Optimism and Speculation in Bayesian Games with Mutual Acts*

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Abstract

Various well known agreement theorems show that if players have a common prior and there is common knowledge of posteriors (Aumann 1976), common knowledge of feasible trade (Milgrom & Stokey 1982), or common knowledge of actions (Geanakoplos 1994), then two players cannot agree to disagree or agree to forgo a Pareto optimal outcome simply because of private information. The Nonspeculation Theorem of (Geanakoplos 1994) weakens these common knowledge requirements by only requiring common knowledge of rationality to rule out equilibrium speculation. In this paper we extend this theorem and show that for a broad class of Bayesian games, that include trade and speculation games as special cases, taking mutual acts when such actions lead to a socially inefficient payoff cannot be part of any Bayesian Nash equilibrium.

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1 Introduction

In many situations, from trade to war to contracts to the tango, certain outcomes require that two individuals both be willing to participate. Regardless of the particulars of such "mutual acts," decision-makers in these situations face a similar strategic problem. The mutual act offers the possibility of higher rewards than not participating, but the higher benefit can only go to one of the two participants (i.e., nonparticipation is Pareto optimal). In such situations, informal intuition suggests that the mutual act could occur due to "mutual optimism" or "agreeing to disagree"; because of private information, both decision-makers could believe that they were likely to receive the high benefit of the mutual act.

Of course, it is well known from various agreement and no-trade theorems that such outcomes cannot arise from rational players that share some sort of common knowledge. The literature has shown that if players have a common prior and there is common knowledge of posteriors (Aumann 1976), common knowledge of feasible trades (Milgrom & Stokey 1982), or common knowledge of actions (Geanakoplos 1994), then two players cannot agree to disagree or agree to forgo a Pareto optimal outcome simply because of private information. However, even this weakest condition, common knowledge of actions, seems like a strong requirement. As a theoretical matter, actions need not be common knowledge in a Bayesian Nash equilibrium and as a practical matter, common knowledge is unlikely to occur in decentralized trading systems or in settings like war where there is an incentive to disguise actions.

Taking this concern seriously, Geanakoplos (1994) proved the important but seemingly overlooked "Nonspeculation Theorem" that establishes that if all players in a Bayesian game have a veto action that results in a Pareto optimal outcome, then the veto outcome is the unique Bayesian Nash equilibrium outcome. The only common knowledge condition needed for this result is the standard assumption that the rationality of players is common knowledge. By eliminating the need for players to possess common knowledge of actions, this weaker assumption broadens the scope of the no-trade literature to apply to Bayesian games with veto actions. However, this scope is still limited by the conditions imposed by the Nonspeculation Theorem on the class of such games.

In this paper, we extend the scope of the Nonspeculation Theorem to a broad class of Bayesian games that include trade and speculation games as special cases and offer a different, more intuitive proof of the result that highlights its connection with other results in the no-trade and agreeing to disagree literature. Because our result is quite general with respect to game form, it further extends the ground over which the logic of no-trade and nonspeculation results apply. In our extension of the Nonspeculation Theorem, there can be many outcomes to the game, the mutual act may be the consequence of many different strategy profiles, there may be distributional consequences to different types of mutual acts, there may be coordination considerations at play, and the payoffs to all outcomes can be state dependent. Even though it is possible to prove our result using extensions of the arguments in Geanakoplos (1994), here we offer an alternative logic that makes clear why the weaker condition of common knowledge of rationality is sufficient for the nonspeculation result. We show that because the strategies of players must be measurable with respect to the the $\sigma-$algebra of the information structure determined by the player's information partitions and because the possibility of a mutual act occurring only affects payoffs when a players's choice is pivotal, the expected value of a strategy that might lead to a mutual act will be assessed as if the player "knew" something more than their private information. As in the case of the winner's curse in auctions (Milgrom & Weber 1982), the swing voter's curse in elections (Feddersen & Pesendorfer 1996), and mutual optimism and war (Fey & Ramsay 2007), it turns out that the joint nature of a mutual act makes the set of pivotal states the same for both players. But conditioning on the same states makes it impossible for both players to want to take a socially inferior action, which establishes the result.

The next section lays out the class of models for which the nonspeculation result holds. Section 3 gives our result and proof. The fourth section presents two examples that help understand the conditions and limits of our result. The final section concludes.

