A Two-Variable Approach to the Model Reduction Problem with Hankel Norm Criterion

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Abstract—**A two-variable approach to the model reduction problem with Hankel norm criterion is discussed. The problem is proved to be reducible to obtaining a two-variable all-pass rational function, interpolating a set of parametric values at specified points inside the unit circle. A polynomial formulation and the properties of the optimal Hankel norm approximations are then shown to result directly from the general form of the solution of the interpolation problem considered.**

I. INTRODUCTION

THE THEORY of bounded Hankel operators has been shown, in a masterful paper published by Adamjan, Arov, and Krein [1] in 1971, to provide a unifying approach to seemingly disconnected fundamental mathematical questions, e.g., approximations of Hankel operators in spectral norm, approximations in different norms, the classical Schur and Nevanlinna–Pick problems [2]–[4] and their various refinements, bounded extensions of one-sided Fourier series, decompositions of periodic functions into two analytic functions bounded in the open disk.... The reader is referred to [1] for more details about the history of these problems.

It turns out that, far from being of a pure academic interest, the results described in [1] are of great significance in some important engineering applications. Although their somewhat abstract original derivation in the framework of the theory of operators and Hardy spaces has been hindering their fast dissemination into the engineering community, their practical importance has begun recently to be recognized in two main application areas: 2-D recursive filter design and system model reduction.

The realization of stable 1-D recursive filters based on their reflection coefficients (ladder forms) is well known to

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be an efficient filter implementation, widely used for example in case of stationary stochastic processes [5]. Quite recently [6], [7], it has been put into light that the same formalism could be carried over to accommodate the 2-D situation. As a matter of fact, the key issue in this extension—a striking decomposition of bounded periodic functions into two functions analytic and bounded inside the unit circle—was solved in [1].

Model reduction techniques are obviously crucial in numerous applications. The problem can be stated in different, more or less equivalent forms in the time domain as well as in the frequency domain; it can be approached in various ways, based on state-space techniques, transfer function (impulse response) analysis or spectrum factorization and approximation. Let us consider the problem in its time-domain formulation and assume the impulse response, possibly noise-corrupted, of an unknown stable linear system be given; how can we retrieve from the data a "best" approximate system function of degree as small as possible. Stated in such a general form, the problem naturally raises a sequence of questions:

how to measure the closeness of the approximate model with respect to the original system (*error norm selection*),

for a preassigned error tolerance, how to obtain the simplest possible model (*minimal degree solution*),

for the same model degree expense, what is the best possible model (*minimal norm solution*),

how insensitive to perturbations is the computed model (*robustness*),

last but not least, what is the complexity of the solution algorithm (*computational cost*).

Obviously, these issues have a long history and have been approached, during the last two decades, in various ways leading to suboptimal model reduction techniques, more or less satisfactory from a practical viewpoint (see, e.g., [8], [10] and the references therein for interesting notes and comments regarding their evolution).

The so-called Hankel norm criterion measuring the deviation of the original system with respect to its model has been receiving increasing attention lately [8]–[13]. This error measure can be described as follows. Since the impulse response sequence of a stable linear system is precisely the sequence of the Markov parameters of its transfer function, comparing two impulse responses amounts to comparing two Hankel matrices formed from the Markov parameters. This norm exhibits two important features from a model reduction viewpoint. First, the Hankel norm can be easily shown to lie between the least squares (L_2-) norm and the Chebyshev ($L_\infty-$) norm in the frequency domain and hence appears naturally as a tradeoff between these two popular error criteria. Secondly, this norm is concerned with the singular values of Hankel matrices and the singular values of such matrices are known to be rather insensitive with respect to perturbations, which make the model fairly robust to cope with uncertainties.

It turns out that the subject of Hankel norm approximations is exactly the main theme of the paper by Adamjan, Arov, and Krein. In [1], a complete solution is given to the minimal norm approximation problem as well as to the corresponding minimal degree problem and these results have direct striking applications in the model reduction problem. The relevance of [1] to the model reduction problem was first mentioned by Kung [8], while a discussion of some numerical aspects involved in the Adamjan–Arov–Krein paper was reported in [9]. In [10], [11], connections between the minimal Hankel norm approximations and the balanced state-space realizations introduced by Moore [14] are put into light and shown to lead to an optimal approximation algorithm requiring the solution of two Lyapunov equations and a singular value decomposition. In [12], [13], Kung took a pure one-variable polynomial approach to elucidate the singular value and singular vector properties of real Hankel matrices, which then lead to a simple generalized eigenvalue formulation and other related algorithms for optimal approximation problem, when the system transfer function is available.

The aim of this paper is twofold. First, under the realistic assumption that the available impulse response originates from a rational transfer function (infinite Hankel matrix of finite rank), the results of Adamjan, Arov, and Krein relevant to the model reduction problem are entirely rederived via a *novel simple approach, relying exclusively on standard matrix and polynomial techniques*. Secondly, it will be made clear that the minimal degree approximations can be obtained at very low *computational cost*: solving linear systems of equations and extracting its stable part from a resulting rational function. From an application viewpoint this result is clearly of prominent interest, since the most realistic problem is precisely the minimal degree approximation problem with preassigned tolerance.

The main philosophy of the original approach developed in this paper can be found in the Adamjan–Arov–Krein paper itself. A unique well-defined approximation problem with Hankel norm criterion is considered in [1] for a *fixed* value of a parameter. It turns out that the properties of the solution of this unique problem strongly depends on the localization of the parameter with respect to the singular values of the Hankel matrix; for example, the form itself of the solution is affected when this parameter coincides with a singular value. This obviously suggests a reformulation of the problem as a *parametric* problem admitting a parametric solution exhibiting some type of degeneracy at the singular values. This key observation coupled with the century old Kronecker theorem, that establishes a one-to-one correspondence between the set of infinite Hankel matrices of finite rank and the set of rational functions, lead directly to our two-variable formulation of the problem. This two-variable reformulation not only sheds new lights on the problem but also provides a unifying framework for several relevant mathematical results mentioned above [8]–[13] together with a new direction for future research.

The paper is hopefully self-contained and has been purposely organized in a succession of elementary intermediate steps with the hope of making its content accessible to the widest possible segment of potential users. However, the reader is assumed to have some familiarity with standard matrix and polynomial properties [15]–[17].

In Section II, a simple parametric interpolation problem by an all-pass parametric rational function is stated and the most general form of its minimal degree solutions is obtained with the help of the Nevanlinna algorithm [4], [19]. The same problem is shown to be equivalent to solving a two-variable polynomial equation, which can be associated with any proper stable one-variable rational function.

In Section III, the stability properties of the parametric all-pass solution as a function of the parameter are discussed via a test of the Schur–Cohn type and proved to be related to the signs of the eigenvalues of a parametric copy of the so-called Nevanlinna matrix, classically associated with the Nevanlinna algorithm [4], [19].

In Section IV, the Hankel norm criterion is considered and its relation with other norms provided. The solutions of the parametric interpolation problem are then proved to lead automatically to approximations, bounded in Hankel norm, of the proper stable rational function introduced in Section II. The singular values of the Hankel matrix of the latter function are finally shown to be identical to the zeros of the determinant of the parametric Nevanlinna matrix.

In Section V, the optimality of the approximations considered in the preceding section is established. Whatever the parameter may be, the inverse of its modulus is an upper bound for the Hankel norm of the approximation error and there exists no approximation of smaller degree admitting the same upper bound. Moreover, when the parameter is equal to a singular value, the approximation becomes spontaneously a minimal Hankel norm approximation. The computational cost involved in obtaining these approximations is then briefly considered in connection with the equivalent polynomial formulation of the problem [12].

In Section VI, the structure of the optimal approximations at the singular values is further investigated and possible critical values for the parametric interpolation problem are analyzed in details. As a result, the validity of the fast algorithms based upon the polynomial formulation is established in full generality.

In Section VII, an example is discussed in great details; the general form of all the approximations in Hankel norm corresponding to the rational function considered is obtained and appears to be most illustrative of the various aspects involved in Hankel norm rational approximations.

Throughout the paper, the rational function to be approximated has been assumed to have all its poles distinct. This hypothesis, simplifying the algebraic manipulations, can be removed without conceptual difficulty. In the Appendix, it is shown how the parametric interpolation problem has to be modified so as to cope with the multiple pole situation.

II. A PARAMETRIC INTERPOLATION PROBLEM

Let us consider the following interpolation problem. A set of n distinct points z_i with $|z_i|<1$ and a set of n arbitrary values μ_i^1 are given; it is then required to obtain an all-pass rational function $s_1(\lambda, z)$ depending on the *real* parameter λ:

$$|s_1(\lambda, e^{i\theta})|=1, \qquad 0\leq\theta\leq 2\pi \tag{1}$$

of minimum degree and satisfying the interpolation constraints:

$$s_1(\lambda, z_i)=\lambda\mu_i^1, \qquad 1\leq i\leq n. \tag{2}$$

The problem can equivalently be rephrased in terms of two-variable polynomials by the following argument. With any polynomial $p(z)$ of degree n, let us associate its reciprocal $\hat{p}(z)$ defined by

$$\hat{p}(z)=z^n\bar{p}(1/\bar{z}). \tag{3}$$

Introducing the polynomials $a(z)$ and $b(z)$ of degree n and $n-1$, respectively,

$$a(z)=\prod_{i=1}^{n}(z-z_i) \tag{4}$$

$$b(z)=\sum_{i=1}^{n}\mu_i^1\frac{\hat{a}(z_i)a(z)}{a'(z_i)(z-z_i)} \tag{5}$$

one can easily verify the identities

$$\frac{b(z_i)}{\hat{a}(z_i)}\equiv\mu_i^1, \qquad 1\leq i\leq n. \tag{6}$$

Furthermore, the rational function

$$h(z)=\frac{b(z)}{a(z)} \tag{7}$$

is clearly stable, proper, and of degree n. Since any all-pass rational function has necessarily the form $s_1(\lambda, z)=\epsilon\hat{r}(\lambda, z)/r(\lambda, z)$, the constraints (6) can be rewritten as ($|\epsilon|=1$)

$$\lambda b(z)r(\lambda, z)-\epsilon\hat{r}(\lambda, z)\hat{a}(z)=\pi(\lambda, z)a(z) \tag{8}$$

where $\pi(\lambda, z)$ is some polynomial. From (6), it is also apparent that $r(\lambda, z)$ and $\pi(\lambda, z)$ are essentially two-variable polynomials.

