

$$\begin{aligned}\theta(1-l) &= \frac{c'}{y} = \frac{\beta^2}{q}(1-\beta)\left[1 - \theta + p\lambda\frac{G'(y) + l\theta(p^2 - q)}{1-\beta^3}\right] \\ &\geq \frac{\beta^2}{q}(1-\beta)\left[1 - \theta + \lambda\frac{\beta^3(1-\theta) + \theta pq}{1-\beta^3}\right], \text{ or}\end{aligned}$$

$$l \leq h_0(p) + h_1(q)\frac{1-\theta}{\theta}, \text{ where}$$

$$h_0(p) = 1 - \frac{\lambda\beta^2 p}{1+\beta+\beta^2}, \text{ and } h_1(q) = -\left(1 - \beta + \frac{\lambda\beta^3}{1+\beta+\beta^2}\right)\frac{\beta^2}{q}.$$

In (A22), if $g_1(p,q) \geq 0$, then we have

$$\frac{1-\theta}{\theta} \geq \frac{1+\beta^2}{\beta^2}q \geq 1 + \beta^2,$$

by Lemma 1. This inequality contradicts the assumption $\theta \geq \theta^{**}$.

If, instead, $g_1(p,q) < 0$, then from (A23), we have

$$\frac{1-\theta}{\theta} \geq g_0(q) + g_1(p,q)\left[h_0(p) + h_1(q)\frac{1-\theta}{\theta}\right],$$

Because $g_1(p,q)h_1(q) < 1$, we have

$$\begin{aligned}\frac{1-\theta}{\theta} &\geq \frac{g_0(q)+g_1(p,q)h_0(p)}{1-g_1(p,q)h_1} \\ &= \frac{p(1+p)+\frac{(1+\beta^2)\lambda pq}{1+\beta+\beta^2}-\frac{\lambda\beta^2p^2(1+p)}{1+\beta+\beta^2}}{\beta-\beta^2+\beta^3-\frac{\lambda\beta^3(1+\beta^2)}{1+\beta+\beta^2}+\frac{\beta^2p(1+p)}{q}\left(1-\beta+\frac{\lambda\beta^3}{1+\beta+\beta^2}\right)}.\end{aligned}$$

From (A10) and (A21), we learn

$$\frac{\beta}{p}=\frac{c}{c"}=\frac{c'}{c"}\frac{c}{c'}\geq\frac{\beta^4}{q^2}, \text{ or}$$

$$q \geq \beta^{3/2}p^{1/2}.$$

Since RHS of (A24) is strictly increasing function of q, by substituting $\beta^{3/2}p^{1/2}$ for q in (A24), we get

$$\frac{1-\theta}{\theta} \geq \frac{p(1+p)+\frac{(1+\beta^2)\lambda\beta^{3/2}p^{3/2}}{1+\beta+\beta^2}-\frac{\lambda\beta^2p^2(1+p)}{1+\beta+\beta^2}}{\beta-\beta^2+\beta^3-\frac{\lambda\beta^3(1+\beta^2)}{1+\beta+\beta^2}+\beta^{1/2}p^{1/2}(1+p)\left(1-\beta+\frac{\lambda\beta^3}{1+\beta+\beta^2}\right)}.$$

Since RHS is strictly increasing function of p, and $p > \beta$ by Proposition 1 and Lemma 1 and 2, by substituting β for p, we get

$$\frac{1-\theta}{\theta} \geq \frac{1+\beta+\frac{\lambda\beta(1-\beta)}{1+\beta+\beta^2}}{2-\beta-\frac{\lambda\beta(1-\beta)}{1+\beta+\beta^2}} = \frac{1-\theta^{**}}{\theta^{**}}.$$

This contradicts the assumption of $\theta \geq \theta^{**}$. *Q.E.D.*

Proof of Proposition 3:

From Lemma 5, Lemma 7 and Lemma 8, the flow-of-funds conditions $(7, 8, 9)$ and the paper market clearing conditions $(11, 12)$ are:

$$c + G(y) + \frac{1-\phi}{\phi}G(z) = q\theta(y-z) + p^2\theta z + m" + n' \tag{A26}$$

$$c' + qn' = n" \tag{A26}$$

$$c" + pm" + qn" = (1-\theta)y \tag{A27}$$

$$\theta(y-z) = n' + n" \tag{A28}$$

$$\text{either } \theta(1+p)z = m" \text{ and } p < 1 \tag{A29}$$

$$\text{or, } \theta(1+p)z \leq m" \text{ and } p = 1.$$

Also, Lemma 5 and 8 and the first order conditions of illiquid paper (A10,A11) imply equation (21) and (22) in the text.

