Bezout Space-Time Precoders and Equalizers for MIMO Channels

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***Abstract*—The transmitter/receiver diversities in p-input–q-output multiple-input multiple-output (MIMO) channels will play a key role in future high-rate wireless communication. The major challenge in MIMO signal recovery is the mitigation of the inevitable intersymbol interference (ISI) and interchannel interference (ICI) in multipath MIMO channels. The Bezout system theory provides a simple solution for ISI/ICI cancellation and, thus, can serve as a powerful mathematical foundation for MIMO systems. If $p < q$ and the MIMO system $\mathrm{H}(D)$ is right coprime, the system is perfectly recoverable via a Bezout equalizer. If $p > q$ and the MIMO system $\mathrm{H}(D)$ is left coprime, the system is perfectly recoverable via a Bezout precoder, assuming the channel information is available at the transmitter. For a robust channel design, a quantitative analysis of signal-to-noise ratio (SNR) of the optimal Bezout equalizer/precoder can be derived from the Bezout null space. Compared with the Alamouti space-time coding, the Bezout precoder is shown to be more appealing in slowly time-varying channels, where the feedback of the channel information induces minor overhead.**

The Bezout system can work well jointly with the space-time block coding (STBC). The STBC can be adopted to artificially construct a larger $q' \times p'$ ($q' > p'$) virtual transfer function without the channel information. A necessary and sufficient condition is provided for perfect recoverability (PR) for the STBC-induced virtual MIMO systems. Based on the virtual transfer function, both channel-independent and channel-dependent precoding strategies are feasible. Via a singular value decomposition (SVD) analysis, the Bezout precoder can be shown to outperform the orthogonal frequency division multiplexing (OFDM) precoder in bit-error-rate (BER), transmission rate, and receiver implementation. The interplay of the two design parameters (N and ρ) is analyzed to provide a simple guideline for an optimal configuration. The combined STBC and Bezout equalization techniques offer a broader spectrum of transceiver system configuration to achieve optimal design tradeoff among BER, transmission rate, and implementation complexity.

I. Introduction

The rapid progress in wireless mobile communications has led to an increased demand for more adaptive, efficient signal processing techniques with higher quality [7]. The transmitter/receiver diversities in multiple-input multiple-output (MIMO) channels will play a key role in future high-rate wireless communication. In a practical environment, the impairment introduced by multipath propagation and limited bandwidth can cause severe receiver performance degradation. Intersymbol interference (ISI) has become a very critical problem in high-speed telecommunication systems, such as terrestrial television broadcasting and cellular mobile communication systems [16]. In addition to ISI, yet another kind of interference, i.e., interchannel interference (ICI), will become inevitable in the environment of MIMO channels. Existing approaches to ISI/ICI mitigation include

i) post equalization;
ii) precoding techniques;
iii) multicarrier modulation;
iv) space-time block coding (STBC) techniques [19]–[21], [26].

In this paper, we combine the Bezout inverse theory with the STBC techniques. Based on the theory of Bezout inverse systems, the necessary and sufficient condition of perfect recoverability (PR) can be determined from a generalized resultant matrix [12]. The term perfect recoverability here is equivalent to perfect reconstruction used in [26]. This leads to the construction of optimal Bezout equalizers, which are sometimes called zero-forcing equalizers. This paper extends the same concept to the design of so-called Bezout precoders.

STBC falls into the category of transmitter-side diversity creation techniques [1], [19]–[21]. It is best understood within a multirate polyphase framework. By adding interleaving and deinterleaving operators to the transmission channel, an expanded MIMO system can be created. In fact, it has been shown in [17] that many contemporary diversity creation techniques, such as orthogonal frequency division multiplexing (OFDM) [4], fractional sampling, interleaving, and code division multiple access (CDMA), can be formulated within a unifying polyphase framework. For details regarding the multirate systems, see [13], [24], and [25]. In [26], the recoverability conditions for the virtual MIMO created from a single-input single-output (SISO) channel were established in terms of channel zeros. In this paper, based on the Bezout inverse theory, we provide the recoverability conditions for the virtual MIMO from physical MIMO channels. In addition, we shall provide qualitative and quantitative analysis on the expanded virtual MIMO.

Practically, the design criterion of optimal equalizers should be based on the SNR performance under noisy conditions. The Bezout null space plays an important role in improving the SNR. Traditionally, static or low-order FIR equalizers would be preferred due to their simplicity. However, in modern communication systems, the transmission power and bandwidth are usually very tightly budgeted, while the VLSI digital signal processors have made possible cost-effective and low-power computation.

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This, in turn, will encourage designers to consider more sophisticated high-order equalizers as opposed to simpler static equalizers. Moreover, it will be proved theoretically that a high-order equalizer can yield a superior SNR compared with a static (instantaneous) equalizer. In fact, the higher order the equalizer, the better the SNR. Similarly, we establish the fact that the larger the block size, the higher the SNR. Finally, reduced transmission rate (via transmitter redundancy by inserting zeros into each block) can also significantly enhance the gain. We will discuss the effects of these parameters on the SNR from both the theoretical and experimental perspectives. With the flexibility offered by preprocessing in STBC and postprocessing in Bezout equalization, a whole spectrum of system configurations are available for optimal design tradeoff.

The organization of this paper is outlined as follows. In Section II, we review the basic analysis on the Bezout inverse system, which serves as a useful foundation for the perfect recoverability of MIMO channels. Section III shows that the Bezout inverse system may be effectively applied to the design of Bezout equalizers and/or precoders. A duality relationship between the Bezout equalizer and precoder designs is established. Therefore, the same design optimization scheme may be adopted for both designs. An optimal Bezout solution based on the resultant matrix is also provided. In Section IV, we will discuss the critical role of the well-established polyphase STBC system. From a MIMO transmission channel, the STBC schema can construct an expanded virtual MIMO system. Based on the virtual transfer function, both channel-independent and channel-dependent precoding strategies are feasible. A channel-dependent optimal OFDM precoder is proposed, whose performance is compared with the Bezout precoder. In Section V, based on the Bezout null space theory, the SNR of the Bezout equalizers can be analyzed. In addition, the effects of STBC parameters on the SNR can be derived. The same section demonstrates simulation results on the SNRs and channel capacity for various block sizes, equalizer orders, and transmission rates. The results support both the qualitative and quantitative analysis presented in the paper.

II. Bezout System Theory for MIMO Channels

A. MIMO Channel Models

Consider a MIMO channel with p transmitters and q receivers. Let $s_j(k)$ denote the sequence from transmitter j ($j = 1, \ldots, p$), and let $h_{ij}(k)$ be the channel response from input j to output i ($i = 1, \ldots, q$). Consequently, the output sequence at the receiver i is

$$x_i(k) = \sum_{j=1}^{p} \sum_{l=0}^{d} h_{ij}(l) s_j(k-l) \tag{1}$$

where d denotes the maximal ISI degree. The above convolution can be equivalently expressed in a z-transform notation

$$\mathbf{H}(D)\mathbf{s}(D) = \mathbf{x}(D) \tag{2}$$

where $\mathbf{s}(D) = [s_1(D)s_2(D)\ldots s_p(D)]^T$, $\mathbf{x}(D) = [x_1(D) \ldots x_q(D)]^T$, and $\mathbf{H}(D) = \{h_{ij}(D)\}$ are the z-transform vectors (matrix) of the corresponding sequences or impulse responses. Departing from the traditional z-transform notation, here, we adopt a D-transform with $D = z^{-1}$, cf. [15]. A p-input–q-output MIMO system can then be represented by the $q \times p$ matrix $\mathbf{H}(D)$, which is referred to as the *transfer function* of the MIMO system.

Definition 1—Perfect Recoverability: A transfer function $\mathbf{H}(D)$ is perfectly recoverable (PR) if and only if there exists a polynomial matrix $\mathbf{G}(D)$ such that

$$\mathbf{G}(D)\mathbf{H}(D) = \mathrm{Diag}[D^{k_j}] \tag{3}$$

where k_j denotes the necessary delays incurred on the recovered jth source signal. The equality in (3) is referred to as (generalized) Bezout identity. $\mathbf{G}(D)$ will be called a Bezout inverse of $\mathbf{H}(D)$, which can perfectly recover the original signals in the absence of noise.

Given a MIMO system in (1), the jth source signal is said to be perfectly recoverable (PR) with a ρ_j-tap equalizer (i.e., degree $= \rho_j - 1$) if there exists a (row) polynomial vector $\mathbf{g}(D)$ with degree lower than ρ_j such that

$$\mathbf{g}(D)\mathbf{x}(D) = s_j(D)D^{k_j}. \tag{4}$$

Note that each source signal has its own sufficient order. More treatment on this subject can be found in [12]. ∎

A polynomial matrix $\mathbf{C}(D)$ is said to be a (right) common factor of the columns in $\mathbf{H}(D)$ if $\mathbf{H}(D) = \mathbf{H}'(D)\mathbf{C}(D)$ and $\mathbf{H}'(D)$ is itself a polynomial matrix. Moreover, a polynomial matrix is unimodular if and only if its determinant is a constant. Then, a polynomial matrix is said to be (right) coprime if and only if there exists no nonunimodular right common factor. Obviously, the right coprimeness of $\mathbf{H}(D)$ requires it to have more rows than columns, i.e., $q > p$, except when it is a unimodular matrix.

Theorem 1—Signal Recoverability for MIMO Channel:

1) A MIMO system with transfer function $\mathbf{H}(D)$ is perfectly recoverable (PR) if and only if $\mathbf{H}(D)$ is (right) coprime, except for a common factor with determinant D^k. (This property will be called delay-permissive coprimeness.)
2) A MIMO system with transfer function $\mathbf{H}(D)$ is perfectly recoverable if and only if $\mathbf{H}(\lambda)$ has full column rank for any (complex value) $\lambda \neq 0$.

Proof: See, e.g., [3] and [12]. ∎

B. Minimal Basis (MB)

For any p-input–q-output MIMO transfer function $\mathbf{H}(D)$, there exists a unimodular polynomial matrix $\mathbf{U}(D)$ [9] such that

$$\mathbf{H}(D)\mathbf{U}(D) = \hat{\mathbf{H}}(D)$$

where $\hat{\mathbf{H}}(D)$ is a minimal basis (which wil be described later). The $q \times p$ polynomial matrix $\hat{\mathbf{H}}(D)$ is called a minimal basis if and only if $\hat{\mathbf{H}}(D)$ is column reduced, i.e., its (column-wise) highest-degree coefficient matrix has full column rank. For convenience, we will denote the kth column degree of the MB $\hat{\mathbf{H}}(D)$ by μ_k.

The *McMillan degree* (which is denoted as η) of a $q \times p$ ($p < q$) polynomial matrix $\mathbf{H}(D)$ is defined as the highest degree of the determinants of all the $p \times p$ minors in $\mathbf{H}(D)$. It can be shown [9] that $\eta = \sum_{k=1}^{p} \mu_k$.

C. Null-Space's Minimal Basis (NMB)

Given a $q \times p$ ($p < q$) polynomial matrix $\mathbf{H}(D)$ with full column rank (on the polynomial ring), a $(q-p) \times q$ polynomial matrix $\mathbf{N}(D)$ is called a null-space minimal basis of $\mathbf{H}(D)$ if and only if

1) $\mathbf{N}(D)$ is in the null space of $\mathbf{H}(D)$, i.e., $\mathbf{N}(D)\mathbf{H}(D) = \mathbf{0}$;
2) $\mathbf{N}(D)$ is left-coprime;
3) $\mathbf{N}(D)$ is row reduced, i.e., its (row-wise) highest-degree coefficient matrix has full row rank.

We denote the row degrees of $\mathbf{N}(D)$ by $\{\nu_i\}_{i=1}^{q-p}$. The degree of the minimal basis of the null space of $\mathbf{H}(D)$ is defined as

$$\nu = \max_i\{\nu_i\}.$$

Any polynomial vector in the null subspace of $\mathbf{H}(D)$ can be expressed as a (polynomial-wise) combination of the row vectors in $\mathbf{N}(D)$. For more properties about minimal basis, see [5] and [9, ch. 6].

D. MIMO Resultant Matrix

The MIMO resultant matrix $\mathbf{\Gamma}^\rho[\mathbf{H}]$ is a block Toeplitz matrix (with ρ block-rows, i.e., $\rho \times q$ rows)

$$\mathbf{\Gamma}^\rho[\mathbf{H}] = \begin{bmatrix} \mathbf{H}_d & \mathbf{H}_{d-1} & \cdots & \mathbf{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_d & \mathbf{H}_{d-1} & \cdots & \mathbf{H}_0 & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_d & \mathbf{H}_{d-1} & \cdots & \mathbf{H}_0 \end{bmatrix} \quad (5)$$

where $\mathbf{H}_i$ is the ith degree coefficient matrix of $\mathbf{H}(D)$, i.e., $\mathbf{H}(D) = \mathbf{H}_0 + D\mathbf{H}_1 + \cdots + D^d\mathbf{H}_d$. The resultant matrix can provide an effective tool for testing the coprimeness and, hence, the recoverability (PR) of the MIMO system.