We shall now approach the problem under its first formulation, i.e., (1) and (2), with the help of the so-called Nevanlinna algorithm, which amounts to computing its solution in a recursive way [4], [9].

Let indeed $s_1(\lambda, z)$ be a solution; in particular, one must have

$$s_1(\lambda, z_1)=\lambda\mu_1^1 \tag{9}$$

so that the rational function $s_1(\lambda, z)-\lambda\mu_1^1$ has a zero in $z=z_1$, independent of λ. Hence, the rational function $s_2(\lambda, z)$ defined as

$$s_2(\lambda, z)=\frac{1-\bar{z}_1 z}{z-z_1}\cdot\frac{s_1(\lambda, z)-\lambda\mu_1^1}{1-\lambda\bar{\mu}_1^1 s_1(\lambda, z)} \tag{10}$$

is rational, has no pole independent of λ in $z=z_1$ and moreover, is an all-pass as it is apparent from the identity:

$$1-|s_2(\lambda, e^{i\theta})|^2=\frac{(1-\lambda^2|\mu_1^1|^2)(1-|s_1(\lambda, e^{i\theta})|^2)}{|1-\lambda\bar{\mu}_1^1 s_1(\lambda, e^{i\theta})|^2}. \tag{11}$$

In terms of $s_2(\lambda, z)$, the remaining interpolation constraints are found with the help of (9) to be

$$s_2(\lambda, z_i)=\lambda\mu_i^2(\lambda^2), \qquad 2\leqslant i\leqslant n$$
$$=\frac{1-\bar{z}_1 z_i}{z_i-z_1}\frac{\lambda(\mu_i^1-\mu_1^1)}{1-\lambda^2\bar{\mu}_1^1\mu_i^1}. \tag{12}$$

As a consequence, there must exist an all-pass rational function $s_2(\lambda, z)$ interpolating the $(n-1)$ values $\lambda\mu_i^2(\lambda^2)$ at the $(n-1)$ points $z_2, z_3, \cdots, z_n$. In other words, the problem of computing $s_2(\lambda, z)$ has the same form as the original one, except that there is now one less interpolation point.

The above procedure can obviously be iterated on the successive interpolation points and this yields the Nevanlinna recursion scheme, that can be described as follows [19]:

a) Deduce recursively from (2) the so-called Fenyves array of data, i.e.,

$$\lambda\mu_i^k(\lambda^2)=\frac{1-\bar{z}_{k-1}z_i}{z_i-z_{k-1}}\cdot\frac{\lambda\mu_i^{k-1}(\lambda^2)-\lambda\mu_{k-1}^{k-1}(\lambda^2)}{1-\lambda^2\bar{\mu}_{k-1}^{k-1}(\lambda^2)\mu_i^{k-1}(\lambda^2)},$$
$$\text{for } k=2,3,\cdots,n \text{ and } i\geqslant k. \tag{13}$$

b) Compute iteratively the solution $s_1(\lambda, z)$ from the diagonal entries of the Fenyves array with the help of the formula

$$s_k(\lambda, z)=\frac{(1-\bar{z}_k z)\lambda\mu_k^k(\lambda^2)+(z-z_k)s_{k+1}(\lambda, z)}{(1-\bar{z}_k z)+(z-z_k)\lambda\bar{\mu}_k^k(\lambda^2)s_{k+1}(\lambda, z)} \tag{14}$$

for $k=n, n-1, \cdots, 1$ and where $s_{n+1}(\lambda, z)=s_{n+1}(z)$ is an arbitrary one-variable all-pass rational function without pole at $z=z_n$.

It is most convenient to use the formalism of homographic transformations associated with J-unitary matrices [18] to describe the above algorithm. Let us indeed define $L_k(\lambda, z)$ as the matrix:

$$L_k(\lambda, z)=\frac{1}{\sqrt{1-|\lambda\mu_k^k(\lambda^2)|^2}}$$
$$\cdot\begin{bmatrix} 1 & \lambda\mu_k^k(\lambda^2) \\ \lambda\bar{\mu}_k^k(\lambda^2) & 1 \end{bmatrix}\begin{bmatrix} \frac{z-z_k}{1-\bar{z}_k z} & \\ & 1 \end{bmatrix}. \tag{15}$$

If $\hat{L}_k(\lambda, z)$ stands for $z\hat{L}_k(\lambda, 1/\bar{z})$, it turns out that $L_k(\lambda, z)J\hat{L}_k(\lambda, z)=\pm J$ where $J=1+-1$ and that the product of such J-unitary matrices is again J-unitary as it can be easily verified. Let us define the homographic transformation $y=L[x]$ where $L=\{l_{ij}\}$ is any J-unitary matrix as $y=(l_{11}x+l_{12})/(l_{21}x+l_{22})$. It turns out that (14) can be rewritten as the homographic transformation

$$s_k(\lambda, z)=L_k(\lambda, z)[s_{k+1}(\lambda, z)] \tag{16}$$

and by using the multiplicative property of J-unitary matrices as

$$s_k(\lambda, z)=T_{n-k+1}(\lambda, z)[s_{n+1}(z)] \tag{17}$$

where

$$T_{n-k+1}(\lambda, z)=\prod_{j=k}^{n}L_j(\lambda, z).$$

Theorem 1. The parametric interpolation problem (1),(2) can always be solved. The minimum degree of the solution is n, in general, and there exists an infinite family of such solutions depending on a parameter ϵ of unit modulus.

Proof: In (17), the degrees in z of $T_{n-k+1}(\lambda, z)$ and $s_{n+1}(z)$ clearly add up so that a solution of minimal degree will be achieved by selecting $s_{n+1}(z)$ as an all-pass function of degree zero, i.e., a constant ϵ of unit modulus. Hence, the solutions of minimal degree n will be of the form

$$s_1(\lambda, z)=T_n(\lambda, z)[\epsilon] \tag{18}$$

whatever the labeling of the data pairs (z_i, μ_i^1) may be. On the other hand, by construction, one has

$$s_1(\lambda, z_i)=\lambda\mu_i^1, \qquad 1\leqslant i\leqslant n \tag{19}$$

for all values of λ with, however, the following exception. The algorithm does not yield a solution of type (18) if one has

$$\lambda^2|\mu_k^k(\lambda^2)|=1 \tag{20}$$

for some k and some value of λ, since (14) degenerates into

$$s_k(\lambda, z)\equiv\lambda\mu_k^k(\lambda^2). \tag{21}$$

However, since the $\mu_k^k(\lambda^2)$ are rational functions of λ^2, as it is apparent from (7), such situations are only possible for a finite number of values of λ^2. As a result, (19) holds for almost all values of λ and hence must be an identity since $s_1(\lambda, z_i)$ is a one-variable rational function. □

It should be noted that the only values of λ where a degeneracy of the form (21) may occur are indeed real in view of (20).

Let us now define the polynomials $p_k(z, \lambda)$ and $q_k(z, \lambda)$ of degree k in z and whose coefficients are rational functions of λ by

$$\begin{bmatrix} p_k(\lambda, z) \\ q_k(\lambda, z) \end{bmatrix}=\left[\prod_{j=n-k+1}^{n}(1-\bar{z}_j z)\left(1-|\lambda\mu_j^j(\lambda^2)|^2\right)^{1/2}\right]$$
$$\cdot T_k(\lambda, z)\begin{bmatrix} \epsilon \\ 1 \end{bmatrix}. \tag{22}$$

From the very definition of $T_k(\lambda, z)$, it turns out that one has the identity $p_k(\lambda, z)=\epsilon\hat{q}_k(\lambda, z)$ and hence $s_1(\lambda, z)=\epsilon\hat{q}_n(\lambda, z)/q_n(\lambda, z)$. Since $q_n(\lambda, z)$ is a polynomial in z, whose coefficients are rational functions of λ, it can be factorized as

$$q_n(\lambda, z)=\frac{n(\lambda)}{d(\lambda)}r(\lambda, z) \tag{23}$$

where $r(\lambda, z)$ is an irreducible two-variable polynomial linearly depending on ϵ, i.e.,

$$r(\lambda, z)=r_1(\lambda, z)\epsilon+r_2(\lambda, z) \tag{24}$$

while $d(\lambda)$ is an even polynomial and $n(\lambda)$ an odd or even polynomial depending on λ only. Indeed, the only values of λ for which the solution (18) does not hold are real and symmetric with respect to $\lambda=0$ in view of (20); moreover, $d(\lambda)$ is even as it is apparent from (13) while the form (24) is obvious due to (18).

Theorem 2. The whole family of the minimal degree solutions of the parametric interpolation problem (1),(2) can be described by the irreducible two-variable rational function

$$s_1(\lambda, z)=\frac{\hat{r}_1(\lambda, z)+\epsilon\hat{r}_2(\lambda, z)}{\epsilon r_1(\lambda, z)+r_2(\lambda, z)} \tag{25}$$

where $r_1(\lambda, z)$ and $r_2(\lambda, z)$ are two-variable polynomials uniquely defined up to a constant of unit modulus while ϵ runs over the unit circle $|\epsilon|=1$. Moreover, the polynomials $r_1(\lambda, z)$ and $r_2(\lambda, z)$ satisfy the relation

$$\hat{r}_1(\lambda, z)r_1(\lambda, z)-\hat{r}_2(\lambda, z)r_2(\lambda, z)=a(z)\hat{a}(z)p(\lambda^2) \tag{26}$$

where $p(\lambda^2)$ is a real polynomial.