From Lemma 3, (A13) holds with equality as (16) in the text. From Proposition 2 and Lemma 1 and 2, for $\theta < \theta^*$, we learn

$$p > \beta, \text{ and} \tag{A30}$$

$$q > \beta^2. \tag{A31}$$

From (21) and (22) and (A7), we learn

$$\frac{\beta^4}{q^2} = \frac{c}{c"} \geq \frac{\beta}{p}.$$

Then from (A31),

$$p^2 - q \geq q\left(\frac{q^3}{\beta^6} - 1\right) > 0.$$

Then from (A29) and (16), we learn both z and $m"$ are strictly positive, and together with (A7), we know (23) in the text. Then (A12) implies (17) in the text. Combining (21), (22) and (23), we have (19), which implies $q = p^{1/2}\beta^{3/2}$. Equations (17) and (19) also imply (18) in the text. From (18), we learn the marginal return on investment with maximum borrowing $[(1-\theta)/(G'(y) - \theta q)]$ exceeds the rate of return on illiquid paper $1/q$, because $q > \beta^2$ by (A31). Thus the borrowing constraints bind at the time of investment.

From (2) and (16), (A25) can be written as:

$$\left[G'(y) - \theta q\right] y + c = \lambda G'(y)y + \lambda\theta(p^2 - q)z + m" + n'. \tag{A32}$$

By adding (A27), p times (A32) and q times (A26), together with the first order conditions (17), (21), (22) and (23), we get:

$$c" = (1-\beta)\left\{\left[1 - \theta + p\lambda \frac{G'(y) + l\theta(p^2 - q)}{1-\beta^3}\right] y + pn'\right\} \tag{A33}$$

Then, from (22), (A26), (A28) and (19), we have

$$n' = \frac{1}{1+(\beta p)^{1/2}}\left\{\theta(1-l) - (\beta p)^{1/2}(1-\beta)\left[1 - \theta + p\lambda\frac{G'(y) + l\theta(p^2-q)}{1-\beta^3}\right]\right\} y. \tag{A34}$$

Then, from (A27) and (A29), we get:

$$\theta p(1+p)z \leq pm"$$

$$= \frac{1-\beta+\beta^2}{1+(\beta p)^{\frac{1}{2}}}\left\{\left[\beta + (\beta p)^{\frac{1}{2}}\right](1-\theta) - p(1-l)\theta - p\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta^2+\beta^4}\left[G'(y) + l\theta(p^2-q)\right]\right\} y. \tag{A35}$$

Then, combining this with (A29), (17) and (19), we have "paper market equilibrium" schedule:

$$\Psi(p,\theta,\phi) \geq 0, \text{ and } (p-1)\Psi(p,\theta,\phi) = 0, \text{ where} \tag{A36}$$

$$\Psi(p,\theta,\phi) = -p(1+p)\theta l +$$

$$\frac{1-\beta+\beta^2}{1+(\beta p)^{\frac{1}{2}}}\left\{\left[\beta + (\beta p)^{\frac{1}{2}}\right](1-\theta) - p(1-l)\theta - \lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta^2+\beta^4}\left[\theta(\beta p)^{\frac{3}{2}} + (1-\theta)\beta^3 + l\theta p^{\frac{3}{2}}(p^{\frac{3}{2}} - \beta^{\frac{3}{2}})\right]\right\}.$$

Using $(2), (16)$ and (17), we have

$$l = \frac{z}{y} = \left[\frac{\theta\phi}{1-\phi} \frac{p^3 - (\beta p)^{\frac{3}{2}}}{(1-\theta)\beta^3 + \theta(\beta p)^{\frac{3}{2}}} \right]^{\frac{1-\lambda}{\lambda}} = l^S(p, \theta, \phi), \tag{A37}$$

where $\frac{\partial l^S}{\partial p} > 0$, $\frac{\partial l^S}{\partial \theta} > 0$ and $\frac{\partial l^S}{\partial \phi} > 0$. From (A36) and (A37), we have:

$$\frac{\partial \Psi}{\partial p} < 0, \ \frac{\partial \Psi}{\partial \theta} < 0, \text{ and } \frac{\partial \Psi}{\partial \phi} < 0, \text{ for } \theta^{**} \leq \theta < \theta^*. \tag{A38}$$

Here, we observe that $\Psi(p, \theta, \phi) = 0$ at $p = \beta$, if and only if $\theta = \theta^*$. Thus we have

$$p > \beta, \text{ if and only if } \theta < \theta^*.$$

Then, from (A38), we learn p and q are decreasing functions of θ and ϕ, for $\theta^{**} \leq \theta < \theta^*$. Also, from (A31) (21) and (22), we have $c" > c' > c$, and $c"/c'$ and c'/c are decreasing functions of θ and ϕ.

