

# Parting Thoughts on the Final

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## Abstract

Instead of an answer sheet, I thought I will leave some parting thoughts on the final exam with you.

I thoroughly enjoyed my ME351B experience, and I thank you all for your courtesy and attendance.

## 1 Fully coupled compressible BL code

I was disappointed that not everybody was able to submit a working code for this problem.

Some obviously had programming problems. But the most notable conceptual “error” was that the Crocco Energy Integral can be used after the first 100 steps. Many students abandoned the fully coupled “raw” equations, and incorporated the Crocco energy integral in their codes.

The important lesson is: just because you have an exact solution (e.g. the Busemann or the Crocco energy integral) to the governing PDE of your problem, it does not mean it can be used for your problem. The initial and boundary conditions must also be consistent with it. In the first 100 steps, the Busemann energy integral is valid, not only because it satisfies the PDE (when  $Pr = 1$ ), but also because the upstream “in-flow” condition and the adiabatic wall condition are satisfied by it. Once we switched the wall to a constant wall temperature boundary condition, then we (apparently) have the Crocco energy integral available, EXCEPT that it cannot be used, because the “initial condition” (at the moment you switched over) the temperature distribution is not compatible with the Crocco energy integral. So it can’t be used—until far, far downstream when the boundary layer has forgotten when the switch was made.

## 2 Implicit versus Explicit algorithms

Consider the simple ODE:

$$\frac{d\phi}{dt} = -\lambda\phi, \quad \phi(0) = C. \quad (1)$$

The analytical solution is obvious:  $\phi = C \exp(-\lambda t)$ .

It is useful to recall the definition of the  $e$ :

$$e \equiv \lim_{\epsilon \rightarrow 0} (1 + \epsilon)^{1/\epsilon}. \quad (2)$$

### 2.1 Explicit method

The finite-difference equation for the explicit method is:

$$\frac{\phi(t + \Delta t) - \phi(t)}{\Delta t} = -\lambda\phi(t), \quad (3)$$

or,

$$\phi(t + \Delta t) = (1 - \lambda\Delta t)\phi(t). \quad (4)$$

So, after  $K$  steps ( $t = K\Delta t$ ), we have:

$$\phi(t) = C(1 - \lambda\Delta t)^K = C(1 - \lambda\Delta t)^{t/\Delta t} = C \left[ (1 - \lambda\Delta t)^{1/(\lambda\Delta t)} \right]^{\lambda t}. \quad (5)$$

Hence, in the limit of  $\lambda\Delta t \rightarrow 0$ , the square bracket goes to  $1/e$ , and this algorithm yields the correct exact solution.

### 2.2 Implicit method

The finite-difference equation for the explicit method is:

$$\frac{\phi(t + \Delta t) - \phi(t)}{\Delta t} = -\lambda\phi(t + \Delta t), \quad (6)$$

or,

$$\phi(t + \Delta t) = \frac{1}{1 + \lambda\Delta t} \phi(t), \quad (7)$$

So, after  $K$  steps ( $t = K\Delta t$ ), we have:

$$\phi(t) = \frac{C}{(1 + \lambda\Delta t)^K} = \frac{C}{(1 + \lambda\Delta t)^{t/\Delta t}} = C \left[ (1 + \lambda\Delta t)^{1/(\lambda\Delta t)} \right]^{-\lambda t}. \quad (8)$$

Again, in the limit of  $\lambda\Delta t \rightarrow 0$ , the square bracket goes to  $e$ , this algorithm yields the correct exact solution.

## 2.3 What happens if $\lambda\Delta t$ is not small?

When  $\lambda\Delta t$  is not small, the square brackets in the above exposition can no longer be related to  $1/e$  or  $e$ . They are some other numbers.

- For the explicit method, when  $\lambda$  is positive and  $\Delta t$  is chosen with  $\lambda\Delta t > 1$ , the computed solution goes haywire. It changes sign at every time step.
- For the implicit method, when  $\lambda$  is positive and  $\Delta t$  is chosen with  $\lambda\Delta t > 1$ , the computed solution “decays.” It does not decay at the right rate, but it goes to zero, just like the exact solution.

Thus arises the reputation of the implicit method—it is very forgiving in taking unwarranted big marching steps. It will go toward the right answer—when  $\lambda$  is positive.

Consider the case when  $\lambda$  is negative. In this case the explicit scheme (when you take an unwarranted large  $\Delta t$ ), the solution blows up, just like it ought to, but at the wrong rate (the square bracket does not equal to  $e$ ). The implicit scheme now switches sign at every time step, and goes generally haywire. Who is more forgiving now?

So, be careful in singing the praises of the implicit scheme.