Proof: The irreducible form (25) is an obvious consequence of (18), (22)–(24). The uniqueness of the polynomials $r_1(\lambda, z)$ and $r_2(\lambda, z)$ results from the following argument. Let $s_1'(\lambda, z)=\epsilon'\hat{r}_1'(\lambda, z)/r_1'(\lambda, z)$ be another irreducible solution different from (25). Since two all-pass rational functions of minimum degree interpolating the same points are necessarily identical, one has $r(\lambda, z)\equiv e^{i\alpha}r'(\lambda, z)$ for some real α. Hence, $r_1'(\lambda, z)\equiv e^{-i\alpha}r_1(\lambda, z)$ and $r_2'(\lambda, z)\equiv e^{-i\alpha}r_2(\lambda, z)$ since the above identity must hold for all ϵ of unit modulus. Finally, the left-hand member of (22) for $k=n$ can be written with the help of (23) and (24) as

$$\frac{n(\lambda)}{d(\lambda)}\begin{bmatrix}\hat{r}_2(\lambda, z) & \hat{r}_1(\lambda, z)\\ r_1(\lambda, z) & r_2(\lambda, z)\end{bmatrix}\begin{bmatrix}\epsilon\\ 1\end{bmatrix} = \prod_{j=1}^{n}(1-\bar{z}_j z)\left(1-|\lambda\mu_j^j(\lambda^2)|^2\right)^{1/2}T_n(\lambda, z)\begin{bmatrix}\epsilon\\ 1\end{bmatrix}. \tag{27}$$

Since (27) must hold for all ϵ of unit modulus, the two system matrices must be identical and the J-unitary property of $T_n(\lambda, z)$ produces

$$r_1(\lambda, z)\hat{r}_1(\lambda, z)-r_2(\lambda, z)\hat{r}_2(\lambda, z) = a(z)\hat{a}(z)\frac{d^2(\lambda)}{n^2(\lambda)}\prod_{j=1}^{n}\left(1-|\lambda\mu_j^j(\lambda^2)|^2\right). \tag{28}$$

Finally, as the left-hand member of (26) is unique and polynomial, so is the right-hand member and this completes the proof since the rational function $d(\lambda)/n(\lambda)$ is known to be even or odd. □

By plugging (24) into the polynomial form (8) of the parametric interpolation problem, one has

$$\pi(\lambda, z)=\pi_1(\lambda, z)+\epsilon\pi_2(\lambda, z) \tag{29}$$

so that the polynomial $p(\lambda^2)$ in (28) can be alternatively written as

$$p(\lambda^2)=\frac{\pi_1(\lambda, z)\hat{\pi}_1(\lambda, z)-\pi_2(\lambda, z)\hat{\pi}_2(\lambda, z)}{\lambda^2 z b(z)\hat{b}(z)-a(z)\hat{a}(z)} \tag{30}$$

Furthermore, by comparing the degrees in (8), one is led to the following observation.

Corollary 1. The two-variable polynomial $\pi(\lambda, z)$ has degree n in z.

It should be pointed out that $s_1(\lambda, z)$ as given by (25) has not necessarily degree n for all fixed values of λ. On the contrary, there may obviously exist a finite number of real values of λ for which $r(\lambda, z)$ and $\hat{r}(\lambda, z)$ may have common zeros, leading to a degree reduction of $s_1(\lambda, z)$. Those "irregular" values of λ will be discussed later on in detail.

It is also worth noting that at such an irregular value of λ, say λ_α, one has not necessarily

$$\lim_{z\to z_i}\left[\lim_{\lambda\to\lambda_\alpha} s_1(\lambda, z)\right]=\lambda_\alpha\mu_i^1 \tag{31}$$

for all i, although interchanging the limit in (31) yields the correct value on the strength of Theorem 1. If (31) is not satisfied for some i, the rational function $s_1(\lambda, z)$ has then an essential singularity in $\lambda=\lambda_\alpha$, $z=z_i$; in other words, one has necessarily $r(\lambda_\alpha, z_i)=\hat{r}(\lambda_\alpha, z_i)=0$ (see Theorem 14).

III. Stability Properties

Let us define $\lambda=\lambda_\alpha$ as a *regular* value of the parameter of the interpolation problem if there exists at least one labeling of the data pairs (z_i, μ_i^1) such that none of the $\lambda_\alpha\mu_k^k(\lambda_\alpha^2)$ has unit modulus.

From the definition of $T_k(\lambda, z)$ and (22), the polynomials $q_k(\lambda, z)$ are readily seen to satisfy the following recurrence:

$$q_{k+1}(\lambda, z)=(z-z_{n-k})\lambda\bar{\mu}_{n-k}^{n-k}(\lambda^2)\epsilon\hat{q}_k(\lambda, z) +(1-\bar{z}_{n-k}z)q_k(\lambda, z) \tag{32}$$

for $k=0,1,\cdots,n-1$ and with the initialization $q_0(\lambda, z)\equiv 1$. Let us consider the two sequences of integers δ_i and π_i, actually taking the values $+1$ and -1, defined by

$$\delta_i=\operatorname{sgn}\left[1-|\lambda\mu_i^i(\lambda^2)|^2\right], \qquad 1\leqslant i\leqslant n \tag{33}$$

$$\pi_i=\prod_{j=1}^{i}\delta_j, \qquad 1\leqslant i\leqslant n. \tag{34}$$

Theorem 3. At any regular value $\lambda=\lambda_\alpha$, the polynomial $r(\lambda_\alpha, z)$ has no zero on the unit circle and the number $\nu(\lambda_\alpha)$ of its zeros inside the unit circle ($|z|<1$) is equal to the number of negative π_i in the sequence $\pi_1, \pi_2, \cdots, \pi_n$.

Proof: Let us denote by $\nu(k)$ the number of zeros of $q_k(\lambda_\alpha, z)$ inside the unit circle. Applying Rouché's theorem [17] on (32) yields

$$\begin{aligned}\nu(k+1)&=\nu(k), &&\text{if } \delta_{n-k}=+1\\ \nu(k+1)&=k+1-\nu(k), &&\text{if } \delta_{n-k}=-1\end{aligned} \tag{35}$$

which makes the proof obvious. Note that the same argument still applies if λ_α is a pole of $\lambda\mu_k^k(\lambda^2)$, in which case $q_{k+1}(\lambda, z)$ as well as the subsequent $q_{k+i}(\lambda, z)$ have necessarily the same pole independent of z so that $(\lambda^2-\lambda_\alpha^2)$ turns out to be a factor of $d(\lambda)$ in (23). □

As an immediate consequence of the above theorem, one has:

Corollary 2. At any regular value $\lambda=\lambda_\alpha$, any minimal degree irreducible solution $s_1(\lambda, z)$ of the parametric interpolation problem has precisely $\nu(\lambda_\alpha)$ poles inside the unit circle.

Proof: If $s_1(\lambda_\alpha, z)=\epsilon\hat{r}(\lambda_\alpha, z)/r(\lambda_\alpha, z)$ has degree less than n, then either $r(\lambda_\alpha, z)$ has a zero z_μ on the unit circle or the factor $(z-z_\mu)(1-\bar{z}_\mu z)$ if $|z_\mu|\neq 1$. The first situation is impossible due to Theorem 3, and the second possibility has to be ruled out since $\lambda_\alpha\mu_{n-2}^{n-2}(\lambda_\alpha^2)$ has then necessarily unit modulus, as it can be easily verified. □

Although Theorem 3 and its corollary provide us with a way of computing the number of stable poles of $s_1(\lambda, z)$ at any regular value λ, they do not give any indication on how this number evolves with λ. It turns out that the latter goal can be achieved by considering the so-called Nevanlinna matrix $N(\lambda^2)$, classically associated with the Nevanlinna algorithm and defined as follows [4], [19]:

$$N(\lambda^2)=\left\{\frac{1-\lambda^2\bar{\mu}_i^1\mu_j^1}{1-\bar{z}_i z_j}, 1\leqslant i, j\leqslant n\right\}. \tag{36}$$

We next prove the following fundamental theorem.

Theorem 4. For any regular value $\lambda=\lambda_\alpha$, the number $\nu(\lambda_\alpha)$ of stable poles of $s_1(\lambda_\alpha, z)$ is equal to the number of negative eigenvalues of N.

Proof: The proof essentially amounts to showing that $N(\lambda_\alpha^2)$ can be reduced by a nonsingular conjunctive transformation to the diagonal matrix $\Delta(\pi_i)$ [19]; a direct application of Theorem 3 and its corollary will then complete the proof.

