

ME 351B - Answers to Homework #3

Problem 1: (Ethan Barbour)

Find $\epsilon_{r\theta}$ in cylindrical coordinates.

$$\nabla \bar{V} = \bar{e}_{\xi_1} \frac{\partial \bar{V}}{h_1 \partial \xi_1} + \bar{e}_{\xi_2} \frac{\partial \bar{V}}{h_2 \partial \xi_2} + \bar{e}_{\xi_3} \frac{\partial \bar{V}}{h_3 \partial \xi_3}$$

$$h_1 = 1 \quad h_2 = r \quad h_3 = 1$$

$$\xi_1 = r \quad \xi_2 = \theta \quad \xi_3 = z$$

and $\bar{V} = V_r \bar{e}_r + V_\theta \bar{e}_\theta + V_z \bar{e}_z$

$$\begin{aligned} \therefore \nabla \bar{V} &= \bar{e}_r \frac{\partial}{\partial r} (V_r \bar{e}_r + V_\theta \bar{e}_\theta + V_z \bar{e}_z) \\ &+ \bar{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (V_r \bar{e}_r + V_\theta \bar{e}_\theta + V_z \bar{e}_z) \\ &+ \bar{e}_z \frac{\partial}{\partial z} (V_r \bar{e}_r + V_\theta \bar{e}_\theta + V_z \bar{e}_z) \\ &= \bar{e}_r \left(\bar{e}_r \frac{\partial V_r}{\partial r} + V_r \frac{\partial \bar{e}_r}{\partial r} + \bar{e}_\theta \frac{\partial V_\theta}{\partial r} + V_\theta \frac{\partial \bar{e}_\theta}{\partial r} + \bar{e}_z \frac{\partial V_z}{\partial r} + V_z \frac{\partial \bar{e}_z}{\partial r} \right) \\ &+ \frac{1}{r} \bar{e}_\theta \left(\bar{e}_r \frac{\partial V_r}{\partial \theta} + V_r \frac{\partial \bar{e}_r}{\partial \theta} + \bar{e}_\theta \frac{\partial V_\theta}{\partial \theta} + V_\theta \frac{\partial \bar{e}_\theta}{\partial \theta} + \bar{e}_z \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial \bar{e}_z}{\partial \theta} \right) \\ &+ \bar{e}_z \left(\bar{e}_r \frac{\partial V_r}{\partial z} + V_r \frac{\partial \bar{e}_r}{\partial z} + \bar{e}_\theta \frac{\partial V_\theta}{\partial z} + V_\theta \frac{\partial \bar{e}_\theta}{\partial z} + \bar{e}_z \frac{\partial V_z}{\partial z} + V_z \frac{\partial \bar{e}_z}{\partial z} \right) \\ &= \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_z}{\partial r} \\ \frac{1}{r} \left(\frac{\partial V_r}{\partial \theta} - V_\theta \right) & \frac{1}{r} \left(V_r + \frac{\partial V_\theta}{\partial \theta} \right) & \frac{1}{r} \frac{\partial V_z}{\partial \theta} \\ \frac{\partial V_r}{\partial z} & \frac{\partial V_\theta}{\partial z} & \frac{\partial V_z}{\partial z} \end{bmatrix} \end{aligned}$$

$$\therefore \epsilon_{r\theta} = (\nabla \bar{V} + (\nabla \bar{V})^T)_{r\theta}$$

$$\boxed{\epsilon_{r\theta} = \frac{\partial V_\theta}{\partial r} + \frac{1}{r} \left(\frac{\partial V_r}{\partial \theta} - V_\theta \right)}$$

Recall that in curvilinear coordinate systems the basis unit vectors can change orientation and therefore can have non-zero derivatives. In cylindrical polar coordinates, remember the relations:

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$$

Problem 2: (Ethan Barbour)

