Proceedings: IEEE Conference on Decision and Control
New Orleans, LA., December 1977

TP6 - 3:30

INVERSE SYSTEMS AND AN INVERSION RULE‡

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1. Introduction

In recent years, the concept of inverse systems was found to be very useful in many areas such as modern control theory, coding and estimation theories; hence it has received increasing attention from authors in various areas.

Historically, Brockett and Mesarovic [1] were the first to study the properties of inverse systems (see also [2]). Massey & Sain [3] then introduced the notion of "invertibility" and the so-called "q-delay" inverse system. Invertibility for various systems was also discussed in [4, 5, 6]. Silverman developed a "structure algorithm" for solving inverse systems problems. His algorithm can be used to efficiently compute an inverse system and simultaneously test for its existence; furthermore, it can be extended to time-varying cases. Porter [7, 8, 9] had several interesting results on inverse systems [7] discusses the time-varying cases, which seems to be related to Silverman's method, [9] made the first extension to the nonlinear system case.

Most results were obtained using two approaches: one uses state-space models, the other uses polynomial matrix transfer functions. It seems that the state-space approach has the potential advantage that it is more easily extended to the problem of inverting non-relaxed systems (systems with nonzero initial state). However, the transfer function approach has its own merits.

For a relaxed or initially rest system, represented by a rational transfer function matrix $P(s)$, the (left) inverse system must have a rational representation $H(s)$ satisfying

$$H(s)P(s) = I. \tag{1}$$

Inverting a square nonsingular rational matrix is relatively trivial; however it is more challenging to find a left inverse for a non-square transfer matrix as in (1) [3, 10, 11].

In recent years, the inversion problem has been treated as a special case of so-called minimal design problems by Wang and Davison [12] (see also [13]). This is a novel idea which deals with the problem of solving $H(s)$ in the equation

$$H(s)P(s) = Q(s)$$

where $P(s)$ and $Q(s)$ are given rational matrices.

Although this minimal design approach provides very important theoretical results, an efficient algorithm for solving $H(s)$ is still lacking. Kung, Kailath and Morf [11] proposed an efficient algorithm to calculate a minimal order causal solution. In the same paper, a new and simple test for the existence of the causal solution is also discussed.

It is rather surprising that the inversion problem has not received much attention in network theory literature. Nonetheless, we shall attack the problem, adopting a circuit point of view. In some sense this viewpoint is quite practical since the circuit representations often appear to be simpler and more intuitive. Our circuit approach makes it possible to avoid both state-space models and transfer functions. Moreover, our results are valid for a very large class of systems, in actuality, the only constraint is the system linearity.

2. An Inversion Rule

We first consider a linear relaxed system. Here, "relaxed" refers to systems with zero initial state. In other words, the inputs are identically zero for the entire past. This assumption will simplify the problem considerably because we only have to consider the input/output behavior, without going into the internal descriptions. However, in §3, we shall consider the extension to "nonrelaxed" systems.

The class of systems considered here is quite general. They may be time-varying or time-invariant. Their memory elements may be time-delays, integrators, distributed types, or a mix of them. Thus the function of each element will be represented by a linear operator (rather than a transfer function as in the time-invariant case).

However, for reasons of realizability we confine our discussion to causal systems. Recall that a system is causal if no response may happen prior to the excitation. We also assume that the system is causally invertible, i.e., there is an inverse system which is also causal.

There is a special feature on the representation of the system considered in this paper, i.e., the system must be given in a circuit or flow graph

†The author was with the Information Systems Laboratory, Stanford University, when this research was conducted.

‡This work was supported by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF44-620-74-C-0068, by the National Science Foundation under Contract NSF-ENG-75-18952, and the Joint Service Electronics Program under Contract N00014-75-C-0601.

form. More precisely, this special form requires that every node having more than one incoming branch is restricted to have at most one outgoing branch. This implies (and is implied by) that every node having more than one outgoing branch is restricted to have at most one incoming branch. In fact, this representation still covers a wide range of applications. For example, the block diagrams of automatic control, signal processing systems, etc. are very commonly constructed in just this form. Note that the system is composed by 2 different types of nodes, which may be termed distinctively (see [14]). Referring to Fig. 1, the node where a number of branches converge but only one branch radiates may be termed as a "circle"; and the node where a single entering branch separates into several outgoing paths may be termed as a "dot."