For writing convenience, let us use the notation $y_i=(z_i-z_1)/(1-\bar{z}_1 z_i)$. Since the following identity is available from (13):

$$\frac{1-\bar{y}_i y_j\lambda_\alpha^2\bar{\mu}_i^2(\lambda_\alpha^2)\mu_j^2(\lambda_\alpha^2)}{1-\bar{z}_i z_j} = \frac{(1-\lambda_\alpha^2|\mu_1^1|^2)(1-\lambda_\alpha^2\bar{\mu}_i^1\mu_j^1)}{(1-\lambda_\alpha^2\mu_1^1\bar{\mu}_i^1)(1-\bar{z}_i z_j)(1-\lambda_\alpha^2\bar{\mu}_1^1\mu_j^1)} \tag{37}$$

one is led to

$$N(\lambda_\alpha^2)=\frac{1}{(1-\lambda_\alpha^2|\mu_1^1|^2)}\Delta(1-\lambda_\alpha^2\mu_1^1\bar{\mu}_i^1)X\Delta(1-\lambda_\alpha^2\bar{\mu}_1^1\mu_j^1) \tag{38}$$

where X is the matrix

$$X=\left\{\frac{1-\bar{y}_i y_j\lambda_\alpha^2\bar{\mu}_i^2(\lambda_\alpha^2)\mu_j^2(\lambda_\alpha^2)}{1-\bar{z}_i z_j}, 1\leqslant i, j\leqslant n\right\}. \tag{39}$$

Note that $y_1\equiv 0$ so that if P^{-1} stands for the matrix

$$P^{-1}=\begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -\dfrac{1-|z_1|^2}{1-\bar{z}_2 z_1} & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -\dfrac{1-|z_1|^2}{1-\bar{z}_n z_1} & 0 & \cdot & \cdot & 0 & 1 \end{bmatrix} \tag{40}$$

one can verify after some elementary algebraic manipulations the identity

$$P^{-1}X\tilde{P}^{-1}=\frac{1}{(1-\bar{z}_1 z_1)}\dotplus\Delta\left(\frac{\bar{z}_i-\bar{z}_1}{1-z_1\bar{z}_i}\right)N'(\lambda_\alpha^2)\left(\frac{z_j-z_1}{1-\bar{z}_1 z_j}\right) \tag{41}$$

where $N'(\lambda_\alpha^2)$ is the $(n-1)\times(n-1)$ updated Nevanlinna matrix

$$N'(\lambda_\alpha^2)=\left\{\frac{1-\lambda_\alpha^2\bar{\mu}_i^2(\lambda_\alpha^2)\mu_j^2(\lambda_\alpha^2)}{1-\bar{z}_i z_j}, 2\leqslant i, j\leqslant n\right\}. \tag{42}$$

By introducing the real number x_1 defined from the equation $x_1^2=\delta_1(1-\lambda_\alpha^2|\mu_1^1|^2)$ and with Q standing for the nonsingular matrix

$$Q=\frac{1}{x_1}\Delta(1-\lambda_\alpha^2\mu_1^1\bar{\mu}_i^1)P\left[(1-\bar{z}_1 z_1)^{-1/2}\dotplus\Delta\left(\frac{\bar{z}_i-\bar{z}_1}{1-\bar{z}_i z_1}\right)\right] \tag{43}$$

one deduces the identity

$$N(\lambda_\alpha^2)=Q[\delta_1\dotplus\delta_1 N'(\lambda_\alpha^2)]\tilde{Q}. \tag{44}$$

Iterating the above procedure on $N'(\lambda_\alpha^2)$ and so on up to reducing $N(\lambda_\alpha^2)$ to its diagonal form clearly shows that $N(\lambda_\alpha^2)$ can be conjunctively transformed into $\Delta(\pi_i)$ in view of the very definition (34) of the π_i.

It should be noted that the above decomposition procedure can still be used in case $1-\lambda^2\mu_1^1\bar{\mu}_1^i$ would have a zero in $\lambda^2=\lambda_\alpha^2$. In such a case, $1-\lambda^2\mu_1^1\bar{\mu}_1^i$ would have a zero in $\lambda^2=\lambda_\alpha^2$. Then, $X_{1,j}$ is readily seen with the help of (13) to have a pole at $\lambda^2=\lambda_\alpha^2$ for $j=2,\cdots,n$ so that this singularity can be removed without difficulty. The same argument can obviously be applied, if needed, at any stage of the decomposition. □

Let us now consider the case $\lambda=0$; the Nevanlinna matrix $N(0)$ takes the form

$$N(0)=\left\{\frac{1}{1-\bar{z}_i z_j}, 1\leqslant i, j\leqslant n\right\} \tag{45}$$

and is clearly positive definite since all $\lambda\mu_k^k(\lambda^2)$ vanish for $\lambda=0$, which makes all δ_i in (33) equal to $+1$.

Theorem 5. The number of negative eigenvalues of $N(\lambda^2)$ is a nondecreasing function of λ^2 and actually increases only at the n positive real values of λ^2 which make $N(\lambda^2)$ singular.

Proof: With the help of $N(0)$, $N(\lambda^2)$ can be rewritten as

$$N(\lambda^2)=N(0)-\lambda^2\Delta(\bar{\mu}_i^1)N(0)\Delta(\mu_j). \tag{46}$$

Since both $N(0)$ and $\Delta(\bar{\mu}_i^1)N(0)\Delta(\mu_j)$ are positive definite, they can be simultaneously diagonalized by the same conjunctive transformation T to yield

$$N(\lambda^2)=T\left[I-\lambda^2\Delta(\alpha_i)\right]\tilde{T} \tag{47}$$

where the α_i are positive real numbers. ☐

Let us label the n positive real zeros, not necessarily distinct, of det $N(\lambda^2)$ in ascending order so as to have

$$0<\lambda_1^2\leqslant\lambda_2^2\leqslant\cdots\leqslant\lambda_i^2\leqslant\cdots\leqslant\lambda_n^2<\infty \tag{48}$$

with det $N(\lambda_i^2)=0$ for $i=1,2,\cdots,n$. Combining Theorems 4 and 5 yields:

Theorem 6. Let λ_s^2 and λ_{s+1}^2 be two successive distinct zeros of det $N(\lambda^2)$. For any λ satisfying $\lambda_s^2<\lambda^2\leqslant\lambda_{s+1}^2$, the number of stable poles $\nu(\lambda)$ of $s_1(\lambda, z)$ is at most s.

Proof: The proof is trivial if $\lambda=\lambda_\alpha$ is a regular value, since one has $\nu(\lambda_\alpha)=s$ from Theorem 4 and 5. Let then $\lambda=\lambda_\beta$ be a nonregular value and let us consider the denominator of $s_1(\lambda, z)$, i.e., the two-variable polynomial $r(\lambda, z)$, as a polynomial of z, whose coefficients depend on a parameter. Now, if $r(\lambda_\beta, z)$ is assumed to have $s+k$ stable zeros ($k>0$), so is necessarily the case for the polynomial $r(\lambda, z)$ on the interval $\lambda_\beta-\delta\leqslant\lambda\leqslant\lambda_\beta+\delta$ for some $\delta>0$, since the zeros of a one-variable polynomial are continuous functions of their coefficients. However, the number of nonregular values of λ is known to be finite so that the above assumption leads to a contradiction and, hence, one has $\nu(\lambda_\beta)\leqslant s$. ☐

The values $\lambda=\pm\lambda_i$, satisfying det $N(\lambda_i^2)=0$, are easily seen to be nonregular, otherwise $N(\lambda_i^2)$ could be reduced by a nonsingular conjunctive transformation to a diagonal matrix of ±1, by using the argument contained in the proof of Theorem 4. It should be pointed out that there may exist other nonregular values of λ, as shown by the example ($\mu_1^1=1, \mu_2^1=-1$) leading to the Nevanlinna matrix

$$N(\lambda^2)=\begin{bmatrix}\dfrac{1-\lambda^2}{1-\bar{z}_1z_1} & \dfrac{1+\lambda^2}{1-\bar{z}_1z_2}\\[2ex] \dfrac{1+\lambda^2}{1-\bar{z}_2z_1} & \dfrac{1-\lambda^2}{1-\bar{z}_2z_2}\end{bmatrix} \tag{49}$$

clearly nonsingular for the nonregular values $\lambda=\pm1$.

The nonregular values λ_β of the parametric interpolation problem will be called in the sequel *singular*, if one has det $N(\lambda_\beta^2)=0$, and *critical*[1] otherwise.

Let us now give a characterization as well as some preliminary properties of the finite set of nonregular values λ_β.

Theorem 7. The value $\lambda=\lambda_\beta$ is nonregular if and only if $p(\lambda_\beta^2)=0$. Moreover, the infinite family of solutions of the parametric interpolation problem degenerates into a unique solution for any nonregular value λ_β.

Proof: The first part of the theorem is an immediate consequence of comparing (26) with (28), if one remembers that the zeros of $d^2(\lambda)$ cancel with the poles of the product of the $(1-|\lambda\mu_j^j(\lambda^2)|^2)$. The proof is then completed by observing from (26) that $p(\lambda_\beta^2)=0$ implies

$$\frac{\hat{r}_1(\lambda_\beta, z)}{r_2(\lambda_\beta, z)}=\frac{\hat{r}_2(\lambda_\beta, z)}{r_1(\lambda_\beta, z)} \tag{50}$$

so that $s_1(\lambda_\beta, z)$, as given by (26), degenerates into the unique function

$$s_1(\lambda_\beta, z)=\epsilon_\beta\frac{\hat{r}_1(\lambda_\beta, z)}{r_1(\lambda_\beta, z)} \tag{51}$$

with $\epsilon_\beta=\hat{r}_2(\lambda_\beta, z)/\hat{r}_1(\lambda_\beta, z)$. ☐

From (26) and the above theorem, it is obvious that the nonregular values of λ are the only values where the family of solutions reduces to a unique function.

Corollary 3. If $\lambda=\lambda_\beta$ is a nonregular value, then $s_1(\lambda_\beta, z)$ has degree smaller than n and conversely.

Proof: Let $n-k$ be the largest integer such that one has $|\lambda_\beta\mu_{n-k}^{n-k}(\lambda_\beta^2)|=1$ after all possible relabelings of the data pairs. From (32), the identity $\epsilon\hat{q}_{k+1}(\lambda_\beta, z)=\lambda_\beta\mu_{n-k}^{n-k}(\lambda_\beta^2)q_{k+1}(\lambda_\beta, z)$ is easily deduced, which proves that the self-reciprocal polynomial $q_{k+1}(\lambda_\beta, z)$, obviously a factor of $q_n(\lambda_\beta, z)$, cancels out in the ratio $s_1(\lambda_\beta, z)=\epsilon\hat{q}_n(\lambda_\beta, z)/q_n(\lambda_\beta, z)$. The converse results directly from the observation that, if there is a cancellation in an all-pass rational function, the common factor is necessarily self-reciprocal. ☐

IV. RATIONAL APPROXIMATIONS IN HANKEL NORM

It turns out that the parametric interpolation problem considered so far is intimately connected with the rational approximation problem of $b(z)/a(z)$ in Hankel norm.