Theorem 2—Resultant Matrix Test for MIMO Recoverability: a) A MIMO system $\mathbf{H}(D)$ is coprime if and only if there exists an integer ρ such that

$$\operatorname{rank}\{\mathbf{\Gamma}^\rho[\mathbf{H}]\} = \eta + p \times \rho \quad (6)$$

where η is the McMillan degree of $\mathbf{H}(D)$. (The smallest integer ρ to meet the above equality is equal to the NMB degree of $\mathbf{H}(D)$, i.e., ν.)

b) A MIMO system $\mathbf{H}(D)$ is PR if and only if there exists an integer ρ such that

$$\operatorname{rank}\{\mathbf{\Gamma}^\rho[\mathbf{H}]\} = \eta' + p \times \rho \quad (7)$$

where η' is the reduced McMillan degree of $\mathbf{H}(D)$, which can be obtained by simply subtracting the degree contributed by those pure delay factors from the full McMillan degree. More precisely

$$\eta' = \eta - \text{degree associated with pure delay factors.}$$

Again, the smallest integer to meet the above equality is equal to the NMB degree ν.

Proof: For part a), see [10] and [11]. For part b), if $\mathbf{H}_0 = \mathbf{H}(0)$ has full column rank, from Theorem 1, $\mathbf{H}(D)$ is PR if and only if it is right coprime. Even if $\mathbf{H}_0$ does not have full column rank, one can simply subtract the degree contributed by those pure delay factors from the full McMillan degree. Statement a) can then be modified to test the perfect recoverability of the MIMO. The following is a simple procedure to compute the delay degree, thus deriving the *reduced McMillan degree* from the *McMillan degree*. The delay degree will be zero if $\mathbf{H}_0$ has full column rank. If it has rank deficiency, then there must exist a nonsingular matrix denoted as $\mathbf{U}$ such that $\mathbf{H}_0\mathbf{U}$ has its first column equal to zero. Consequently, we can remove one delay factor from the matrix $\mathbf{H}(D)\mathbf{U}$. The new matrix will be denoted by $\mathbf{H}'(D) \equiv \mathbf{H}(D)\mathbf{U}\mathrm{Diag}[D^{-1}, 1, \ldots, 1]$. Obviously, $\mathbf{H}'(D)$ is itself a polynomial matrix. If there is still rank deficiency in $\mathbf{H}'(0)$, then the same procedure can be applied to remove more delay factors. The process can continue on until there is no more such rank deficiency. If $\mathbf{H}(D)$ has full column rank (on the polynomial ring), the process is assured to terminate in finite steps. The delay degree is equal to the total number of such reduction steps since it is the same as the number of D-factors removed from $\mathbf{H}(D)$ in the entire process. ∎

III. Bezout Equalizer and Precoder

The Bezout system theory serves as a theoretical foundation for flexible transceiver design of MIMO systems. Its potential applications can be logically divided into the following categories:

a) $p < q$, and the channel is known to receiver.
b) $p > q$, and the channel is fed back to the transmitter.
c) $p \geq q$, and the channel is known to the receiver.
d) When $p < q$, the channel is known to the transmitter and the receiver.

Note that d) is usually handled without using transmitter channel knowledge, and hence, it can be treated as a). As outlined in the flowchart in Fig. 1, case c) will be treated in Section IV, while cases a) and b) are treated subsequently.

a) **When $p < q$ and the channel is known to receiver:** According to Theorem 1, there exists a PR Bezout equalizer if and only if the $q \times p$ transfer function $\mathbf{H}(D)$ is delay-permissive right coprime. More exactly, $\mathbf{G}(D)$ is a PR Bezout equalizer if and only if $\mathbf{G}(D)$ satisfies

$$\mathbf{G}(D)\mathbf{H}(D) = \mathrm{Diag}[D^{k_j}]$$

b) **When $p > q$ and the channel is known to transmitter:** According to Theorem 1, there exists a PR Bezout precoder if and only if the $q \times p$ transfer function $\mathbf{H}(D)$ is delay-permissive left coprime. More exactly, $\mathbf{F}(D)$ is a PR Bezout precoder if and only if $\mathbf{F}(D)$ satisfies

$$\mathbf{H}(D)\mathbf{F}(D) = \mathrm{Diag}[D^{k_j}].$$

With such a Bezout precoder, the symbols received by the receiver will not only be ISI-free but also ICI-free (containing only the desired stream's information).

A. Bezout Equalizer When $p < q$ and Channel Is Known to Receiver

We have established the theoretical basis for the Bezout inverse system under an idealistic assumption that the system is

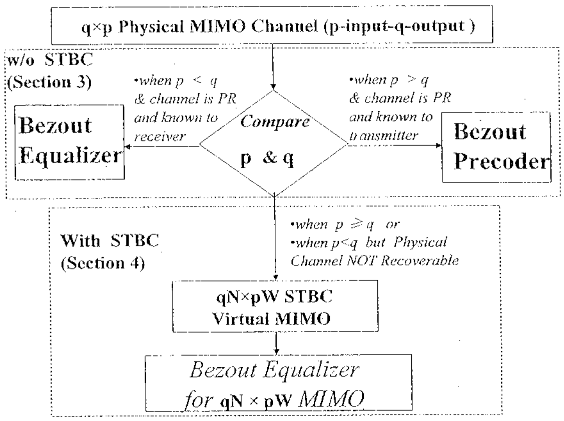


Fig. 1. Design strategy is outlined in the flowchart. The pure Bezout solutions are outlined in the upper dashed box. The joint Bezout-STBC solutions will be treated in Section IV as shown in the lower dashed box.

noise free. However, noise is pervasive in all practical applications. To analyze the system performance under noisy conditions, we need to investigate the notion of SNR pertaining to the Bezout inverse system.

Definition 2—Post-Processing SNR and BER: The post-processing SNR is defined as the SNR at the output of the equalizer. Bit-error-rate (BER) is an equally popular criterion of the system performance. Assuming BPSK constellation and that the bit is decided by zero thresholding, the BER can be determined as $\text{BER} = Q(\sqrt{\text{SNR}})$. ■

The Bezout equalization is illustrated in Fig. 2. For a Bezout equalizer system $\mathbf{G}(D)\mathbf{H}(D) = \text{Diag}[D^{k_j}]$, let us denote one row of $\mathbf{G}(D)$ as $\mathrm{g}(D) = \mathrm{g}_0 + D\mathrm{g}_1 + \cdots + D^{\rho-1}\mathrm{g}_{\rho-1}$, and let $\vec{\mathrm{g}}$ denote the expanded row vector

$$\vec{\mathrm{g}} \equiv [\mathrm{g}_{\rho-1} \quad \cdots \quad \mathrm{g}_1 \quad \mathrm{g}_0]. \tag{8}$$

Obviously, the equalizer system can be designed separately for each individual stream. For the design of an individual equalizer $\mathrm{g}(D)$, note that the i.i.d. additive-white-Gaussian noise (AWGN) will be filtered by $\mathrm{g}(D)$ before its effect appears at the *output* of the equalizer. This leads to the post-processing noise power: $\sigma_n^2\|\vec{\mathrm{g}}\|^2 = N_0/2\|\vec{\mathrm{g}}\|^2$, where N_0 is the noise spectral density. Therefore, the SNR pertaining to a Bezout equalizer can be derived as

$$\text{equalizer SNR} = \frac{2E_b}{N_0\|\vec{\mathrm{g}}\|^2} \tag{9}$$

where E_b is the transmit energy per bit. This immediately suggests a design criterion of minimizing the 2-norm of the equalizer.

From Fig. 2, we note that the problem of designing an optimal Bezout equalizer for all the source inputs can be decoupled into the task of separately designing many individual equalizers, one for each input. The strategy of designing an optimal (individual) Bezout equalizer (for the jth input) is to find $\mathrm{g}(D)$ with minimal two-norm $\|\vec{\mathrm{g}}\|$ such that

$$\mathrm{g}(D)\mathbf{H}(D) = [0 \quad \ldots \quad D^{k_j} \quad \ldots \quad 0], \quad j = 1, \ldots, p.$$

Equivalently, in a resultant matrix notation:

$$\vec{\mathrm{g}}\mathbf{\Gamma}^{\rho}[\mathbf{H}] = \mathrm{e}_i \tag{10}$$

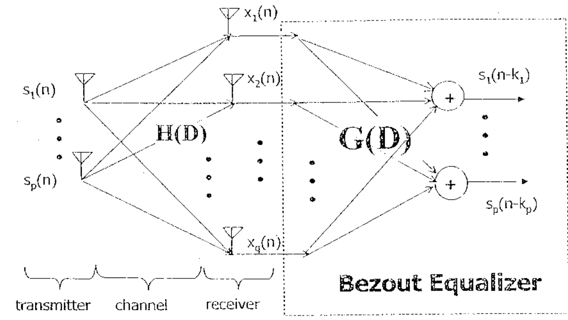


Fig. 2. Bezout equalizer $\mathbf{G}(D)\mathbf{H}(D) = \text{Diag}[D^{k_j}]$ with $p < q$.

where e_i is a row vector with all elements being zero except an entry 1 at $i = j + p(d + \rho - 1 - k_j)$, $k_j = 0, \ldots, d + \rho - 1$.

The derivation of Bezout inverse depends not only on the equalizer order but also on the system delay k_j. Recall that the solution of an individual Bezout inverse is given in (10). Take singular value decomposition (SVD) on $\mathbf{\Gamma}^{\rho}[\mathbf{H}] : \mathbf{\Gamma}^{\rho}[\mathbf{H}] = U\Sigma V$, where Σ is a square diagonal matrix of positive singular values. Then, a Bezout inverse for (10) exists if (and only if) there is a solution b for $\mathrm{b}V = \mathrm{e}_i$, and thereafter, the Bezout solution

$$\vec{\mathrm{g}} = \mathrm{b}\Sigma^{-1}U^H = \mathrm{e}_i V^H \Sigma^{-1} U^H$$

yields a minimum-norm solution with the optimal 2-norm

$$\|\vec{\mathrm{g}}\|^2 = \mathrm{e}_i V^H \Sigma^{-2} V \mathrm{e}_i^H. \tag{11}$$

Note that the system delay k_j provides an extra (and important) degree of freedom for the Bezout inverse. Selection of k_j is just the same as selecting i since $i = j + p(d + \rho - 1 - k_j)$. The best integer i minimizing the 2-norm in (11) is

$$i^* = \arg\min_i \left\{(V^H\Sigma^{-2}V)_{ii} | i = j(\text{mod } p),\ \mathrm{e}_i \in \text{Row Span}\{V\}\right\}. \tag{12}$$

B. Bezout Precoder When $p > q$ and Channel is Known to Transmitter

A Bezout precoder system is illustrated in Fig. 3. Mathematically

$$\mathbf{H}(D)\mathbf{F}(D) = \text{Diag}[D^{k_j}] \tag{13}$$

Let us denote one column of $\mathbf{F}(D)$ as $\mathrm{f}(D) = \mathrm{f}_0 + D\mathrm{f}_1 + \cdots + D^{\rho-1}\mathrm{f}_{\rho-1}$, and its expanded column vector $\vec{\mathrm{f}} \equiv [\mathrm{f}_{\rho-1}^T \ \ldots \ \mathrm{f}_1^T \ \mathrm{f}_0^T]^T$. Let σ_s^2 be the input variance for one stream. Each precoding column vector $\mathrm{f}(D)$ amplifies the transmitter's power to $\sigma_s^2\|\vec{\mathrm{f}}\|^2$.

In order to have an ISI-free communication, we must have

$$\mathbf{H}(D)\mathrm{f}(D) = [0 \quad \ldots \quad D^{k_j} \quad \ldots \quad 0]^T$$

which produces a signal power of σ_s^2 at each receiver. Moreover, we must set $\sigma_s^2 = E_b/\|\vec{\mathrm{f}}\|^2$ in order to normalize the transmit energy per bit to E_b. In this case, the SNR for the Bezout precoder becomes

$$\text{precoder SNR} = \frac{2E_b}{N_0\|\vec{\mathrm{f}}\|^2}. \tag{14}$$

Note that the noise power at each receiver is $N_0/2$.

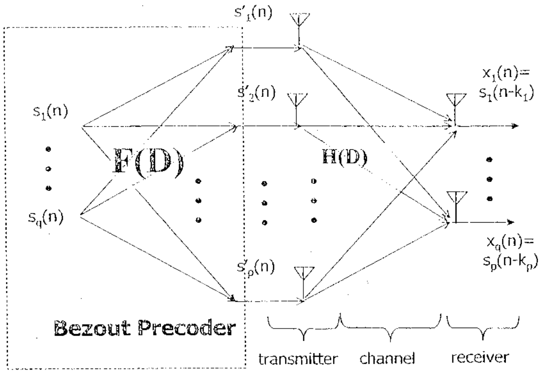


Fig. 3. Bezout precoder: $\mathbf{H}(D)\mathbf{F}(D) = \mathrm{Diag}[D^{k_j}]$ with $p > q$.