Find $\bar{\bar{\mathbf{I}}} : \bar{\bar{\mathbf{I}}}$

$$\begin{aligned} \bar{\bar{\mathbf{I}}} : \bar{\bar{\mathbf{I}}} &= \sum_{i,j=1}^3 \mathbf{I}_{ij} \mathbf{I}_{ij} \\ &= \sum_{i,j=1}^3 \delta_{ij} \delta_{ij} \\ &= 1 + 1 + 1 \end{aligned}$$

$$\boxed{\bar{\bar{\mathbf{I}}} : \bar{\bar{\mathbf{I}}} = 3}$$

Gibbs notation ($\bar{\bar{\mathbf{I}}} : \bar{\bar{\mathbf{I}}}$) is very elegant but its meaning is not unambiguous unless the proximity rule is agreed upon. Under the proximity rule, the definition of the double inner product between two tensors is:

$$\bar{\bar{\mathbf{A}}} : \bar{\bar{\mathbf{B}}} = A_{ij} B_{ji}$$

For non-symmetric tensors, this differs from $A_{ij} B_{ij}$, which in Gibbs notation would be written $\bar{\bar{\mathbf{A}}} : \bar{\bar{\mathbf{B}}}^T$.

Problem 3: (Beverly Tang)

Incompressible: \bar{V} & p

Equations: $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{V} = 0$ (mass conservation)

$\rho \frac{D\bar{V}}{Dt} = -\nabla p + \nabla \cdot \bar{E} + \rho \bar{F}$ (momentum)

Compressible: \bar{V} , ρ , T , p , e

Equations: $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{V} = 0$ (mass)

$\rho \frac{D\bar{V}}{Dt} = -\nabla p + \nabla \cdot \bar{E} + \rho \bar{F}$ (momentum)
 $+ \underline{E} \cdot \underline{V} + \nabla \cdot (-p\underline{V} + \bar{E} \cdot \underline{V})$

$\rho \frac{D}{Dt} \left(\frac{e}{\rho} + \frac{1}{2} V^2 \right) = \nabla \cdot \kappa \nabla T$ (energy)

$p = \rho R T$ (some equation of state)

$e = e(\rho, T)$ (some relationship for internal energy - can be found in a table)

this is the incompressible case. $\left\{ \begin{array}{l} \text{assume } p \text{ is constant in } x \text{ \& } t \\ \text{if } \mu \text{ is constant, then the energy equation} \\ \text{is not needed} \end{array} \right.$

If the flow is incompressible, the momentum equation is uncoupled from the energy equation and can be solved first without any thermodynamical considerations. Once the flow field is known, the temperature field can be solved for from the energy equation.

However in the compressible case momentum and energy equations are coupled and must be solved simultaneously. An additional equation of state (typically $p = p(\rho, T)$) is needed.

Problem 4: (Ethan Barbour)

Gibbs equation:

$$\begin{aligned} T ds &= de + p d\left(\frac{1}{\rho}\right) \\ &= de - \frac{p}{\rho^2} d\rho \end{aligned}$$

Following a particle:

$$T \frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (1)$$

$$\text{Continuity: } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \bar{u} + \bar{u} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{u} = 0 \quad (2)$$

"Counterfeit" energy (i.e. momentum dotted with \bar{u}):

$$\rho \frac{D(u^2/2)}{Dt} = -\bar{u} \cdot \nabla p + (\nabla \cdot \bar{E}) \cdot \bar{u} + \bar{b} \cdot \bar{u} \quad (3)$$

$$\text{Energy: } \rho \frac{D}{Dt} \left(e + \frac{u^2}{2} \right) = -\nabla \cdot (\rho \bar{u}) + \bar{b} \cdot \bar{u} + \nabla \cdot (\bar{E} \cdot \bar{u}) + \nabla \cdot (k \nabla T) \quad (4)$$

$$(4) - (3): \rho \frac{De}{Dt} = -\nabla \cdot (\rho \bar{u}) + \nabla \cdot (\bar{E} \cdot \bar{u}) + \nabla \cdot (k \nabla T) + \bar{u} \cdot \nabla p - (\nabla \cdot \bar{E}) \cdot \bar{u}$$

$$\rho \frac{De}{Dt} = -\frac{\partial}{\partial x_i} (\rho u_i) + \frac{\partial}{\partial x_i} (E_{ij} u_j) + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + u_i \frac{\partial p}{\partial x_i} - \left(\frac{\partial E_{ij}}{\partial x_i} \right) u_j$$

$$= -u_i \frac{\partial \rho}{\partial x_i} - \rho \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial E_{ij}}{\partial x_i} + E_{ij} \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + u_i \frac{\partial p}{\partial x_i} - u_j \frac{\partial E_{ij}}{\partial x_i}$$

$$= -\rho \frac{\partial u_i}{\partial x_i} + E_{ij} \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) \quad (5)$$

Sub (5) & (2) into (1):

$$T \frac{Ds}{Dt} = \frac{1}{\rho} \left(-\rho \frac{\partial u_i}{\partial x_i} + E_{ij} \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) \right) - \frac{p}{\rho^2} \left(-\rho \nabla \cdot \bar{u} \right)$$

$$\boxed{\frac{Ds}{Dt} = \frac{1}{\rho} \left[E_{ij} \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) \right]}$$

The production of entropy has two origins: thermal diffusion and viscous dissipation, which are both irreversible processes.