It should be clear now that, in this system representation, there is no node with multiple incoming branches and multiple outgoing branches. It is certainly more restrictive than the general flow graph representation. However, this restriction is paid off by a nice topological property, which is very useful in the derivation of inverse systems. In fact, given a system in this representation, an inverse system may be readily constructed by a very straightforward rule stated in

Theorem 1. Inversion Rule Theorem. Let a linear, relaxed, causal and causally invertible system S be given in its circuit form with G as its linear system operator. Then, by the following steps, one can construct a linear relaxed system $\hat{S}$ which has system operator $\hat{G} = G^{-1}$.

(1) Find a path, say T, which connects input and output. T must contain only causally invertible branches.

(2) Reverse the directions and invert the gains of all the branches along the path T, and interchange input and output.

(3) Change the signs of all "entering branches," (i. e., the branches entering the path T). □

This result indicates that the input signal may be recovered by processing the output signal through the inverse system in a very parallel manner (but in a reversed direction) comparing to the original system. This theorem was given by Mason [14,1960] for the case where the branches are represented by scalar gains. However, in the Appendix, we provide another proof which is valid for a more general case where the branches are linear, time-varying operators. This generalization turns out be be substantial in our later effort of further extending this theorem to systems with multiple inputs and multiple outputs (§5).

To illustrate the construction rule, we consider a simple example.

Example 1. Inversion of a time-invariant system (see Fig. 2).

Obviously, if g(s) is invertible, we shall choose path T as input – g(s) – output and out inverse system, when constructed by Theorem 1, is as shown in Fig. 3. It can be easily checked that the transfer function of S is

$$H(s) = [1 + g(s)f(s)]^{-1}g(s)$$

and that of $\hat{S}$ is

$$\hat{H}(s) = g^{-1}(s) + f(s)$$

and therefore

$$\hat{H}(s) = H^{-1}(s).$$

3. Inversion of Non-relaxed Systems

If a system is not initially rest, then part of the output is contributed from its non-zero initial condition, and this part may be termed as free-response. For linear systems, the free response is additive to the other part of response caused by the input. It is therefore convenient to denote the output of a linear system S by

$$\text{output}[S, X(t_o), u(t)] ,$$

i. e., the output of a system S with initial state $X(t_o)$ at $t = t_o$, and input u(t) for $t \geq t_o$.

Our new problem is to find an appropriate system $\bar{S}$ and an appropriate initial state for $\bar{S}$ such that

$$\text{output}[\bar{S}, \bar{X}(t_o), y(t)] = u(t)$$

where

$$y(t) = \text{output}[S, X(t_o), u(t)] \quad \text{for } t \geq t_o .$$

In other words, we want to find a system $\bar{S}$ and an initial state $\bar{X}(t_o)$ such that the cascaded system of S and $\bar{S}$, with initial conditions $X(t_o)$ and $\bar{X}(t_o)$, can reproduce the input u(t) for $t \geq t_o$.

Theorem 2. Let S be a linear, causal and causally invertible system with nonzero initial condition $X(t_o)$, and $Y(t) = \text{output}[S, X(0), u(t)]$ then the inverse system S is the same as that in Theorem 1 and if the initial state of $\bar{S}$ is

$$\bar{X}(t_o) = X(t_o)$$

then

$$\text{output}[\bar{S}, X(t_0), y(t)] = u(t) . \quad (2)$$

A complete proof will be given in
However, the outline of the proof is presented here. It is obvious that y(t) is the sum of two parts

$$y(t) = y_u(t) + y_x(t)$$

where y_u represents the portion contributed from the input and y_x that from the initial state. Or in equations

$$\text{output}[\bar{S}, X(t_o), y(t)] = \text{output}[\bar{S}, 0, y_u(t)]$$
$$+ \text{output}[\bar{S}, X(t_o), y_x(t)] = u(t) + \text{output}[\bar{S}, X(t_o), y_x(t)].$$

Hence, to prove (2) is the same as to prove

$$\text{output}[\bar{S}, X(t_o), y_x(t)] = 0 \quad (3)$$

where

$$y_x(t) = \text{output}[S, X(t_o), 0] .$$

Or using the cascaded system, we need only prove

$$\text{output}[S \ominus \bar{S}, X(t_o) \ominus \bar{X}(t_o), 0] = 0 \text{ if } X(t_o) = X(t_o) . \quad (4)$$

The detailed proof of (4) can be found in the Appendix. Here we shall only explain the idea by an illustrative example.