2 Model

Let $i = 1, 2$ be two players and let Ω be a finite set of *states*, where each $\omega \in \Omega$ describe a possible state of the world. We assume that the players share a *common prior* $\pi(\omega)$ over the state space. Let P_i be player i's *information partition* which we assume is partitional. As is standard, we interpret P_i as representing knowledge in the sense that for any event $E \subset \Omega$, if $P_i(\omega) \subseteq E$, then player i knows E has occurred at ω.

Now suppose there is one or more mutual acts that could result from the combined choices of the two players. The process by which one of these mutual acts occurs or not is captured by a two-player finite strategic-form game G. The set of *actions* for player i is given by a finite set A_i. Depending on the choice of actions by the two players, the *outcome* of the game is that one of the possible mutual acts occurs or none do. That is, the range of the

		Player 2		
		Trade in NY	Trade in NJ	No Trade
	Trade in NY	J_1	N	N
Player 1	Trade in NJ	N	J_2	N
	No Trade	N	N	N

Figure 1: A Trading Game with Coordination.

outcome function of G is given by a set of outcomes $O = \{J_1, J_2, J_3, \ldots, J_K, N\}$, where J_k represents a given mutual act and N represents the the outcome in which no mutual act occurs. A particular outcome function of a game G, therefore, is a map $g : A_1 \times A_2 \to O$.

We are completely general about the form of this interaction and which choices lead to which outcome, with the following caveat. As each player always has the choice of not participating in any mutual act, we assume that for each player i there is an action $\tilde{a}_i \in A_i$ such that $g(\tilde{a}_i, a_j) = N$ for all $a_j \in A_j$, $j \neq i$. One can think of this as a veto action in which the player can guarantee the outcome in which no mutual act occurs. This outcome may correspond to no trade, acquittal under unanimity rule in a jury, or refusing to agree to a contract. Without loss of generality, we assume there is exactly one strategy for player i with this property. We will denote the set of games that satisfy these conditions by $\mathcal{G}$.

In order to understand how our result broadens the scope of the Nonspeculation Theorem, note that Geanakoplos' result assumes that the Pareto optimal "status quo" outcome occurs if and only if *all* players choose the veto action. In our result, however, the no mutual act outcome occurs whenever a single player chooses the veto action and can also arise in the absence of veto actions as dictated by the game form under consideration. A useful illustration of this difference is a game in which trade requires coordination. For example, Figure 1 gives a game in which players must coordinate on their location in order for trade to occur. In this game the no-trade outcome occurs not only when one of the players refuses to trade, but also when both players choose to pursue a trade but fail to coordinate on their locations. This additional possibility is not covered by the Nonspeculation Theorem. The importance of coordination is strengthened because it may be more beneficial for one of the players to trade in one location, say New York, than in the other location.

As the payoffs of the game to the two players can depend on the state of world when a mutual act occurs, as well as when none of the mutual acts occur, we denote these payoffs by $u_i(J_k, \omega)$ and $u_i(N, \omega)$ for $i = 1, 2$. In keeping with our motivation, we assume that the outcome of any mutual act is socially inefficient. For example, war is always costly, in every state of the world. Formally, we say that the outcome N *socially dominates* the outcome J_k if $\sum u_i(N, \omega) > \sum u_i(J_k, \omega)$ for all $\omega \in \Omega$. In addition, we say that the outcome N is *socially efficient* if it socially dominates every outcome J_k.

We now define *strategies* for each player. We reflect the fact that players can condition their choice of action on their private information by defining a (mixed) strategy s_i as a function $s_i : \Omega \to \Delta A_i$ with the restriction that

$$P_i(\omega) = P_i(\omega') \quad \Rightarrow \quad s_i(\omega) = s_i(\omega').$$

This condition states that if a player cannot distinguish state ω from state ω', then its action distribution must be the same in both states. The set of all strategies for player i is denoted S_i.

3 Result

We now give our main result.

Theorem 1 *Suppose that for a game $G \in \mathcal{G}$, players have a common prior, P_i is partitional for $i = 1, 2$, and the outcome N is socially efficient. Then there is no Bayesian-Nash equilibrium of G in which a socially dominated mutual act occurs with positive probability.*

Proof: Suppose not. Let the strategy profile (s_1^*, s_2^*) be such a Bayesian Nash equilibrium, where some socially dominated mutual act J_k occurs with positive probability. Let T be the non-empty set of states where the outcome of $g(s_1^*(\omega), s_2^*(\omega))$ is something other than N with positive probability. For every $\hat{\omega} \in T$, it must be that the strategy $s_i^*(\hat{\omega})$ places positive probability on some action that is not $\tilde{a}_i$.