Equation (14) establishes a useful duality between the optimal designs of Bezout equalizer and precoder systems. The optimal precoder design for a MIMO system $\mathbf{H}(D)$ is equivalent to the optimal equalizer design for a MIMO with transfer function $\mathbf{H}(D)^H$. Similar to the previous equalizer design, design of an optimal Bezout precoder can be decoupled into the task of separately designing many individual precoders: one for each output.

An optimal (individual) Bezout precoder can be obtained by finding $\mathbf{f}(D)$, with minimal two-norm such that $\mathbf{H}(D)\mathbf{f}(D) = [0 \ \ldots \ D^{k_j} \ \ldots \ 0]^H$. Equivalently, in a resultant matrix notation

$$\mathbf{\Gamma}^{\rho}[\mathbf{H}^H]^H \vec{\mathbf{f}} = \mathbf{e}_i \tag{15}$$

where $\mathbf{e}_i$ is now a column vector with all zeros, except an entry 0.1 at $i = j + q(d + \rho - 1 - k_j)$, $k_j = 0, \ldots, d + \rho - 1$. Take SVD on $\mathbf{\Gamma}^{\rho}[\mathbf{H}^H]^H$:

$$\mathbf{\Gamma}^{\rho}[\mathbf{H}^H]^H = U\Sigma V. \tag{16}$$

For a given index i, a Bezout precoder $\vec{\mathbf{f}}$ for (15) exists iff a solution $\mathbf{b}$ for $U\mathbf{b} = \mathbf{e}_i$ exists. Similarly, the optimal Bezout precoder is $\vec{\mathbf{f}} = V^H\Sigma^{-1}\mathbf{b} = V^H\Sigma^{-1}U^H\mathbf{e}_i$ with 2-norm

$$\|\vec{\mathbf{f}}\|^2 = (U\Sigma^{-2}U^H)_{ii}. \tag{17}$$

The optimal integer i^* corresponding to the optimal delay k_j is

$$i^* = \arg\min_i \left\{ (U\Sigma^{-2}U^H)_{ii} | i = j (\mathrm{mod}\ q) \right.$$
$$\left. \mathbf{e}_i \in \text{Column Span}\ \{U\} \right\}. \tag{18}$$

C. *Bezout Precoder versus Alamouti STBC*

It was pointed out in Alamouti's space-time coding scheme [1] that by a joint transmitter-receiver strategy, the system with two transmitters and one receiver can reach the same SNR as the system with one transmitter and two receivers under the instantaneous channel model. However, Alamouti's scheme would double the total radiated power. Via the Bezout precoding scheme proposed here, it is possible to deliver the same SNR as Alamouti's while cutting the transmission power to one half.

Assuming a two-input one-output instantaneous channel with transfer function $\mathbf{H}(D) = [h_1 \ \ h_2]$, it can be verified [1] that the output SNR under Alamouti's scheme is $(\sigma_s^2/\sigma_n^2)(\|h_1\|^2 + \|h_2\|^2)$ with the total transmission power $2\sigma_s^2$. In our scheme, we design a transmitter end precoder $\mathbf{F}(D) = 1/\sqrt{\|h_1\|^2 + \|h_2\|^2}[h_1^* \ \ h_2^*]^T$, which implies $\mathbf{H}(D)\mathbf{F}(D) = \sqrt{\|h_1\|^2 + \|h_2\|^2}$. Then, the output SNR would be the same as in Alamouti's case, whereas the transmission power is now reduced to σ_s^2 since $\|\mathbf{f}\| = 1$.

However, we should also note that Alamouti's space-time coding has a major advantage (over the Bezout precoding) in that it does not require the transmitter to know the channel information. In contrast, the Bezout precoding requires channel information to be conveyed to the transmitter from the receiver (where the channel estimation is usually performed). According to our classification in Section IV-B, Alamouti's scheme belongs to channel-independent designs, whereas Bezout precoding belongs to channel-dependent designs. In fast time-varying channels, such feedback could incur a costly overhead, rendering it less appealing. On the other hand, the Bezout precoding will be very attractive in fixed or slowly time-varying channels, where the feedback of the channel information would induce little overhead.

IV. Precoder and Equalizer Design for STBC Systems

In many down-link communication scenarios, we face a size deficiency problem with fewer receivers than transmitters (i.e., $p \geq q$). Then, the system cannot be perfectly recoverable via a Bezout equalizer unless sufficient redundancy is incorporated. Moreover, the channel information may not be available to the transmitter, and then, the Bezout precoder will not be viable. Even if the channel information is available and the channel is theoretically PR, it may not be robustly recoverable. To handle all these concerns, it may be effective to adopt a design strategy combining the Bezout and STBC techniques.

A. *STBC MIMO Systems*

STBC has been introduced as an efficient diversity creation technique. It can be regarded as a combination of channel coding techniques and equalization techniques. The performance can be enhanced by incorporating some structured redundancy at the transmitter side, e.g., with zero-padding or cyclic-prefixing. The basic STBC system used in this paper is illustrated in Fig. 4(a). Given a $q \times p$ transfer function of a (p-input–q-output) physical channel model, the STBC system can generate an expanded virtual $q' \times p'$ transfer function with $p' = pW$ and $q' = qN$. Namely, it results in a $qN \times pW$ virtual transfer function. Here, N denotes the block size, and W is the number of symbols sent per transmitter per block. For convenience, *this will be referred to as a (N, W)-STBC MIMO system.*

Virtual Transfer Function for (N, N)-STBC Systems: If the original channel is originally PR, there is no need to incorporate redundancy; then, we can set $W = N$. Such a (N, N)-STBC system is obtained by adding interleaving and deinterleaving operators to the transmission channel inputs and outputs, respectively. The virtual transfer function corresponding to a physical SISO channel has been well studied [17], [24], [26]. Given a transfer function $\mathbf{H}(D)$ of a p-input q-output MIMO physical channel $\mathbf{H}(D) = \sum_{n=0}^{d} \mathbf{H}_n D^n$ and $\mathbf{H}_{j/N}(D)$ its jth forward polyphase component out of N

$$\mathbf{H}_{j/N}(D) = \sum_{n=0}^{\lceil d/N \rceil} \mathbf{H}_{Nn+j} D^n \tag{19}$$

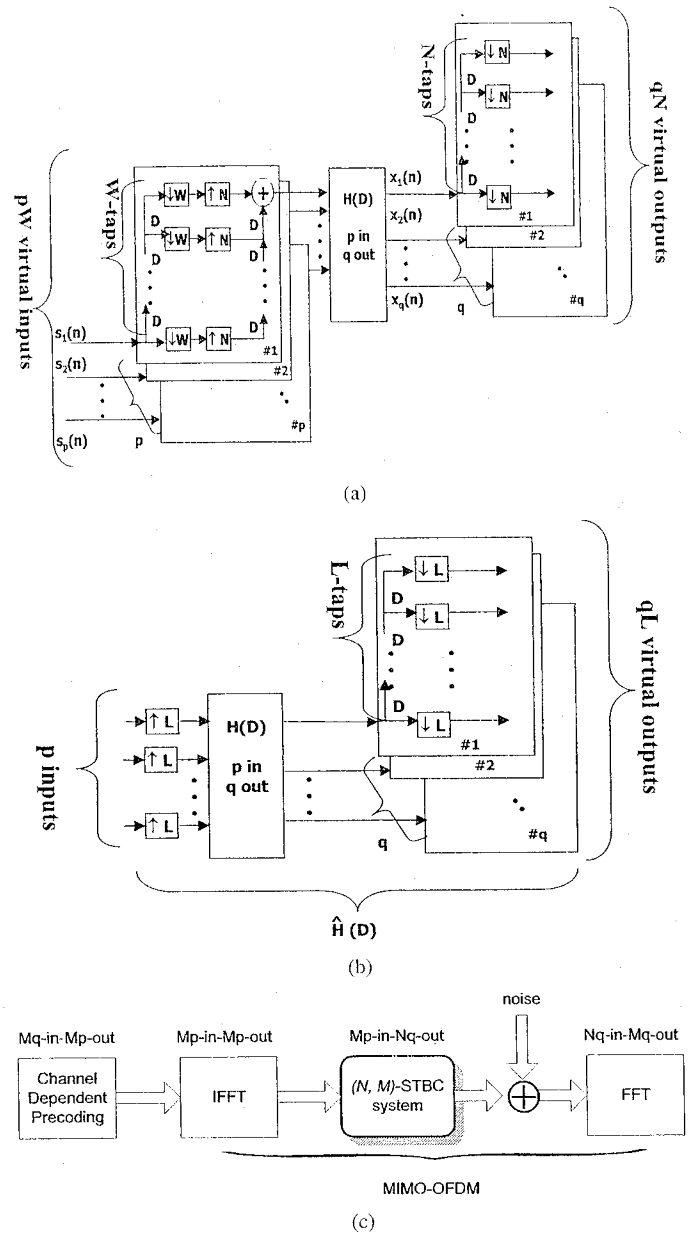


Fig. 4. (a) Basic (N, W)-STBC system. It creates a $Nq \times pW$ virtual transfer function from a p-input–q-output physical MIMO channel. (b) $\hat{\mathbf{H}}(D)$ can be created by upsampling $\mathbf{H}(D)$, whose output number is expanded by a factor of L. The interleaving precoding $(N, W)_L$-STBC is equivalent to applying the (N, W)-STBC to an L-upsampled channel $\hat{\mathbf{H}}(D)$. (c) (N, M)-OFDM system can be built on the basic (N, M)-STBC system [see (a)], with additional precoding and post-processing operations.

where $0 \leq j \leq N-1$. Following [26], we can obtain a $qN \times pN$ *virtual transfer function* denoted as $\bar{\mathbf{H}}(D)$:

$$\bar{\mathbf{H}}(D) = \begin{bmatrix} \mathbf{H}_{0/N}(D) & D\mathbf{H}_{N-1/N}(D) & \cdots & D\mathbf{H}_{1/N}(D) \\ \mathbf{H}_{1/N}(D) & \mathbf{H}_{0/N}(D) & \cdots & D\mathbf{H}_{2/N}(D) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{H}_{N-2/N}(D) & \mathbf{H}_{N-3/N}(D) & \cdots & D\mathbf{H}_{N-1/N}(D) \\ \mathbf{H}_{N-1/N}(D) & \mathbf{H}_{N-2/N}(D) & \cdots & \mathbf{H}_{0/N}(D) \end{bmatrix}. \tag{20}$$

Lemma 1: Let $\mathbf{H}(D)$ be a $q \times p$ (assuming $p \leq q$) transfer function of a physical MIMO channel and $\bar{\mathbf{H}}(D)$ be the virtual transfer function of the (N, N)-STBC system, which is given in (20); then, there is a one-to-one correspondence between the common zeros of $\mathbf{H}(D)$ (which is denoted by $\{\lambda_i\}$) and that of $\bar{\mathbf{H}}(D)$ (which is denoted by $\{\eta_i\}$). More exactly

$$\eta_i = \lambda_i^N. \tag{21}$$

Proof: Let $\mathbf{H}(D)$ be factorized as $\mathbf{H}(D) = \mathbf{H}'(D)\mathbf{C}(D)$, where $\mathbf{H}'(D)$ is coprime, and $\mathbf{C}(D)$ is the greatest (right) common divisor of $\mathbf{H}(D)$. Then, for the (N, N)-STBC coded system, we have $\bar{\mathbf{H}}(D) = \bar{\mathbf{H}}'(D)\bar{\mathbf{C}}(D)$. Therefore, $\bar{\mathbf{C}}(D)$ is the greatest (right) common divisor of $\bar{\mathbf{H}}(D)$. The roots of $\det(\mathbf{C}(D))$ and $\det(\bar{\mathbf{C}}(D))$ are the common zeros of $\mathbf{H}(D)$ and $\bar{\mathbf{H}}(D)$, respectively. Note that $\bar{\mathbf{C}}(D^N)$ can be diagonalized as $\bar{\mathbf{C}}(D^N) = M(D)^{-1}\mathrm{Diag}\left[\mathbf{C}(D), \mathbf{C}(DW_N), \ldots, \mathbf{C}\left(DW_N^{N-1}\right)\right] M(D)$ [26], where $W_N \equiv e^{-2\pi j/N}$. Hence, we have

$$\begin{aligned} \det(\bar{\mathbf{C}}(D^N)) &= \det(\mathbf{C}(D))\det(\mathbf{C}(DW_N)) \\ &\quad \cdots \det\left(\mathbf{C}\left(DW_N^{N-1}\right)\right) && (22) \\ &= \prod_i (D - \lambda_i) \cdot \prod_i (DW_N - \lambda_i) \\ &\quad \cdots \prod_i \left(DW_N^{N-1} - \lambda_i\right) && (23) \\ &= \prod_i \left(D^N - \lambda_i^N\right). && (24) \end{aligned}$$

Since $\det(\bar{\mathbf{C}}(D^N)) = \prod_i (D^N - \eta_i)$, we have $\eta_i = \lambda_i^N$. ∎

It is obvious from Lemma 1 that the (N, N)-STBC preserves the PR property of the original transfer function [27].