Problem 5: (Long but detailed derivation by Sangwook Park)

$$\bar{\omega} \equiv \nabla \times \bar{V}$$

momentum equation.

$$\rho \frac{D\bar{u}}{Dt} = \rho \cdot \bar{f}_b + \nabla \cdot \bar{\tau}$$

$$(\bar{\tau} = \mu (\nabla \bar{u} + (\nabla \bar{u})^T) + \lambda \bar{I} (\nabla \cdot \bar{u})) \quad \leftarrow -p\bar{I}$$

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u}$$

$$= \frac{\partial \bar{u}}{\partial t} + \nabla \left(\frac{\bar{u} \cdot \bar{u}}{2} \right) + \bar{\omega} \times \bar{u}$$

$$\ast \Rightarrow \nabla \left(\frac{\bar{u} \cdot \bar{u}}{2} \right)_i + (\bar{\omega} \times \bar{u})_i$$

$$= \frac{1}{2} \frac{\partial}{\partial x_i} (u_j^2) + \left[\left(\frac{\partial u_q}{\partial x_p} \epsilon_{pqr} \right)_r u_j \epsilon_{rja} \right]_i$$

$$= \frac{\partial u_q}{\partial x_p} \cdot u_j \cdot \epsilon_{pqr} \cdot \epsilon_{rja}$$

$$\stackrel{(p=j \text{ or } i)}{=} \frac{\partial u_q}{\partial x_j} u_j (\delta_{qj} - \delta_{ji} \cdot \delta_{qj}) + \frac{\partial u_q}{\partial x_i} u_j (\delta_{ij} \delta_{qj} - \delta_{qj})$$

$$\stackrel{(q=i \text{ or } j)}{=} \frac{\partial u_i}{\partial x_j} u_j (1 - \delta_{ij}) + \frac{\partial u_j}{\partial x_j} u_j (\delta_{ji} - \delta_{ji})$$

$$+ \frac{\partial u_i}{\partial x_i} u_i (\delta_{ij} - \delta_{ij}) + \frac{\partial u_i}{\partial x_i} u_j (\delta_{ij} \delta_{ji} - 1)$$

$$= \frac{\partial u_i}{\partial x_j} u_j + \underbrace{\left(\frac{\partial u_i}{\partial x_j} u_j \right) \delta_{ij} + \left(\frac{\partial u_j}{\partial x_i} u_j \right) \delta_{ij}}_{\leftarrow 0} - \frac{\partial u_j}{\partial x_i} u_j$$

$$= \frac{\partial u_i}{\partial x_j} u_j - \frac{\partial u_j}{\partial x_i} u_j$$

$$= u_j \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \cdot u_j - \frac{\partial u_j}{\partial x_i} u_j = \frac{\partial u_i}{\partial x_j} \cdot u_j = \bar{u} \cdot \nabla \bar{u}$$

$$\therefore \frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \nabla \left(\frac{\bar{u} \cdot \bar{u}}{2} \right) + \bar{\omega} \times \bar{u}$$

∴ momentum equation

$$\Rightarrow \rho \left(\frac{\partial \bar{u}}{\partial t} + \nabla \left(\frac{\bar{u} \cdot \bar{u}}{2} \right) + \bar{\omega} \times \bar{u} \right) = -\bar{f}_b + \frac{1}{\rho} \nabla \cdot \bar{\tau}$$