To begin with, let us denote by H the infinite Hankel matrix associated with the rational function $h(z)=b(z)/a(z)$,

$$H=\begin{bmatrix}h_1 & h_2 & h_3 & \cdot & \cdot\\ h_2 & h_3 & \cdot & \cdot & \cdot\\ h_3 & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\end{bmatrix} \tag{52}$$

where the h_i are the Markov parameters of $h(z)$, i.e.,

$$h(z)=\sum_{i=1}^{\infty}h_iz^{-i}. \tag{53}$$

For any fixed value of λ, let us decompose the rational function $\pi(\lambda, z)/\lambda r(\lambda, z)$ as

$$\frac{\pi(\lambda, z)}{\lambda r(\lambda, z)}=\frac{\pi_u(\lambda, z)}{\lambda r_u(\lambda, z)}+\frac{\pi_{st}(\lambda, z)}{\lambda r_{st}(\lambda, z)} \tag{54}$$

where $\pi_u(\lambda, z)/\lambda r_u(\lambda, z)$ is the resulting rational function after extracting from $\pi(\lambda, z)/\lambda r(\lambda, z)$ all its poles inside

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[1] It can be proved that there exists at most one pair $\pm\lambda_\beta$ of critical values [1].

the unique circle. The second term in the right-hand member of (54) is obviously stable and proper and will be referred to as the *stable projection* of $\pi(\lambda, z)/\lambda r(\lambda, z)$; similarly to (52), (53), let us call $G(\lambda)$ its associated infinite Hankel matrix.

Since $h(z)$ as well as $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$ are bounded in the closed unit disk, H and $G(\lambda)$ can be considered as bounded linear operators acting on the Hilbert space of square summable sequences.

Theorem 8. The L_2-norm of $H-G(\lambda)$ is bounded by $|\lambda|^{-1}$ for all real values of λ,

$$\|H-G(\lambda)\|_2 \leq \frac{1}{|\lambda|}. \tag{55}$$

Proof: With the help of (54), the polynomial relation (8) can be rewritten as

$$-\frac{\pi_u(\lambda, z)}{\lambda r_u(\lambda, z)}+\left[\frac{b(z)}{a(z)}-\frac{\pi_{st}(\lambda, z)}{\lambda r_{st}(\lambda, z)}\right]=\frac{1}{\lambda} s_1(\lambda, z)\frac{\hat{a}(z)}{a(z)}. \tag{56}$$

We first note that the right-hand member of (56) has constant modulus $|\lambda|^{-1}$ on the unit circle and this implies in particular that the left-hand member has no pole on the unit circle. Next, denoting by $\vec{x}$ the infinite vector of the Fourier coefficients of any L_2^+-function $x(e^{i\theta})$ of unit norm and taking into account that both terms in the left-hand member of (56) have one-sided Fourier expansion, we immediately deduce the inequality

$$\tilde{\vec{x}}[\tilde{H}-\tilde{G}(\lambda)][H-G(\lambda)]\vec{x} \leq |\lambda|^{-2}\|x(e^{i\theta})\|_2^2=|\lambda|^{-2} \tag{57}$$

and this obviously completes the proof. □

As a result, it turns out that $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$ is some approximation of $b(z)/a(z)$ in the sense of the norm (55), called the *Hankel norm*. This approximation feature will be emphasized in the sequel by using the notation

$$\|e(\lambda, z)\|_H \leq |\lambda|^{-1} \tag{58}$$

instead of (55), where $e(\lambda, z)$ stands for the approximation error $b(z)/a(z)-\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$. An interesting property of the Hankel norm is that it appears naturally as some intermediate norm between the popular least squares and Chebyshev norms

$$\|e(\lambda, z)\|_2 \leq \|e(\lambda, z)\|_H \leq \|e(\lambda, z)\|_\infty \tag{59}$$

as it can be easily verified by setting $\vec{x}=[1,0,0,\cdots]$ in (57) on the one hand and from an obvious argument on the other hand.

It should be remarked that, although the degree of the approximation $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$ is known to be s at most on the interval $|\lambda_s| \leq \lambda \leq |\lambda_{s+1}|$ in view of Theorem 6, we do not know much so far concerning the quality of this approximation. In the next section, this important issue will be considered in detail with the help of the following important theorem [1].

Theorem 9. The singular values of the parametric interpolation problem are identical to the inverses of the nonzero singular values of the infinite Hankel matrix H:

$$\det[I-\lambda^2\tilde{H}H]=\frac{\det N(\lambda^2)}{\det N(0)}. \tag{60}$$

Proof: Let V be the infinite Vandermonde matrix

$$V=\begin{bmatrix} 1 & z_1 & z_1^2 & \cdot & \cdot & \cdot \\ 1 & z_2 & z_2^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & z_n & z_n^2 & \cdot & \cdot & \cdot \end{bmatrix}.$$

Since the residue of $h(z)$ at the pole $z=z_i$ is equal to $\mu_i^1 \hat{a}(z_i)/a'(z_i)$ in view of (6) and (7), one has the identity

$$H=V^T\Delta\left(\frac{\mu_i^1\hat{a}(z_i)}{a'(z_i)}\right)V. \tag{61}$$

On the other hand, the matrix $N(0)$ is easily verified to admit the factorization

$$N(0)=\bar{\tilde{V}}V^T. \tag{62}$$

We next prove the identity

$$N^T(0)\bar{\Delta}\left(\frac{\hat{a}(z_i)}{a'(z_i)}\right)N(0)\Delta\left(\frac{\hat{a}(z_j)}{a'(z_j)}\right)=I. \tag{63}$$

To see this, let us consider the jth column of the inverse of $N^T(0)$ and denote by $\nu_1, \nu_2, \cdots, \nu_n$ its successive entries. By definition, one must have

$$\sum_{i=1}^{n} \frac{\nu_i}{1-z\bar{z}_i}=\begin{cases} 1, & \text{if } z=z_j \\ 0, & \text{if } z=z_i \quad (i \neq j) \end{cases} \tag{64}$$

so that the numerator of the left-hand member of (64) can be expressed as $c\Pi_{i\neq j}(z-z_i)$ with c a constant, actually determined by the first condition (64) as $\hat{a}(z_j)/a'(z_j)$. As a result, one has

$$\sum_{i=1}^{n} \frac{\nu_i}{1-z\bar{z}_i}=\frac{\bar{\hat{a}}(z_j)a(z)}{a'(z_j)(z-z_j)\hat{a}(z)} \tag{65}$$

which yields the ν_i by computing the residues of the right-hand member at the poles $z=1/\bar{z}_i$, i.e.,

$$\nu_i=\frac{\bar{\hat{a}}(z_i)}{\bar{a}'(z_i)}\cdot\frac{1}{1-\bar{z}_i z_j}\cdot\frac{\hat{a}(z_j)}{a'(z_j)} \tag{66}$$

so that the identity (63) is established. By using (61), (62), and (63), one then easily deduces the following matrix identity in λ^2:

$$V(I-\lambda^2\tilde{H}H)\tilde{V}=N^T(0)\bar{\Delta}\left(\frac{\hat{a}(z_i)}{a'(z_i)}\right) \cdot N(\lambda^2)\Delta\left(\frac{\hat{a}(z_j)}{a'(z_j)}\right)\bar{N}(0). \tag{67}$$

The right-hand member of the above equation has clearly rank n for almost all values of λ^2 and hence, so has the left-hand member. On the other hand, H has rank n too and consequently has n nonzero singular values only.

Hence, $\det(I-\lambda^2 H\tilde{H})$ is a polynomial of degree n in λ^2, whose zeros must coincide with those of $\det N(\lambda^2)$ in view of (67). ☐

V. Optimality Properties

Let $g(z)$ be any proper stable rational approximation of $h(z)$ satisfying

$$\|h(z)-g(z)\|_H \leq |\lambda|^{-1}. \tag{68}$$

If G stands for the infinite Hankel matrix associated with $g(z)$, let us denote $\nu_1^{-1} \geq \nu_2^{-1} \geq \nu_3^{-1} \geq \cdots \geq 0$ its successive singular values.

Theorem 10. If $|\lambda_s|<|\lambda|$, where λ_s is the sth singular value of the parametric interpolation problem such that $|\lambda_s|<|\lambda_{s+1}|$, the degree of $g(z)$ is at least s. Moreover, if $g(z)$ has degree s, the following lower bound of the Hankel norm exists:

$$|\lambda_{s+1}|^{-1} \leq \|h(z)-g(z)\|_H. \tag{69}$$

Proof: Since $\|H\|_2=\|H-G+G\|_2 \leq \|H-G\|_2+\|G\|_2$ by the triangle inequality, one deduces from (68)

$$\|H\|_2 \leq |\lambda|^{-1}+\|G\|_2. \tag{70}$$

More generally, in view of the min–max property of the eigenvalues of symmetric matrices [16] one has

$$|\lambda_i|^{-1} \leq |\lambda|^{-1}+\nu_i^{-1}, \qquad \text{all } i. \tag{71}$$

Since the above first s inequalities cannot be satisfied with $\nu_i^{-1}=0$ on the one hand and as the degree of $g(z)$ is equal to the rank of G from Kronecker's theorem on the other hand, the degree of $g(z)$ is at least s. The lower bound (69) follows trivially from the first part of the theorem. ☐

We are now in a position to prove the following two fundamental theorems, regarding the optimality of the approximations $\pi_{st}(\lambda,z)/\lambda r_{st}(\lambda,z)$ in Hankel norm.

Theorem 11. (Minimal Degree Approximation Theorem). Let $|\lambda|^{-1}$, the tolerance of the approximation of $h(z)$ in Hankel norm, be given; then $\pi_s(\lambda,z)/\lambda r_{st}(\lambda,z)$ is an approximation of $h(z)$ of minimum degree.