From (A29) and (A36), we see there is storage or valued fiat money, if and only if $\Psi(p, \theta, \phi) > 0$, or $\theta < \Theta(\phi)$ where $\Theta(\phi)$ is defined as a solution of:

$$\Psi(1, \Theta(\phi), \phi) = 0, \text{ where}$$

$$\Psi(1, \theta, \phi) = -\frac{1}{1+\beta^{\frac{1}{2}}} \left[1 + 2\beta^{\frac{1}{2}} + \beta - \beta^2 - \lambda\left(1 - \beta^{\frac{3}{2}}\right) \frac{1 + \beta^{\frac{3}{2}} + \beta^2}{1 + \beta + \beta^2} \right] \theta l^S(1, \theta, \phi)$$
$$+ \frac{1 - \beta + \beta^2}{1 + \beta^{\frac{1}{2}}} \left[\left(\beta + \beta^{\frac{1}{2}}\right)(1-\theta) - \theta - \lambda\left[(1-\theta)\beta^3 + \theta\beta^{\frac{3}{2}}\right] \frac{1 + \beta^{\frac{3}{2}} + \beta^2}{1 + \beta^2 + \beta^4} \right]. \tag{A39}$$

Because $\Psi(1, \theta, \phi)$ is decreasing function of θ and ϕ, we learn $\Theta(\phi)$ is a decreasing function of ϕ. For $\theta < \Theta(\phi)$, $p = 1$ from (A34) and $q = \beta^{3/2}$ from (19).

It only remains to show that output is lower than the first best for $\theta < \theta^*$. First from (17) and

(19), we learn:

$$\theta\left(p^{\frac{3}{2}}-\beta^{\frac{3}{2}}\right)=\frac{p}{\beta^{\frac{3}{2}}}G'(y)-\beta^{\frac{3}{2}}. \tag{A40}$$

First consider the case of $\theta > \Theta(\phi)$. From the above argument, we know $\beta < p < 1$, and

$$\Psi(p,\theta,\phi)=0$$

in (A36). Multiplying both sides by $\left(p^{\frac{1}{2}}-\beta^{\frac{1}{2}}\right)\left[1+(\beta p)^{\frac{1}{2}}\right]$, and using (17), we get:

$$0=-p\left(p^{\frac{3}{2}}-\beta^{\frac{3}{2}}\right)\left[1-\beta+(\beta p)^{\frac{1}{2}}\right]\theta l+\left(1-\beta+\beta^2\right)\left[\beta^{\frac{1}{2}}(p-\beta)-\theta\left(p^{\frac{3}{2}}-\beta^{\frac{3}{2}}\right)\right]$$

$$-\lambda p\left(p^{\frac{1}{2}}-\beta^{\frac{1}{2}}\right)\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta+\beta^2}\left[G'(y)+l\theta\left(p^{\frac{3}{2}}-\beta^{\frac{3}{2}}\right)p^{\frac{1}{2}}\right].$$

Substituting (A40) into this, we can solve for $G'(y)$ as:

$$G'(y)=\beta^2\frac{B}{A},\ \text{where}$$

$$A=pl\left[1-\beta+(\beta p)^{\frac{1}{2}}\right]+1-\beta+\beta^2+\lambda\left(p^{\frac{1}{2}}-\beta^{\frac{1}{2}}\right)\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta+\beta^2}\left(\beta^{\frac{3}{2}}+lp^{\frac{3}{2}}\right)$$

$$B=\beta l\left[1-\beta+(\beta p)^{\frac{1}{2}}\right]+1-\beta+\beta^2+\lambda\left(p^{\frac{1}{2}}-\beta^{\frac{1}{2}}\right)\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta+\beta^2}p^{\frac{1}{2}}\beta l.$$

Because we know $p > \beta$, we learn $A > B$, or $G'(y) < \beta^2$ which implies output is lower than the first best level y^* which satisfies $G'(y^*) = \beta^2$.

Also from (A40), we have:

$$G'(y)=\theta\beta^{\frac{3}{2}}p^{\frac{1}{2}}+(1-\theta)\frac{\beta^3}{p}. \tag{A41}$$

If RHS is a decreasing function of p, then we learn $\frac{\partial y}{\partial \phi} > 0$, because we already know $\frac{\partial p}{\partial \phi} < 0$. Differentiating RHS of (A41) with respect to p, the condition for $G'(y)$ being a decreasing function of p is:

$$\frac{1-\theta}{\theta} > \frac{1}{2}\left(p/\beta\right)^{\frac{3}{2}}. \tag{A42}$$

Because RHS of (A41) is an increasing function of θ, we also know that (A42) is a sufficient condition of $\frac{\partial y}{\partial \phi} > 0$.

Suppose that (A42) is not true. Substituting into (A36), we get:

$$\frac{1}{\theta}\Psi(p,\theta,\phi) \leq -pl\left\{1+p-\frac{1-\beta+\beta^2}{1+(\beta p)^{\frac{1}{2}}}\left[1-\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta^2+\beta^4}(p^2-\beta^{\frac{3}{2}}p^{\frac{1}{2}})\right]\right\}$$

$$+\frac{1-\beta+\beta^2}{1+(\beta p)^{\frac{1}{2}}}\frac{p}{2\beta}\left\{\left[p+(\beta p)^{\frac{1}{2}}\right]-2\beta-3\beta^{\frac{5}{2}}p^{\frac{1}{2}}\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta^2+\beta^4}\right\}.$$

Because the expression in the bracket in the second line is increasing of p, we learn $\lambda \geq \underline{\lambda}$ leads to $l < 0$ from (A36). Because l is non-negative, this is contradiction. Thus we prove (A42), and thus $\frac{\partial y}{\partial \phi} > 0$ and $\frac{\partial y}{\partial \theta} > 0$.

