ME 351B – Answers to Homework #2

Problem 1: (Ed Knudsen)

1) Prove identities (12) - (15) & (22) in the notes.

(12): $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$

$(B \times C)_i = \varepsilon_{ijk} B_j C_k \qquad (A \times (B \times C))_l = \varepsilon_{lmi} A_m (\varepsilon_{ijk} B_j C_k)$

$(A \times (B \times C))_l = \varepsilon_{ilm} \varepsilon_{ijk} A_m B_j C_k$

Use equation (22): $\varepsilon_{ilm} \varepsilon_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$

$(A \times (B \times C))_l = \delta_{lj}\delta_{mk} A_m B_j C_k - \delta_{lk}\delta_{mj} A_m B_j C_k = A_m C_m B_l - A_m B_m C_l$

Since m is a dummy index, $A_m C_m = A \cdot C$ and $A_m B_m = A \cdot B$

$\Longrightarrow \boxed{A \times B \times C = (A \cdot C) B - (A \cdot B) C}$ ✓

(13): $\nabla \times (B \times C) = C \cdot \nabla B - B \cdot \nabla C + B(\nabla \cdot C) - C(\nabla \cdot B)$

$(\nabla \times (B \times C))_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} B_l C_m) = \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} (B_l C_m)$

Use (22): $= \delta_{il}\delta_{jm} \frac{\partial}{\partial x_j}(B_l C_m) - \delta_{im}\delta_{jl} \frac{\partial}{\partial x_j}(B_l C_m)$

$= \frac{\partial}{\partial x_j}(B_i C_j) - \frac{\partial}{\partial x_j}(B_j C_i)$

$= B_i \frac{\partial C_j}{\partial x_j} + C_j \frac{\partial B_i}{\partial x_j} - C_i \frac{\partial B_j}{\partial x_j} - B_j \frac{\partial C_i}{\partial x_j}$

By definition, $\nabla \cdot A = \frac{\partial A_j}{\partial x_j}$. Also, define $(\nabla A)_i = \frac{\partial A_i}{\partial x_j}$

$\Longrightarrow \boxed{\nabla \times (B \times C) = B(\nabla \cdot C) + C \cdot \nabla B - C(\nabla \cdot B) - B \cdot \nabla C}$ ✓

(14): $A \times (\nabla \times A) = \nabla \frac{A^2}{2} - A \cdot \nabla A$

$$(A \times (\nabla \times A))_i = \varepsilon_{ijk} A_j \left(\varepsilon_{klm} \frac{\partial A_m}{\partial x_l} \right) = \varepsilon_{kij} \varepsilon_{klm} A_j \frac{\partial A_m}{\partial x_l}$$

Use (22): $$= \delta_{il} \delta_{jm} A_j \frac{\partial A_m}{\partial x_l} - \delta_{im} \delta_{jl} A_j \frac{\partial A_m}{\partial x_l}$$

$$= A_j \frac{\partial A_j}{\partial x_i} - A_j \frac{\partial A_i}{\partial x_j}$$

Using above definition, $A_j \frac{\partial A_i}{\partial x_j} = A \cdot \nabla A$

For $A_j \frac{\partial A_j}{\partial x_i}$ term, consider $\nabla \frac{A^2}{2} = \nabla \frac{A \cdot A}{2} = \frac{1}{2} \nabla A_j A_j$

$$\frac{\nabla A^2}{2} = \frac{1}{2} \frac{\partial}{\partial x_i} (A_j A_j) = \frac{1}{2} \left[A_j \frac{\partial A_j}{\partial x_i} + A_j \frac{\partial A_j}{\partial x_i} \right] = \frac{1}{2} \left(2 A_j \frac{\partial A_j}{\partial x_i} \right) = A_j \frac{\partial A_j}{\partial x_i}$$

$$\Longrightarrow \boxed{A \times (\nabla \times A) = \frac{1}{2} \nabla A^2 - A \cdot \nabla A} \quad \checkmark$$

(15): $\nabla \times (\nabla \times C) = \nabla (\nabla \cdot C) - \nabla^2 C$

$$(\nabla \times (\nabla \times C))_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} \frac{\partial}{\partial x_l} C_m \right) = \varepsilon_{kij} \varepsilon_{klm} \frac{\partial^2 C_m}{\partial x_j \partial x_l}$$