To simplify notation, let $\sigma_k^*(\omega)$ be the probability that outcome $o_k \in O$ occurs under the equilibrium strategy (s_1^*, s_2^*). Then the expected utility to player i of the strategy profile (s_1^*, s_2^*) for a given ω is

$$E[u_i(s_1^*(\omega), s_2^*(\omega)) \mid \omega] = \sum_{o_k \in O} \sigma_k^*(\omega) u_i(o_k, \omega)$$

As each player can impose outcome N by playing $\tilde{a}_i$ but this deviation changes the payoff to player i only if N would not have occurred anyway, equilibrium requires that

$$E[E[u_i(s_1^*(\omega), s_2^*(\omega)) \mid \omega] \mid P_i(\omega') \cap T] \geq E[u_i(N, \omega) \mid P_i(\omega') \cap T] \tag{1}$$

for every $\omega' \in T$.

As the correspondence P_i is partitional, we can define a set of states D_i^* with $D_i^* \subseteq T$ such that the sets $\{P_i(\omega)\}_{\omega \in D_i^*}$ are disjoint and their union is equal to $\bigcup_{\hat{\omega} \in T} P_i(\hat{\omega})$. Since $D_i^* \subseteq T$, we have from inequality (1) that

$$E[E[u_i(s_1^*(\omega), s_2^*(\omega)) \mid \omega] \mid P_i(\hat{\omega}) \cap T] \geq E[u_i(N, \omega) \mid P_i(\hat{\omega}) \cap T]$$

for every $\hat{\omega} \in D_i^*$. As this holds for each disjoint set $P_1(\hat{\omega})$, then the same conditional expectation inequality holds over the union of these disjoint sets. Therefore,

$$E[E[u_i(s_1^*(\omega), s_2^*(\omega)) \mid \omega] \mid \bigcup_{\hat{\omega} \in D_i^*} P_i(\hat{\omega}) \cap T] \geq E[u_i(N, \omega) \mid \bigcup_{\hat{\omega} \in D_i^*} P_i(\hat{\omega}) \cap T].$$

Because $\bigcup_{\hat{\omega} \in D_i^*} P_i(\hat{\omega}) = \bigcup_{\hat{\omega} \in T} P_i(\hat{\omega})$, it follows that $\bigcup_{\hat{\omega} \in D_i^*} P_i(\hat{\omega}) \cap T = T$. Thus, we have

$$E[E[u_i(s_1^*(\omega), s_2^*(\omega)) \mid \omega] \mid T] \geq E[u_i(N, \omega) \mid T].$$

From this we conclude that

$$E[\sum_{o_k \in O} \sigma_k^*(\omega) u_i(o_k, \omega) \mid T] \geq E[u_i(N, \omega) \mid T] \quad \text{for } i = 1, 2. \tag{2}$$

To arrive at a contradiction, recall that for every $\omega' \in T$, $\sigma_k^*(\omega')$ must be positive for some socially dominated J_k. Because the outcome N socially dominates every J_k, it follows that

$$\sum_i \sum_{o_k \in O} \sigma_k^*(\omega') u_i(o_k, \omega') < \sum_i u_i(N, \omega')$$

for every $\omega' \in T$. Therefore, it must be that

$$\sum_i E[\sum_{o_k \in O} \sigma_k^*(\omega) u_i(o_k, \omega) \mid T] < \sum_i E[u_i(N, \omega) \mid T].$$

This contradicts inequality (2) and establishes the result. ∎

This result is easily extended to the case with $n > 2$ players. The proof turns on the intuitive fact that a player's optimal choice is conditioned on a set of states that are a subset of those their private information tells them are possible. This set of choice-relevant, or "pivotal," states are those where the decision not to select the player's veto action will have consequence. The joint nature of the players' actions, and the fact that information sets are partitions, then imply that the set of states that are decision-relevant are the same for both players. As there is a common prior, both players cannot simultaneously be optimistic about the resulting outcome of the game given the common decision-relevant set of states.

The key step in this argument is that the set of decision-relevant states is the same for both players. In other "agreement" theorems in the literature, this step is reached by a common knowledge assumption, for example, common knowledge of actions or posteriors. In our proof, however, this step is reached by the equilibrium requirement that each player chooses her action as if she were pivotal. In this way, our proof clearly illustrates how the usual common knowledge assumptions can be weakened in the case of Bayesian games with veto actions.

4 The Necessity of Social Efficiency

A basic assumption of the result in the previous section and in the no-trade literature in general is that the no mutual act outcome is socially efficient. But with multiple possible mutual acts and general game forms, it is natural to wonder if this condition can be relaxed. In this section we give two brief examples that illustrate the importance of social efficiency for our result.