For $\theta \leq \Theta(\phi)$, we know $p = 1$ and $q = \beta^{3/2}$. Thus from (20), we have:

$$G'(y) = \theta\beta^{\frac{3}{2}} + (1-\theta)\beta^3. \tag{A43}$$

We already showed $y < y^*$ for $\Theta(\phi) < \theta < \theta^*$. Then, because RHS of (A41) is an increasing function of θ and does not depend upon ϕ, we learn y is an increasing function of θ and does not depend upon ϕ, for $\theta \in [\theta^{**}, \Theta(\phi)]$. *Q.E.D.*

Proof of Proposition 4:

< The Economy with Outside Money: $\theta \in [\theta^{**}, \Theta(\phi)]$ >

Because $p = 1$ here, we learn from (A37) that the ratio of inside money to GDP, $p(1+p)\theta z/y$ $= p(1+p)\theta l$ is ah increasing function of θ and ϕ. From (A35), the ratio of outside money to GDP (if there is no storage) is equal to $\Psi(1,\theta,\phi)$ in (A39). From (A37), we learn the ratio of outside money to GDP, $\Psi(1,\theta,\phi)$ is a decreasing function of θ and ϕ. The ratio of total money to GDP is given from (A35) and (A37) as:

$$\frac{pm"}{y} = \frac{1-\beta+\beta^2}{1+\beta^{\frac{1}{2}}}\left\{\left[\beta+\beta^{\frac{1}{2}}\right](1-\theta)-(1-l)\theta-\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}}{1+\beta^2+\beta^4}\left[\beta^3+\theta\left(1-\beta^{\frac{3}{2}}\right)\left(\beta^{\frac{3}{2}}+l\right)\right]\right\} \tag{A44}$$

$$\text{where } l = \left[\frac{\theta\phi}{1-\phi}\frac{1-\beta^{\frac{3}{2}}}{(1-\theta)\beta^3+\theta\beta^{\frac{3}{2}}}\right]^{\frac{1-\lambda}{\lambda}}.$$

Because RHS of (A44) is an increasing function of l, and l is an increasing function of ϕ, we learn the ratio of total money to GDP is an increasing function of ϕ. Differentiating RHS with respect to θ, we also get:

$$\begin{aligned}
&\frac{1+\beta^{\frac{1}{2}}}{1-\beta+\beta^2}\frac{\partial(pm"/y)}{\partial\theta} \\
= \quad & -\left(1+\beta^{\frac{1}{2}}+\beta\right)-\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}}{1+\beta^2+\beta^4}\left(\beta^{\frac{3}{2}}-\beta^3\right)+\left[1-\lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}}{1+\beta^2+\beta^4}\left(1-\beta^{\frac{3}{2}}\right)\right]\frac{l}{\lambda}\frac{\beta^{\frac{3}{2}}+\lambda\theta\left(1-\beta^{\frac{3}{2}}\right)}{\beta^{\frac{3}{2}}+\theta\left(1-\beta^{\frac{3}{2}}\right)}.
\end{aligned}$$

The first two negative terms are the direct effect of θ on $(pm"/y)$ while the last positive term is the effect through l. The last term can dominate the first two terms if the elasticity of l with respect to θ is large (or λ is small). But, we can show RHS is negative if $\lambda \geq 1/3$, because $l \leq 1$.

< The Economy without Outside Money: $\theta \in [\Theta(\phi),\ \theta^*]$ >

Here, we know $\Psi(p,\theta,\phi)=0$ in (A36). Solving this for l, the ratio of inside money to GDP is:

$$\begin{aligned}
L &= p(1+p)\theta l \\
&= \frac{1+p}{\beta+(\beta p)^{\frac{1}{2}}+p}\frac{\left(1-\beta+\beta^2\right)\left\{\beta+(\beta p)^{\frac{1}{2}}-\theta\left[\beta+(\beta p)^{\frac{1}{2}}+p\right]\right\}-\left[\beta^3+\theta\left(\beta^{\frac{3}{2}}-\beta^3\right)\right]\Lambda(p)}{1-\beta+(\beta p)^{\frac{1}{2}}+\left[p-(\beta p)^{\frac{1}{2}}\right]\Lambda(p)}
\end{aligned}$$

$$\text{where } \Lambda(p) = \lambda\frac{1+\beta^2+\beta^{\frac{3}{2}}p^{\frac{1}{2}}}{1+\beta+\beta^2}.$$