Use (22): $$= \delta_{il} \delta_{jm} \frac{\partial^2 C_m}{\partial x_j \partial x_l} - \delta_{im} \delta_{jl} \frac{\partial^2 C_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial^2 C_j}{\partial x_j \partial x_i} - \frac{\partial^2 C_i}{\partial x_j^2} = \frac{\partial}{\partial x_i} \left(\frac{\partial C_j}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial C_i}{\partial x_j} \right)$$

Using $\nabla \cdot A = \frac{\partial A_j}{\partial x_j}$ and $\nabla \phi = \frac{\partial \phi}{\partial x_i}$,

$$\boxed{\nabla \times (\nabla \times C) = \nabla (\nabla \cdot C) - \nabla \cdot (\nabla C) = \nabla (\nabla \cdot C) - \nabla^2 C} \quad \checkmark$$

A few remarks on the use of indicial notation:

- Remember that an index present only once is a free index, and can take on any value from 1 to 3, e.g. V_i means $V_i,\ i = 1, 2, 3$. An index repeated twice is implicitly summed over: $A_j B_j$ by convention means $\sum_{j=1}^{3} A_j B_j$. An index should never be repeated more than twice: expression such as $a_k b_k c_k$ or $u_k \frac{\partial v_k}{\partial x_k}$ make no sense: if you end up with terms containing three or more same indices, check your work: you must have a mistake.
- Avoid writing expressions such as A_i^2 or $\frac{\partial^2 \phi}{\partial x_j^2}$. Even though it may be clear to you what you mean, they are confusing because the index which should be summed over appears only once. Instead, write $A_i A_i$ and $\frac{\partial^2 \phi}{\partial x_j \partial x_j}$ for which there is no ambiguity.
- When you add up terms written in index notation, they should all have the same free indices: $a_i + b_k$, $\frac{\partial \phi}{\partial x_m} + u_j v_j w_l$ make no sense. Correct expressions would be $a_k + b_k$ and $\frac{\partial \phi}{\partial x_m} + u_j v_j w_m$ for instance.

(22): $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

Prove by a few examples:

if $j=k$, $\varepsilon_{ijk}=0$ $\therefore$ $\delta_{jm}\delta_{jn} - \delta_{jn}\delta_{jm} = 0$ ✓
(same if $m=n$)

if $i=j$, $\varepsilon_{ijk}=0$ $\therefore$ (let $i=j=1$, $k=2$, $m=2$, $n=3$)
(same if $i=k$, $i=m$, $i=n$)
$\delta_{12}\delta_{13} - \delta_{13}\delta_{12} = 0$ (no double indexes)

if $i=1$, $j=m=2$, $k=n=3$, $\varepsilon_{ijk}\varepsilon_{imn}=1$ $\therefore$ $\delta_{22}\delta_{33} - \delta_{23}\delta_{32} = 1$
(same if $i=2$, $j=m=3$, $k=n=1$
same if $i=3$, $j=m=1$, $k=n=2$)

if $i=1$, $j=m=3$, $k=n=2$, $\varepsilon_{ijk}\varepsilon_{imn}=1$ $\therefore$ $\delta_{33}\delta_{22} - \delta_{32}\delta_{23} = 1$
(same if $i=2$, $j=m=1$, $k=n=3$
" " $i=3$, $j=m=2$, $k=n=1$)

if $i=1$, $j=n=2$, $k=m=3$, $\varepsilon_{ijk}\varepsilon_{imn}=-1$ $\therefore$ $\delta_{23}\delta_{32} - \delta_{22}\delta_{33} = -1$
(same if $i=2$, $j=n=3$, $k=m=1$
" " $i=3$, $j=n=1$, $k=m=2$)

These examples prove the identity holds.

More on the proof of $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$**:**

It is possible, as done above, to show that the relation is true just by looking at the different values that j, k, m and n can take. However, a more rigorous proof exists based on some knowledge of tensor analysis:

$\epsilon_{ijk}\epsilon_{imn}$ is an example of a fourth-order tensor (indeed, it has four free indices since i is summed over). It is also isotropic, i.e. it has no preferred direction in space (another way of saying this is that its representation in a basis is the same whichever orthonormal basis you choose). It can be shown that the most general form for an isotropic fourth order tensor is $\lambda\delta_{jk}\delta_{mn} + \lambda'\delta_{jm}\delta_{kn} + \lambda''\delta_{jn}\delta_{km}$ where the three constants are to be determined. Therefore:

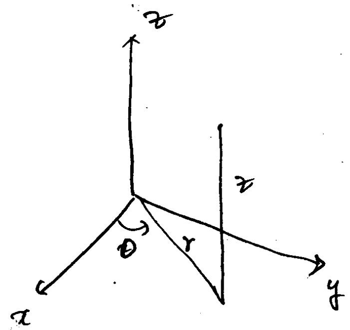
$$\epsilon_{ijk}\epsilon_{imn} = \lambda\delta_{jk}\delta_{mn} + \lambda'\delta_{jm}\delta_{kn} + \lambda''\delta_{jn}\delta_{km}$$

and all we have to do is find the values of λ, λ' and λ''. This easily done by plugging values for j, k, m and n. For instance:

- doing $j=k$ and $m=n$ leads to $\lambda = 0$.
- $j=m=1$ and $k=n=2$ gives $\lambda' = 1$.
- $j=n=1$ and $k=m=2$ gives $\lambda'' = -1$.

Problem 2: (Joo Hyun Lee)

(h_1, h_2, h_3) in cylindrical polar (r, θ, z)



$$(dl)^2 = (h_1\, d\xi_1)^2 + (h_2\, d\xi_2)^2 + (h_3\, d\xi_3)^2$$

$h_1 = 1 \qquad \xi_1 = r$

$h_2 = r \qquad \xi_2 = \theta$

$h_3 = 1 \qquad \xi_3 = z$

$h_1 = 1, \quad h_2 = r, \quad h_3 = 1$ ✓

Problem 3: (Joo Hyun Lee)

$\nabla^2\phi = ?$

$$\nabla^2\phi = \nabla\cdot\nabla\phi$$

$$\nabla\phi = e_1\frac{\partial\phi}{h_1\partial\xi_1} + e_2\frac{\partial\phi}{h_2\partial\xi_2} + e_3\frac{\partial\phi}{h_3\partial\xi_3} = e_r\frac{\partial\phi}{\partial r} + e_\theta\frac{\partial\phi}{r\,\partial\theta} + e_z\frac{\partial\phi}{\partial z}$$

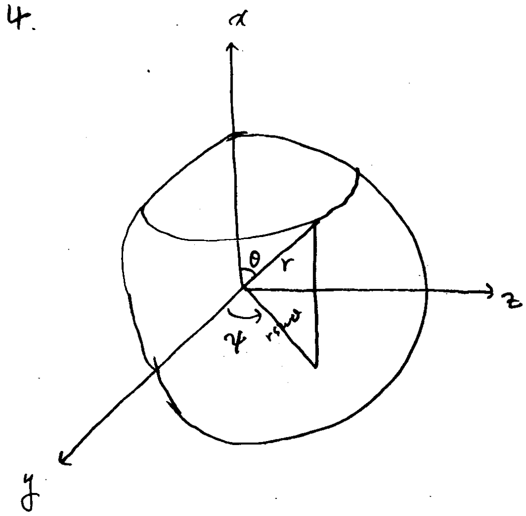
$$\nabla\cdot\nabla\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial h_2h_3(\nabla\phi)_1}{\partial\xi_1} + \frac{\partial h_3h_1(\nabla\phi)_2}{\partial\xi_2} + \frac{\partial h_1h_2(\nabla\phi)_3}{\partial\xi_3}\right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{r\,\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\,\frac{\partial\phi}{\partial z}\right)\right]$$

$$= \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$ ✓

Problem 4: **(Joo Hyun Lee)**

4.



$$(dl)^2 = (h_1 d\xi_1)^2 + (h_2 d\xi_2)^2 + (h_3 d\xi_3)^2$$

$h_1 = 1$ $\quad \xi_1 = r$

$h_2 = r$ $\quad \xi_2 = \theta$

$h_3 = r\sin\theta$ $\quad \xi_3 = \psi$

$h_1 = 1, \quad h_2 = r \quad h_3 = r\sin\theta$ ✓

Some students used a different convention for spherical coordinates (they used the inverse definition for θ and ψ). This is fine as long as you are consistent throughout. Note that the most usual definition is the one adopted above, where $\theta \in [0, \pi]$, and $\psi \in [0, 2\pi)$.