Precoding for STBC Systems: Problems arise for (N, N)-STBC systems when the original $\bar{\mathbf{H}}(D)$ is not PR or when it is PR but not robustly recoverable. An effective solution is via a redundant *precoding matrix* $\mathbf{F}$

$$\tilde{\mathbf{H}}(D) = \bar{\mathbf{H}}(D)\mathbf{F}. \tag{25}$$

The simplest (N, W)-STBC precoding is represented by a *precoding matrix*

$$\mathbf{F} = \left[\mathbf{I}_{pW \times pW} | \mathbf{0}_{pW \times p(N-W)}\right]^T. \tag{26}$$

This will lead to a $qN \times pW$ *virtual transfer function*, which is denoted as $\tilde{\mathbf{H}}(D)$ in (27), shown at the bottom of the next page. Since W is the number of symbols sent per transmitter per block, the number of symbols being transmitted during one block period is pW; thus, the data transmission rate is pW/N. By using fewer columns than rows in the *precoding matrix* $\mathbf{F}$, some redundancy can be incorporated for the purpose of enhancing the robust recoverability of the new transfer function $\tilde{\mathbf{H}}(D)$.

Lemma 2—Perfect Recoverability of Precoded STBC Systems: If $\mathbf{H}(D)$ is PR, then $\tilde{\mathbf{H}}(D)$ in (25) is PR if and only if the precoding matrix $\mathbf{F}$ has full column rank. In particular, for the (N, W)-STBC system, the $qN \times pW$ virtual transfer function $\tilde{\mathbf{H}}(D)$ is PR, provided that $\mathbf{H}(D)$ is PR. ∎

Proof: If $\mathbf{H}(D)$ is PR, then $\mathbf{H}(\lambda)$ has full column rank for all $\lambda \neq 0$. This, in turn, implies that $\bar{\mathbf{H}}(\lambda)$ has full column rank for all $\lambda \neq 0$. If $\mathbf{F}$ has full column rank, then $\tilde{\mathbf{H}}(\lambda)$ has full column rank for all $\lambda \neq 0$. Thus, $\tilde{\mathbf{H}}(\lambda)$ is PR according to Theorem 1(b). Note that the precoding matrix for (N, W)-STBC system is obviously of full rank, cf. (26). ∎

As shown in the following example, even if $\mathbf{H}(D)$ is not PR, it is possible to create a PR $\tilde{\mathbf{H}}(D)$ by proper precoding. Some channel-independent and channel-dependent precoding design examples are also exemplified.

Example 1—Create PR STBC System From Originally Non-PR Channel: Given a 2×2 transfer function

$$\mathbf{H}(D) = \begin{bmatrix} D & 2+D \\ 1+D & 1+0.5D \end{bmatrix}.$$

This is not (right) coprime since there exists a common zero at $\lambda = -2$. If $N = 2$, the virtual transfer function for the (2,2)-STBC system is

$$\tilde{\mathbf{H}}(D) = \begin{bmatrix} 0 & 2 & D & D \\ 1 & 1 & D & 0.5D \\ 1 & 1 & 0 & 2 \\ 1 & 0.5 & 1 & 1 \end{bmatrix}$$

which remains noncoprime. Note that according to (21), the common zero for $\tilde{\mathbf{H}}(D)$ is now located at $\lambda = (-2)^N = 4$, which can be verified by observing the rank deficiency of $\tilde{\mathbf{H}}(4)$. By a channel-independent precoding matrix $\mathbf{F} = [\mathbf{I}_{2\times 2} | \mathbf{0}_{2\times 2}]^T$, we can obtain a coprime (2,1)-STBC system

$$\tilde{\mathbf{H}}(D) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

However, this precoding scheme will result in a 50% loss of the possible achievable rate.

B. PR Test of STBC Systems

For virtual transfer functions derived from STBC precoding, the existence of a Bezout equalizer can be tested by applying Theorem 1. More specifically, we have the following lemma.

Lemma 3—Perfect Recoverability of STBC MIMO: There exists a PR Bezout equalizer for the STBC MIMO transfer function $\tilde{\mathbf{H}}(D)$ if and only if we have the following.

1) The basic size requirement is satisfied:

$$pW \leq qN. \tag{28}$$

2) The virtual transfer function $\tilde{\mathbf{H}}(D)$ is delay-permissive right coprime.

The coprimeness of $\tilde{\mathbf{H}}(D)$ can be checked by the rank properties of the resultant matrix $bf\Gamma^{\rho}[\tilde{\mathbf{H}}]$, cf. Theorem 2.

PR Test for First-Order STBC Systems for SISO Channel: An important application of (N, W)-STBC systems is for physical SISO channels. When $N > d$, most of the elements of the virtual transfer function are constants and the first-order terms only fall in the upper-right $(d+W-N) \times (d+W-N)$ triangular corner of $\tilde{\mathbf{H}}(D)$.

1) **Predictable (reduced) McMillan degree:** Assuming $h_0 \neq 0$, it can be easily checked that $\tilde{\mathbf{H}}(D)$ is already column reduced by examining the rank of the column-wise highest degree coefficient matrix. Therefore, the McMillan degree of $\tilde{\mathbf{H}}(D)$ is $\eta = d + W - N$. (Note that $\eta = \eta'$ since we assume $h_0 \neq 0$.)
2) **Bound on the NMB degree** ν: It can be shown that $\nu \leq \eta = d + W - N$; therefore, according to Theorem 2, for PR test of $\tilde{\mathbf{H}}(D)$, we need only to check the rank property of the $\eta N \times (\eta+1)W$ resultant matrix $\Gamma^{\eta}[\tilde{\mathbf{H}}]$.
3) **Reduced Resultant Matrix:** Since the upper-right $\eta \times \eta$ triangular corner of $\tilde{\mathbf{H}}_1$ is nonsingular, it provides η pivotal elements to eliminate (by Gaussian elimination) all the first η rows in $\Gamma^{\eta}[\tilde{\mathbf{H}}]$. Removal of the first η rows and the entire first column-block from $\Gamma^{\eta}[\tilde{\mathbf{H}}]$ leads to a reduced resultant matrix of size $\eta(N-1) \times \eta W$, which is denoted by $\mathbf{R}^{\eta}$:

$$\mathbf{R}^{\eta} \equiv E_1 \Gamma^{\eta}[\tilde{\mathbf{H}}] E_2^T \tag{29}$$

where $E_1 = [\mathbf{0}_{\eta(N-1)\times\eta)} | \mathbf{I}_{\eta(N-1)\times\eta(N-1)}]$, and $E_2 = [\mathbf{0}_{\eta W \times W} | \mathbf{I}_{\eta W \times \eta W}]$.

The above discussion is summarized by the following theorem.

Lemma 4—First-Order (N, W)-STBC System for Physical SISO: Let $\tilde{\mathbf{H}}(D)$ denote the first-order pseudo-circulant virtual transfer function derived by applying the (N, W)-STBC coding to a physical SISO channel. Suppose that $N > d$ and $h_0 \neq 0$; then, $\tilde{\mathbf{H}}(D)$ is PR if and only if the reduced resultant matrix $\mathbf{R}^{\eta}$ defined in (29) has full column rank. ∎

Example 2-PR Test for a (N, W)-STBC SISO System: Given a physical SISO channel with the transfer function

$$h(D) = [1 + 0.8D + 0.6D^2 + 0.3D^3 + 0.1D^4]$$

$$\tilde{\mathbf{H}}(D) = \begin{bmatrix} \mathbf{H}_{0/N}(D) & D\mathbf{H}_{N-1/N}(D) & \cdots & D\mathbf{H}_{N-W+1/N}(D) \\ \mathbf{H}_{1/N}(D) & \mathbf{H}_{0/N}(D) & \cdots & D\mathbf{H}_{N-W+2/N}(D) \\ \vdots & \vdots & \vdots & \vdots \\ & & & D\mathbf{H}_{N-1/N}(D) \\ \vdots & \vdots & \vdots & \mathbf{H}_{0/N}(D) \\ & & & \vdots \\ \mathbf{H}_{N-2/N}(D) & \mathbf{H}_{N-3/N}(D) & \cdots & \mathbf{H}_{N-W-1/N}(D) \\ \mathbf{H}_{N-1/N}(D) & \mathbf{H}_{N-2/N}(D) & \cdots & \mathbf{H}_{N-W/N}(D) \end{bmatrix} \tag{27}$$

the virtual transfer function after (5,3)-STBC is

$$\tilde{\mathbf{H}}(D) = \begin{bmatrix} 1 & 0.1D & 0.3D \\ 0.8 & 1 & 0.1D \\ 0.6 & 0.8 & 1 \\ 0.3 & 0.6 & 0.8 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}. \tag{30}$$

Note that the McMillan degree $\eta = d + W - N = 2$. Removal of $\eta = 2$ rows and the first block-column (= 3 columns) from $\Gamma^\eta[\tilde{\mathbf{H}}]$ leads to the following reduced resultant matrix $\mathbf{R}^\eta = \mathbf{R}^2$:

$$\begin{aligned}\mathbf{R}^2 &= [\mathbf{0}_{8\times 2}|\mathbf{I}_{8\times 8}]\Gamma^\eta[\tilde{\mathbf{H}}][\mathbf{0}_{6\times 3}|\mathbf{I}_{6\times 6}]^T \\ &= \left[\begin{array}{ccc|ccc} 0.6 & 0.8 & 1 & 0 & 0 & 0 \\ 0.3 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0.1 & 0.3 & 0.6 & 0 & 0 & 0 \\ 0 & 0.1 & 0.3 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0.8 & 1 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 1 \\ 0 & 0 & 0 & 0.3 & 0.6 & 0.8 \\ 0 & 0 & 0 & 0.1 & 0.3 & 0.6 \end{array}\right].\end{aligned} \tag{31}$$

$\tilde{\mathbf{H}}(D)$ is PR if and only if the reduced resultant matrix $\mathbf{R}^2$ has full column rank. ■

Comparison With Existing Test [17]: Note that our rank test (for virtual MIMO created from SISO channel) is slightly different from an earlier test proposed in [17]. The recoverability condition in [17] is based on checking the full-column-rank of a submatrix of $\mathbf{R}^\eta$. In fact, this submatrix comprises exactly the lower $\eta N - d$ rows of $\mathbf{R}^\eta$. Obviously, the submatrix rank test can only provide a sufficient (but not necessary) condition for testing the perfect recoverability of the (N, W)-STBC MIMO. However, we should note that [17] offers an important feature of demonstrating the existence of a family of channel-independent redundant precoders which guarantee the PR of the symbols.

C. Channel-Independent Precoding: Interleaving Zero-Padding Scheme

Assume $N > d$. Then, all the polyphase components $\mathbf{H}_{j/N}(D)$ in (19) are either constant matrices or zeros. Moreover, $\bar{\mathbf{H}}(D)$ is almost an instantaneous channel. More exactly

$$\mathbf{H}_{j/N}(D) = \mathbf{H}_j = \begin{Bmatrix} \text{constant}, & \text{if } j \le d \\ 0, & \text{if } j > d \end{Bmatrix}. \tag{32}$$

This will yield a *first-order* virtual transfer function

$$\bar{\mathbf{H}}(D) = \begin{bmatrix} \mathbf{H}_0 & 0 & \dots & 0 & D\mathbf{H}_d & \dots & D\mathbf{H}_1 \\ \vdots & \mathbf{H}_0 & \ddots & \cdot & \ddots & \ddots & \vdots \\ \vdots & \cdot & \ddots & \ddots & \cdot & \ddots & D\mathbf{H}_d \\ \mathbf{H}_d & \cdot & \cdot & \mathbf{H}_0 & \ddots & \cdot & 0 \\ \vdots & \cdot & \cdot & \vdots & \mathbf{H}_0 & \ddots & \vdots \\ \cdot & \cdot & \cdot & \vdots & \cdot & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{H}_d & \mathbf{H}_{d-1} & \cdots & \mathbf{H}_0 \end{bmatrix}. \tag{33}$$

Two categories of precoding matrix $\mathbf{F}$ are of interest: 1) channel-independent precoding and 2) channel-dependent precoding. In the following, we will discuss one example for each category.