Rearranging this, we get:

$$A(p) - B(p)L = p(1+p)\theta \tag{A45}$$

$$\text{where } A(p) = \frac{p(1+p)}{\beta + (\beta p)^{\frac{1}{2}} + p} \frac{\left(1-\beta+\beta^2\right)\left[\beta + (\beta p)^{\frac{1}{2}}\right] - \beta^3 \Lambda(p)}{1 - \beta + \beta^2 + \beta^{\frac{3}{2}}\left(p^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right)\Lambda(p)}$$

$$\text{and } B(p) = p\frac{1 - \beta + (\beta p)^{\frac{1}{2}} + p^{\frac{1}{2}}\left(p^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right)\Lambda(p)}{1 - \beta + \beta^2 + \beta^{\frac{3}{2}}\left(p^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right)\Lambda(p)}.$$

Substituting this into (A37), we have:

$$\begin{aligned} 0 &= \Phi(p, L) \\ &= L\left\{1 + \frac{\beta^{\frac{3}{2}} p(1+p)}{\left(p^{\frac{3}{2}} - \beta^{\frac{3}{2}}\right)[A(p) - B(p)L]}\right\}^{\frac{1-\lambda}{\lambda}} - [A(p) - B(p)L]\left(\frac{\phi}{1-\phi}\frac{p^{\frac{3}{2}}}{\beta^{\frac{3}{2}}}\right)^{\frac{1-\lambda}{\lambda}}. \end{aligned} \tag{A46}$$

Because $A(p) - B(p)L > 0$ by (A45), we learn $\frac{\partial \Phi(p,L)}{\partial L} > 0$. We also know $\frac{\partial p}{\partial \theta} < 0$ from Proposition 3. Thus the necessary and sufficient condition for $\frac{\partial L}{\partial \theta} < 0$ is $\frac{\partial \Phi(p,L)}{\partial p} < 0$, and, from (A46), the sufficient condition for this is:

$$A'(p) - B'(p)L > 0. \tag{A47}$$

From (A45), we know $B'(p) > 0$ and $L \leq [A(p) - p(1+p)\theta^{**}]/B(p)$ by $\theta \geq \theta^{**}$. Thus, the sufficient condition for (A47) is

$$\frac{B(p)}{A(p)}\frac{A'(p)}{B'(p)} > 1 - \frac{p(1+p)\theta^{**}}{A(p)}. \tag{A48}$$

Since RHS is a decreasing function of p and $p \geq \beta$ by Lemma 1, the sufficient condition of (A48) is

$$\frac{B(p)}{A(p)}\frac{A'(p)}{B'(p)} > 1 - \frac{B(p)(1+\beta)\theta^{**}}{A(p)} = \frac{\beta(1+\beta^2+\beta^4) - \lambda\beta^3(2+\beta^2)}{2(1+\beta^2+\beta^4) - \lambda\beta^2(1+2\beta^2)} \tag{A49}$$

Because RHS is a decreasing function of λ, the sufficient condition for (A49) and thus for $\frac{\partial L}{\partial \theta} < 0$ is:

$$\frac{\partial}{\partial p}\left[\log A(p)\right] - \frac{\beta}{2}\frac{\partial}{\partial p}\left[\log B(p)\right] > 0. \tag{A50}$$

A tedious but straightforward calculation shows that (A50) is true, and thus we learn $\frac{\partial L}{\partial \theta} < 0$.

In order to prove $\frac{\partial L}{\partial \phi} > 0$, we rewrite (A46) as:

$$L = \frac{A(p)}{B(p)} - \frac{\theta}{\frac{B(p)}{p(1+p)}} = \frac{\frac{A(p)}{p(1+p)} - \theta}{\frac{B(p)}{p(1+p)}}. \tag{A51}$$

Since we know $\frac{\partial p}{\partial \phi} < 0$ by Proposition 3, we learn $\frac{\partial L}{\partial \phi} > 0$ if and only if RHS of (A51) is a decreasing function of p. We can check $[A(p)/p(1+p)]$ is a strictly decreasing function of p. Thus, if $[B(p)/p(1+p)]$ is an increasing function of p, then we learn RHS is a decreasing function of p. If $[B(p)/p(1+p)]$ is an increasing function of p, then

$$\frac{\partial}{\partial p}[\log A(p)] - \frac{\partial}{\partial p}[\log B(p)] < 0. \tag{A52}$$

is the sufficient condition for RHS of (A51) to be a decreasing function of p. Therefore, (A52) is the sufficient condition for $\frac{\partial L}{\partial \phi} > 0$. Again, a tedious and straightforward calculation shows (A52) holds. *Q.E.D.*

7.2.1 Appendix C: Adverse Selection of Second-Hand Paper

In order to prove that Assumption 2 is sufficient for the market for unbundled second-hand paper to breakdown, we describe the economy with active market for unbundled second-hand paper. Let r be the price of unbundled second-hand paper that is to mature in the next period. Let n, n' and n" be the holding of unbundled papers agents hold until maturity unless the collateral projects turn out to be failing, and let m, m' and m" be the planned liquid saving at the time of investment, growing and harvesting. The flow-of-funds conditions are:

$$c + G(y) + \frac{1-\phi}{\phi}G(z) + pm + qn = q\theta(y-z) + p^2\theta z + m" + n' + r\alpha n" \tag{A53}$$

$$c' + pm' + qn' = m + n" + r\alpha n \tag{A54}$$

$$c" + pm" + qn" = (1-\theta)y + m' + n + r\alpha n' \tag{A55}$$

The difference from the model of the text is that people sell the unbundled papers which is secured by the failing parts of project (fraction α) before maturity at price r.