Proof: In view of the preceding theorem, the proof amounts to establishing that the degree of $\pi_{st}(\lambda,z)/\lambda r_{st}(\lambda,z)$ is s for $|\lambda_s|<|\lambda|\leq|\lambda_{s+1}|$. If λ is a regular value, the proof is obvious in view of Theorems 6 and 9 and Corollary 3, since $s_1(\lambda,z)$ and $\pi(\lambda,z)/\lambda r(\lambda,z)$ have the same degree. If λ is nonregular, the proof requires a little more attention since degree $s_1(\lambda,z)$ is reduced in view of Corollary 3. However, the arguments leading to the proof of Theorem 6 have actually established that $r(\lambda,z)$ has s stable zeros at most in any case, so that the above argument can still be applied on $\pi(\lambda,z)/\lambda r(\lambda,z)$. It is still worth noting that the number of stable poles of $s_1(\lambda,z)$ can be less than that of $\pi(\lambda,z)/\lambda r(\lambda,z)$ if λ is nonregular (see example).

Theorem 12. (Minimal Norm Approximation Theorem). Let $\lambda=\lambda_{s+1}$ with $|\lambda_{s+1}|>|\lambda_s|$ be a singular value of the parametric interpolation problem. Then, $g(z)=\pi_{st}(\lambda_{s+1},z)/\lambda_{s+1}r_{st}(\lambda_{s+1},z)$ minimizes $\|h(z)-g(z)\|_H$ over the set of all proper stable rational functions $g(z)$ of degree s; moreover, the value of the norm is precisely $|\lambda_{s+1}^{-1}|$.

Proof: From Theorem 11 and its proof, the degree of $\pi_{st}(\lambda,z)/\lambda r_{st}(\lambda,z)$ is known to be s. The proof is then completed by comparing (69) and (58) with $\lambda=\lambda_{s+1}$. ☐

The practical significance of the above two theorems must be strongly emphasized. Indeed, for a given tolerance (say $|\lambda_\mu|^{-1}$) and an arbitrarily fixed value ϵ of unit modulus, solving (8) clearly amounts to solving a linear system of equations in the unknown coefficients of the one-variable polynomials $r(\lambda_\mu,z)$ and $\pi(\lambda_\mu,z)$. A *minimal degree approximation* is then achieved by removing from $r(\lambda_\mu,z)/\lambda r(\lambda_\mu,z)$ its unstable part and hence can be obtained at low computational cost. The *minimal norm approximation* problem for a given degree (say s) of the approximation involves the additional calculation of the first $s+1$ singular values of H; however, the latter problem can be shown with the help of the polynomial equation (8) to be reducible to the calculation of the eigenvalues of an $n\times n$ symmetric matrix. Moreover, the above two problems can be put in various forms, for which fast algorithms are available. A detailed discussion of the computational aspects of the Hankel norm approximation problem as well as of the algorithms specifically adapted to its numerical solution will be published elsewhere.

VI. Singular and Critical Values

This section is devoted to elucidating the behavior of the solution of the parametric interpolation problem at both the singular and the critical values. Moreover, the structure of the eigenvectors associated with the singular values of the infinite Hankel matrix H will be put into light. Last but not least, the validity of the fast algorithms based on solving the polynomial equation (8) to obtain Hankel norm approximations will be established in full generality.

Theorem 13. If the polynomial $r(\lambda_s,z)$ has k zeros on the unit circle, λ_s is a singular value of multiplicity k ($|\lambda_s|=|\lambda_{s+1}|=\cdots=|\lambda_{s+k-1}|$) of the parametric interpolation problem and conversely.

Proof: A factor $(z-\alpha)$ with $|\alpha|=1$ is clearly self-reciprocal since $(z-\alpha)=-\alpha(1-\bar{\alpha}z)$. Hence, $r(\lambda_s,z)$, $\hat{r}(\lambda_s,z)$ and $\pi(\lambda_s,z)$ have necessarily k common zeros on the unit circle, so that (8) can be rewritten as

$$\frac{\lambda_s b(z)}{a(z)}\,\frac{r_*(\lambda_s,z)}{\hat{a}(z)}-\epsilon\frac{\hat{r}_*(\lambda_s,z)}{a(z)}=\frac{\pi_*(\lambda_s,z)}{\hat{a}(z)} \tag{72}$$

where $r_*(\lambda_s,z)$ and $\pi_*(\lambda_s,z)$ have degree $n-k$. Since $\hat{a}(z)$ is devoid of zeros inside the closed unit disk, the Fourier expansion of $r_*(\lambda_s,e^{i\theta})/\hat{a}(e^{i\theta})$ has the form

$$\frac{r_*(\lambda_s,e^{i\theta})}{\hat{a}(e^{i\theta})}=\sum_{p=0}^{\infty}\xi_p e^{ip\theta} \tag{73}$$

so that the expansion of $\hat{r}_*(\lambda_s,e^{i\theta})/a(e^{i\theta})$ turns out to be

$$\frac{\hat{r}_*(\lambda_s,e^{i\theta})}{a(e^{i\theta})}=e^{-ik\theta}\sum_{p=0}^{\infty}\bar{\xi}_p e^{-ip\theta}. \tag{74}$$

As $\pi_*(\lambda_s, e^{i\theta})/\hat{a}(e^{i\theta})$ has clearly a one-sided L_2^+-Fourier expansion while

$$b(e^{i\theta})/a(e^{i\theta}) = \sum_{q=1}^{\infty} h_q e^{-iq\theta}$$

expressing that the left-hand member of (72) has no Fourier coefficients of negative index, leads to

$$\lambda_s H\vec{\xi} = \epsilon\vec{\bar{\xi}}_{(k-1)} \tag{75}$$

where $\vec{\xi}$ is the infinite vector of the Fourier coefficients of (73) while the notation $\vec{\xi}_{(v)}$ is used for the v-shifted copy of $\vec{\xi}$, i.e.,

$$\vec{\xi}^T = \left[\overbrace{0,0,\cdots,0,}^{v}\vec{\xi}^T\right].$$

In view of the shift-invariant property of the Hankel operator H, one easily verifies

$$\lambda_s H\left[\vec{\xi}, \vec{\xi}_{(1)}, \cdots, \vec{\xi}_{(k-1)}\right] = \left[\vec{\bar{\xi}}_{(k-1)}, \vec{\bar{\xi}}_{(k-2)}, \cdots, \vec{\bar{\xi}}\right] \tag{76}$$

and hence

$$\left[I - \lambda_s^2 \tilde{H}H\right]\left[\vec{\xi}, \vec{\xi}_{(1)}, \cdots, \vec{\xi}_{(k-1)}\right] = 0 \tag{77}$$

which shows that $\vec{\xi}, \vec{\xi}_{(1)}, \cdots, \vec{\xi}_{(k-1)}$ are k independent eigenvectors associated with the eigenvalue $1/\lambda_s^2$ of $\tilde{H}H$. Consequently, $|\lambda_s|$ is a singular value of multiplicity k of the parametric interpolation problem. The proof of the converse can be obtained in a straightforward manner. For a sufficiently small positive nonzero number δ, the polynomial $r(\lambda, z)$ is known to have on the interval $\lambda_s - \delta \leq \lambda \leq \lambda_s + \delta$, respectively, $(s-1)$ and $(s+k-1)$ stable zeros for $\lambda < \lambda_s$ and $\lambda > \lambda_s$. As the zeros of a polynomial are continuous functions of its coefficients, it must have at least k zeros on the unit circle for $\lambda = \lambda_s$ and this number is exactly k, otherwise λ_s would be a singular value of multiplicity larger than k in view of the first part of the theorem. □

The behavior of the solution of the parametric interpolation when $\lambda = \lambda_\beta$ is a critical value can now be elucidated.

Theorem 14. If $\lambda = \lambda_\beta$ is a critical value of the parametric interpolation problem, $r(\lambda, z)$ contains at least a self-reciprocal factor of the form $(z - z_i)(1 - \bar{z}_i z)$ where z_i is an interpolation point. Moreover, the interpolation constraint is not satisfied at that point, i.e., $s_1(\lambda_\beta, z_i) \neq \lambda_\beta \mu_i^1$.

Proof: Since $\lambda = \lambda_\beta$ is a nonregular value, $s_1(\lambda_\beta, z)$ is known from Corollary 3 to have degree smaller than n. Hence $\hat{r}(\lambda_\beta, z)$ and $r(\lambda_\beta, z)$ have a common factor, necessarily self-reciprocal, that may not have zeros on the unit circle in view of the preceding theorem. As a result, this common factor has the form

$$\prod_{u=1}^{v} (z - z_u)(1 - \bar{z}_u z)$$

with $v \geq 1$ and $|z_u| < 1$ for all u. Let us now assume that $\pi(\lambda_\beta, z)$ contains the same factor. Then (8) can be rewritten as (72) where $r_*(\lambda_\beta, z)$ and $\pi_*(\lambda_\beta, z)$ have degree $n - 2v$ and the argument used in the proof of Theorem 13 leads to the conclusion that λ_β is a singular value of multiplicity $2v$. This contradiction clearly forces the z_u to be identical to v interpolation points z_i at which one has obviously $s_1(\lambda_\beta, z) \neq \lambda_\beta \mu_u^1$ for all u. Note that $\pi(\lambda_\beta, z)$ contains necessarily the factor

$$\prod_{u=1}^{v} (1 - \bar{z}_u z). \qquad \square$$

It should be noted that the above theorem admits the following weak converse only.

Corollary 4. If, for $\lambda = \lambda_\beta$, all interpolation constraints are not satisfied, then λ_β is either a singular or a critical value of the parametric interpolation problem.