Without having to know the exact channel information, sufficient redundancy can be obtained by periodically inserting zeros into the transmission sequences. Note that the size constraint requires $Wp \le qN$, cf. (28), i.e., $W/N \le q/p$. For SISO channels or MIMO channels with (almost) equal number of transmitters and receivers, we can select $W \approx N$ block columns because $q \approx p$. Since most of the block columns of $\bar{\mathbf{H}}(D)$ are selected, a simple precoding with $\mathbf{F} = [\mathbf{I} \quad \mathbf{0}]^T$ should be as good as other schemes. However, this argument does not hold when q is much smaller than p. In this case, W will be a small fraction of N. For simplicity, let us assume $L \equiv p/q$ and N/L are both integers. Since only a small fraction of the columns of $\bar{\mathbf{H}}(D)$ can be retained, an optimal selection strategy is critical. This motivates us to extend $\mathbf{F}$ in (26) to a more general *interleaving precoding* form

$$\mathbf{F} = \mathbf{P}_L = \mathbf{\Pi} \otimes I_p \tag{34}$$

where $\mathbf{\Pi}$ is a $N \times W$ matrix ($W \le N/L$) with its ith column having all the zero entries, except an entry "1" at the $[L(i-1)+1]$th location. For example, for the four-input two-output (4I2O) channel, $p = 4$ and $q = 2$; hence, $L = 2$. Thus, the first column of $\mathbf{\Pi}$ is $[1 \quad 0 \quad 0 \quad \dots \quad 0]^T$, and the second column $[0 \quad 0 \quad 1 \quad 0 \quad \dots \quad 0]^T$, and so on.

The scheme in (34) is termed $(N, W)_L$-STBC precoding, i.e., W (out of N) block columns of $\bar{\mathbf{H}}(D)$ will be selected with a uniform sampling interval L. Note that $(N, W)_L$-STBC is equivalent to an L-upsampling operation, as shown in Fig. 4(b), followed by the (N, W)-STBC. In particular, when $L = 1$, the two STBC schemes become the same. For example, for a 4I2O channel, it is preferable to use interleaving precoder $(N, W)_2$, instead of (N, W)-STBC. In fact, choosing $L = 2$ leads to a 4I4O channel with ISI degree $d = 2$. This is explained as follows. Suppose the original channel transfer function is

$$\mathbf{H}(D) = \mathbf{H}_0 + \mathbf{H}_1 D + \cdots + \mathbf{H}_5 D^5.$$

Upsampling the original system by a factor of 2 leads to the following equivalent 4I4O transfer function:

$$\hat{\mathbf{H}}(D) = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{H}_2 \\ \mathbf{H}_3 \end{bmatrix} D + \begin{bmatrix} \mathbf{H}_4 \\ \mathbf{H}_5 \end{bmatrix} D^2.$$

Note that $(N, W)_L$-STBC, while enlarging the number of output, reduces the effective ISI-length by a factor of L. Just like (N, W)-STBC, $(N, W)_L$-STBC will not require the exact channel information. Knowledge of the ISI length should suffice to determine the appropriate range of L and W.

Two intriguing examples can help illustrate the role of the *interleaving precoding matrix* $\mathbf{P}_L$. The first is when $L \equiv p/q = d+1$, i.e., the diversity created by the ISI-length is just enough to compensate the deficiency of the receivers. Then, the $W = N/L$ block columns, as selected by (34), will be mutually orthogonal among themselves, resulting in a high post-processing SNR for the virtual transfer function $\tilde{\mathbf{H}}(D) = \bar{\mathbf{H}}(D)\mathbf{P}_L$. Another interesting scenario is when $L > d+1$, i.e., the ISI diversity is now insufficient to compensate the receiver deficiency. In this case, there exists no solution in (34) to yield a PR virtual transfer function. Other precoding techniques, e.g., orthogonal precoding [1], [19], [20], should be considered.

D. Channel-Dependent Precoding: Optimal OFDM Precoder

Assume $N = M + d$ and $N \gg d$. Let the virtual transfer function $\tilde{\mathbf{H}}(D)$ of the (N, M)-STBC system be an instantaneous matrix obtained by truncating the last dp columns from $\tilde{\mathbf{H}}(D)$

$$C \equiv \tilde{\mathbf{H}}(D)F$$

where $F = [\mathbf{I}_{pM \times pM} \mid \mathbf{0}_{pM \times p(N-M))}]^T$.

Note that C is itself a resultant matrix, more exactly, $C = P_1 \Gamma^M [\mathbf{H}^H]^H P_2$, where P_1 and P_2 are block permutation matrices. Although C is generally not a full row rank matrix, this problem can be rectified by the OFDM technique [4]. The instantaneous channel C can be transformed by OFDM into a simple block-diagonal (frequency-response) matrix C_f.

$$\begin{aligned} C_f \equiv& \mathrm{Diag}\left[\mathbf{H}\left(W_M^0\right), \mathbf{H}\left(W_M^{-1}\right), \ldots, \mathbf{H}\left(W_M^{-(M-1)}\right)\right] \\ =& [\tilde{U} \otimes I_q] C [\tilde{V} \otimes I_p] \end{aligned} \quad (35)$$

where $\tilde{U}$ is a $M \times N$ matrix with $W_M^{(i-1)(j-1)}$ as its (i,j)th element, and $\tilde{V}$ is a $M \times M$ matrix with $(1/M)W_M^{-(i-1)(j-1)}$ as its (i,j)th element. (Here, $W_M \equiv e^{-2\pi j/M}$, and $\otimes$ denotes the Kronecker product.) Note that if the physical channel $\mathbf{H}(D)$ is (left) coprime, then C_f has full row rank since each block diagonal matrix in C_f must have full row rank, cf. Theorem 1.

Assume w.l.o.g. that the (uncorrelated) AWGN has equal power among all the (broadcast) receivers. The objective is to design an optimal *OFDM precoder* so that all the receivers get the same post-processing SNR, or they are served according to a preassigned differential QoS. This problem can be mathematically formulated as a "ZF Constrained Output SNR Criterion" [17].

Definition 3—Optimal Equi-SNR or Differential-QoS Precoder Design: Given a full-row-rank instantaneous matrix C_f, the equi-SNR precoder design is to find F such that $\mathrm{tr}(F^H F)$ is minimized under the equi-SNR constraint $C_f F = I$. For the differential-QoS precoder design, the constraint is generalized to become $C_f F = \Lambda \otimes I_M$, where $\Lambda = \mathrm{Diag}[\lambda_i]$, $i = 1, \ldots, q$ and λ_i reflects the level of QoS allocated to the ith receiver.

Given the SVD $C_f = \tilde{U}\tilde{\Sigma}\tilde{V}$, the optimal equi-SNR precoder for the matrix C_f is [17]

$$\tilde{F}^* = \tilde{V}^H \tilde{\Sigma}^{-1} \tilde{U}^H. \quad (36)$$

Note that $\tilde{U}$ is a nonsingular unitary matrix since C_f has full row rank. As illustrated in Fig. 4(c), the total OFDM precoding system is a concatenation of three different precoding stages:

$$\mathbf{F} = F_1 F_2 F_3$$

where $F_1 = [\mathbf{I}_{pM \times pM} \mid \mathbf{0}_{pM \times p(N-M))}]^T$ [for (N, M)-precoding], $F_2 = [\tilde{V} \otimes I_p]$ (for IFFT), and $F_3 = \tilde{F}^*$. For signal recovery in the OFDM system, there will be corresponding post-processing at the receiver's end, represented by $[\tilde{U} \otimes I_q]$ (for FFT), cf. Fig. 4(b).

Theorem 3—Comparison of Bezout Precoder and OFDM Precoder: Suppose that a MIMO channel $\mathbf{H}(D)$ $(q < p)$ is delay-permissive left coprime. The optimal equi-SNR bezout precoder (with $\rho = M$ and system delay optimally selected) is given in (13) and (18), and the optimal equi-SNR OFDM precoder is given in (36). When $M \gg d$, the optimal equi-SNR Bezout precoder requires less or equal total transmitting power than the optimal equi-SNR OFDM precoder for the same post-processing SNR. Equivalently, for the same total transmission power, the optimal Bezout precoder deliver the same or better BER than the optimal OFDM precoder. Furthermore, the same performance advantage of the Bezout precoder over OFDM precoder also holds for the differential-QoS applications.

In addition, the Bezout precoder does not compromise any transmission rate, unlike the OFDM approach, which has to endure a $1 - M/N$ rate reduction. Moreover, compared with the OFDM precoder, the Bezout approach incurs less implementation cost at the receiver end.

Proof: For the equi-SNR case, the proof is based on some approximation that becomes valid when $M \gg d$. Since $\mathbf{H}(D)$ is delay-permissive left coprime, then according to Theorem 2, we must have $\mathrm{rank}\{\mathbf{\Gamma}^M[\mathbf{H}^H]^H\} = \eta' + q \times M$, cf. (7). When $M \gg d$, this means that U nearly spans the entire column space, and then, according to (18), the transmit power for the jth output for the optimal Bezout precoder is

$$p(j) = \min_i \left\{ (U\Sigma^{-2}U^H)_{ii} \mid i = j \pmod{q} \right\}. \quad (37)$$

The total transmit power for all the channels is $P_{BP} = \sum_j^q p(j)$.

Recall that $C = P_1 \Gamma^M [\mathbf{H}^H]^H P_2$, where P_1 and P_2 are block permutation matrices. Therefore, by setting $\rho = M$ in (16), $C = P_1 \Gamma^M [\mathbf{H}^H]^H P_2 = P_1 U \Sigma V P_2$. By (35), we have

$$C_f = [\tilde{U} \otimes I_q] C [\tilde{V} \otimes I_p] = ([\tilde{U} \otimes I_q] P_1 U) \Sigma (V P_2 [\tilde{V} \otimes I_p]). \quad (38)$$

Asymptotically speaking, when $N \gg d$, $\tilde{U}$ will approach a unitary matrix, and then, (38) will represent a SVD factorization. By (36), the optimal precoder for C_f is

$$\tilde{F}^* = [\tilde{V} \otimes I_p]^H P_2^H V^H \Sigma^{-1} U^H P_1^H [\tilde{U} \otimes I_q]^H.$$

The average transmit power for the optimal OFDM precoder is

$$P_{\mathrm{OFDM}} = \frac{1}{M}\mathrm{tr}\left((\tilde{F}^*)^H(\tilde{F}^*)\right) = \frac{1}{M}\mathrm{tr}(\Sigma^{-2}).$$

Since the trace norm will not change by *similarity transformation* with matrix U in (37), we have

$$P_{\mathrm{OFDM}} = \frac{1}{M}\mathrm{tr}(U\Sigma^{-2}U^H). \quad (39)$$

Note that both P_{BP} (37) and P_{OFDM} (39) are expressed in terms of the diagonal entries of $U\Sigma^{-2}U^H$, where the only difference lies in that P_{BP} is based on the minimal value (indicated by the index i), whereas P_{OFDM} is taking the average value. Therefore, it is obvious that

$$P_{BP} \le P_{\mathrm{OFDM}}.$$

For differential-QoS design, the optimal Bezout precoder is again obtained from (13) and (18), except that $\mathbf{H}(D)$ first has to be replaced by $\Lambda^{-1}\mathbf{H}(D)$. In addition, the

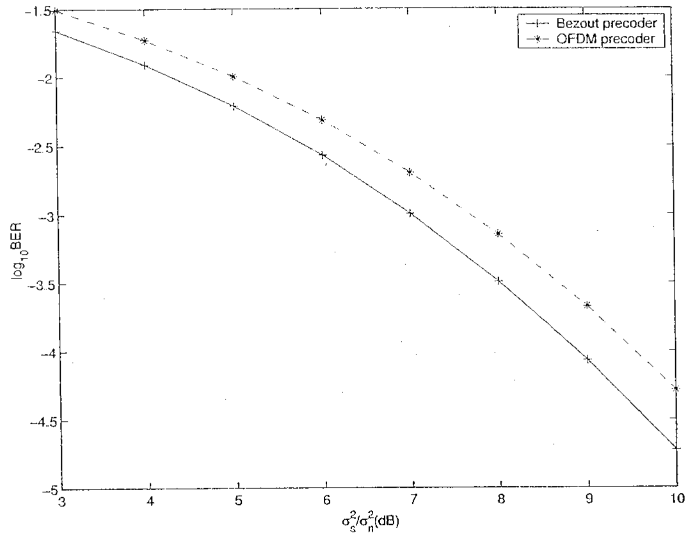


Fig. 5. Comparison of BER performance: Bezout precoder versus the MIMO-OFDM precoder. (Here, $M = \rho = 32$.) The simulation demonstrates that for the same total transmission power, the optimal Bezout precoder has BER superior to the optimal OFDM precoder.

optimal differential-QoS OFDM precoder can be obtained from the SVD: $[\Lambda^{-1} \otimes I_M]C_f = \tilde{U}\tilde{\Sigma}\tilde{V}$. It is easy to verify that the optimal precoder given in (36) yields $C_f\tilde{F}^* = \tilde{U}\tilde{\Sigma}\tilde{V}\tilde{V}^H\tilde{\Sigma}^{-1}\tilde{U}^H = \Lambda \otimes I_M$. The rest of proof follows the equi-SNR case very closely. ■

The performance advantage of the Bezout precoder (over OFDM precoder) is further confirmed by the simulation study. The space-time expansion parameter M and the Bezout precoder order ρ are both set to 32. The ISI degree is $d = 5$; thus, $N = 32 + 5 = 37$. Simulations are carried out for 100 randomly generated channels. From Fig. 5, it is observed that the Bezout precoder outperforms the MIMO-OFDM precoder, and Theorem 3 is confirmed. It is worth noting that the transmission rate of the Bezout precoder is also higher than that of the MIMO-OFDM precoder by $\frac{N-M}{M} = 5/32$.