Take curl

$$\rightarrow \left(\underbrace{\nabla \times \frac{\partial \bar{u}}{\partial t}}_{\frac{\partial}{\partial t} (\nabla \times \bar{u})} + \underbrace{\nabla \times \nabla \left(\frac{\bar{u} \cdot \bar{u}}{2} \right)}_{\text{curl (grad } f) = 0} + \underbrace{\nabla \times (\bar{\omega} \times \bar{u})}_{\textcircled{1}} \right) = \underbrace{-\nabla \times \bar{f}_b}_{\textcircled{2}} + \underbrace{\nabla \times \frac{1}{\rho} \nabla \cdot \bar{\tau}}_{\textcircled{3}}$$

$$\downarrow$$

$$\left(\nabla \frac{1}{\rho} \right) \times \nabla \cdot \bar{\tau} + \frac{1}{\rho} (\nabla \times \nabla \cdot \bar{\tau}) \quad \textcircled{4}$$

$$\textcircled{1} \nabla \times (\bar{\omega} \times \bar{u})_i$$

$$= \frac{\partial}{\partial x_j} (\omega_p \cdot u_q \epsilon_{pqr}) \epsilon_{jri}$$

$$= \frac{\partial}{\partial x_j} (\omega_p \cdot u_q) \cdot \epsilon_{pqr} \cdot \epsilon_{ijr}$$

$$= \left(u_q \frac{\partial}{\partial x_j} \omega_p + \omega_p \frac{\partial}{\partial x_j} u_q \right) (\delta_{pi} \delta_{qj} - \delta_{pj} \delta_{qi})$$

$$= u_j \frac{\partial}{\partial x_j} \omega_i + \omega_i \frac{\partial}{\partial x_j} u_j - u_i \frac{\partial}{\partial x_j} \omega_j - \omega_j \frac{\partial}{\partial x_j} u_i$$

$$= (\bar{u} \cdot \nabla) \bar{\omega} + (\nabla \cdot \bar{u}) \bar{\omega} - (\nabla \cdot \bar{\omega}) \bar{u} - (\bar{\omega} \cdot \nabla) \bar{u}$$

② if. conservative force

$$\Rightarrow \bar{F}_b = \nabla G$$

$$\therefore \nabla \times (\nabla G) = 0$$

$$\textcircled{2} \nabla \times \nabla \cdot \left(\mu (\nabla \bar{u} + (\nabla \bar{u})^T) \right) + \cancel{\nabla \cdot (\nabla \cdot \bar{u})} = \cancel{\bar{f}_b}$$

\downarrow scalar \downarrow scalar \downarrow scalar
 $\left(\because \text{curl (grad } G) = 0 \right)$

$$= \left(\begin{aligned} &= \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \\ &= \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right) = \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) \right) \\ &= \mu \left(\nabla^2 \bar{u} + \nabla \left(\nabla \cdot \bar{u} \right) \right) \quad \left(\because \text{curl (grad (scalar))} = 0 \right) \end{aligned} \right)$$

\downarrow scalar

$$\textcircled{3} \quad \therefore \frac{1}{\rho} (\nabla \times \nabla \cdot \vec{c}) = \frac{1}{\rho} \nabla \times (\mu \nabla^2 \vec{u})$$

$$= \frac{\mu}{\rho} \nabla^2 (\nabla \times \vec{u}) = \nu \nabla^2 \vec{\omega}$$

$$\therefore (*) : = \underbrace{\left(\frac{D}{Dt} \vec{\omega} + (\vec{u} \cdot \nabla) \vec{\omega} + (\nabla \cdot \vec{u}) \vec{\omega} - (\nabla \cdot \vec{\omega}) \vec{u} - (\vec{\omega} \cdot \nabla) \vec{u} \right)}_{\rightarrow \frac{D\vec{\omega}}{Dt}}$$

$$= \nabla \times \vec{f}_b + \nu \nabla^2 \vec{\omega} + \textcircled{4}$$

$$\therefore \frac{D}{Dt} \vec{\omega} + (\nabla \cdot \vec{u}) \vec{\omega} = (\nabla \cdot \vec{\omega}) \vec{u} + (\vec{\omega} \cdot \nabla) \vec{u} + \nabla \times \vec{f}_b + \nu \nabla^2 \vec{\omega} + \textcircled{4}$$

$$\downarrow$$

$$\nabla \cdot (\nabla \times \vec{u})$$

$$= \frac{\partial}{\partial x_3} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \frac{\partial}{\partial x_1} \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right)$$

$$= 0$$

$$\Rightarrow \frac{D}{Dt} \vec{\omega} + (\nabla \cdot \vec{u}) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} + \nabla \times \vec{f}_b + \nu \nabla^2 \vec{\omega} + \textcircled{4}$$

if f_b is conservative force

$$\Rightarrow \frac{D}{Dt} \vec{\omega} + (\nabla \cdot \vec{u}) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} + \nu \nabla^2 \vec{\omega} + \textcircled{4}$$

