

# Compressible Steady Laminar Boundary Layers

Harvey S. H. Lam

February 18, 2004  
(typos edited 2/29/2004)

## 1 The PDEs

We have, for two-dimensional problems, the following three boundary layer PDEs:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad (1)$$

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp_e}{dx} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right), \quad (2)$$

$$\rho \left( u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) = u \frac{dp_e}{dx} + \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2, \quad (3)$$

where  $h$  is specific enthalpy ( $h = e + p/\rho$ ), and subscript 'e' denotes conditions at the edge of the boundary layer. Thus  $\rho_e(x)$ ,  $p_e(x)$  and  $U_e(x)$  (and also  $h_e(x)$ ) are assumed given. In addition, we assume an equation of state is given, and  $h$ ,  $\mu$  and  $\kappa$  are known functions of the state variables of the fluid.

Note that the energy equation is needed only to account for the temperature dependence of  $\rho$  and  $\mu$  in the continuity and momentum equation. It is not needed if these two entities can adequately be approximated by constants. In that case, the energy equation can be solved later to study the temperature distribution (and heat transfer). In the general case, the three equations (plus equation of state, etc.) need to be solved simultaneously.

## 1.1 Prandtl Number

We assume  $h$  to be a function of temperature only:

$$h = \int^T C_p(T')dT' + h_{ref} \quad (4)$$

where  $C_p(T)$  is the specific heat at constant pressure and  $h_{ref}$  is a “constant” (for non-reacting flows). The dimensionless Prandtl Number  $P_r$  is defined by:

$$P_r \equiv \frac{\mu C_p}{\kappa}. \quad (5)$$

Using it in eq.(3), we obtain an alternative form as follows:

$$\rho \left( u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) = u \frac{dp_e}{dx} + \frac{\partial}{\partial y} \left( \frac{\mu}{P_r} \frac{\partial h}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2. \quad (6)$$

Let us multiply eq.(2) by  $u$  to obtain our “counterfeit” energy equation:

$$\rho \left( u \frac{\partial (u^2/2)}{\partial x} + v \frac{\partial (u^2/2)}{\partial y} \right) = -u \frac{dp_e}{dx} + u \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right). \quad (7)$$

Adding the above two equations, we obtain<sup>1</sup> without any approximation:

$$\rho \left( u \frac{\partial h^o}{\partial x} + v \frac{\partial h^o}{\partial y} \right) = \frac{\partial}{\partial y} \left( \mu \frac{\partial h^o}{\partial y} \right) + \frac{\partial}{\partial y} \left( \mu \left( \frac{1}{P_r} - 1 \right) \frac{\partial h}{\partial y} \right). \quad (8)$$

where  $h^o$  is the *stagnation enthalpy* defined by:

$$h^o \equiv h + \frac{u^2}{2}. \quad (9)$$

It is obvious that wonderful simplifications are available if  $P_r = 1$  is a good approximation for the fluids in question.

## 2 Energy Integrals when $P_r = 1$

In this section, we adopt the assumption that  $P_r = 1$ .

---

<sup>1</sup>Actually, when the energy equation is derived from first principles, the kinetic energy was included in the “internal energy” and is later removed via the counterfeit energy equation. So, we are undoing here what we did then.

## 2.1 The Busemann Integral

Busemann was the first to point out that  $h^o = \text{constant}$  is an exact solution to eq.(8)—for any viscosity/specific heat/heat conductivity laws, so long as the Prandtl Number is 1.0. Note that for steady inviscid flows,  $h^o = \text{constant}$  is an exact solution (boundary conditions permitting).

Differentiating  $h_o$  with respect to  $y$ , we obtain:

$$\frac{\partial h^o}{\partial y} = \frac{\partial h}{\partial y} + u \frac{\partial u}{\partial y} = 0. \quad (10)$$

Evaluating eq.(10) at the wall ( $y = 0$ ) and applying the no-slip condition, we found that the Busemann energy integral is valid only when the heat transfer rate at the wall is zero—insulated wall.