V. SNR Optimization via Bezout Null Space: Quantitative Analysis

Sections II–IV treated only the theoretical and qualitative analysis on the system's recoverability. This section will provide a *quantitative analysis* dealing with post-processing SNR. Such analysis will be instrumental for achieving resilient precoder/equalizer designs and assuring satisfactory performance in BER and channel capacity. The analysis is further supported by the simulation study presented in the section.

A. Roles of Bezout Null Space on Optimal SNR

The Bezout null space is an effective tool for SNR analysis of Bezout equalizers. The analysis on SNR hinges upon the interplay between two subspaces:

1) The Bezout signal subspace is based on the Bezout inverse in (3), from which we extract one row for the user j: $\mathbf{g}(D)\mathbf{H}(D) = [\,0 \ldots 0 \quad D^{k_j} \ldots 0\,]$, where the delay term is located at the jth entry. Recall that in (8), we use $\vec{\mathbf{g}}$ to denote the

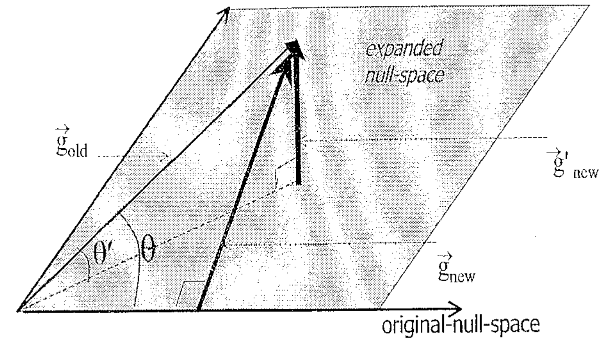


Fig. 6. Role of the Bezout null subspaces for optimizing SNR. Geometric illustration of how the 2-norm of combiner filter may be reduced by a process of the least-square-error projection. The process can be used to reduce the vector 2-norm by a reduction factor of $\sin^2\theta$. Two different projections are shown. 1) Project the vector $\vec{\mathbf{g}}_{\rm old}$ onto the old null subspace (shown as a horizontal line here), and $\vec{\mathbf{g}}_{\rm new}$ shows the reduced vector based on the original (i.e., more limited) null subspace. 2) $\vec{\mathbf{g}}'_{\rm new}$ shows a further reduction of the vector norm when the null subspace is expanded (shown as a 2-D plane).

expanded row vector $[\,\mathbf{g}_{\rho-1} \quad \cdots \quad \mathbf{g}_1 \quad \mathbf{g}_0\,]$. The vector $\vec{\mathbf{g}}$ can be derived by solving

$$\vec{\mathbf{g}}\boldsymbol{\Gamma}^{\rho}[\mathbf{H}] = \mathbf{e}_i. \tag{40}$$

2) The Bezout null space

$$\aleph\boldsymbol{\Gamma}^{\rho}[\mathbf{H}] = \mathbf{0}. \tag{41}$$

Let $\aleph$ denote a matrix that spans the Bezout null subspace; then, any vector in the null space of $\boldsymbol{\Gamma}^{\rho}[\mathbf{H}]$ can be expressed as $\mathbf{v}\aleph$ for some row vector $\mathbf{v}$.

Let $\vec{\mathbf{g}}_{\rm old}$ be an arbitrary Bezout inverse solution for (40). Since the Bezout null subspace (which is denoted as $\aleph$) by definition induces no effect on the output of any new filter $\vec{\mathbf{g}}_{\rm new} = \vec{\mathbf{g}}_{\rm old} - \mathbf{v}\aleph$ for any vector $\mathbf{v}$, $\vec{\mathbf{g}}_{\rm new}$ will also qualify as a Bezout equalizer (qualitatively), while delivering a different SNR performance from $\vec{\mathbf{g}}_{\rm old}$ (quantitatively). For optimal SNR, the vector $\mathbf{v}$ is chosen to minimize $\|\vec{\mathbf{g}}_{\rm new}\|^2 = \min_{\mathbf{v}} \|\vec{\mathbf{g}}_{\rm old} - \mathbf{v}\aleph\|^2$. The least-square-error solution can be obtained by projecting the vector $\vec{\mathbf{g}}_{\rm old}$ onto the null subspace (i.e., the subspace spanned by $\aleph$), as depicted in Fig. 6. The process reduces the squared vector 2-norm to a ratio of $\sin^2\theta$.

SNR Enhancement Strategy: Note that the smaller the projection angle θ is, the smaller the 2-norm can result, which can in turn yield higher post-processing SNR. Thus, the output SNR is closely related with the null-space property. As illustrated by Fig. 6, when the null space is expanded from the one-dimensional (1-D) (old) subspace to a two–dimensional (2-D) (new) subspace, correspondingly, the projection angle is reduced from θ to θ', and the projection component is shown as a dashed line lying on the 2-D plane. The newly updated vector shown as $\vec{\mathbf{g}}'_{\rm new}$ (pertaining to the new null-space) should has smaller norm than $\vec{\mathbf{g}}_{\rm new}$ (corresponding to the old null-space).

The analysis above leads to the following *SNR enhancement strategy*.

1) Expand the null subspace.
2) Project the same old vector to an expanded null subspace.
3) Remove the projection component; then, a new (norm-reduced) vector can be produced.

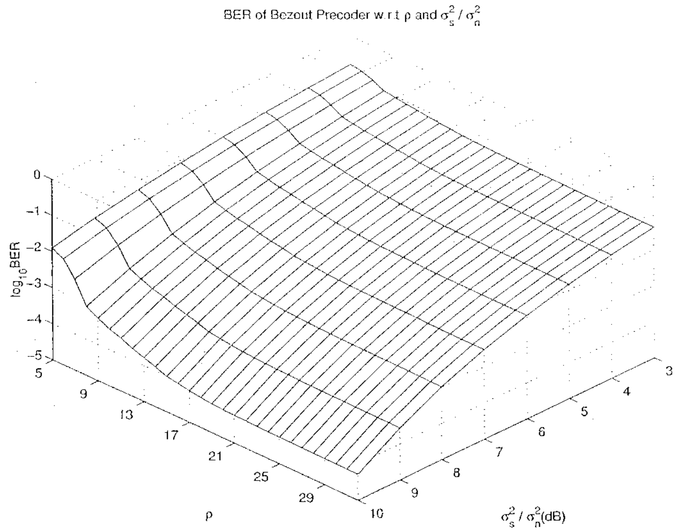


Fig. 7 BER performance for Bezout precoder w.r.t. various ρ and σ_s^2/σ_n^2.

The most critical step in the above strategy is how to expand the null space. Three effective null-space expansion techniques can be adopted:

1) increasing the equalizer order (i.e., larger ρ);
2) increasing the block size (i.e., larger N);
3) decreasing the transmission rate (i.e., smaller W).

B. Roles of Equalizer Order on SNR

Via a Bezout null-space analysis, we will shortly show that a higher order Bezout equalizer will deliver higher post-processing SNR.

Lemma 5—Roles of Equalizer Order on SNR: The 2-norm $\|\vec{g}\|$ is a monotonically decreasing function with respect to the equalizer's order.

Proof: The proof basically follows the steps in the SNR enhancement strategy. Since a larger ρ yields a more expanded null subspace of $\Gamma^\rho[\mathbf{H}]$, the SNR can be enhanced by increasing the equalizer order. It can also be shown theoretically that noticeable norm reduction will continue with increasing order until $\rho \geq k_j + \nu$, where k_j denotes the system delay. ∎

Fig. 7 shows a significant improvement on the BER performance. The experiment is based on a randomly generated four-input two-output channel with ISI degree $d = 5$. The BER is reduced from 10^{-2} for $\rho = 5$ to 10^{-4} for $\rho = 12$. Hence, the precoder order plays an effective role in the BER performance.

Example 3—Enhancing SNR by Increasing Equalizer's Order: Design an optimal-gain Bezout equalizer for a single-input two-output transfer function $\mathbf{H}(D)$ with $h_1(D) = 2 + D$ and $h_2(D) = 1 - D$. We have $\Gamma^1[\mathbf{H}] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, which has full column rank$(2 = \eta + p \times \rho = 1 + 1 \times 1)$. According to Theorem 2, the NMB degree is $\nu = 1$

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \Gamma^1[\mathbf{H}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The optimal one-tap Bezout equalizers with no-delay and one-delay are $\mathrm{g}_{\mathrm{old}}^{(0)}(D) = [1/3 \quad 1/3]$ and $\mathrm{g}_{\mathrm{old}}^{(1)}(D) = [1/3 \quad -2/3]$, respectively. Now, we show how to improve SNR by increasing the order of the Bezout equalizers. Note that

$$\vec{g}_{\mathrm{old}}^{(0)} \Gamma^2[\mathbf{H}] = [0 \quad 0 \quad \tfrac{1}{3} \quad \tfrac{1}{3}] \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = [0 \quad 0 \quad 1].$$

With the increased order, the optimal solution can be derived as

$$\min_{\mathbf{v}} \left\| \vec{g}_{\mathrm{old}}^{(0)} - \mathbf{v}\aleph \right\|^2. \tag{42}$$

For this, we project $\vec{g}_{\mathrm{old}}^{0}$ onto the space spanned by $\aleph$, where $\aleph = [1 \quad 1 \quad -1 \quad 2]$. The optimal LSE solution leads to

$$\vec{g}_{\mathrm{new}}^{(0)} = [-\tfrac{1}{21} \quad -\tfrac{1}{21} \quad \tfrac{8}{21} \quad \tfrac{5}{21}].$$

Thus, the optimal two-tap zero-delay Bezout equalizer is $\mathrm{g}_{\mathrm{new}}^{(0)}(D) = [(8/21) - (1/21)D \quad (5/21) - (1/21)D]$. This represents only a small enhancement of SNR to a ratio of 14/13 compared with the optimal static equalizer. However, by the same process, one can obtain a two-tap one-delay equalizer

$$\vec{g}_{\mathrm{new}}^{(1)} = [\tfrac{5}{21} \quad \tfrac{5}{21} \quad \tfrac{2}{21} \quad -\tfrac{4}{21}]$$

and a two-tap two-delay equalizer

$$\vec{g}_{\mathrm{new}}^{(2)} = [\tfrac{8}{21} \quad -\tfrac{13}{21} \quad -\tfrac{1}{21} \quad \tfrac{2}{21}].$$

Thus, the optimal two-tap Bezout equalizer is $\mathrm{g}_{\mathrm{new}}^{(1)}(D) = [(2/21) + (5/21)D \quad -(4/21) + (5/21)D]$. This represents an enhancement of SNR to a ratio of 7/5 compared with the optimal static equalizer. Obviously, such SNRs can be further improved if we allow ρ to go even higher.

C. Roles of STBC Block-Size on SNR

Lemma 6—Resultant Matrix for (N, N)-STBC: The resultant matrix for $\bar{\mathbf{H}}(D)$, i.e., the (N, N)-STBC of $\mathbf{H}(D)$, is equivalent to a (higher order) resultant matrix for $\mathbf{H}(D)$, except for some leading columns of zeros. More exactly

$$\Gamma^\rho[\bar{\mathbf{H}}(D)] = [\mathbf{0} \quad \Gamma^{\rho N}[\mathbf{H}(D)]]. \tag{43}$$

Proof: Referring to (20), the jth-degree coefficient matrix of $\bar{\mathbf{H}}(D)$ is

$$\bar{\mathbf{H}}_j = \begin{bmatrix} \mathbf{H}_{jN} & \mathbf{H}_{jN-1} & \cdots & \mathbf{H}_{jN-N+1} \\ \mathbf{H}_{jN+1} & \mathbf{H}_{jN} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_{jN-1} \\ \mathbf{H}_{jN+N-1} & \cdots & \mathbf{H}_{jN+1} & \mathbf{H}_{jN} \end{bmatrix}. \tag{44}$$

The lemma can be verified by substituting $\mathbf{H}_j$ with (44) in (5). ∎

For notation simplicity and w.l.o.g., the focus of the quantitative analysis in this section is placed on the channel-independent (N, W)-STBC precoding scheme.