Let S be the total quantity of newly issued unbundled papers people plan to sell before maturity even if the collateral projects turn out to be successful. S must be strictly positive for market of the unbundled second-hand paper to be active, because, otherwise, there are only unbundled papers against failing parts (lemons) traded. Since all the means of liquid saving must earn the same rate of return in equilibrium, we have:

$$\frac{1}{p} = \frac{r}{q} \tag{A56}$$

$$= \frac{\frac{S}{S+\alpha(n+n'+n")}}{r}. \tag{A57}$$

The first term is the rate of return of liquid paper secured by the bundled projects. The second term is the rate of return of unbundled paper between the issue date and the middle date, when agents choose to always sell entirely in the middle date, even if a fraction of the paper's collateral projects will be successful. The last term is the rate of return on the unbundled second-hand paper between the middle date and the maturity date: The total payoff of goods in maturity date is S, while the total number of papers in the market is equal to $S + \alpha(n + n' + n")$. The market clearing conditions for new unbundled papers and liquid saving are:

$$\theta(y - z) = n + n' + n" + S \tag{A58}$$

$$(1+p)\theta z + (1+r)S \leq m + m' + m", \text{ and equality holds if } p < 1. \tag{A59}$$

In (A58), the demand for new unbundled papers includes S, the total quantity of new unbundled papers people always sell entirely in the next period. In (A59), the supply of liquidity includes the maturity value of S and the resale value of S in the unbundled second-hand paper market.

The first order conditions for the choice of new unbundled papers, liquid papers, output and the bundling are:

$$\frac{q}{c} \geq \frac{r\alpha\beta}{c'} + \frac{\beta^2}{c"}, \text{ and } \left(\frac{q}{c} - \frac{r\alpha\beta}{c'} - \frac{\beta^2}{c"}\right) n = 0, \tag{A60}$$

$$\frac{q}{c'} \geq \frac{r\alpha\beta}{c"} + \frac{\beta^2}{c}, \text{ and } \left(\frac{q}{c'} - \frac{r\alpha\beta}{c"} - \frac{\beta^2}{c}\right) n' = 0, \tag{A61}$$

$$\frac{q}{c"} \geq \frac{r\alpha\beta}{c} + \frac{\beta^2}{c'}, \text{ and } \left(\frac{q}{c"} - \frac{r\alpha\beta}{c} - \frac{\beta^2}{c'}\right) n" = 0, \tag{A62}$$

and (A5,A6,A7,Al2,A13). The competitive equilibrium is characterized by 15 variables $(y, z, S, p, q, r, c, c', c", m, m', m", n, n', n")$ which satisfy 15 independent equilibrium conditions (A53-A62) and (A5,A6,A7,Al2,Al3).

Because $S > 0$, at least one of m, $m'$and $m"$ is strictly positive by (A59). Suppose that $m" > 0$ without loss of generality. Then from (A7), we have $c = \beta c"/p$. Substituting this into (A61) and (A62), using $q = pr$ from (A56), we obtain:

$$\frac{\beta\,(\alpha r + p)}{pr} \leq \frac{c"}{c'} \leq \frac{pr(1-\alpha)}{\beta^2}, \text{ or}$$

$$F(p, r) \equiv (1-\alpha)p^2r^2 - \alpha\beta^3 r - \beta^3 p \geq 0.$$

For (p, r) which satisfies $F(p, r) \geq 0$, we learn:

$$p\frac{\partial F(p,r)}{\partial p} = 2(1-\alpha)p^2r^2 - \beta^3 p \geq 2\alpha\beta^3 r + \beta^3 p > 0,$$

$$r\frac{\partial F(p,r)}{\partial r} = 2(1-\alpha)p^2r^2 - \alpha\beta^3 r \geq \alpha\beta^3 r + 2\beta^3 p > 0.$$

We know $p \in [\beta, 1]$ and $r \leq p$ from (A56) and (A57). Thus we learn

$$0 \leq F(1, 1) = 1 - \alpha - \alpha\beta^3 - \beta^3. \tag{A64}$$

This contradicts Assumption 2. Therefore, there is no equilibrium in which unbundled papers are actively traded in the second-hand market under Assumption 2. *Q.E.D.*