So when  $Pr = 1$  and the walls are insulating,  $h^o = \text{constant}$  is a valid solution both in the inviscid and in the boundary layer flows—boundary condition permitting.

## 2.2 The Crocco Integral

If in addition to  $Pr = 1$ , the prevailing pressure gradient  $dp_e/dx$  is also negligible from the momentum equation, then  $h^o = Ku + \text{constant}$  is also seen to be an exact solution, where  $K$  is some constant.<sup>2</sup>

For this case, the equation analogous to eq.(10) is:

$$\frac{\partial h^o}{\partial y} = \frac{\partial h}{\partial y} + u \frac{\partial u}{\partial y} = K \frac{\partial u}{\partial y}. \quad (11)$$

We can find the heat transfer rate at the wall by evaluating this at the wall:

$$\left( \kappa \frac{\partial T}{\partial y} \right)_{y=0} = \left( \frac{\mu}{Pr} \frac{\partial h}{\partial y} \right)_{y=0} = \left( \frac{K}{Pr} \mu \frac{\partial u}{\partial y} \right)_{y=0} = K (\tau_{yx})_{y=0}. \quad (12)$$

Aha! Crocco's integral is valid when there is heat transfer (but no pressure gradient). And it says the heat transfer rate is proportional to the viscous wall friction. This is called "Reynolds' Analogy" in the heat transfer community as an empirical rule of thumb (used by many even when there is pressure

---

<sup>2</sup>Note that at the edge of the boundary layer, the inviscid theory says  $h_e^o$  is a constant. The two constants in  $h^o$  can be determined by the values of  $h_e^o$  and the value of  $h_{wall}$ —the latter must be a constant. What is the physical dimension of  $K$ ?

gradient and  $P_r$  is not precisely unity). One must not apply this analogy blindly when there is a pressure gradient. For example, in a separated flow region where the wall friction is negative, we do not expect the local heat transfer to reverse direction. At a stagnation point, the wall friction is zero, but certainly not the local heat transfer rate.

### 3 Recovery Temperature

You have a flow over some solid surface which is thermally insulated. What is the temperature of the wall surface?

Physically, the viscous friction is active inside the thin boundary layer, and heat is generated by the  $\mu(\partial u/\partial y)^2$  term in the energy equation. This heating is called aerodynamic heating.

The Busemann Integral can be used when  $P_r = 1$  is a good approximation. It says the wall value of enthalpy is simply the stagnation enthalpy of the freestream. As simple as that! It is seen that the amount of viscous heating is proportional to the square of the characteristic Mach Number of the flow. Hence it is not a big practical issue unless the Mach Number is high.

What happens when Prandtl Number is not unity? The usual procedure is to introduce a *recovery factor*, usually denoted by 'r' which depends on  $P_r$ :

$$h_{wall} = h_e^o + (r - 1)\frac{U_e^2}{2}. \quad (13)$$

In general,  $r = r(P_r)$ , and  $r = 1$  for  $P_r = 1$ .<sup>3</sup> Instead of enthalpy, this equation can be expressed in terms of temperature. Assuming constant specific heat  $C_p$  (and  $h_{ref} = 0$  in eq.(4)) and a perfect gas law ( $p = \rho RT$ ), we have:

$$T_{wall} = T_e^o \left(1 + \frac{(r - 1)(\gamma - 1)U_e^2}{2\gamma RT_e^o}\right) \quad (14)$$

There is a Mach Number squared in this equation staring back at you.

### 4 The Howarth Transformation

We first introduce the concept of a stream function (for two-dimensional or axi-symmetric flows). We will deal only with two-dimensional flows here.

---

<sup>3</sup>For  $P_r$  near unity, good recovery factor empirical formula (curve-fitted) for laminar flow are given in White on top of page 510 and in Fig.7-1 on page 509.