## 2.4 Watching the blow-up of a linearly unstable problem

We now consider the problem in the final exam:

$$\frac{\partial\phi}{\partial t} = \frac{\sin(\phi)}{\tau} + \mu\frac{\partial^2\phi}{\partial x^2} \quad (9)$$

where  $\tau$  and  $\mu$  are both positive. The boundary conditions are:

$$\phi(0, t) = \phi(L, t) = 0. \quad (10)$$

The initial condition is:

$$\phi(x, 0) = \epsilon f(x) \quad (11)$$

where  $|\epsilon| \ll 1$  but the sign of  $\epsilon$  is unknown. Now what happens to this tiny, weenie disturbance of uncertain sign?

We linearize about  $\phi_{ss} = 0$ , and study the linear stability problem using separation of variables. Nearly everybody got the stability

criterion: the teenie weenie initial disturbance will die off, regardless of the sign of  $\epsilon$ , when  $\tau$  honors this inequality:

$$\tau > \frac{L^2}{\mu\pi^2}. \quad (12)$$

Now what happens when  $\tau$  is smaller than that?

The answer is, of course, the teenie weenie initial disturbance—being linearly unstable, is going to grow, or blow up. That is the mathematics of the situation. What happens to the growing solution? Well, it will be “caught” and become stabilized at another steady-state solution! Lets find it! Setting the time derivative to zero, we have:

$$\mu \frac{\partial^2 \phi_{ss}}{\partial x^2} + \frac{\sin(\phi_{ss})}{\tau} = 0. \quad (13)$$

Introducing a new independent variable  $\eta$  by:

$$\eta = \frac{x}{\sqrt{\mu\tau}}. \quad (14)$$

The ODE becomes:

$$\frac{\partial^2 \phi_{ss}}{\partial \eta^2} + \sin(\phi_{ss}) = 0 \quad (15)$$

which can be integrated once to yield:

$$\frac{d\phi_{ss}}{d\eta} = \pm \sqrt{2 \cos(\phi_{ss}) - c} \quad (16)$$

where  $c$  is an integration constant—to be determined by the boundary conditions. For the sake of simplicity of presentation, let us assume the system is violently unstable:

$$\tau \ll \frac{L^2}{\mu\pi^2}. \quad (17)$$

It is easy to show that for this limiting case  $c \rightarrow 0$ .<sup>1</sup> It is a trivial matter to integrate once more to get  $\phi_{ss}(\eta)$ . Note that we have the choice of the plus or the minus signs. So there are two steady-state solutions staring back at you: a positive one, and a negative one (note that you have to switch sign whenever the square-root touches zero).

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<sup>1</sup>The solution for this case is made up of two “boundary layers” (at  $x \approx 0$  and  $x \approx L$  each with thickness  $O(\sqrt{\mu\tau})$  separated by a middle flat region with  $\phi_{ss} = \pi/2$ .

Very messy stuff. It should be clear that, depending on the sign of  $\epsilon$ , the disturbance could get caught by either the positive or the negative  $\phi_{ss}$  derived above.

But we are the modern engineers, the Matlab kind. We want to do this thing numerically, and make a color movie of the mess. Now the choice of the computational time step is the one being addressed. What do we do?

Our analysis shows that the implicit scheme no longer has the assurance that it is the more tolerant algorithm when you take an unwarranted large  $\Delta t$ . You want to see the mess (and to figure out which steady-state solution your particular mess settles on eventually), you pick a small  $\Delta t/\tau$ .

### 3 Dimensional/dimensionless answers

There is no question that dimensionless formulas and graphs are the educated way to present data and answers. But from the engineering application point of view, dimensional answers are usually more useful. If you increase your characteristic velocity, your Reynolds number goes up, and your dimensionless friction coefficient goes down. But your dimensional frictional force goes up. One must be able to do the conversions.

What happens if the characteristic velocity is quadrupled? The first issue that needs to be resolved is: what dimensionless parameters are affected? Obviously, the Reynolds Number  $R_e$ . The other obvious one is the Mach Number. So the question can be addressed for two distinct cases: low Mach Numbers (can be represented by constant properties models), and significant Mach Number compressible flows.

In the low Mach Number case, the question is on the influence of (dimensionless) Reynolds Number.

In the finite Mach Number case, the new Mach Number is now quadrupled. The most significant impact is the increase of the recovery temperature. The heat transfer for a fixed wall temperature problem *can change sign* as a consequence of this! (e.g. old wall temperature was below old recovery temperature, and the new situation is reversed). What are the other major impacts? Well, they depend on the viscosity-temperature law of the gas. For example, if viscosity is approximately linearly proportional to temperature, the impacts on all the dimensionless variables are small.