Example 3.1. Again, we consider a simple feedback system (Fig. 2) and assume g is memoryless and invertible, i. e., g is a nonzero gain. Then our cascaded original and inverse system are shown in Fig. 5.

The output affected by initial state $X(t_o)$ and $X(t_o)$ is the same as the output by the input $\varphi_r(X(t_o))$ at the two "circles" (see Fig. 5) where

$$\varphi_r(X(t_o)) = \text{the open loop response of } f \text{ with initial state } X(t_o) . \quad (5)$$

The overall output can then be calculated as (assuming stable systems)

$$(g^{-1}+f)\left(\sum_{n=0}^{\infty}(gf)^n g\varphi_r(X(t_o)) - \varphi_r(X(t_o))\right)$$

$$= \sum_{n=0}^{\infty}(fg)^n\varphi_r(X(t_o)) - \sum_{n=0}^{\infty}(fg)^{n+1}\varphi_r(X(t_o)) - \varphi_r(X(t_o)) = 0 .$$

In this example, we have shown that (4) is justified.

4. Observability, Controllability and Reproducibility of States

A linear, time-varying system is said to be observable at t_o, if for any state $X(t_o)$ at time t_o in the state space, there exists a finite $t_1 > t_0$ such that the output with knowledge of the free-response (i.e., the output with zero input) over the time interval $[t_o, t_1]$ suffices to determine the state $X(t_o)$. It is easy to see that this definition may be rephrased in a different form:

A system S is observable at t_o if and only if there exist no nonzero state $X(t_o)$ such that

$$\text{output}[S, X(t_o), 0] = 0 \quad \text{for all } t \geq t_o .$$

Remark: The concept of controllability is dual to that of observability, and is therefore omitted here.

Theorem 3. With reference to Theorem 1, the inverse system $\bar{S}$ is observable at t_o if and only if the original system S is observable at t_o. □

Proof: Let us assume that S is unobservable at t_o then there exists $X(t_o)$ such that

$$\text{output}[S, X(t_o), 0] = 0 \quad \text{for } t \geq t_o$$

that is, by definition, $\bar{S}$ is not observable. This proves the "if" part of the theorem. By switching S and $\bar{S}$, we can also prove the "only if" part of the theorem.

The discussion for controllability can be carried out by applying the same reasoning to the dual system and the dual inverse system.

Remark: It should be noted that the observability of the finite dimensional system (continuous or discrete time) can be checked by the observability matrix (see e.g., [15]). However, the present result is valid for a much more general class of systems.

Reproducibility of States

Note the observability (for all $t \geq 0$) of the inverse system has another interesting viewpoint, namely: it is possible to use the inverse system as a state follower, a special case of the state reconstruction problem.

Theorem 4. Let us assume the observable system S is relaxed at $t = 0$, and its inverse system $\bar{S}$ is cascaded to the original system S at $t = 0$. Then not only is the output $\bar{y}(t)$ of S equal to u(t) but also

$$\bar{X}(t) = X(t), \quad t \geq 0 . \quad □$$

Proof: Note that for any $\tau > t \geq 0$, we have

$$\text{output}[\bar{S}, \bar{X}(t), y(\tau)] = \text{output}[\bar{S}, X(t), y(\tau)] + \text{output}[\bar{S}, \bar{X}(t) - X(t), 0] ,$$

and by Theorem 2 we know that

$$\bar{y}(\tau) = u(\tau) = u(\tau) + \text{output}[\bar{S}, \bar{X}(t) - X(t), 0] .$$

Since the system is observable at t, then the fact that

$$\text{output}[\bar{S}, \bar{X}(t) - X(t), 0] = 0$$

implies

$$\bar{X}(t) = x(t) . \quad □$$

5. Multi-Input Multi-Output Systems

We now note that all the results above carry through to multi-input mulit-output systems. The only "new" difficulty for the MIMO system is the "noncommutavity" of the system operators. However, it is easy to check that, in our proof, the ordering of the operators is carefully preserved. Hence our previous results are valid for both SISO and MIMO systems. This is the reason that there is no restriction specifically mentioned in the previous sections.

Nonetheless, for the MIMO case we use a slightly modified fule:

Theorem 5. Assume S represents a linear, causal and causally invertible system with m inputs and m outputs. Consider its circuit representation and let G be its system operator. Then the inverse system $\bar{S}$ with operator $\bar{G} = G^{-1}$ can be constructed as follows.