if ρ is constant, this term will be zero because of continuity equation. $(\nabla \cdot \vec{u} = 0)$

vortex tilting and stretching term

diffusion term (diffusion of the vortex)

$$\rightarrow \omega_j \frac{\partial u_i}{\partial x_j} = \underbrace{\omega_i \frac{\partial u_i}{\partial x_i}}_{\text{vortex stretching}} + \underbrace{\omega_j \frac{\partial u_i}{\partial x_j}}_{\text{vortex tilting}} \Big|_{j \neq i}$$

$$\begin{aligned}
 \textcircled{4} \quad \nabla \frac{1}{\rho} \times \nabla \cdot \bar{\epsilon} &= \frac{1}{\rho} \nabla \rho \times \nabla (\nabla \cdot \bar{u}) \\
 &= \frac{1}{\rho^2} \left(\nabla \rho \times \nabla^2 \bar{u} + \nabla \rho \times \nabla (\nabla \cdot \bar{u}) \right) = \frac{\lambda}{\rho^2} \nabla \rho \times \nabla (\nabla \cdot \bar{u}) - \frac{1}{\rho^2} \nabla \rho \times \nabla p
 \end{aligned}$$

$$\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \nabla) \bar{u} + \nu \nabla^2 \bar{\omega} - (\nabla \cdot \bar{u}) \bar{\omega} - \frac{1}{\rho^2} \left[\nu (\nabla \rho \times \nabla^2 \bar{u} + \nabla \rho \times \nabla (\nabla \cdot \bar{u})) - \lambda \nabla \rho \times \nabla (\nabla \cdot \bar{u}) - \nabla \rho \times \nabla p \right]$$

vortex tilting & stretching diffusion if $\rho = \text{const.}$ $\nabla \cdot \bar{u} = 0$ misalignment of $\nabla \rho$ and ∇p generate $\bar{\omega}$

In summary, the vorticity equation can be written:

$$\frac{D\bar{\omega}}{Dt} = \bar{\omega} \cdot \nabla \bar{u} - \bar{\omega} \nabla \cdot \bar{u} + \nabla \times \bar{f} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \times \left(\frac{\nabla \cdot \bar{\epsilon}}{\rho} \right)$$

The viscous term $\nabla \times (\nabla \cdot \bar{\epsilon} / \rho)$ can be expanded as done above. The different terms in the equation have the following interpretation:

- $\bar{\omega} \cdot \nabla \bar{u} - \bar{\omega} \nabla \cdot \bar{u}$: this term, which reduces to $\bar{\omega} \cdot \nabla \bar{u}$ in an incompressible flow, expresses the elongation and twisting of the vortex lines/tubes by the velocity field.
- $\nabla \times \bar{f}$: \bar{f} , the body force per unit mass, can create vorticity. However if \bar{f} is conservative, i.e. if there is a potential ϕ such that $\bar{f} = -\nabla \phi$, this term is identically zero.
- $(1/\rho^2) \nabla \rho \times \nabla p$: this term, called baroclinic production term, only occurs when the two vectors $\nabla \rho$ and ∇p are not parallel, i.e. when the isobars (surfaces of constant pressure) do not coincide with the isochores (surfaces of constant density). When it is possible to conclude that $\rho = \rho(p)$ is valid for the flow field of interest (such flow fields are called barotropic; e.g. isothermal or uniform entropy flow fields in which the same equation of state holds everywhere), then this term is zero.
- $\nabla \times (\nabla \cdot \bar{\epsilon} / \rho)$: this is a viscous term, which is zero in an inviscid fluid. In the incompressible case it becomes $\nu \nabla^2 \bar{\omega}$, and expresses the diffusion of vorticity by the viscosity of the fluid.

In an incompressible, barotropic, inviscid flow with conservative body forces, the vorticity equation becomes:

$$\frac{D\bar{\omega}}{Dt} = \bar{\omega} \cdot \nabla \bar{u}$$

which shows that if at an initial time the vorticity is zero everywhere, then it remains so at later times (there is no production of vorticity in such fluids). This says that an irrotational flow remains irrotational, and is sometimes called Lagrange's theorem. It justifies the use of the irrotational assumption in potential flow.