Proof: If $s_1(\lambda_\beta, z_i) \neq \lambda_\beta \mu_i^1$, $r(\lambda_\beta, z)$ has clearly a factor $(z - z_i)(1 - \bar{z}_i z)$ by the same argument as above. However, $r(\lambda_\beta, z)$ can have in addition a zero on the unit circle, in which case λ_β would be a singular value. Finally, from Corollary 3, $s_1(\lambda_\beta, z)$ is known to have a degree smaller than n if and only if λ_β is a nonregular value and this obviously completes the proof. □

In Theorems 1 and 2, the Nevanlinna algorithm has been proved to produce the most general form of the minimal degree parametric all-pass function $s_1(\lambda, z)$ satisfying all the parametric constraints $s_1(\lambda, z_i) = \lambda \mu_i^1$. However, it has just been shown that such is not the case for the interpolation problem when λ has a *fixed* value, if this value is critical or even perhaps singular. On the other hand, any interpolation problem by an all-pass rational function always admits a solution of degree not larger than n and satisfying n arbitrary constraints (z_i, μ_i^1). Thus one is left with the following last but crucial question. Will the algorithms solving the polynomial equation (8) for a fixed value of λ lead to a Hankel norm approximation of minimum degree for all values of λ? The answer is affirmative on the basis of the following final theorem.

Theorem 15. For any fixed value of λ, say λ_μ, all solutions of the polynomial equation $\lambda_\mu b(z) r(z) - \epsilon \hat{r}(z) a(z) = \pi(z) a(z)$, where ϵ is any constant of unit modulus, produce rational functions $\pi(z)/r(z)$ with the same number of stable poles.

Proof: Let $r_1(z)$, $\pi_1(z)$ be the solution $r(\lambda_\mu, z)\pi(\lambda_\mu, z)$ and $r_2(z)$, $\pi_2(z)$ be any other solution. Writing ϵ_1 and ϵ_2 as $e^{2i\varphi_1}$ and $e^{2i\varphi_2}$ and multiplying the two polynomial equations by $e^{-i\varphi_2}$ and $e^{-i\varphi_2}$, respectively, one can rewrite these equations as $(i = 1, 2)$

$$\lambda_\mu b(z) r_i(z) - \hat{r}_i(z) \hat{a}(z) = \pi_i(z) a(z) \tag{78}$$

at the cost of a renormalization of the pairs of polynomials $(r_i(z), \pi_i(z))$. Let us first assume λ_μ not be singular. Then $r_3(z) = r_1(z) + k r_2(z)$ and $\pi_3(z) = \pi_1(z) + k\pi_2(z)$ form clearly another solution pair for any k in the interval $0 \leq k \leq 1$. If $r_1(z)$ and $r_2(z)$ have not the same number of stable zeros, there must exist by continuity a value of k for which $r_3(z)$ will have at least one zero on the unit circle. In view of Theorem 13, λ_μ is then a singular value in contradiction with our assumption. If λ_μ is a singular value of multiplicity k, $r_1(z)$ and $r_2(z)$ are known on the strength of Theorem 13 to have the same number k of zeros on the unit circle. Hence, the same argument can still be applied after deleting these zeros from both equations (78).

VII. EXAMPLE

Let us consider the parametric interpolation problem

$$s_1(\lambda, z_i)=\lambda\mu_i^1, \qquad i=1,2 \tag{79}$$

with $z_1=1/2$, $z_2=-1/2$ and $\mu_1^1=1$, $\mu_2^1=-1$. Equivalently, since from (4) and (5) one deduces $a(z)=(z^2-1/4)$, $b(z)=152z/8$, let us investigate the Hankel norm approximations of the rational function

$$h(z)=\frac{15z}{2(4z^2-1)}. \tag{80}$$

In spite of the obvious simplicity of its formulation, this problem will appear as an excellent illustration of the various questions discussed in this paper.

The diagonal entries of the Fenyves array (13) turn out to be $\lambda\mu_1^1=\lambda$, $\lambda\mu_2^2(\lambda^2)=5\lambda/2(1+\lambda^2)$ and not much effort is needed to compute from (32) the polynomial $q_2(\lambda, z)$ as

$$q_2(\lambda, z)=\frac{1}{8(1+\lambda^2)}\big[(1+\lambda\epsilon)(8\lambda^2-2)z^2 + 15\lambda(\lambda+\epsilon)z+(1+\lambda\epsilon)(8-2\lambda^2)\big] \tag{81}$$

and hence via (23), (24), and (25) the most general form of the irreducible minimal degree solution as

$$s_1(\lambda, z)=\frac{\big[\lambda(8-2\lambda^2)z^2+15\lambda z+\lambda(8\lambda^2-2)\big]+\epsilon\big[(8-2\lambda^2)z^2+15\lambda^2 z+(8\lambda^2-2)\big]}{\epsilon\big[\lambda(8\lambda^2-2)z^2+15\lambda z+\lambda(8-2\lambda^2)\big]+\big[(8\lambda^2-2)z^2+15\lambda^2 z+(8-2\lambda^2)\big]}. \tag{82}$$

Setting $z=\pm 1/2$ in (82), one easily verifies in accordance with Theorem 1 the two identities

$$s_1(\lambda, \pm 1/2)\equiv\pm\lambda. \tag{83}$$

The polynomial $p(\lambda^2)$ resulting from (26) is found to be

$$p(\lambda^2)=(\lambda^2-1)(4\lambda^2-1)(\lambda^2-4) \tag{84}$$

and hence according to Theorem 7, $\lambda_\beta=\pm 1, \pm 2, \pm 1/2$ are either singular or critical values for the parametric interpolation problem considered.

The Nevanlinna matrix (36) has the form

$$N(\lambda^2)=\begin{bmatrix} \dfrac{1-\lambda^2}{3/4} & \dfrac{1+\lambda^2}{5/4} \\[2ex] \dfrac{1+\lambda^2}{5/4} & \dfrac{1-\lambda^2}{3/4} \end{bmatrix} \tag{85}$$

and, since its determinant has the value $(64/225)(4\lambda^2-1)(\lambda^2-4)$, the pairs $\lambda_\beta=\pm 2$ and $\pm 1/2$ are singular values, while the pair $\lambda_\beta=\pm 1$ is critical as it is obvious by setting $\lambda^2=1$ in (86).

Deducing the polynomial $\pi(\lambda, z)$ from (8), one obtains

$$\pi(\lambda, z)=\tfrac{1}{2}(\lambda+\epsilon)(4-\lambda^2)z^2+15(1+\lambda\epsilon)\lambda^3 z + 8(\lambda+\epsilon)(4\lambda^2-1) \tag{86}$$

so that the minimal degree approximations of $h(z)$ in Hankel norm will be achieved due to Theorem 11 by extracting from

$$\frac{\pi(\lambda, z)}{\lambda r(\lambda, z)}=\frac{(\lambda+\epsilon)(4-\lambda^2)z^2+30(1+\lambda\epsilon)\lambda^3 z+16(\lambda+\epsilon)(4\lambda^2-1)}{4(1+\lambda\epsilon)(4\lambda^3-\lambda)z^2+30(\lambda+\epsilon)\lambda^2 z+4(1+\lambda\epsilon)(4\lambda-\lambda^3)} \tag{87}$$

its stable projection $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$.

As a result of Theorems 11 and 12, one has:

a) *In the interval* $0<|\lambda|\leqslant 1/2$, $r(\lambda, z)$ has its zeros outside the unit circle. Hence, $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)\equiv 0$ is the minimum degree approximation in Hankel norm of $h(z)$ for any fixed tolerance $|\lambda|^{-1}\geqslant 2$. Moreover, $g(z)=0$ trivially minimizes $\|h(z)-g(z)\|_H$ over the set of all proper stable rational functions of degree 0 and one has $\|h(z)-0\|_H=|\lambda_1|^{-1}=2$. Note that at the singular value $\lambda_1=\pm 1/2$, $s_1(\pm 1/2, z)$ and $\pi(\pm 1/2, z)/\pm 1/2r(\pm 1/2, z)$ degenerate, respectively, into z and $z/2$.

b) *In the interval* $1/2<|\lambda|\leqslant 2$, $r(\lambda, z)$ has one zero inside the unit circle. Let z_α be this zero for a fixed value of λ, say λ_α, and h_α the residue of $\pi(\lambda_\alpha, z)/\lambda_\alpha r(\lambda_\alpha, z)$ at the pole $z=z_\alpha$. The stable projection of $\pi(\lambda_\alpha, z)/\lambda_\alpha r(\lambda_\alpha, z)$ is clearly $h_\alpha/(z-z_\alpha)$ and this rational function is a minimal degree approximation of $h(z)$ satisfying $\|h(z)-h_\alpha/(z-z_\alpha)\|_H\leqslant|\lambda_\alpha|^{-1}$. The minimal Hankel norm approximation of degree one is reached at the second singular value $\lambda_\alpha=\pm 2$ where $\pm\pi(\pm 2, z)/2r(\pm 2, z)$ degenerates into its stable projection $2/z$. As a result, one has $\|h(z)-2/z\|_H=|\lambda_2|^{-1}=1/2$.

c) *In the interval* $2<\lambda\leqslant\infty$, $r(\lambda, z)$ has all its zeros inside the unit circle so that $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$ is obtained by removing from (83) its value at infinity and hence has the form $n(\lambda, z)/4(1+\lambda\epsilon)(4\lambda^2-1)\lambda r(\lambda, z)$ where $n(\lambda, z)$ is the first degree polynomial

$$n(\lambda, z)=30\lambda\big[4\lambda^2(4\lambda^2-1)\lambda^2(1+\lambda\epsilon)^2-(4-\lambda^2)(\lambda+\epsilon)^2\big]z + 4(\lambda+\epsilon)(1+\lambda\epsilon)(256\lambda^4-127\lambda^2+12). \tag{88}$$

For any value of λ, $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$ is a minimal degree approximation of $h(z)$ whose Hankel norm is smaller than or equal to $|\lambda|^{-1}$. It can be easily verified with the help of (88) that the minimal Hankel norm approximation of degree two is indeed $h(z)$ itself, by letting λ tend to infinity in $\pi_{st}(\lambda, z)/\lambda r_{st}(\lambda, z)$.