1) Find m transmission paths such that each of the m inputs is connected to one of the outputs. These transmission paths must not overlap and must contain causally invertible branches.

2) Reverse the directions and invert the gains of all the branches on the transmission paths and interchange input and output.

3) Change the signs of all the "entering branches." □

Remark: At this point, one may argue that our chosen paths ought to have the same number of branches for all the paths so that the proof used for Theorem 1 can be applied directly. However, this is not a problem since some trivial branches (i.e., those with unit gain) can be artificially inserted so that all the paths have the same number of branches. It is easy to see that, once the inverse system is formed as in Theorem 5, removing (or ignoring) thos artificially inserted branches does not change the inverse system.

Remark: It should also be noted that the invertibility of the system guarantees that there exist m non-overlapping, causally invertible paths. Namely, m such paths always exist if the overall system admits a causal inverse. (This is verified by a straightforward "information sufficience" argument.) If the given circuit has a complicated structure, then some computer-aided search technique may be utilized to find and trace the appropriate paths. (See Fig. 5 for example).

6. An Application: Speech Processing Model

The so-called inverse filtering approach is concerned with identifying system structures and parameters. In speech processing, the problem of modeling and estimation of the vocal tract shape has received a lot of attention. It is known [16] that the vocal tract may be modeled by an auto-regressive (AR) model with a transfer function

$$H(z) = 1/a(z),$$

where

$$a(z) = 1 + a_1 z^{-1} + \cdots + a_N z^{-N}.$$

It can be shown that identifying the parameters (a_i, $i = 1, \ldots, N$) is equivalent to identifying the reflection coefficients $\{k_i, i = 1, \ldots, N\}$ which are determined by variations in the cross-sectional area of the vocal tract. In general, the AR model can be realized in a latter form, using only the reflection coefficients as parameters (see Fig. 6).

It can be shown that an efficient way to identify the model is by utilizing an optimal inverse filter as shown in Fig. 7, such that the error is as small as possible.

In fact, it is desirable that the inverse filter be found in a similar ladder form as in Fig. 6 so that the vocal tract configuration can be derived.

The most direct method to write the inverse filter of Fig. 7 is to apply our inversion rule and the inverse filter can be shown as in Fig. 8.

It should be noted that if such an inverse system is cascaded with the original AR system then, by Theorem 4 and the fact that the system is observable, the state at each stage, as well as the input, is reproduced in the inverse system. This is an important observation since we in fact do not know the reflection coefficients (k_i, $i = 1, \ldots, N$) a priori; however this important information is contained in the states of the ladder form realization. It can be shown that if the input to the speech producing model (Fig. 6) is a stationary white Gaussian noise (WGN) then the correlation between the "forward wave" X_i^+ and "backward wave" X_i^- is proportional to the reflection coefficient k_i. The derivation of this relationship is quite involved and we shall simply state the result here,

$$\langle X_i^+, X_i^- \rangle / \langle X_i^+, X_i^+ \rangle = -k_i. \qquad (6)$$

By (6), our inverse system is already close to the well-known optimal inverse filter [16] in which the gains $\{k_i\text{'s}\}$ in Fig. 8 are replaced by the so-called partial correlation coefficients. Note that if the true k_i's were used then the output of the (perfect) inverse filter would be (perfectly) white. Consequently, the optimum inverse filter gives an output which is as "white" as possible.

We shall not go into detailed discussion since it involves the orthogonal properties of the states using the orthogonal polynomial theory and "maximum entropy" spectral analysis [16-20]. After all, our main purpose here is not to find new inverse filtering methods but to demonstrate how the inversion rule is related to the well-known inverse filtering technique and its stochastic interpretation.

It is worth mentioning that the inversion rule still holds if the delay elements z^{-1} are replaced by other elements, for example, $z^{-1} \to z_i^{-1}$, $i=1, \ldots, N$, where z_i^{-1} represent delays with different time duration. This situation arises when a layered system has different thickness for each layer. Thus, this modified representation may have some potential applications.

Acknowledgement

The author wishes to thank Profs. T. Kailath and M. Morf of the Information Systems Laboratory, Stanford University, for many useful discussions and suggestions.