Definition 4—SNR for (N, W)-STBC: For the ith stream, let $\mathbf{SNR(N, W, \rho, i)}$ denote the best SNR achievable by a ρ-tap Bezout equalizer for the virtual transfer function corresponding to (N, W)-STBC. (If a Bezout equalizer does not exist, the gain is set to be 0.) The gain of the original system without STBC will be treated as a special case, i.e., $\mathbf{SNR(1, 1, \rho, i)}$.

Theorem 4—SNR Comparison With or Without STBC: The following provides an analysis of the achievable SNRs with or without STBC.

a) **Interchangeable roles between block size N and tap size ρ:**

a1) $\mathrm{SNR}(1,1,\rho N-N+1,i)\leq \mathrm{SNR}(N,N,\rho,kp+i)\leq \mathrm{SNR}(1,1,\rho N,i)$, for $i=1,\ldots,p$, $k=0,\ldots,N-1$.

a2) In fact, $\max_{k=0}^{N-1}\mathrm{SNR}(N,N,\rho,kp+i)=\mathrm{SNR}(1,1,\rho N,i)$.

In terms of SNRs, there will be no net advantage of applying (N,N)-STBC unless there is a reduction in transmission rate. The equation clearly suggests that the role of N is replaceable by using longer tap-length ρ.

b) **SNR Enhancement via Rate Reduction:**

$\mathrm{SNR}(1,1,\rho N-N+1,i)\leq \mathrm{SNR}(N,N,\rho,kp+i)\leq \mathrm{SNR}(N,W,\rho,kp+i)$, for $i=1,\ldots,p$, $k=0,\ldots,W-1$.

In other words, (N,W)-STBC yields advantage in diversity-gains at the expense of reduced transmission rate (i.e., setting $W<N$). Such improvements can be very substantial, especially for the MIMO systems, cf. Fig. 9(b).

Proof: **Part a1):** $\mathrm{SNR}(1,1,\rho N-N+1,i)\leq \mathrm{SNR}(N,N,\rho,kp+i)$.

Suppose the best $(\rho N-N+1)$-tap Bezout equalizer of $\mathbf{H}(D)$ is $\mathbf{G}(D)$; then, its corresponding virtual transfer function $\bar{\mathbf{G}}(D)$ via (N,N)-STBC will be a Bezout equalizer for the virtual transfer function $\bar{\mathbf{H}}(D)$. (Note that $\bar{\mathbf{G}}(D)$ has ρ taps.) Comparing the ith row of $\mathbf{G}(D)$ with the $(kp+i)$th row of $\bar{\mathbf{G}}(D)$, we can observe that the two rows have exactly the same 2-norm. Thus, we have the proof.

Part a2): $\max_{k=0}^{N-1}\mathrm{SNR}(N,N,\rho,kp+i)=\mathrm{SNR}(1,1,\rho N,i)$.

For $\mathbf{G}(D)$, the optimal ρN-tap equalizer $\vec{\mathrm{g}}$ for the ith stream is the one satisfying the following equation with the least 2-norm:

$$\vec{\mathrm{g}}\Gamma^{\rho N}[\mathbf{H}]=\mathbf{e}_j$$

where $\mathbf{e}_j$ is all zero except at $j=i+(d+\rho N-1-k_i)p$, $k_i=0,\ldots,d+\rho N-1$.

For $\bar{\mathbf{G}}(D)$, the optimal ρ-tap equalizer $\vec{\mathrm{g}}$ for the $(kp+i)$th stream is the one satisfying the following equation with the least 2-norm:

$$\vec{\mathrm{g}}\Gamma^{\rho}[\bar{\mathbf{H}}]=\mathbf{e}_j$$

where $\mathbf{e}_j$ is all zero except at $j=kp+i+(\lfloor d/N\rfloor+\rho-1-k_i)pN$, $k_i=0,\ldots,\lfloor d/N\rfloor+\rho-1$.

By Lemma 6, $\Gamma^{\rho}[\bar{\mathbf{H}}]=[\,\mathbf{0}\quad \Gamma^{\rho N}[\mathbf{H}]\,]$. Note that the step size with a different k_i is pN for $\bar{\mathbf{G}}(D)$ and p for $\mathbf{G}(D)$. Therefore, for $\mathbf{G}(D)$, the set of possible delay patterns $\mathbf{e}_j$ for the ith stream is comprised of mutually exclusive subsets of delay patterns, where each subset corresponds to the $(kp+i)$th stream of $\bar{\mathbf{G}}(D)$, $k=0,\ldots,N-1$.

Part b): The (N,W)-STBC MIMO $\tilde{\mathbf{H}}(D)$ can be derived by extracting the first pW columns of $\bar{\mathbf{H}}(D)$. Since $\tilde{\mathbf{H}}(D)$ now has an expanded null subspace than $\bar{\mathbf{H}}(D)$, therefore, the *gain enhancement strategy* can be applied, and the norm of the Bezout equalizer can be further reduced. ∎

Strategy on Choice of N Versus ρ: The optimal selection of block-size N and equalizer's tap-length ρ will depend on a delicate tradeoff among

1) maximizing the transmission rate;
2) retaining a high post-processing SNR;
3) maintaining a low implementation complexity, etc.

In practice, the tradeoff between N and ρ could very well depend on whether it is down-link or up-link wireless communication. The following distinctive roles of N and ρ could become an important design factor for achieving optimal capacity/QoS and implemention/power tradeoff. Note that the block-size N is a preprocessing parameter that is determined at the transmitter side. A larger N incurs higher implementation complexity on both the transmitter and receiver. In contrast, the equalizer's tap-length ρ is a post-processing parameter, i.e., it is a pure receiver parameter. A larger ρ incurs more costly implementation on the receiver only. A larger ρ usually delivers improved SNR. Without the benefit of the transmitter's knowledge of the channel condition, it will be difficult to determine in advance the best block size N. As a result, the lack of channel status may leads an excessively large and unnecessary N for the actual channel condition. In contrast, a suitable ρ can be optimally determined by the receiver since the channel status is known by the receiver. In short, ρ is the more flexible of the two design parameters.

D. Simulations

We have conducted various simulations to verify the quantitative analysis in terms of bit-error rate (BER) and channel capacity. Equation (12), which yields the optimal-gain Bezout equalizer with the best system delay k_j, provides the computational basis for the performance simulations.

1) Performance in Terms of BER: For a MIMO channel which is PR, the system is decoupled into several single-user channels after applying Bezout equalization. Assuming BPSK modulation and detection is done by taking the sign of the equalizer output, the BER for each individual channel is directly related to its SNR by $\mathrm{BER}(i)=Q(\sqrt{\mathrm{SNR}(i)})$. Thus, the average BER is adopted as a criterion:

$$\mathrm{BER}=\frac{1}{\#\text{channels}}\sum_{i=1}^{\#\text{channels}}Q\left(\sqrt{\mathrm{SNR}(i)}\right). \qquad (45)$$

BER Performance of (N,N)-STBC for a Two-Input Four-Output Channel: In this simulation, we evaluate the performance impacts of (N,N)-STBC on an originally PR channel. A two-input four-output physical channel with ISI degree $d=5$ is randomly generated. Fig. 8 gives the 3-D "terrain" plot of the BER with different (N,N,ρ). For a fixed N, the performance monotonically improves with higher ρ. As stated in Theorem 4, we have $\mathrm{SNR}(1,1,\rho N-N+1,i)\leq \mathrm{SNR}(1,1,\rho N,i)$. However, a further null-subspace analysis shows that SNR enhancement will saturate when $N\rho$ becomes very large: SNR(1, 1, ρ

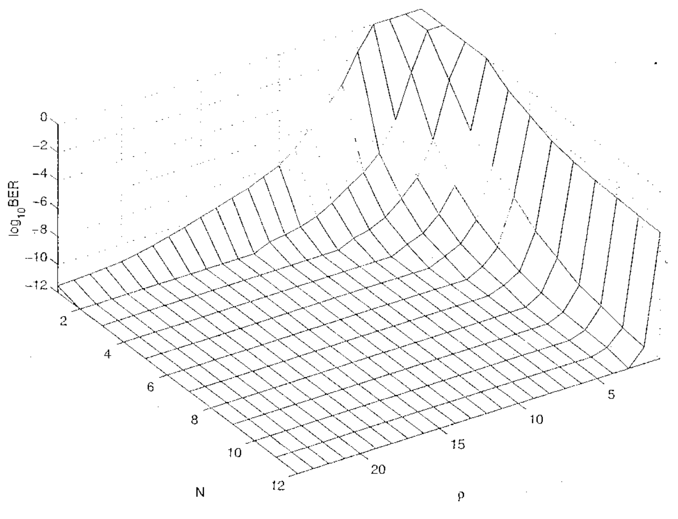


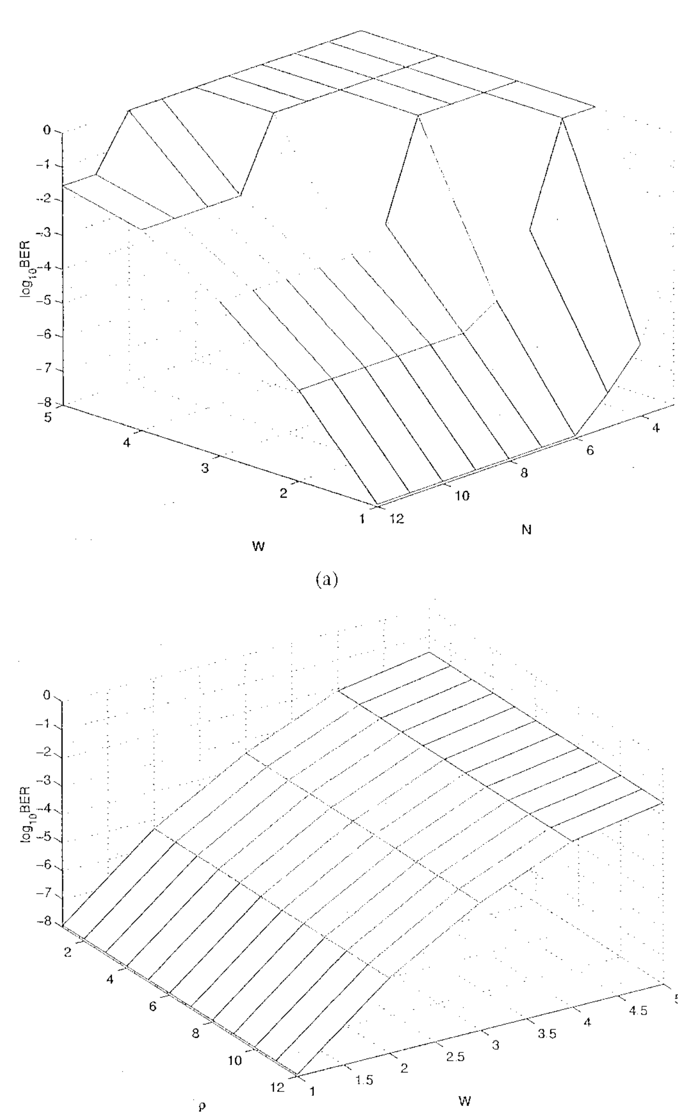
Fig. 8. BER performance w.r.t. N and ρ for a two-input four-output channel with $\sigma_s^2/\sigma_n^2 = 13$ dB.

$N - N + 1, i) \approx \mathrm{SNR}(N, N, \rho, kp + i) \approx \mathrm{SNR}(1, 1, \rho N, i)$. This shows that N and ρ play a complimentary role with each other. This is verified by the fact that the equi-potential curves are roughly $N\rho =$ constant in Fig. 8.

BER Performance of (N, W)*-STBC for 4-Input–2-Output Channels:* In this simulation, we collect the $\mathrm{BER}(N, W, \rho)$ data for a randomly generated 4I2O MIMO physical channel. The ISI degree is $d = 5$. Fig. 9(a) shows the performance with respect to the STBC parameters N and W for a given combiner with tap length $\rho = 12$. (The top flat region corresponds to non-PR systems, which is plotted with BER $= 1$ only for the convenience of display.) We can verify that for a given N, the smaller the W, the better the performance, as stated in Theorem 4(b). We observe that for a given W, the performance monotonically increases with N and saturates after N reaches a certain point. This can be simply checked by examining (27). After N reaches $W + d$, the performance cannot improve further because only rows of zeros will be added to $\tilde{\mathbf{H}}(D)$ with larger N. The points near the foothill in Fig. 9(a) represent good candidates for optimal equalizers because their rates are higher than the neighboring points and the quality only degrades by a negligible amount. *Empirically speaking, a good receiver configuration should adopt an almost saturating rate near the foothill and an adequate equalizer order.*Fig. 9(b) gives the BER performance with respect to W and ρ, for a given block size $N = 12$. Note that higher order equalizers consistently yield better results. *In general, for a fixed N , it is advantageous to use higher order equalizers to achieve better rate/quality tradeoff.*

2) *Performance in Terms of Channel Capacity:* Now, let us examine the transmission rate for the Bezout precoder and the combined STBC/Bezout equalization. Our goal is to give a quantitative understanding of the capacity loss (or gain) due to the following factors:

- the transmitter/receiver knowledge of the channel;
- the particular linear structure of the precoders and equalizers;
- processing procedures such as forcing a parallel/serial structure;



(a)

(b)

Fig. 9. (a) Performance w.r.t. N and W for a 4I2O channel with $\rho = 12$, $d = 5$. (b) Performance w.r.t. ρ with $N = 12$ and $d = 5$.