Let us finally consider the behavior of the solutions (82), (87) when λ takes the critical value ± 1. One easily obtains

$$r(\pm 1, z)=(1\pm\epsilon)12(z+\tfrac{1}{2})(z+2) \tag{89}$$

$$\pi(\pm 1, z)=(\epsilon\pm 1)3(z+8)(z+2) \tag{90}$$

$$s(\pm 1, z)=\pm 1. \tag{91}$$

In agreement with Corollary 3 and Theorem 14, it is easily verified that $s(\pm 1, z)$ has degree smaller than $n=2$ and does not interpolate the second data pair ($z_2=-1/2$, $\mu_2^1=-\lambda=\mp 1$); moreover, $r(\pm 1, z)$ contains the self-reciprocal factor $(z-z_2)(7-\bar{z}_2 z)$. It should be noted, however, that $\pm\pi(\pm 1, z)/r(\pm 1, z)$ remains well defined

$$\pm\frac{\pi(\pm 1, z)}{r(\pm 1, z)}=\frac{z+8}{2(2z+1)} \tag{92}$$

and that its stable projection $15/4(2z+1)$ is still a minimal degree Hankel norm approximation in accordance with Theorem 11. Let us finally remark that the point $\lambda=\pm 1$, $z=-1/2$ is an essential singularity of the two-variable rational function $s_1(\lambda, z)$ as indicated in the end of Section II.

It is not difficult to compute the most general form of the all-pass rational function interpolating the pairs $(z_1=1/2, \mu_1^1=1)$, $(z_2=-1/2, \mu_2^1=-1)$. The solution of this *non*parametric problem turns out to be $(|\epsilon|=1)$

$$s_1(z)=\frac{2z^2+5\epsilon z+2}{2\epsilon z^2+5z+2\epsilon} \tag{93}$$

whose denominator has one stable pole only (see Theorem 15), so that the stable projection of $\pi(z)/r(z)=(z^2+10\epsilon z+16)/(4z^2+5\epsilon z+4)$ is a minimal degree Hankel norm approximation as well as (92); clearly, the Hankel norms of both solutions admit the same upper bound 1.

APPENDIX

The assumption that all the poles of the rational function $h(z)=b(z)/a(z)$ are distinct is by no means essential and can be removed without conceptual difficulty. All the theorems remain the same except for a few obvious details but their proofs require much more algebraic, tedious although elementary, sophistication. We shall not consider these matters in detail but rather briefly indicate how the parametric interpolation problem and its solution algorithm can be modified so as to cope with the multiple poles situation.

Let the problem be to find a minimal degree all-pass rational function $s_1(\lambda, z)$ such that around the point $z_i (|z_i|<1,\ i=1,2,\cdots,n)$, the first k_i coefficients of its Taylor expansion take prescribed parametric values $\lambda\mu_{i,j}^1$ $(j=1,2,\cdots,k_i)$. In other words, one must have

$$s_1(\lambda, z)=\lambda\mu_{i,1}^1+\lambda\mu_{i,2}^1(z-z_i)+\cdots$$
$$+\lambda\mu_{i,k_i}^1(z-z_i)^{k_i-1}+O\left(\overline{z-z_i}^{\,k_i}\right). \tag{A.1}$$

There obviously exists a set of k_i constants $v_{i,j}^1$, uniquely determined from the $\mu_{i,j}^1$ such that the constraint (A.1) can be replaced by

$$s_1(\lambda, z)=\lambda v_{i,1}^1+\lambda v_{i,2}^1\left(\frac{z-z_i}{1-\bar{z}_i z}\right)+\cdots$$
$$+\lambda v_{i,i}^1\left(\frac{z-z_i}{1-\bar{z}_i z}\right)^{k_i-1}+O\left[\left(\frac{z-z_i}{1-\bar{z}_i z}\right)^{k_i}\right]. \tag{A.2}$$

The transformation $z=(y+z_i)/(1+\bar{z}_i y)$ is well known to map the unit disk onto itself on the one hand and is readily verified to be degree preserving for $s_1(\lambda, z)$. Hence, as far as the point z_i is concerned, the problem can be rephrased once more as: to find an all-pass rational function

$$\sigma_1(\lambda, y)=s_1\left(\lambda, \frac{z-z_i}{1-\bar{z}_i z}\right)$$

of minimal degree and which admits around the point $y=0$ the expansion

$$\sigma_1(\lambda, y)=\lambda v_{i,1}^1+\lambda v_{i,2}^1 y+\cdots$$
$$+\lambda v_{i,k_i}^1 y^{k_i-1}+O(\bar{y}^{k_i}). \tag{A.3}$$

Let us consider the interpolation problem at the point z_1 only for the moment. It turns out that the problem of finding the most general form of all parametric all-pass rational functions, satisfying the constraint (A.3) with $i=1$, can be solved with the help of the Schur algorithm [2], closely related to the Nevalinna algorithm, as shown hereafter. Let $\sigma_1(\lambda, y)$ be a solution of the problem (see formulas (9)–(12)); in particular, one must have $\sigma_1(\lambda, 0)=\lambda v_{1,1}^1$ so that the rational function $\sigma_1(\lambda, y)-\lambda v_{1,1}^1$ has a zero in $y=0$, independent of λ. Hence, the rational function $\sigma_2(\lambda, y)$ defined as

$$\sigma_2(\lambda, y)=\frac{1}{y}\cdot\frac{\sigma_1(\lambda, y)-\lambda v_{1,1}^1}{1-\lambda\bar{v}_{1,1}^1\sigma_1(\lambda, y)} \tag{A.4}$$

is rational, has no poles independent of λ in $z=z_1$ and is clearly an all-pass function. In terms of $\sigma_2(\lambda, y)$, the remaining interpolation constraints have the form

$$\sigma_2(\lambda, y)=\lambda v_{1,2}^2(\lambda^2)+\lambda v_{1,3}^2(\lambda^2)y+\cdots$$
$$+\lambda v_{1,k_1}^2 y^{k_1-2}+O(y^{k_1-1}) \tag{A.5}$$

where the $v_{1,j}^2(\lambda^2)$ are uniquely determined from the $v_{1,i}^1$ with the help of (A.4). As in the Nevanlinna algorithm, the above procedure can be iterated up to exhausting all the constraints at the point z_1. The final result can clearly be expressed in compact form with the help of a homographic transformation (see (15)–(18)) as

$$\sigma_1(\lambda, y)=T_{k_1}(\lambda, y)[\sigma(\lambda, y)] \tag{A.6}$$

where $T_{k_1}(\lambda, y)$ is the J-unitary matrix defined from the $v_{1,u}^u (1\leqslant u\leqslant k_1)$ by

$$T_{k_1}(\lambda, y)=\prod_{u=1}^{k_1}\frac{1}{\sqrt{1-|\lambda v_{1,u}^u(\lambda^2)|^2}}\cdot\begin{bmatrix}1 & \lambda v_{1,u}^u(\lambda^2)\\ \lambda\bar{v}_{1,u}^u(\lambda^2) & 1\end{bmatrix}\begin{bmatrix}y & \\ & 1\end{bmatrix} \tag{A.7}$$

and with $\sigma(\lambda, y)$ an arbitrary all-pass rational function without pole at $y=0$. Performing the backward substitution $y\to z$ produces

$$s_1(\lambda, z)=T_{k_1}\left(\lambda, \frac{z-z_1}{1-\bar{z}_1 z}\right)[s_2(\lambda, z)]$$

where $s_2(\lambda, z)$ is an arbitrary all-pass rational function without pole at $z=z_1$. The constraints at the remaining points $z_2, z_3, \cdots, z_n$ can then be translated in terms of $s_2(\lambda, z)$ by inverting the homographic transformation

$$s_2(\lambda, z)=T_{k_1}^{-1}\left(\lambda, \frac{z-z_1}{1-\bar{z}_1 z}\right)[s_1(\lambda, z)] \tag{A.8}$$

and clearly will be of the form (A.1) except that the $\mu_{i,j}^1$ will now be rational functions of λ^2. Since the above procedure can be repeated at the point z_2 and so on, the minimal degree parametric all-pass rational function $s_1(\lambda, z)$ satisfying all the interpolation constraints will be expressed as

$$s_1(\lambda, z)=\prod_{i=1}^{n} T_{k_i}\left(\lambda, \frac{z-z_i}{1-\bar{z}_i z}\right)[\epsilon] \tag{A.9}$$

where the arbitrary all-pass function is chosen as a constant ϵ of unit modulus so as to minimize the degree of $s_1(\lambda, z)$, clearly

$$\sum_{i=1}^{n} k_i.$$

Note finally that the multiple poles problem admits a polynomial formulation identical to (8) with the definition

$$a(z)=\prod_{i=1}^{n}(z-z_i)^{k_i} \tag{A.10}$$

while the relation between the $\mu^1_{i,j}$ and the residues $r_{i,j}$ at the poles of $b(z)/a(z)$, i.e.,

$$\frac{b(z)}{a(z)}=\sum_{i=1}^{n}\sum_{j=1}^{k_i}\frac{r_{i,j}}{(z-z_i)^j} \tag{A.11}$$

is given by

$$r_{i,k_i-\mu}=\sum_{v=1}^{u+1}\frac{1}{(u-v+1)!}u_{i,v}D^{(u-v+1)}m_i(z_i) \tag{A.12}$$

where $D^{(u-v+1)}m_i(z_i)$ stands for the $(u-v+1)$th derivative of $m_i(z)\equiv(z-z_i)^{k_i}\hat{a}(z)/a(z)$ at the point $z=z_i$.

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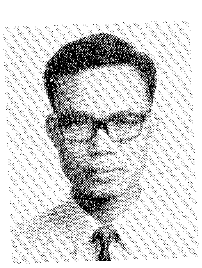

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