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Appendix – Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. Assume a path T is chosen, then by the "linearity" property, the system can be reduced to the one with the form shown in Fig. 9 (assuming there are M "dots" and M "circles" on the path T), where $\{g_{i,j}, i=1,\ldots,M, j=1,\ldots,M\}$ are new operators after proper reduction and each operator $g_{i,j}$ connects the i-th "dot" and the j-th "circle." On the path T, there are (causally) invertible branches $\{\alpha_i, i=1,\ldots,M-1\}$ and $\{\beta_i, i=1,\ldots,M\}$ where α_i connects the i-th circle and i-th dot and β_i connects i-th dot and (i+1)-th circle.

In the same manner, the constructed inverse system $\bar{S}$ can be reduced to a similar form (Fig. 10).

Our next step is then to prove that $\bar{S}$ is indeed an inverse system of S. Here, we shall verify this relation by an induction procedure. That is, we shall establish the inverse relation stage by stage. Let us first work on the circuit in Fig. 9, it can be reduced to S^* as shown in Fig. 11, where, for i=1,..., M-1, j=1, ..., M

$$g^*_{i,j} = g_{i,j} + g_{M,j}(\alpha_M^{-1} - g_{M,M})^{-1}(\delta_{i,M-1}\beta_M + g_{i,M})$$

with $\delta_{i,k} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$. Then the system S^* can be further reduced to $S^\#$ as in Fig. 12, where for i = M-1,

$$g^\#_{M-1,j} = g^*_{M-1,j} = g_{M-1,j} + g_{M,j}(\alpha_M^{-1} - g_{M,M})^{-1}(\beta_M + g_{M-1,M})$$

and for i ≠ M-1,

$$g^\#_{i,j} = g^*_{i,j} + [\delta_{M-1,j}\alpha_{M-1}^{-1} - g^*_{M-1,M}][\beta_M + g_{M-1,M}]^{-1} g_{i,M}$$

$$= g_{i,j} + g_{M,j}(\alpha_M^{-1} - g_{M,M})^{-1} g_{i,M} + [\delta_{M-1,j}\alpha_{M-1}^{-1} - g_{M-1,j}$$

$$- g_{M,j}(\alpha_M^{-1} - g_{M,M})^{-1}(\beta_M + g_{M-1,M})] \cdot (\beta_M + g_{M-1,M})^{-1} g_{i,M}$$

$$= g_{i,j} + [\delta_{M-1,j}\alpha_{M-1}^{-1} - g_{M-1,j}][\beta_M + g_{M-1,M}]^{-1} g_{i,M} .$$

Our next step is to reduce the system $\bar{S}$ as in Fig. 10. By almost the same procedure, $\bar{S}$ can be reduced to $\bar{S}^\#$ as shown in Fig. 13, where for i = M-1

$$\bar{g}^\#_{M-1,j} = g_{M-1,j} + g_{M,j}(\alpha_M^{-1} - g_{M,M})^{-1}(\beta_M + g_{M-1,M})$$

for i ≠ M-1

$$\bar{g}^\#_{i,j} = g_{i,j} + [\delta_{M-1,j}\alpha_{M-1}^{-1} - g_{M-1,j}][\beta_M + g_{M-1,M}]^{-1} g_{i,M} .$$

Now it is clear that the first (rightmost) two sections of $\bar{S}^\#$ (Fig. 13) is the inverse of the last two sections of $S^\#$ (Fig. 12). Therefore, their effects cancel each other. And also note that

$$\bar{g}^\#_{i,j} = g^\#_{i,j} \quad \text{for } i=1,\ldots,M-1,\ j=1,\ldots,M-1.$$

Therefore, the same reduction can go on until the complete inverse relation is established. Hence the proof. □

Proof of Theorem 2. Following the discussion in §3, our goal here is to prove Eq. (4).

$$\text{output}[S \oplus \bar{S}, X(t_o) \oplus \bar{X}(t_o), 0] = 0, \quad \text{if } \bar{X}(t_o) = X(t_o) . \quad (A.1)$$

Let us consider the reduced circuits in Fig. 9. For simplicity, we shall assume that only branch $g_{i,j}$ has nonzero initial state. This assumption is without loss of generality since our system is linear and the contribution from the initial state of each branch can be treated separately. Let

$\varphi_{i,j}(X_{i,j}(t_o))$ = the open loop response of $g_{i,j}$ with initial state $X_{i,j}(t_o)$

(as defined in Eq. (5), §3). Then the output affected by $X(t_o)$ and $\bar{X}(t_o)$ is the same as the output produced by the local inputs $\varphi_{i,j}(X_{i,j}(t_o))$ at the two circles shown in Fig. 14 (refer to Ex. 3.1).