7.2.2 Appendix D: Proof of No Delay of Investment

In our economy under liquidity shortage, $\theta < \theta^*$, we know $p > \beta$ and $\beta^2 < q$ from Proposition 3. When an agent invests every 3 periods, fully engaging in some stage of production every period as in our equilibrium, the present value of utility at the time of harvest is:

$$\begin{aligned} V &= \frac{1}{1-\beta^3}\left(\log c" + \beta \log c + \beta^2 \log c'\right) \\ &= \frac{1}{1-\beta}\log c" + \frac{1}{1-\beta^3}\left[\beta \log\left(\frac{c}{c"}\right) + \beta^2 \log\left(\frac{c'}{c"}\right)\right] \\ &= \frac{1}{1-\beta}[\log(1-\beta) + \log W] - \frac{\beta + \frac{\beta^2}{2}}{1-\beta^3}\ln\left(\frac{p}{\beta}\right), \end{aligned} \tag{A65}$$

where

$$\begin{aligned} W &= (1-\theta)y + pn' + \frac{p\pi}{1-\beta^3}, \text{ and} \\ \pi &= \lambda\left[G'(y) + z\theta\left(p^2 - q\right)y\right], \text{ and} \end{aligned}$$

$G'(y)$ satisfies equation (17). Here π is the "profit", the reward to the human capital of the agent, and W is the total wealth at harvest time including the present value of profit.

Suppose that an agent at harvest time (period 0) considers an alternative sequence of production in which the agent will not necessarily invests every 3 periods, i.e., he may take rests between harvest and investment for one or more periods some time in future. We give such agent a simpler and more favorable environment than that of the text as:

(i) only in the harvest period, the agent can sell nonresaleable paper that matures in the next period at price p;

(ii) if the agent does not take a rest after the previous harvest, he can produce at constant marginal cost of $G'(y)$ in equation (17) with no bundling technology, and he is endowed with π (in (A65)) unit of goods at the time of investment;

{iii) if the agent takes a rest after the previous harvest, he can produce at the constant marginal cost $G'(\widetilde{y})$ with no bundling technology, and he is endowed with $\widetilde{\pi}$ units of goods at the time of investment, where

$$G'(\widetilde{y}) = \theta q + (1-\theta)\frac{\beta^4}{q}, \text{ and} \tag{A66}$$

$$\widetilde{\pi} = \lambda[\widetilde{y}G'(\widetilde{y}) + z\theta\left(p^2 - q\right)]. \tag{A67}$$

Under this modified environment, the cost of investment net of endowment is equal or less than the original cost function (2). Also, when the agent saves by non-resaleable paper at harvest time and invests in two periods later, the compound rate of return for 4 period cycle from a harvest to the next harvest is equal the time preference rate for 4 periods as:

$$\frac{1}{q}\frac{1-\theta}{G'(\widetilde{y}) - \theta q} = \frac{1}{\beta^4}.$$

If the agent takes a rest of more than one period (so that it takes more than 4 periods to go from a harvest to the next harvest), the compound rate of return would be less than the time preference rate, because the rates of returns on saleable paper and non-resaleable paper are both lower than the time preference rate. Thus, the agent would be worse off by taking rests of more than one period, because the present value of endowment would be lower than taking rest for just one period, and it is more difficult to smooth consumption. Therefore, we only have to show that the agent would be worse off, if he takes rests of one period after some harvest (so that producing in 4 period cycle instead of 3 period cycle) some time in future.

Consider an arbitrary sequence which consists of a finite {possibly 0) number of 4 period production cycles and infinite number of 3 period production cycles. Change to a new sequence with an additional 4 period cycle, positioned later than any other 4 period cycles. Suppose this additional 4 period cycle starts at harvest time, period T. From (17) and (A68), we know the compound rate of return from a harvest to the next harvest is always equal to the time preference, and thus consumption at any harvest time is the same as date 0 consumption:

$$c_t = c_0, \text{ for any } t \text{ in the harvest period.}$$

For the new sequence, the first order conditions for consumption and saving choice implies:

$$c_{T+1} = \frac{\beta}{p} c_T = \frac{\beta}{p} c_0,$$

$$c_{T+2} = \frac{\beta^2}{q} c_T = \left(\frac{\beta}{p}\right)^{\frac{1}{2}} c_0,$$

$$c_{T+3} = \frac{\beta^3}{pq} c_T = \left(\frac{\beta}{p}\right)^{\frac{3}{2}} c_0,$$

Now we can compare the discounted utility of the old sequence and the new sequence of production (with an additional 4 period cycle), V^{old} and V^{new}:

$$\begin{aligned} V^{old} &= \frac{1}{1-\beta}\log c_0 + \sum_{t=0}^{\infty} \beta^t \log\left(\frac{c_t}{c_0}\right) \\ &= \frac{1}{1-\beta}[\log(1-\beta) + \log W^{old}] + \sum_{t=0}^{T} \beta^t \log\left(\frac{c_t}{c_0}\right) - \beta^T \frac{\beta + \frac{\beta^2}{2}}{1-\beta^3} \log\left(\frac{p}{\beta}\right). \end{aligned} \tag{A70}$$

$$\begin{aligned} V^{new} &= \frac{1}{1-\beta}[\log(1-\beta) + \log W^{new}] + \sum_{t=0}^{T} \beta^t \log\left(\frac{c_t}{c_0}\right) \\ &\quad -\beta^T \left(\beta + \frac{\beta^2}{2} + \frac{3\beta^3}{2}\right) \log\left(\frac{p}{\beta}\right) - \beta^{T+4} \frac{\beta + \frac{\beta^2}{2}}{1-\beta^3} \log\left(\frac{p}{\beta}\right), \end{aligned} \tag{A71}$$