The stream function  $\psi(x, y)$  is defined by:

$$\frac{\partial \psi}{\partial y} = \rho u, \quad (15)$$

$$\frac{\partial \psi}{\partial x} = -\rho v, \quad (16)$$

It is readily seen that the continuity equation is identically satisfied.

We now formally introduce a new set of independent (dimensional) variables:

$$X = x, \quad (17)$$

$$Y = \int_0^y \rho(x, y') dy' \quad (18)$$

Note that  $X$  and  $Y$  are dimensional and have different physical dimensions.

The derivatives transform as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial}{\partial Y} \quad (19)$$

$$\frac{\partial}{\partial y} = \rho \frac{\partial}{\partial Y} \quad (20)$$

The momentum equations eq.(2) become:

$$\rho \left( u \left( \frac{\partial u}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial u}{\partial Y} \right) + \rho v \frac{\partial u}{\partial Y} \right) = -\frac{dp_e}{dX} + \rho \frac{\partial}{\partial Y} \left( \rho \mu \frac{\partial u}{\partial Y} \right) \quad (21)$$

and (dimensional)  $u$  and  $v$  are now related to  $\psi$  by:

$$u = \frac{\partial \psi}{\partial Y}, \quad (22)$$

$$v = -\frac{1}{\rho} \left( \frac{\partial \psi}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial \psi}{\partial Y} \right). \quad (23)$$

Note: the terms involving  $\partial Y/\partial x$  is very unpleasant. Substituting these into eq.(21), we find that these unpleasant terms cancelled out:

$$\rho \left( u \frac{\partial u}{\partial X} - \frac{\partial \psi}{\partial X} \frac{\partial u}{\partial Y} \right) = -\frac{dp_e}{dX} + \rho \frac{\partial}{\partial Y} \left( \rho \mu \frac{\partial u}{\partial Y} \right). \quad (24)$$

Now, using subscripts to indicate partial derivatives, we have:

$$(\psi_Y \psi_{XY} - \psi_X \psi_{YY}) = -\frac{(p_x)_X}{\rho} + (\rho\mu\psi_{YY})_Y. \quad (25)$$

Doing the same thing to the energy equation eq.(6), we obtain:

$$\psi_Y h_X - \psi_X h_Y = \frac{u}{\rho} \frac{dp_e}{dX} + \left(\frac{\rho\mu}{P_r} h_Y\right)_Y + \rho\mu(\psi_{YY})^2. \quad (26)$$

Since pressure  $p(x)$  is known from its edge value, then  $\rho$ ,  $\rho\mu$  and  $P_r$  are now known functions of  $h$  (or temperature). It is clear from this equation that the variation of the product  $\rho\mu$  is something to keep our eyes on. You guessed it: the *ad hoc* approximation of  $\rho\mu=\text{constant}$  was very popular one in the last century. White has a good collection of well-known similar solutions in the book.<sup>4</sup> You can look at the collection of data in White's book on the effects of Mach Numbers and wall temperatures on the compressible Blasius solution and the influence of viscosity laws.

## 4.1 The Illingworth Transformation

By inspecting the above equations, you can see the hints Professor Illingworth must have seen in proposing his Transformation (see eq.(7-19), particularly his definition of  $\xi$ ). Note that his  $\xi$  is dimensional (with physical unit of viscosity squared), and his  $\eta$  is dimensionless).

## 5 Shock Wave Structure

The Newtonian shear viscosity is intuitively very easy to accept. If  $u$  is a function of  $y$  in a parallel flow, then on a  $y = \text{constant}$  surface there is a viscous shear stress acting in the  $x$ -direction between the layers of fluid. Fine. When we study boundary layers, this viscous shear stress is all we need to know.

But the Navier-Stokes viscous stress tensor analysis tells us that for the same flow, there is also a viscous shear stress acting in the  $y$ -direction on any  $x$ -constant surface. It doesn't do anything to the fluid, but it is there.