By a more involved reduction procedure, similar to the one used in the proof of Theorem 1, it is possible to reduce the circuit to a simpler form and then prove Eq. (A.1).

However, we shall use another very simple proof. Let us modify the circuit in Fig. 14 to the one in Fig. 15. Now, let u'(t) be

$$u'(t) = \varphi_{i,j}(X_{i,j}(t_o))$$

then, using Theorem 1,

$$\bar{u}'(t) = u'(t) = \varphi_{i,j}(X_{i,j}(t_o)) .$$

However, if we let $\varepsilon \to 0$, then Figs. 14 & 15 become equivalent. Note that

$$u(t) = \lim_{\varepsilon \to 0} \varepsilon u'(t) = 0$$

and since $\bar{u}'(t) = \varepsilon^{-1}\bar{u}(t) = \varphi_{i,j}(X_{i,j}(t_o))$; this implies that (assuming $\varphi_{i,j}(X_{i,j}(t_o))$ is bounded), when $\varepsilon \to 0$,

$$\bar{u}(t) = 0 ,$$

i.e., the output of the inverse system is zero, hence the proof. □

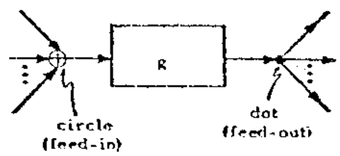

Fig. 1. A typical memory branch, the branch "g" may be delay, integrator, or other type of memory element, i.e., it represents a linear operator which is again denoted by "g."

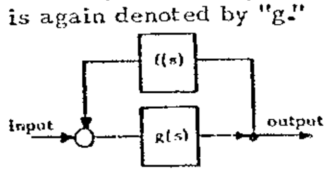

Fig. 2. A simple feedback system S.

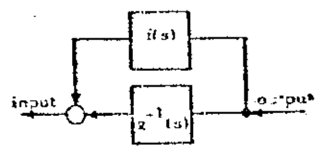

Fig. 3. An inverse system $\bar{S}$.

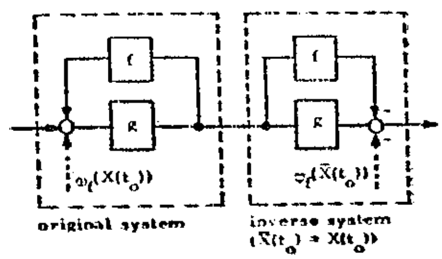

Figure 4

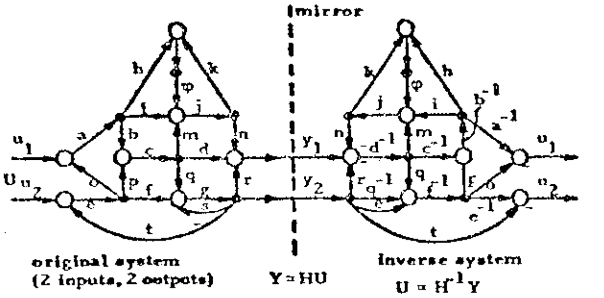

Fig. 5. Example for constructing MIMO inverse systems.

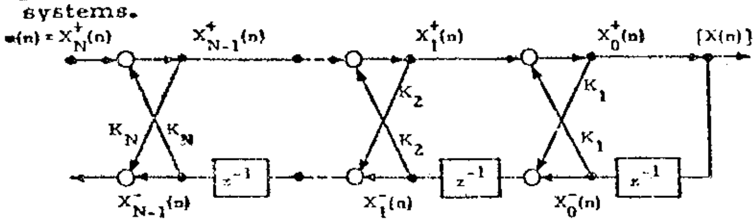

Fig. 6. A ladder form representation of a speech producing model.