Problem 5: **(Joo Hyun Lee)**

5. $\nabla^2\phi = \nabla \cdot \nabla\phi$

$$\nabla\phi = e_1 \frac{\partial\phi}{h_1 \partial\xi_1} + e_2 \frac{\partial\phi}{h_2 \partial\xi_2} + e_3 \frac{\partial\phi}{h_3 \partial\xi_3} = e_r \frac{\partial\phi}{\partial r} + e_\theta \frac{\partial\phi}{r\,\partial\theta} + e_\psi \frac{\partial\phi}{r\sin\theta\,\partial\psi}$$

$$\nabla\cdot\nabla\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial h_2h_3(\nabla\phi)_1}{\partial\xi_1} + \frac{\partial h_3h_1(\nabla\phi)_2}{\partial\xi_2} + \frac{\partial h_1h_2(\nabla\phi)_3}{\partial\xi_3}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta \cdot \frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\cancel{r}\sin\theta \cdot \frac{\partial\phi}{\cancel{r}\,\partial\theta}\right) + \frac{\partial}{\partial\psi}\left(\cancel{r}\, \frac{\partial\phi}{\cancel{r}\sin\theta\,\partial\psi}\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\phi}{\partial\psi^2}$$ ✓

Problem 6: (Yaser Khalighi)

$$\vec{A}\cdot\nabla\vec{A} = \underbrace{\nabla\left(\frac{A^2}{2}\right)}_{(*)} - \underbrace{\vec{A}\times(\nabla\times\vec{A})}_{(**)} \qquad \vec{A} = (A_r, A_\theta, A_z)$$

$$(*)\quad \nabla\left(\frac{A^2}{2}\right) = \frac{1}{2}\nabla(A^2) = \frac{1}{2}\left[\vec{e}_r\frac{\partial}{\partial r}(A^2) + \vec{e}_\theta\frac{\partial}{r\partial\theta}(A^2) + \vec{e}_z\frac{\partial}{\partial z}(A^2)\right]$$

$$= \frac{1}{2}\Big[\vec{e}_r\left(2A_r\frac{\partial A_r}{\partial r} + 2A_\theta\frac{\partial A_\theta}{\partial r} + 2A_z\frac{\partial A_z}{\partial r}\right)$$

$$\frac{\vec{e}_\theta}{r}\left(2A_r\frac{\partial A_r}{\partial\theta} + 2A_\theta\frac{\partial A_\theta}{\partial\theta} + 2A_z\frac{\partial A_z}{\partial\theta}\right)$$

$$+\vec{e}_z(\text{———})\Big]$$

$$(**)\quad \nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & \vec{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix} = \frac{1}{r}\Big[\vec{e}_r\left(\frac{\partial}{\partial\theta}A_z - \frac{\partial}{\partial z}(rA_\theta)\right)$$

$$r\vec{e}_\theta\left(\frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z\right)$$

$$\vec{e}_z\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right)\Big]$$

$$= \left(\frac{\partial}{r\partial\theta}A_z - \frac{\partial}{\partial z}A_\theta\right)\vec{e}_r + \left(\frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z\right)\vec{e}_\theta + \left(\frac{\partial}{\partial r}A_\theta + \frac{A_\theta}{r} - \frac{\partial A_r}{r\partial\theta}\right)\vec{e}_z$$

$$\left(\vec{A}\times(\nabla\times\vec{A})\right)_r = \left[A_\theta\left(\frac{\partial}{\partial r}A_\theta + \frac{A_\theta}{r} - \frac{\partial A_r}{r\partial\theta}\right) - A_z\left(\frac{\partial}{\partial z}A_r - \frac{\partial}{\partial r}A_z\right)\right]$$

$$\left(\vec{A}\times(\nabla\times\vec{A})\right)_\theta = \left[A_z\left(\frac{\partial}{r\partial\theta}A_z - \frac{\partial}{\partial z}A_\theta\right) - A_r\left(\frac{\partial}{\partial r}A_\theta + \frac{A_\theta}{r} - \frac{\partial A_r}{r\partial\theta}\right)\right]$$

$$\Rightarrow (\vec{A}\cdot\nabla\vec{A})_r = A_r\frac{\partial A_r}{\partial r} + \cancel{A_\theta\frac{\partial A_\theta}{\partial r}} + \cancel{A_z\frac{\partial A_z}{\partial r}} - \cancel{A_\theta\frac{\partial A_\theta}{\partial r}} - \frac{A_\theta^2}{r} + A_\theta\frac{\partial A_r}{r\partial\theta}$$