---

<sup>4</sup>Look at White's eq.(7-20) on page 505. He said it is "more or less" an ordinary differential equation. Why did he say that?

A really impressive demonstration that the Navier-Stokes viscous stress tensor is good stuff is its ability to describe the structure of a shock wave. You can measure the viscosity of a gas using a simply viscometer (by measuring the shearing force between two parallel surfaces sliding with respect to each other a fixed distance apart), and use the measured viscosity to calculate the detailed structure of a shock wave. Remember, in inviscid fluid mechanics, a shock wave is a discontinuity. Now, with viscous fluid mechanics, shock wave is a thin region where viscous effects smooths out the otherwise discontinuous jump.

We remember that in boundary layer theory, useful simplifications are available when  $P_r = 1$ . For shock wave structures, similar simplifications occur when the Prandtl Number is  $3/4$  (under the Stokes' hypothesis for the second coefficient of viscosity which is a player here). In any case, the shock wave structure problem is one-dimensional, and is therefore governed by an ODE. Within this framework, the problem is easily solved for any equation of state and viscosity/heat conductivity laws (for a "weak" normal shock, it can even be solved analytically). How thick is a shock wave? No surprise here. The characteristic Reynolds Number of a shock wave, using the thickness as the characteristic length, is of order unity. Physically, this means it is a few mean-free-paths. It also means that the continuum assumption which underlies the Navier-Stokes equations is in trouble.

## 6 Homeworks

1. You already have a working computer program from last week's homework to solve any steady, laminar, two-dimensional constant  $\rho$  and  $\mu$  boundary layer problem.

Lets make it a compressible (non-constant  $\rho$  and  $\mu$ ) code. Instead of handling the energy equation numerically, we shall assume  $P_r = 1$  and that the solid wall is perfectly insulating. So the Busemann Energy Integral is available to express the specific enthalpy in the boundary layer as a function of  $u$ , the streamwise velocity component.

We shall adopt the following assumptions:

- a: At the edge of the boundary layer, the inviscid compressible theory says the stagnation enthalpy ( $h_e + U_e^2/2$ ) and the specific entropy  $s$  are known constants. So once  $U_e(x)$  is known,  $h_e(x)$ ,  $\rho_e(x)$  and

$p_e(x)$  are known and can be calculated (with the help of the next two assumptions).

- b:** The fluid is a perfect gas ( $p = \rho RT$ ) where  $R$  is the gas constant and  $T$  is absolute temperature. We shall assume the gas is air.
- c:** The specific enthalpy  $h$  is a function of  $T$  only. In fact, we shall assume  $h = C_p T$  where  $C_p = \gamma R / (\gamma - 1)$ . You are reminded that  $\gamma RT$  is the square of the speed of sound. You can now compute  $-dp_e/dx$  by  $\rho_e U_e dU_e/dx$  to be used in the x-momentum equation.
- d:** The viscosity  $\mu$  is assumed to depend only on  $T$  by the empirical formula:

$$\mu = \beta T^n$$

where  $\beta$  is a dimensional constant. (pick  $\beta = \mu_*/T_*^n$  where  $\mu$  and  $T_*$  are characteristic values for your problem.)

So all you need to do is to edit your working boundary layer code so that  $\rho$  and  $\mu$  are not constants, but explicit algebraic functions of  $u$ .

Do a diagnostic run for the Blasius case ( $U_e = \text{constant}$  with freestream Mach Number  $M_e = 5$ ). Plot the velocity and temperature profiles. Assume  $n = 1$  for your power-law viscosity.

Do one run with  $U_e(x/L) = U_*(1 + x/(4L))$  with  $M_* = U_*/a_* = 1.5$  where  $U_*$  and  $a_*$  are incoming flow velocity and speed of sound at  $x = 0$ . Plot some results. Assume  $n = 2/3$  for your power-law viscosity.

Note that on a computer, using the empirical power-law viscosity yields very little practical benefits. The “simplifications” made possible for the  $n = 1$  case is not obvious from the numerical answer.