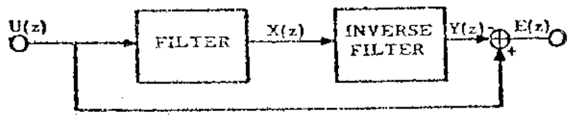

Figure 7

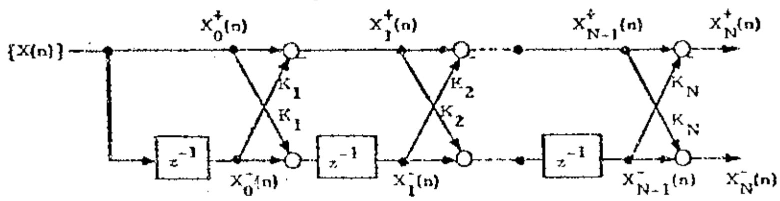

Fig. 8. Inverse filter of the speech producing model; in the optimal inverse filter [16] k_t can be found as $k_n = -\Sigma[X^+_{n-1}(n)X^-_{n-1}(n)]/(\Sigma[X^+_{n-1}(n)]^2\Sigma[X^-_{n-1}(n)]^2)^{1/2}$.

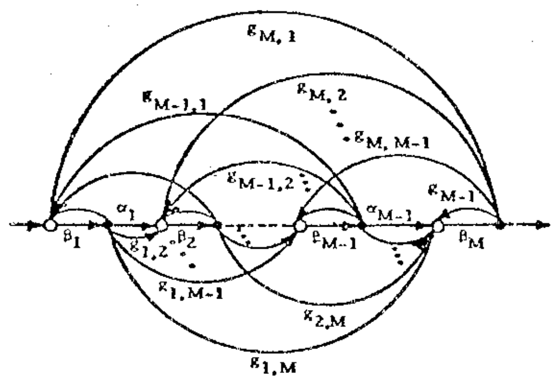

Figure 9

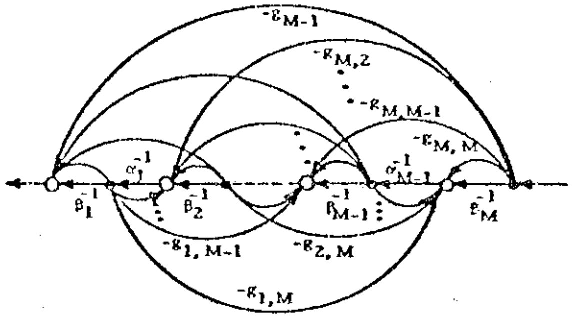

Figure 10

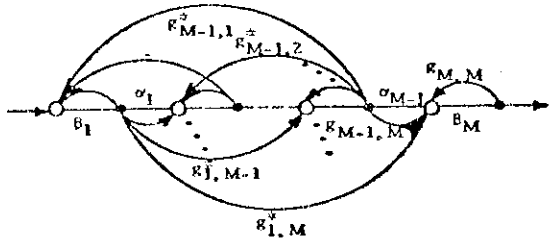

Figure 11

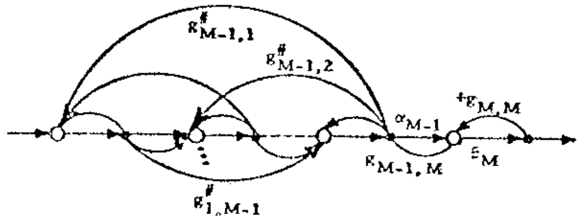

Figure 12

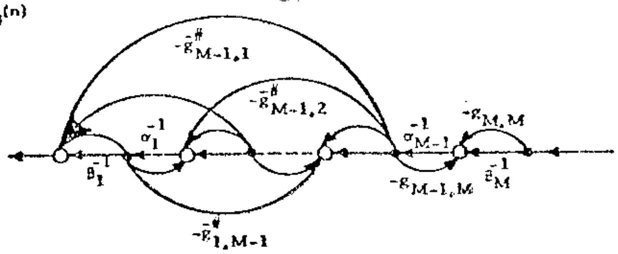

Figure 13

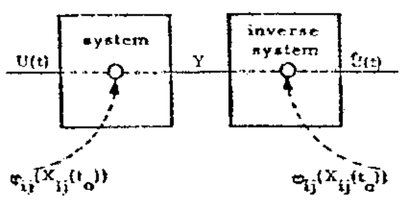

Figure 14

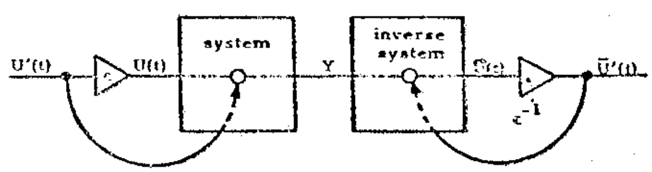

Figure 15