Specifically, suppose a game has multiple mutual acts $\{J_1, J_2, J_3, \ldots, J_K\}$, but not all of them are socially dominated by the no mutual act outcome N. While it is not surprising that these undominated outcomes may occur with positive probability in equilibrium, both of the following examples show that a socially dominated outcome can also occur with positive probability in equilibrium.

Consider the game illustrated in Figure 2. This game can be interpreted as similar to the trading game with coordination given in Figure 1 where a completed trade provides a mutual benefit of 3 to each player but attempting to trade has a cost of 1, whether or not the trade is completed. Obviously, the completed trade outcomes socially dominate the veto outcome. Assume that $\Omega = \{\omega\}$ so there is no private information. It is clear that this game has a mixed strategy in which both players play their first two strategies with probability 1/2. In

		Player 2		
		b_1	b_2	$\tilde{b}$
	a_1	$2, 2$	$-1, -1$	$0, 0$
Player 1	a_2	$-1, -1$	$2, 2$	$0, 0$
	$\tilde{a}$	$0, 0$	$0, 0$	$0, 0$

Figure 2: A Trading Game with Mutually Beneficial Trade

this equilibrium, an outcome that is socially dominated outcome by the veto outcome occurs with probability 1/2.

Our second example makes a similar point but with private information and avoiding mixed strategies. Suppose there are three states of the world, ω_1, ω_2, and ω_3 each corresponding to one of the normal form games shown in Figure 3. Note that the mutual act corresponding to the action pair (a_1, b_1) is socially dominated by the veto outcome in every state of the world and in state ω_2 every mutual act is socially dominated by the veto outcome. Suppose the common prior probabilities on the states are $\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = 1/3$. Also assume that the players' information structure is

$$P_1 : \{\omega_1\}\{\omega_2, \omega_3\}$$
$$P_2 : \{\omega_1, \omega_2\}\{\omega_3\}.$$

Consider the following strategy profile: player 1 plays a_2 at $P_1(\omega_1)$ and a_1 at $P_1(\omega_2)$ and $P_1(\omega_3)$ while player 2 plays b_1 at $P_2(\omega_1)$ and $P_2(\omega_2)$, but b_2 at $P_2(\omega_3)$. It is easy to check that this strategy profile is a Bayesian equilibrium and in this equilibrium, the socially dominated outcome corresponding to the action pair (a_1, b_1) occurs at state ω_2.

These two examples illustrate the importance of social efficiency of the no mutual act outcome in a setting with multiple possible mutual acts. Without this assumption, it is possible that a socially dominated mutual act can occur in equilibrium.

5 Conclusion

In this paper we have extended the Nonspeculation Theorem to show that equilibrium play cannot lead to a socially dominated mutual act in games with private information. Our result is quite general with respect to game form and it extends the ground over which the logic of no-trade and nonspeculation results apply. Thus, in a large class of games, mutual optimism

	Player 2			
	ω_1	b_1	b_2	$\tilde{b}$
	a_1	$-1,-1$	$0,0$	$0,0$
Player 1	a_2	$1,5$	$1,1$	$0,0$
	$\tilde{a}$	$0,0$	$0,0$	$0,0$

	Player 2			
	ω_2	b_1	b_2	$\tilde{b}$
	a_1	$-1,-1$	$-1,-1$	$0,0$
Player 1	a_2	$-1,-1$	$-1,-1$	$0,0$
	$\tilde{a}$	$0,0$	$0,0$	$0,0$

	Player 2			
	ω_3	b_1	b_2	$\tilde{b}$
	a_1	$-1,-1$	$5,1$	$0,0$
Player 1	a_2	$0,0$	$1,1$	$0,0$
	$\tilde{a}$	$0,0$	$0,0$	$0,0$

Figure 3: A Game with Private Information.

due to private information cannot be a cause of suboptimal outcomes such as inefficient trade or costly war. The proof of our result also provides a clearer intuition for why common knowledge of some event is not required for the nonspeculation result. Specifically, we show that because players must make their choices as if their actions were pivotal, they act as if they "knew" something more than their private information. Much like the case in the literature on swing voters (Feddersen & Pesendorfer 1996), the winner's curse in auctions (Milgrom & Weber 1982), and mutual optimism and war (Fey & Ramsay 2007), it turns out that the joint nature of the acts makes the set of pivotal states the same for both players. Thus, conditioning on the same states, it is impossible for both players to want to take a socially inferior action.

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