$$+ A_z\frac{\partial A_r}{\partial z} - \cancel{A_z\frac{\partial}{\partial r}A_z} = A_r\frac{\partial A_r}{\partial r} - \frac{A_\theta^2}{r} + A_\theta\frac{\partial A_r}{r\partial\theta} + A_z\frac{\partial A_r}{\partial z} \checkmark$$

$$(\vec{A}\cdot\nabla\vec{A})_\theta = \cancel{\frac{A_r}{r}\frac{\partial A_r}{\partial\theta}} + \frac{A_\theta}{r}\frac{\partial A_\theta}{\partial\theta} + \cancel{\frac{A_z}{r}\frac{\partial A_z}{\partial\theta}} - \cancel{\frac{A_z}{r}\frac{\partial A_z}{\partial\theta}} + A_z\frac{\partial A_\theta}{\partial z} + A_r\frac{\partial A_\theta}{\partial r} + \frac{A_rA_\theta}{r} - \cancel{A_r\frac{\partial A_r}{r\partial\theta}}$$

$$= \frac{A_\theta}{r}\frac{\partial A_\theta}{\partial\theta} + A_z\frac{\partial A_\theta}{\partial z} + A_r\frac{\partial A_\theta}{\partial r} + \frac{A_rA_\theta}{r} \checkmark$$

5

Problem 7: (Tarun Khurana)

A) $\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\nabla \cdot A) - \nabla^2 A$

$\therefore \nabla^2 A = -\nabla \times (\nabla \times A) + \nabla(\nabla \cdot \bar{A})$

$$\nabla \cdot \bar{A} = \frac{1}{R} \cdot \frac{\partial}{\partial R}(R A_r) + \frac{1}{R} \cdot \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla(\nabla \cdot \bar{A}) = \frac{\partial}{\partial R}[\nabla \cdot \bar{A}] \cdot \hat{e}_r + \frac{1}{R} \cdot \frac{\partial}{\partial \theta}[\nabla \cdot \bar{A}]\, \hat{e}_\theta + \frac{\partial}{\partial z}[\nabla \cdot \bar{A}]\, \hat{e}_z$$

$$(\nabla \times \bar{A}) = \left(\frac{1}{R} \cdot \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left[\frac{1}{R}\frac{\partial (R A_\theta)}{\partial R} - \frac{1}{R} \cdot \frac{\partial A_r}{\partial \theta}\right] \cdot \hat{e}_z$$

$$\bar{\nabla} \times (\nabla \times \bar{A}) = \frac{1}{R}\begin{bmatrix} \hat{e}_r & R\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{R}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} & R\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial R}\right) & \frac{1}{R}\frac{\partial}{\partial R}(R A_\theta) - \frac{1}{R}\frac{\partial A_r}{\partial \theta} \end{bmatrix}$$

e_θ component of $\nabla^2 A$

$$= \frac{1}{R}\frac{\partial}{\partial \theta}[\nabla \cdot \bar{A}] - \left[\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}(R A_\theta) - \frac{1}{R}\frac{\partial A_r}{\partial \theta}\right) - \frac{\partial}{\partial z}\left(\frac{1}{R}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\right]$$

$$= \frac{1}{R}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{R}\frac{\partial}{\partial R}(R A_r)\right) + \frac{1}{R} \cdot \frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial \theta \partial z}\right]$$
$$+ \left[\frac{\partial}{\partial R}\left(\frac{A_\theta}{R} + \frac{\partial A_\theta}{\partial R} - \frac{1}{R} \cdot \frac{\partial A_r}{\partial \theta}\right) - \frac{\partial^2 A_z}{\partial z\, \partial \theta} \times \frac{1}{R} + \frac{\partial^2 A_\theta}{\partial z^2}\right]$$

$$= \frac{1}{R}\left[\frac{\partial^2 A_r}{\partial R \partial \theta} + \frac{1}{R}\frac{\partial A_r}{\partial \theta} + \frac{1}{R}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial \theta \partial z}\right] - \frac{1}{R} \cdot \frac{\partial^2 A_z}{\partial z \partial \theta} + \frac{\partial^2 A}{\partial z^2}$$
$$+ \left[-\frac{A_\theta}{R^2} + \frac{1}{R}\frac{\partial A_\theta}{\partial R} + \frac{\partial^2 A_\theta}{\partial R^2} + \frac{1}{R^2} \cdot \frac{\partial A_r}{\partial \theta} - \frac{1}{R}\frac{\partial^2 A_r}{\partial R \partial \theta}\right]$$