- implementation complexity such as the order of the FIR equalizer/precoder.

Fig. 10(a) shows the rate comparison for the SISO channel with ISI degree $d = 5$: $\mathbf{H}(D) = 1 - 0.3D + 0.5D^{-2} - 0.4D^{-3} + 0.1D^{-4} - 0.02D^{-5}$. The Shannon capacity and the mutual information for i.i.d. Gaussian input are the top two curves, which are shown as solid lines.[1] The Shannon capacity for the equivalent LTI system created with (70,65)-STBC sits just below the two curves. Therefore, imposing the STBC structure incurs minor capacity loss for SISO channels when N is large. The virtual transfer function for (70,65)-STBC is a constant matrix, and the Bezout equalizer for this case is $\left(\tilde{\mathbf{H}}^{\mathrm{H}}\tilde{\mathbf{H}}\right)^{-1}\tilde{\mathbf{H}}^{\mathrm{H}}$. Since the

[1]The Shannon capacity for the MIMO-ISI channel, assuming the channel is known at the transmitter, gives the ultimate bound for the error-free transmission rate. The capacity is achieved with a water-filling scheme where the signal power is jointly and optimally allocated in the spatial and frequency domain with respect to the channel frequency response [2], [8]. When the channel is unknown at the transmitter, the mutual information for i.i.d. Gaussian input, corresponding to uniform power allocation, provides a reasonable yardstick for the performance upper bound.

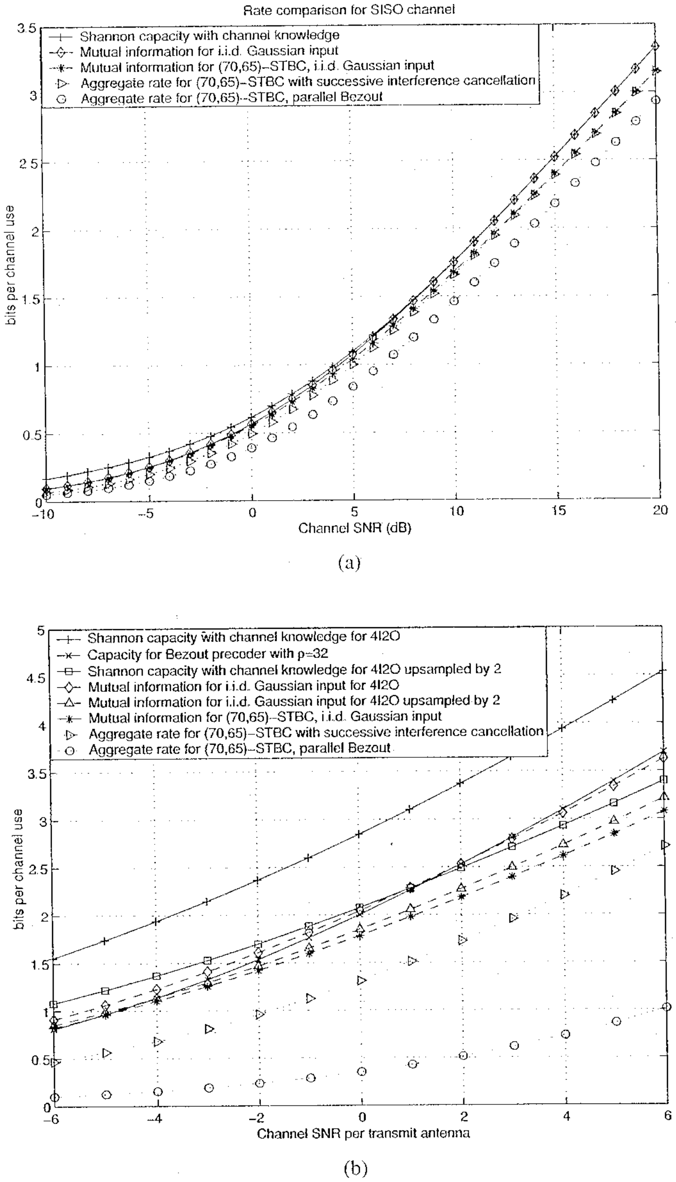


Fig. 10. (a) Rate comparison for SISO. (b) Rate comparison for 4I2O

equalizer applies 1) matched filtering and 2) an invertible transformation, both being information lossless, there should be no loss in capacity by Bezout equalization. Although the signals from different inputs are decoupled at the equalizer output, the noise contributions are spatially correlated and possibly colored in the frequency domain for (N, W)-STBC. By neglecting the spatial correlations, i.e., constraining to only independent processing after Bezout equalization, we arrive at a parallel independent Gaussian channel with colored noise in the frequency domain. Then, the capacity for each channel can be reached by temporal coding only. The rate for such parallel independent Gaussian channel with uniform transmission power is termed *aggregate rate*

$$\text{Aggregate Rate} \equiv \sum_{k=1}^{\#\text{channels}} \frac{1}{4\pi} \times \int_{-\pi}^{\pi} \log_2\left(1 + \text{SNR}\|g_k(e^{j\theta})\|^{-2}\right) d\theta \quad (46)$$

where $g_k(e^{j\theta})$ is the frequency response for the kth equalizer. The lowest curve in Fig. 10(a) shows the aggregate rate for the (70,65)-STBC with Bezout equalization. The gap with the ideal optimal joint processing mutual information is about 0.5 bits per channel use. To further reduce this gap, a layered architecture [6] or successive interference cancellation techniques [29] can be utilized to enhance the achievable rate. In Fig. 10(a), the aggregate rate for the serial processing is shown in triangles. It can be seen that the curve matches closely with the mutual information for the (70,65)-STBC system, which is an upper bound.

Fig. 10(b) shows the rate comparison for a randomly generated 4I2O channel with ISI degree $d = 5$. The abscissa is the channel SNR per transmit antenna, i.e., the total transmission power at all the antennas is $p\text{SNR} \cdot \sigma_n^2$. Depending on whether transmitter knowledge is present at the transmitter, we classify the rate curves into two categories (channel-dependent and channel-independent) and show the channel-dependent curves with solid lines. Among the eight curves shown in Fig. 10(b), three curves (the top 3 at SNR $= 6$ dB) refer to the original 4I2O channel without upsampling, and the remaining five refer to the 4I2O channel with upsampling.

1) *4I2O Channel without Upsampling:* The difference between the Shannon capacity and the Bezout precoder with $\rho = 32$ is about 1 bit per channel use. This demonstrates the effectiveness of Bezout precoder, especially in downlink communications, where output collaborations are not allowed. With Bezout precoding, a parallel independent AWGN channel without ISI and with uniform SNR is created. Achieving the capacity for this parallel channel can be easily done with only temporal coding. Hence, for slowly time-varying channels, where channel feedback does not incur much overhead, it is desirable to best employ that information at the transmitter side.

2) *4I2O Channel with upsampling:* As discussed earlier, for the 4I2O channel, it is preferable to use an interleaving precoder $(N, W)_2$ rather than (N, W)-STBC, cf. Fig. 4(b). (Choosing $L = 2$ leads to a equivalent 4I4O channel with ISI degree $d = 2$.) In Fig. 10(b), the "square" and "upright-triangle" curves are the normalized capacity and mutual information for i.i.d. Gaussian input of the 4I2O channel upsampled by 2. The plotted curves are $\text{Rate}(2 * \text{SNR})/2$ since the system has inserted a zero between every two input symbols in the unsampling process, and thus, the available energy per symbol is doubled, and the system rate is halved. At SNR $= 6$ dB, the upsampling operation trails the Shannon capacity by about 1.2 bits per channel use and mutual information for i.i.d. Gaussian inputs by about 0.5 bits per channel use.

The five curves referring to the 4I2O channel with upsampling are the MIMO counterparts for the SISO case in Fig. 10(a). Similar to the SISO case, we observe the following.

1) Not having channel knowledge decreases the rate by about 0.2 bit per channel use.
2) The structural constraints of (70,65)-STBC decreases capacity by another 0.1 bit.
3) Lacking output collaboration results in a loss around 2 bits for SNR $= 6$ dB; however, the loss can be reduced to 0.4 bits by applying successive interference cancellation [29].

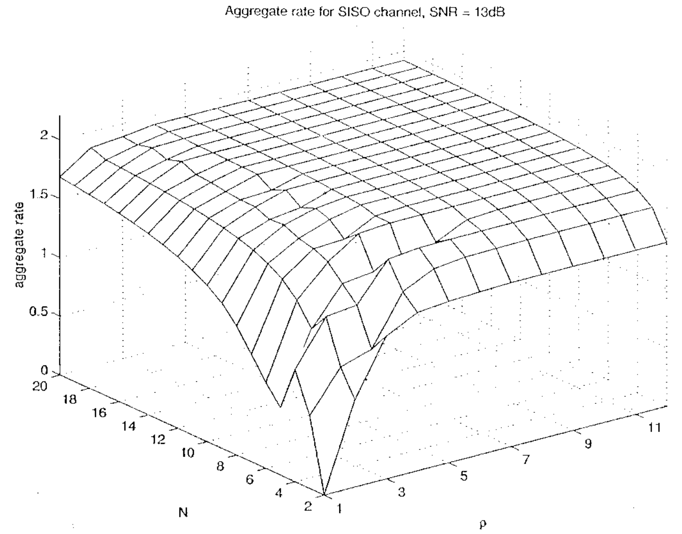


Fig. 11. Aggregate rate for SISO channel w.r.t. N and ρ, SNR = 13 dB.

To further investigate the effects of the STBC parameters on the capacity, in Fig. 11, we provide the maximum aggregate rate for different values of N and ρ with opitmal W. It is observed that the rate is generally increasing with larger N and ρ. Furthermore, from the previous discussion, the capacity loss due to (N, W)-STBC is small for large N. Since larger ρ and N imply higher implementation complexity, the figure also demonstrates the tradeoff between capacity and implementation complexity. In addition, for low-complexity designs, both N and ρ should be jointly considered to achieve a high rate rather than relying on one parameter exclusively.

VI. Conclusion

In this paper, we have provided qualitative (algebraic) and quantitative (numeric) analyses on the design of PR equalizer/precoder. The Bezout systems for the virtual STBC systems are studied. The Bezout theory serves as the algebraic basis for various situations and leads to some interesting and promising applications. With the flexibility offered by STBC, a spectrum of system configurations are available with tradeoff among the BER, transmission rate, and implementation complexities.

The paper offers several new results. Qualitatively speaking, results on the reduced McMillan degree give necessary and sufficient condition for testing the PR property of the channel. Quantitatively speaking, a Bezout null space analysis provides the dependence of post-processing SNRs on various design parameters: space-time expansion factor, transmission rate, and combiner tap length. In particular, the interplay of the two design parameters (N and ρ) is analyzed to provide a simple guideline for an optimal configuration. From the application's perspective, compared with the Alamouti space-time coding, the Bezout precoder is shown to be more appealing in slowly time-varying channel, where the feedback of the channel information induces minor overhead. Finally, via an SVD analysis, it can be shown that the Bezout precoder outperforms the OFDM precoder in power, rate, and receiver implementation.

There are many interesting future research topics. Although this paper only treats the case when the channel is known, the Bezout system can also be applied effectively to blind signal recovery where the channel information is unknown [3], [12], [18]. Optimization via joint transceiver incorporating the MMSE criterion also deserves a further exploration [7], [28]. In [30], the asymptotic performance of flat-fading (narrowband) MIMO channels was studied, and a unified analysis is established between so called diversity gain (the asymptotic slope of log-scale BER w.r.t. SNR) and multiplexing gain (the asymptotic slope of log-scale transmission rate w.r.t. SNR). Although this paper deals with frequency-selective (broadband) case and treats the tradeoff between BER and transmission rate for STBC Bezout systems, the analysis is limited to deterministic channel models. Therefore, an important future research topic is to extend the current framework to explore tradeoff between diversity gain and multiplexing gain for random frequency-selective fading channels.

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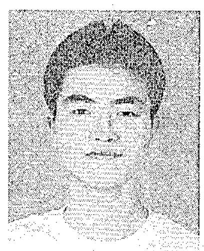

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