Dimensional Analysis*

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Abstract

All fluid mechanics students learned dimensional analysis, and passed midterms and finals which included dimensional analysis problems. But few recognize that it is very useful for everyday common-sense-based engineering estimations.

1 Introduction

Dimensional analysis is not restricted to fluid mechanics problems. We shall use fluid mechanics examples in our discussions.

Given a problem of interest, we usually have a list of unknown dimensional variables which shall be denoted by Z_i, (i=1,..., I), and a list of relevant dimensional parameters (which often include the spatial coordinates and time) which shall be denoted by p_j (j=1,..., J). There are thus a total of $K = I + J$ dimensional entities of interest. The engineering goal is to find

$$Z_i = \mathcal{Z}_i(p_1, \ldots, p_J), \quad i = 1, \ldots, I, \tag{1}$$

one way or the other.

Of course, if the problem can be solved analytically using mathematics, these relations are the "results." For problems for which theory is unable to do the job, these relations can be found by using experimental data. When the experimental route is taken, the issue is how to make as few experiments as possible, and to get as much

*These notes are usually written the evening before the class. So there will be typos. Check the web for the latest revision.

mileage out of the (precious) measured experimental data as possible. In the modern age of computations, often such desired relations are extracted from computer simulations.

For example, suppose we want to design a pipeline to deliver natural gas from Fairbank Alaska to Palo Alto California. We want to know how much pumping power P (horsepower) is needed to deliver Q (cubic feet per day) using round smooth pipes of diameter R centimeters.

Now, how do we do experiments in a laboratory to generate answers for this problem?

Of course, all fluid mechanics students immediately recognize this to be the claim to fame of the famed *Moody Diagram*, which plots the dimensionless pressure drop (per unit pipe length) coefficient against the Reynolds Number with the roughness of the pipe as a parameter. Moody did the experiments in the basement of Green Hall at Princeton Univerisity, using water as his working fluid. Yet his results can be applied to (low speed) flow of any Newtonian fluid (including gases) in long straight pipes anywhere in the universe.

How about the re-entry problem of the nose-cone of a space craft? The issue is to estimate the heat transfer rate during re-entry. There is a diagram somewhere in the NASA library (Professor James Fay of MIT played a role there) that summarized the answers.

Suppose you have a small gas burner trying to boil water in a big pot. You want to boil water faster. Should you get a second small gas burner (doubling the use of fuel), or should you double the fuel flow rate through your present burner? You can give an intelligent answer to this heat transfer problem without using a computer!

Dimensional analysis coupled with informed engineering insights and common sense can go a long way.

2 Review of Dimensional Analysis

In nearly all non-electrical engineering problems, there are three fundamental physical units: mass, length and time. When temperature is a relevant entity, multiplying it with the gas constant immediately converts it into the physical unit of energy per unit mass. So we will stay with the premise that there are only three fundamental units.

The basic theorem (Buckingham's theorem) is: when you have K dimensional variables and 3 fundamental physical units, you can find a minimum of $K - r$ dimensionless numbers, where r is the rank of a certain matrix. Usually, r is 3. Finding r is no big deal. Look into your old fluids book to find how to find r.

Once you know how many dimensionless numbers you are looking for, the next step is to find them. This is the fun part of dimensional analysis. Just remember: a dimensionless number is usually a ratio of two dimensional numbers—both should mean something real and intuitively meaningful to you. You should think about the order magnitude of your candidate dimensionless numbers. For the kind of problems you are interested in studying, some should be "of order unity," some should be very small, and some should be very large.

3 Examples

3.1 Molecular viscosity

Suppose I am interested in μ, the Newtonian viscosity of a fluid. The Newtonian hypothesis is that viscosity is a "material property" of the fluid in question (Look up your favorite fluids book to find out the definition of μ).

The physical unit of μ is mass/length-time. What material property of the fluid can generate this physical unit? Now ρ, the fluid density, is mass/volume. So, one possibility is that μ is proportional to ρ times some velocity v_* times some length ℓ:

$$\mu \propto \rho v_* \ell \tag{2}$$

Now, what material property has the dimension of velocity? For a gas, an intelligent guess is the speed of sound (usually denoted by 'a'). What material property has the dimension of length? The intelligent guess is the mean-free-path λ. So, a good dimensionless parameter Π_1 is:

$$\Pi_1 = \frac{\mu}{\rho a \lambda} \tag{3}$$

If you look into the kinetic theory literature, you will find that elaborate theoretical calculations eventually tells us that Π_1 is a little bit less than 0.5 for simple gases.

3.2 Turbulent "eddy" viscosity

What would you choose for v_* and ℓ for turbulent eddy viscosity?

3.3 Heat conductivity

Now I am interested in the heat conductivity κ. It is also supposed to be a material property. It has a different physical unit than μ—which we now know is a material property. Using the hint that the gas constant times temperature gives energy per unit mass, it is easy to show that the following Π_2 is dimensionless:

$$\Pi_2 = \frac{\mu C_p}{\kappa} \tag{4}$$

where C_p is the specific heat at constant pressure (which has the same physical unit as the gas constant). Experiments and theories show that Π_2 is roughly 0.7-0.8 for most simple gases.

3.4 Bomb explosion

Suppose a powerful bomb suddenly released energy E in a quiescent atmosphere which had uniform density ρ. What can you say about the trajectory of the shock wave after a simple dimensional analysis?

3.5 Other dimensionless numbers

Reynolds number, Mach number, Prandtl number, Peclet number, Schmidt number, Dämkolher number, Froud number, Richardson number, ...

3.6 Homework

1 : Go and look into your favorite fluid mechanics book, and find the disgram of pressure drop coefficient versus Reynolds number. See the difference between laminar and turbulent flows. For a specified volume flow rate, how does the pumping power requirement depend on the pipe radius R for turbulent flows?

2 : Find the diagram of drag coefficient of a sphere as a function of Reynolds number in your favorite text book. See what happens when transition occurs.

3 : It is obvious that an aircraft in flight with no lift will experience a drag which shall be called the zero-lift drag. It is obvious that the actual drag when lift is present should depend on the amount of lift. Can you convince yourself that this lift-induced drag coefficient **cannot** possibly depend on the lift coefficient linearly?

4 : Find the numerical values of viscosity and kinematic viscosity for air and water (under standard conditions).

5 : What speed should a tennis ball travel in air so that the flow Reynolds number is of order unity?

6 : What speed should a baseball travel in air so that the flow is at transition?

7 : What is the order of magnitude of the mean-free-path of air?

8 : What is the mean distance between air molecules in this room (the Avogadro number is 6.022×10^{23}, one mole of gas occupies 22.4 liters under standard condition).

9 : Look at Fig. 1-13 on page 14 of White. This diagram was obviously plotted by someone uneducated in dimensional analysis! After doing all the hard work in getting the experimental and computational data for this air filter system, this diagram is worthless to you if you are in need of information to design a similar but new system (perhaps for a much bigger system which filters dusty methane gas or water)? How would a well educated fluid mechanicist plot the same data (what should be used for the dimensionless ordinate and the dimensionless abscissa)? You are allowed one phone call to the fellow who originally plotted this graph. (I am saying that this fellow forgot to include one crucial (dimensional) number to make his wonderful data more "universally" useful. Shame on him!)