$$= \frac{1}{R^2}\frac{\partial A_r}{\partial \theta} + \frac{1}{R^2}\cdot\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_\theta}{\partial z^2} - \frac{A_\theta}{R^2} + \frac{1}{R}\frac{\partial A_\theta}{\partial R} + \frac{\partial^2 A_\theta}{\partial R^2} + \frac{1}{R^2}\frac{\partial A_r}{\partial \theta}$$

$$= \frac{2}{R^2}\frac{\partial A_r}{\partial \theta} + \frac{1}{R^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_\theta}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial A_\theta}{\partial R}\right) - \frac{A_\theta}{R^2}$$

$$= \nabla^2 A_\theta + \frac{2}{R^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{R^2}$$

$$= \nabla^2 A_\theta + \frac{2}{R^2}\cdot\frac{\partial A_R}{\partial \theta} - \frac{A_\theta}{R^2} = \hat{e}_\theta \text{ component of } \nabla^2 \mathbf{A}.$$

A few remarks on problems 6 and 7:

The $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ basis in cylindrical coordinates (and similarly the $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\psi)$ basis in spherical coordinates) is just another orthonormal basis. Its only peculiarity is that the orientation of the unit vectors changes with position, and therefore this must be taken into account when calculating differential operators such as the gradient, divergence, curl, laplacian etc. This is what equations (24–26) in the handout are all about: they save you the trouble of taking the derivatives yourself and accounting for the change in orientation of the unit vectors. However, all the other operations on vectors that do not involve spatial derivatives (norm, dot product, cross-product...) are performed in exactly the same way as in a Cartesian basis. In particular:

$$A^2 = A_r^2 + A_\theta^2 + A_z^2$$

$$\mathbf{A}\cdot\mathbf{B} = A_r B_r + A_\theta B_\theta + A_z B_z$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$$

This is where you must understand the difference between the two symbols $\times$ in $\mathbf{A}\times\mathbf{B}$ and $\nabla\times\mathbf{A}$. In the first case it denotes an operation between two vectors; in the second case, $\nabla\times$ is a differential operator acting on a vector, but is not a cross product between ∇ and $\mathbf{A}$. The similarity of the notations is convenient in Cartesian coordinate systems for quite obvious reasons, but one must be cautious when using curvilinear coordinate systems.

Problem 6 could also be solved directly by writing $\nabla\mathbf{A} = \nabla(A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z)$,using Equation 24 for the gradient and recognizing that $\partial\mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$ and $\partial\mathbf{e}_\theta/\partial\theta = -\mathbf{e}_r$ (which you can show by writing $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in a Cartesian basis and taking the derivative in this basis). If you use this approach, you must pay attention to the order of the pairs of unit vectors that you obtain in the end (see Problem 1 of Homework 3 for an application of this direct approach).

Problem 8: (Wendy Ong)

Q8. Given a volume in a fluid flow field:

Mass : $m_{CV} = \iiint_{CV} \rho(\vec{x},t)\, dV$

momentum : $P_{CV} = \iiint_{CV} \rho\, V(\vec{x},t)\, dV$

Energy : $E_{CV} = \iiint_{CV} \rho\, e(\vec{x},t)\, dV$

Kinetic Energy : $KE_{CV} = \iiint_{CV} \frac{1}{2}\rho\, \vec{V}\cdot\vec{V}(\vec{x},t)\, dV$

pollutant : $\iiint_{CV} W(\vec{x},t)\, dV$

Gold fish : $G_{CV} = \iiint_{CV} G(\vec{x},t)\, dV$ ✓

Problem 9:

The intention of the problem was for students to recognize the physical meaning of the substantial derivative. What is the substantial derivative of a variable x in English? It is the rate of change of x with respect to time **following** a material parcel. So what is $\mathrm{D}\rho/\mathrm{D}t$ in English? It is the rate of change of density with respect to time following a material parcel. The densities of water and meat balls are not expected to change with respect to time when they flow down the river. So the $\mathrm{D}\rho/\mathrm{D}t$ terms can be dropped of the continuity equation to yield simply $\nabla\cdot\mathbf{V} = 0$. Density discontinuities introduced no conceptual problem. We do not expect velocity discontinuities at the interfaces because liquid soup sticks to the meat ball surface via the no-slip condition.