where W^{old} and W^{new} are date 0 total wealth of the old and the new sequences. Because both date 0 saving and the production sequence up to date T are the same between the old and the new sequences, the total wealth differ by the present value of profits:

$$\begin{aligned} W^{new} - W^{old} &= \beta^T q\widetilde{\pi} + \beta^{T+4}\frac{p\pi}{1-\beta^3} - \beta^T\frac{p\pi}{1-\beta^3} \\ &= \beta^T\left(1-\beta^4\right)\left(\frac{q\widetilde{\pi}}{1-\beta^4} - \frac{p\pi}{1-\beta^3}\right). \end{aligned} \tag{A72}$$

The new sequence has an additional 4-period cycle starting from date T, and the following 3-period cycles starts at date $T+4$ instead of date $T+3$. We also know the compound rate of return on saving and the ratio of consumption (c_t/c_0) are the same up to date T between the old and the new sequences. Therefore, from (A70) and (A71), we learn:

$$V^{new} - V^{old} = \frac{1}{1-\beta}\left(W^{new} - W^{old}\right) - \beta^{T+3}\frac{\frac{3}{2}+\frac{\beta}{2}+\beta^2}{1+\beta+\beta^2}\log\left(\frac{p}{\beta}\right). \tag{A73}$$

Since $\log(p/\beta) > 0$ by $p > \beta$, the sufficient condition for $V^{new} < V^{old}$ is that

$$\frac{q\widetilde{\pi}}{1-\beta^4} \leq \frac{p\pi}{1-\beta^3} \tag{A74}$$

Under this condition, the agent would be worse off by adding another 4-period cycles from any arbitrary sequence of production. Then, he would be worse off by moving from the original sequence of producing every 3 period to any sequence that contains 4-period cycles. Because the agent who produce every 3 period under the modified environment enjoys the same utility with the equilibrium of the original economy in the text, and because the agent who take rests derive lower utility despite he has favorable environment than the original economy, no body will take take any rests in the equilibrium under the condition of (A74).

Using equations (A65),(A66),(A67),(17) and (2), we have

$$\frac{\pi}{\widetilde{\pi}} = \frac{yG'(y) + z\theta\left(p^2-q\right)}{\widetilde{y}G'(\widetilde{y}) + z\theta\left(p^2-q\right)} \geq \frac{yG'(y)}{\widetilde{y}G'(\widetilde{y})} = \left[\frac{\theta q + (1-\theta)\frac{\beta^3}{p}}{\theta q + (1-\theta)\frac{\beta^4}{q}}\right]^{\frac{1}{\lambda}}$$

$$= \left[1 - \frac{1 - \frac{q}{p}}{1 + \frac{\theta}{1-\theta}\frac{q^2}{\beta^4}}\right]^{\frac{1}{\lambda}} > 1 - \frac{1}{\lambda}\frac{1 - \left(\frac{\beta}{p}\right)^{\frac{1}{2}}}{1 + \frac{\theta}{1-\theta}\frac{p}{\beta}} = 1 - \frac{1}{\lambda}\frac{1 - \left(\frac{\beta}{p}\right)^{\frac{1}{2}}}{1 + \frac{\theta^{**}}{1-\theta^{**}}\frac{p}{\beta}}. \tag{A75}$$

The last inequality comes from $\theta \geq \theta^{**}$. Thus the sufficient condition for (A74) is

$$\lambda \geq \frac{1 - \left(\frac{1}{p'}\right)^{\frac{1}{2}}}{1 - \frac{\beta-\beta^4}{1-\beta^4}\left(\frac{1}{p'}\right)^{\frac{1}{2}}} \frac{1}{1 + p'\frac{2-\beta-\lambda\frac{\beta^2(1-\beta)}{1+\beta+\beta^2}}{1+\beta+\lambda\frac{\beta^2(1-\beta)}{1+\beta+\beta^2}}} = H(p', \beta) \tag{A76}$$

where $p' = (p/\beta) \in (1, 1/\beta]$. RHS of (A76) is an increasing function of p', for a given β, if $\beta \geq 7/8$. Thus substituting p' for $1/\beta$, we can derive that a sufficient condition for (A76) is that $\lambda \geq 4(1-\beta)/3$ for $\beta \geq 7/8$. More generally, $H(p', \beta)$ is an increasing function of β for a given p'. Thus substituting for $\beta = 1/p'$, a tedious but straightforward calculation leads us to find that a sufficient condition for (A76) is $\lambda \geq 1/6$. Therefore, if

$$\lambda \geq \min\left[\frac{1}{6}, \frac{4(1-\beta)}{3}\right], \tag{A77}$$

then the condition (A76) and (A74) is satisfied and the agent will not delay investment. $Q.E.D.$