Problem 6: (Beverly Tang)

If the person wants $u(y=0)$ to depend on $\frac{\partial u}{\partial y}$, He/she is now adding a dimensional variable $\left(\frac{\text{length}}{\text{length} \cdot \text{time}}\right) = \left(\frac{1}{\text{time}}\right)$ to the equation. In order to non-dimensionalize, multiply by a characteristic length & divide by a characteristic velocity (mean free path & speed of sound, for example).

$$\Pi_1 = f(\Pi_2)$$

$$\frac{u_{\text{wall}}}{c} = f\left(\frac{\partial u}{\partial y} \cdot \frac{\text{MFP}}{c}\right)$$

$\alpha_2: \frac{u_{\text{wall}}}{\lambda \left. \frac{\partial u}{\partial y} \right|_w} = \text{constant}$
 mfp. \nearrow which is sometimes used as a slip BC (see White pp. 47)

We wish to write a relation between the tangential velocity at the wall u_w and its derivative $(\partial u / \partial y)_w$, which we can write symbolically as:

$$F\left(u_w, \left. \frac{\partial u}{\partial y} \right|_w\right) = 0$$

Dimensional analysis tells us that such a relation must in fact be written in terms of dimensionless parameters constructed from our variables: $\tilde{F}(\Pi_1, \Pi_2, \dots, \Pi_n) = 0$ where n is obtained by looking at the rank of the matrix of dimensions. However it is clear, by looking at the dimensions of u_w and $(\partial u / \partial y)_w$ that no dimensionless parameter can be constructed from these two variables only: we need a characteristic length scale! Since the macroscopic geometric length scales should not influence what happens very near to the wall, we must seek an intrinsic length scale related to the fluid. In a gas a reasonable candidate is the mean free path λ . Using λ as our additional variable we can construct one independent dimensionless parameter ($n = 1$):

$$\Pi_1 = \frac{u_w}{\lambda \left. \frac{\partial u}{\partial y} \right|_w}$$

Our relation becomes $\tilde{F}(\Pi_1) = 0$, which can be alternately written $\Pi_1 = \text{constant}$, or:

$$u_w = \text{constant} \times \lambda \left. \frac{\partial u}{\partial y} \right|_w$$

This relation, which is a good guess but can only be validated by experiments or a more sophisticated theory, is indeed used as a boundary condition for very dilute gases with $\text{constant} \simeq 1$ (see White pp. 47).

Problem 7: (Ethan Barbour)

White defines "energy" as $e + \frac{1}{2} V^2 - \bar{g} \cdot \bar{r}$.

However, this definition leads to a quantity which is balanced by work & heat (i.e. obeys 1st law) only if the rate of work done by the body force is equal to the rate of change of this "potential energy"

Since \bar{g} does not depend on \bar{r} , and $\frac{\partial r_1}{\partial x_1} = \frac{\partial r_2}{\partial x_2} = \frac{\partial r_3}{\partial x_3} = 1$

$$\begin{aligned} \bar{u} \cdot \bar{g} &\stackrel{?}{=} \frac{D}{Dt} (\bar{g} \cdot \bar{r}) \\ (\bar{u} \cdot \nabla) (\bar{g} \cdot \bar{r}) &\stackrel{?}{=} \frac{\partial}{\partial t} (\bar{g} \cdot \bar{r}) + (\bar{u} \cdot \nabla) (\bar{g} \cdot \bar{r}) \\ &\stackrel{?}{=} \frac{\partial}{\partial t} (gz) + (\bar{u} \cdot \nabla) (\bar{g} \cdot \bar{r}) \quad \text{since the ship travels in a straight line} \\ &\stackrel{?}{=} z \frac{\partial g}{\partial t} + (\bar{u} \cdot \nabla) (\bar{g} \cdot \bar{r}) \quad \text{since } z \text{ \& } t \text{ are independent} \end{aligned}$$

∴ if g is only a function of t and the ship travels in a straight line, White's formulation requires an additional term $z \frac{\partial g}{\partial t}$

The aim of this question was to recognize that when the body force depends on time, using a potential energy $E_p = -\bar{g}(t) \cdot \bar{r}$ is not equivalent to including the work done by the body force $W = \bar{g}(t) \cdot \bar{u}$ in the energy equation. Taking the substantial derivative of E_p will recover W , but will introduce additional terms involving $\partial g / \partial t$ which should not be there. In conclusion, beware of potential energy when the body force depends on time: the first law of thermodynamics, which is the underlying principle here, makes no mention of potential energy.